Penney's Game*

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Abstract

In this paper, we analyze Penney's game using formal game theory. Initially, we present a simultaneous game and verify a specific MNE that aligns with a pure probability-level analysis. Subsequently, we construct a reputation model incorporating incomplete information, which considers players' inclination to continue playing when there's a potential for dealer cheating. Finally, we extend the model to analyze how strategic behavior among players changes in response to a specific betting system. Our major finding underscores the significant advantage a cheating dealer holds over a rational player, emphasizing the pivotal role of initial beliefs and risk aversion levels in the game.

Keywords: Penney's game, Simultaneous game, Reputation model.

1. Introduction

In the gambling realm, there exists a class of fascinating games distinguished by their simplicity yet profound mathematical underpinnings, where dice, coins, and cards dance. Among these, Penney's game stands out as a classic example. In a simple coin-flipping game involving one dealer and two players, each player submits a string of length 3 to the dealer before the coin-flips begin. The outcome of the coin flip sequence determines the winner: the player whose predicted sequence appears first in the sequence of coin flips wins the bet. In this game, $Head$ is represented as U and $Tail$ as D.

*We first saw this game in manga, *Kakegurui Twin*.

Here is a simple example. Player A selects the string UUU and Player B selects the string UDU. Then the dealer flips the coin several times and yields the following sequence:

U U D D U **U D U**

Player B's string appears, signaling the immediate end of the game, with Player B declared the winner. What is interesting in this game is that if the dealer only flips the coin 3 times, there are 8 results with the same probability.

Table 1: Probability Flipping 3 times

UUU	UDU	UUD	UDD	DUU	DDU	DUD	DDD
1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8

However, the dealer likely needs to keep flipping more than 3 times until one of the two provided strings appears. There will be some sequence problems and the gamble now is an unequal chance game. As an example, Player A chose UDD and Player B chose UUD, with the revealed sequence being U D U U. If Player A wants to win, the next two coin flips must result in D. However, Player B will secure victory first if the upcoming flip results in D. In other words, there will be a beating relationship (Figure 1) among the 8 strings where $UDD \rightarrow DDU$ means UDD "beats" DDU.

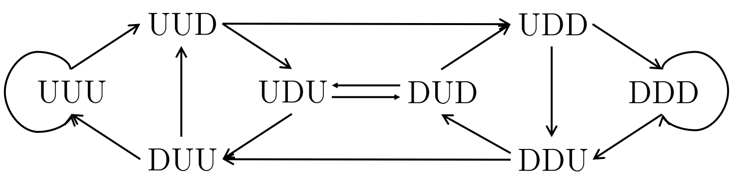


Figure 1: "Beating" Relationships

2. Winning Rates and Expectations

In this section, we approach the game from a purely statistical standpoint, seeking to identify the string that offers the highest probability of winning. To achieve this, we calculate the odds of winning and the expected number of coin flips for each string.

2.1 Probability of Winning

To calculate the winning rate in the pairs, we employ Conway numbers and Conway's algorithm. This involves mathematical principles outlined in the work of Phillips and

Hildebrand (Phillips & Hildebrand, 2021). Here, we demonstrate how to calculate the odds using a simple case and provide the formula and results directly.

Let $A = a_1 a_2 \cdots a_n$ and $B = b_1 b_2 \cdots b_n$ be strings of length n.When the probability of both "U" and "D" appearing is 0.5, the Conway number of A and B is defined by

$$C(A, B) = \sum_{i=1}^{n} \sigma_i 2^{n-i},$$

where

$$\sigma_i = \begin{cases} 1 & \text{if } a_{i+j} = b_{1+j} \text{ for } j = 0, \cdots, n-i, \\ 0 & \text{otherwise.} \end{cases}$$

We assume that Player A selects UDD and Player B selects DDD.

