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CHAPTER01

MLE

Consider the following very simple model for stock pricing. The price at the end of each day is the price of the previous day multiplied by a fixed, but unknown, rate of return, a, with some noise, w. For a two-day period, we can observe the following sequence:

$y_2 = ay_1 + w_1$

$y_1 = ay_0 + w_0$

where the noises w_0, w_1 are iid with the distribution $N(0, \sigma^2)$, $y_0 \sim N(0, \lambda)$ is independent of the noise sequence. σ^2 and λ are known, while a is unknown.

1. Find the MLE of the rate of return, a, given the observed price at the end of each day y_2, y_1, y_0. In other words, compute for the value of a that maximizes $p(y_2, y_1, y_0 \mid a)$.

Hint: This is a Markov process, e.g. y_2 is independent of y_0 given y_1. In general, a process is Markov if $p(y_n \mid y_{n-1}, y_{n-2}, \ldots) = p(y_n \mid y_{n-1})$.

$$p(y_2, y_1, y_0 \mid a) = p(y_2 \mid y_1 ; a) \times p(y_1 \mid y_0 ; a) \times p(y_0)$$

$p(y_2, y_1, y_0 \mid a)$ or $L(a \mid y_2, y_1, y_0)$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{(y_2 - ay_1)^2}{2\sigma^2}\right)} \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{(y_1 - ay_0)^2}{2\sigma^2}\right)} \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{y_0^2}{2\lambda}\right)}$$

take ln

$$\ln(p(y_2, y_1, y_0 \mid a)) = \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{(y_2 - ay_1)^2}{2\sigma^2}\right)}\right) + \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{(y_1 - ay_0)^2}{2\sigma^2}\right)}\right) + \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{y_0^2}{2\lambda}\right)}\right)$$

$$= \left[\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) + \ln\left(e^{\left(-\frac{(y_2 - ay_1)^2}{2\sigma^2}\right)}\right)\right] + \left[\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) + \ln\left(e^{\left(-\frac{(y_1 - ay_0)^2}{2\sigma^2}\right)}\right)\right]$$

$$+ \left[\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) + \ln\left(e^{\left(-\frac{y_0^2}{2\lambda}\right)}\right)\right]$$

$$= \left[\ln(2\pi\sigma^2)^{-1/2} - \frac{(y_2 - ay_1)^2}{2\sigma^2} \overset{1}{\cancel{\ln(e)}}\right] + \left[\ln(2\pi\sigma^2)^{-1/2} - \frac{(y_1 - ay_0)^2}{2\sigma^2} \overset{1}{\cancel{\ln(e)}}\right]$$

$$+ \left[\ln(2\pi\sigma^2)^{-1/2} - \frac{y_0^2}{2\lambda} \overset{1}{\cancel{\ln(e)}}\right]$$

$$= \left[-\frac{1}{2}\left(\ln(2\pi) + \ln(\sigma^2)\right) - \frac{(y_2 - ay_1)^2}{2\sigma^2}\right] + \left[-\frac{1}{2}\left(\ln(2\pi) + \ln(\sigma^2)\right) - \frac{(y_1 - ay_0)^2}{2\sigma^2}\right]$$

$$\left[-\frac{1}{2}\left(\ln(2\pi) + \ln(\sigma^2)\right) - \frac{y_0^2}{2\lambda}\right]$$

$$= \left[-\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma^2) \quad - \frac{(y_2 - ay_1)^2}{2\sigma^2} \right] + \left[-\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma^2) \quad - \frac{(y_1 - ay_0)^2}{2\sigma^2} \right]$$

$$\left[-\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma^2) \quad - \frac{y_0^2}{2\lambda} \right]$$

$$= -\frac{1}{2}\ln(2\pi) - \ln(\sigma) - \frac{(y_2 - ay_1)^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi) - \ln(\sigma) - \frac{(y_1 - ay_0)^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi) - \ln(\sigma) - \frac{y_0^2}{2\lambda}$$

$$\ln(P(y_2, y_1, y_0 | a)) = -\frac{3}{2}\ln(2\pi) - 3\ln(\sigma) - \frac{(y_2 - ay_1)^2}{2\sigma^2} - \frac{(y_1 - ay_0)^2}{2\sigma^2} - \frac{y_0^2}{2\lambda}$$

find derivative

chain rule

$$\frac{\partial}{\partial a}\ln\left(P(y_2, y_1, y_0 | a)\right) = 0 - 0 - \frac{1}{\cancel{2}\sigma^2}\cdot\cancel{2}(y_2 - ay_1)(0 - y_1) - \frac{1}{\cancel{2}\sigma^2}\cdot\cancel{2}(y_1 - ay_0)(0 - y_0)$$

$$= \frac{1}{\sigma^2}(y_2 - ay_1)(y_1) + \frac{1}{\sigma^2}(y_1 - ay_0)(y_0)$$

$$= \frac{1}{\sigma^2}(y_2y_1 - ay_1^2 + y_1y_0 - ay_0^2)$$

$$= \frac{1}{\sigma^2}(y_2y_1 + y_1y_0 - a(y_1^2 + y_0^2))$$

แทน $\frac{\partial}{\partial a} = 0$ เพื่อหา MLE ของ a เพราะ ที่ slope = 0 คือจุดสูงสุดของ curve

$$0 = \frac{1}{\sigma^2}(y_2y_1 + y_1y_0 - a(y_1^2 + y_0^2))$$

$$0 = y_2y_1 + y_1y_0 - a(y_1^2 + y_0^2)$$

$$a = \frac{y_2y_1 + y_1y_0}{y_1^2 + y_0^2}$$

Ans