When $i = 1$,

$$\begin{array}{ccc|c} \mathrm{U} & \mathrm{D} & \mathrm{D} & \mathrm{A} \\ \mathrm{D} & \mathrm{D} & \mathrm{D} & \mathrm{B} \\ \hline 0 & & & \mathrm{C(A,B)} \end{array}$$

When $i = 2$,

$$\begin{array}{cccc|c} \mathrm{U} & \mathrm{D} & \mathrm{D} & & \mathrm{A} \\ & \mathrm{D} & \mathrm{D} & \mathrm{D} & \mathrm{B} \\ \hline 0 & 1 & & & \mathrm{C(A,B)} \end{array}$$

When $i = 3$,

$$\begin{array}{ccccc|c} \mathrm{U} & \mathrm{D} & \mathrm{D} & & & \mathrm{A} \\ & & \mathrm{D} & \mathrm{D} & \mathrm{D} & \mathrm{B} \\ \hline 0 & 1 & 1 & & & \mathrm{C(A,B)} \end{array}$$

Thus, $C(A, B) = 0 * 2^2 + 1 * 2^1 + 1 * 2^0 = 3$[1]

Then the odds, which are the formula of Conway's algorithm, for Player A are

$$\frac{Pr(A \text{ appears before } B)}{Pr(B \text{ appears before } A)} = \frac{C(B, B) - C(B, A)}{C(A, A) - C(A, B)}$$

In this case, Player A's odds are $(7 - 0)/(4 - 3) = 7 : 1$, which means that if Player A has selected UDD and Player B has selected DDD, the winning rate for Player A is 7/8.

[1] Similarly, $C(B, B) = 1 * 2^2 + 1 * 2^1 + 1 * 2^0 = 7$
$C(B, A) = 0 * 2^2 + 0 * 2^1 + 0 * 2^0 = 0$
$C(A, A) = 1 * 2^2 + 0 * 2^1 + 0 * 2^0 = 4$.

By applying this method, we provide the table of probabilities of winning in the game as follows:

Table 2: Player A's Probability of Winning against Player B

A \ B	DDD	DDU	UDD	DUD	UDU	DUU	UUD	UUU
DDD	$*$	1/2	1/8	2/5	5/12	2/5	3/10	1/2
DDU	1/2	$*$	1/4	2/3	5/8	2/3	1/2	7/10
UDD	7/8	3/4	$*$	1/2	1/2	1/2	1/3	3/5
DUD	3/5	1/3	1/2	$*$	1/2	1/2	3/8	7/12
UDU	7/12	3/8	1/2	1/2	$*$	1/2	1/3	3/5
DUU	3/5	1/3	1/2	1/2	1/2	$*$	3/4	7/8
UUD	7/10	1/2	2/3	5/8	2/3	1/4	$*$	1/2
UUU	1/2	3/10	2/5	5/12	2/5	1/8	1/2	$*$

2.2 Expected Number of Coin Flips

In this game, the expected number of coin flips is 2 times the corresponding Conway number, which has been proved by Agarwal's team (Agarwal et al., 2020).

Table 3: Expected Number of Flips

DDD	DDU	UDD	DUD	UDU	DUU	UUD	UUU
14	8	8	10	10	8	8	14

From Table 3, a rational player who plays alone should consistently select among DDU, UDD, DUU, and UUD for the shortest expected wait time. Notably, UDD and DUU form a symmetric pair resembling the format of ABB, while UUD and DDU constitute a symmetric AAB pair. Despite both pairs having equal expected wait times, their winning rates differ when players face off against each other. As shown in Table 2, assuming not knowing the opponent's choice, ABB exhibits the highest average winning rate. To further evaluate the desirability of each pair, we conducted 100,000 simulations in Python to determine $Pr(\text{string } A \text{ wins})$ and $\mathbb{E}(\text{flip times for } A|A \text{ wins})$.

Table 4: Simulation Results

	DDD	DDU	UDD	DUD	UDU	DUU	UUD	UUU
$Pr(\cdot)$	0.377	0.557	0.581	0.486	0.485	0.580	0.558	0.376
$\mathbb{E}(\cdot)$	6.04	5.77	6.23	5.85	5.86	6.21	5.77	6.03

Based on Table 4, it's apparent that for strings other than UDD and DUU, the expected flips should be within approximately 6 flips for a guaranteed victory, regardless of the true expected number of flips. However, for UDD and DUU, we allow for more flips to win, indicating their dominant roles over others. Also notably, the ABB demonstrates a superior chance of winning compared to the AAB. As submissions occur simultaneously, a player is expected to begin with either UDD or DUU.

3. Benchmark: Zero-sum Simultaneous Game

3.1 Setup

In this benchmark game, two players are engaged, each fully aware of the game's mechanics. Both players bet 100, simultaneously submit a string of length three as their guess, and await the outcome. To eliminate the possibility of a draw, an adjustment has been implemented: when both players' guesses coincide, each player has a 50% chance of winning the game.

Based on the table of winning probabilities (Table 2), an expected payoff table is provided:

Table 5: Payoff

A \ B	DDD	DDU	UDD	DUD	UDU	DUU	UUD	UUU
DDD	(0,0)	(0,0)	(-75,75)	(-20,20)	$(-\frac{50}{3},\frac{50}{3})$	(-20,20)	(-40,40)	(0,0)
DDU	(0,0)	(0,0)	(-50,50)	$(\frac{100}{3},-\frac{100}{3})$	(25,-25)	$(\frac{100}{3},-\frac{100}{3})$	(0,0)	(40,-40)
UDD	(75,-75)	(50,-50)	(0,0)	(0,0)	(0,0)	(0,0)	$(-\frac{100}{3},\frac{100}{3})$	(20,-20)
DUD	(20,-20)	$(-\frac{100}{3},\frac{100}{3})$	(0,0)	(0,0)	(0,0)	(0,0)	(-25,25)	$(\frac{50}{3},-\frac{50}{3})$
UDU	$(\frac{50}{3},-\frac{50}{3})$	(-25,25)	(0,0)	(0,0)	(0,0)	(0,0)	$(-\frac{100}{3},\frac{100}{3})$	(20,-20)
DUU	(20,-20)	$(-\frac{100}{3},\frac{100}{3})$	(0,0)	(0,0)	(0,0)	(0,0)	(50,-50)	(75,-75)
UUD	(40,-40)	(0,0)	$(\frac{100}{3},-\frac{100}{3})$	(25,-25)	$(\frac{100}{3},-\frac{100}{3})$	(-50,50)	(0,0)	(0,0)
UUU	(0,0)	(-40,40)	(-20,20)	$(-\frac{50}{3},\frac{50}{3})$	(-20,20)	(-75,75)	(0,0)	(0,0)

It's important to notice that even when players select strategies with seemingly advantageous expected outcomes, the actual payoff in each scenario isn't guaranteed. This uncertainty stems from the uncertainty of realization.[2]. However, without further information, it is reasonable for players to choose a strategy based on the expected payoff.

3.2 One Possible Mixed Strategy Nash Equilibrium

1. Noticing DDD and UUU are strictly dominated by other strategies, we eliminate them at first.

[2]For example, the expected payoff for DDU against DUU is $2/3 * 100 + 1/3 * (-100) = 100/3$.

2. Noticing there is no Pure Strategy Nash Equilibrium, we turn to solve for Mixed Strategy Equilibria.

3. Noticing this game is symmetric, strictly competitive, and zero-sum, we apply the minmax strategy (Von Neumann, 1928) to find a classic MNE which may not be unique. We expect 0 payoff for each player.

Proposition 3.1. *There is a Symmetric Mixed Strategy Nash Equilibrium in which both players play DUU with probability p ($p \in (0.4, 0.6)$), play UDD with probability $1-p$, and play other strategies with probability 0.*

Proof 3.1. We assume that Player 1 deviates to play DDU, UDD, DUD, UDU, DUU, and UUD with strategy $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\}$ other than $\{0, 1-p, 0, 0, p, 0\}$ against $\{0, 1-p, 0, 0, p, 0\}$ for Player 2, his expected payoff is thus given by:

$$\begin{aligned} U(deviation) =& \delta_1 p * \frac{100}{3} + \delta_1(1-p)(-50) + \delta_6 p(-50) + \delta_6(1-p) * \frac{100}{3} \\ &+ \sum_{i \neq 1,6} [\delta_i p * 0 + \delta_i(1-p) * 0] \\ =& \left(\frac{250}{3}p - 50\right)(\delta_1 - \delta_6) - \frac{50}{3}\delta_6 \end{aligned}$$

Table 6: Deviation Payoff

B \ A	DDU	UDD	DUD	UDU	DUU	UUD
UDD	(50,-50)	(0,0)	(0,0)	(0,0)	(0,0)	$(-\frac{100}{3}, \frac{100}{3})$
DUU	$(-\frac{100}{3}, \frac{100}{3})$	(0,0)	(0,0)	(0,0)	(0,0)	(50,-50)

1. If strategy $\{\delta_1, \delta_6\} = \{0, 0\}$ and strategy $\{\delta_2, \delta_3, \delta_4, \delta_5\} \neq \{1-p, 0, 0, p\}$,

$$U(deviation) = 0$$

   There is no profitable deviation.

2. If strategy $\{\delta_1, \delta_6\} = \{\delta, \delta\} \neq \{0, 0\}$ and strategy $\{\delta_2, \delta_3, \delta_4, \delta_5\} \neq \{1-p, 0, 0, p\}$,

$$U(deviation) = -\frac{50}{3}\delta < 0$$

   There is no profitable deviation.

3. If strategy $\{\delta_1, \delta_6\} \neq \{0, 0\}$, $\delta_1 > \delta_6 \geq 0$ and strategy $\{\delta_2, \delta_3, \delta_4, \delta_5\} \neq \{1-p, 0, 0, p\}$, $U(deviation)$ is increasing in p. When $p = 0.6$, the deviation payoff reaches the

maximum value which is given by:

$$U(deviation) = \left(\frac{250}{3} * 0.6 - 50\right)(\delta_1 - \delta_6) - \frac{50}{3}\delta_6 = -\frac{50}{3}\delta_6 \leq 0$$

There is no profitable deviation.

4. If strategy $\{\delta_1, \delta_6\} \neq \{0, 0\}$, $\delta_6 > \delta_1 \geq 0$ and strategy $\{\delta_2, \delta_3, \delta_4, \delta_5\} \neq \{1-p, 0, 0, p\}$, $U(deviation)$ is decreasing in p. When $p = 0.4$, the deviation payoff reaches the maximum value which is given by:

$$U(deviation) = \left(\frac{250}{3} * 0.4 - 50\right)(\delta_1 - \delta_6) - \frac{50}{3}\delta_6 = -\frac{50}{3}\delta_1 \leq 0$$

There is no profitable deviation.

5. There is no need to examine the deviation where $\{\delta_2, \delta_3, \delta_4, \delta_5\} = \{1 - p, 0, 0, p\}$, because necessarily follows $\{\delta_1, \delta_6\} = \{0, 0\}$

□

3.3 Conclusion

In a perfect information simultaneous Penney's game, one can anticipate a special Mixed Nash Equilibrium where both players exhibit nearly equal propensity towards selecting either UDD or DUU. This equilibrium holds practical feasibility in real gameplay scenarios. For instance, if one player possesses knowledge indicating that the average winning rate is highest when choosing either UDD or DUU, they might opt to evenly weight both choices. Consequently, this strategic move compels the opposing player to adopt a similar action, resulting in a balanced equilibrium between the two strategies.

4. Reputation Model

4.1 Setup

So far, We have already provided a possible solution to Penney's game with two rational players who simultaneously propose a string of flips outcomes of length three. However, in real life, it is not always the case. On the one hand, in practice, not all players know the trick of this game because of the lack of ability to solve this game with a firm understanding of advanced probability theory. On the other hand, given that this game in reality is always played in a casino, with incomplete information on the types of opponents and the introduction of a third party in this game—a Dealer, there is a probability for one player to believe that cheating may do prevail in this game. To make Penney's game more intuitive and complete, we apply a classic Bayesian Game—a model of reputation, to refine Penney's game.

Let's set up the game where two players decide to play this simultaneous proposing game. Each player writes down his guess individually and hands it to the dealer before the process of flipping starts. After receiving both messages, the Dealer first publics the guesses and then starts flipping the coin until one of the strings is realized, and then the game stops. The winner can take all the bets from the other one.

Formally, we define:

- ***Types for two players are certain.*** Player 1 is a genius in probability theory but hasn't learned game theory before. He only figures out that the expected wait time for realization of $\{UUD, DUU, UDD, DDU\}$ is indifferently 8. He simulates the game thousands of times which delivers a clear message that UDD and DUU have the highest winning rate which is around 0.58. Since he has never learned game theory before, he doesn't know the following: First, he doesn't notice that it is a game of two. Even though one player picks the string with the highest average winning rate, as long as his opponent picks the 'upstream' string, he still faces a loss with a high probability. Second, he doesn't know how to mix strategies, he will always choose one pure strategy—either UDD or DUU when he plays. Given the above argument, it is reasonable to set up a certain type for Player 1 who always proposes UDD when playing.

  Player 2 is another genius, he knows everything about this game and performs excellently in the game theory course. Additionally, he knows the type of Player 1 and his UDD strategy.

- ***Types for the 'friendship' between Player 1 and the Dealer are chosen by nature.*** The third party—the Dealer in this game introduces uncertainty which makes the game more interesting. Since it is common to observe cheating, especially collusion between one player and the dealer, in real-life gambling, we would like to internalize this possibility in our model. The type of 'friendship' between Player 1 and the Dealer is discretely drawn by nature from the set $\Theta = \{\theta_1, \theta_2\}$, where θ_1 represents the fact of 'collusion', and θ_2 represents a fair game. The joint probability over types $p(\theta_1, \theta_2)$ follows binomial law: θ_1 with probability $q \in (0, 1)$ and θ_2 with probability $1 - q \in (0, 1)$. Here, the Dealer is a potential third player in this game who actually replaces the position of Player 1 when cheating happens.

- ***The cheating method is defined as follows:*** Using different coins with uneven weights for each round to ensure a certain string is not a superior cheating method.

Despite a high requirement regarding the skill of changing coins secretly and the high risk of exposure, this cheating method will drive out the other player in the short term due to the necessary realization of the cheating string in each period which then rules out the possibility of a multi-period game.

We propose an alternative cheating method which, to some extent, guarantees the continuity of the game. The Dealer prepares all the messages in advance and shows Player 2 a certain message instead of what is written by Player 1 after observing the choices. Therefore, if collusion exists, this simultaneous game transfers to a sequential game and the second-mover advantage is ensured. Furthermore, this is a game with natural uncertainty. Even though the 'upstream' strategy is chosen, winning is not an inevitable result. But with enormous games being played in a casino, the expected payoff for cheating is stabilized, making this method plausible. Therefore, by applying this kind of cheating, the 'True Player 1' in this game becomes the Dealer.

- ***The set of strategies is defined as follows:***

  For Player 2: $S_2 = \{Play, Quit\}$.

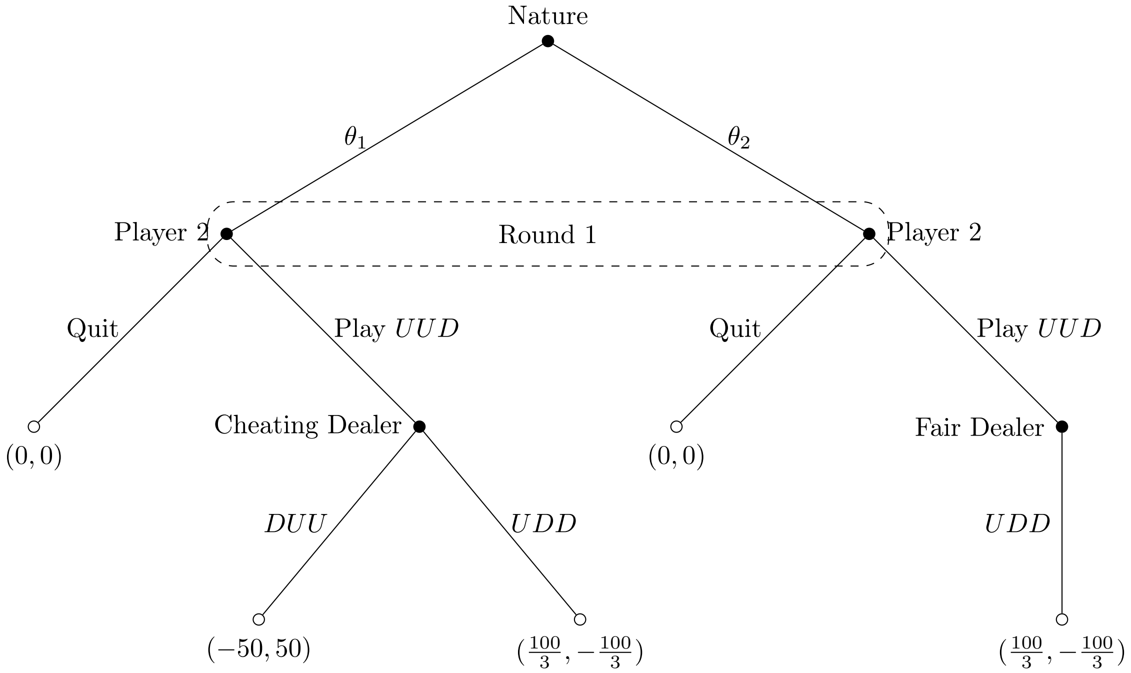
  Player 2 either chooses to play (P) the game or to quit (Q) before the start of each round. Since he knows Player 1 is always playing UDD when there is no cheating, if he decides to play, he will always play UUD (which can be verified in section 2). Here, for simplification, we rule out the action that, after deciding to play, Player 2 can mix among all other strings to offset the possible loss from dealer cheating (e.g. The Dealer later shows DUU against UUD) by picking upstream strings (e.g. Player 2 mixes between UUD and DDU against possible DUU). He just picks the pure strategy that generates the highest expected payoff. And for some thresholds, he simply quits because it is meaningless to play with a cheater. He, however, can mix between *Play* and *Quit*.

  For Player 1: $S_1 = \{UDD$ and $Keep\ Silence$ after observing published messages$\}$
  For Dealer with θ_2: $S_{d,\theta_2} = \{S_1\}$
  For Dealer with θ_1: $S_{d,\theta_1} = \{UDD, DUU\}$

  Here, Player 1 and the Dealer can play only if Player 2 plays. For the cheating type, the Dealer can either play UDD to disguise the truth or play DUU to exploit the highest profit against UUD. One notable point is that the Dealer can only mix between UDD and DUU given the strategies of Player 2. Because if the Dealer wants to disguise, it is meaningless to play strategies other than UDD. If the Dealer wants to reveal the truth and get the highest expected payoff, it is meaningless to mix DUU and other strings which generates less payoff than purely playing DUU given Player 2 only purely plays UUD if he plays.

4.2 One-period Game

The tree of this game is given as below. Depending on the values of q, the resolution of the one-period game is given as follows:



Proposition 4.1. *In the case where $q > 2/5$, there is a unique Perfect Bayesian Nash Equilibrium in which Player 2 quits; (off eq) Player 1 plays UDD and keeps silent, and the Cheating Dealer plays DUU if Player 2 plays.*

In the case where $q < 2/5$, there is a unique Perfect Bayesian Nash Equilibrium in Which Player 2 plays; Player 1 plays UDD and keeps silent, and the Cheating Dealer plays DUU.

Proof 4.1. First, we notice that there is nothing we can do with Player 1, thus can only focus on the Cheating Dealer and Player 2.

By using backward induction, if Player 2 plays, the best response for the Cheating dealer is to play DUU and get an expected payoff:

$$U_{d,\theta_1}(DUU) = 50$$

For Player2, if he decides to play, the corresponding expectation of the expected payoff is given by:

$$U_2(P) = q(-50) + (1-q)\frac{100}{3}$$

For Player2, if he decides to quit at the beginning, the corresponding expectation of the expected payoff is given by:

$$U_2(Q) = 0$$

We define the indifference condition:

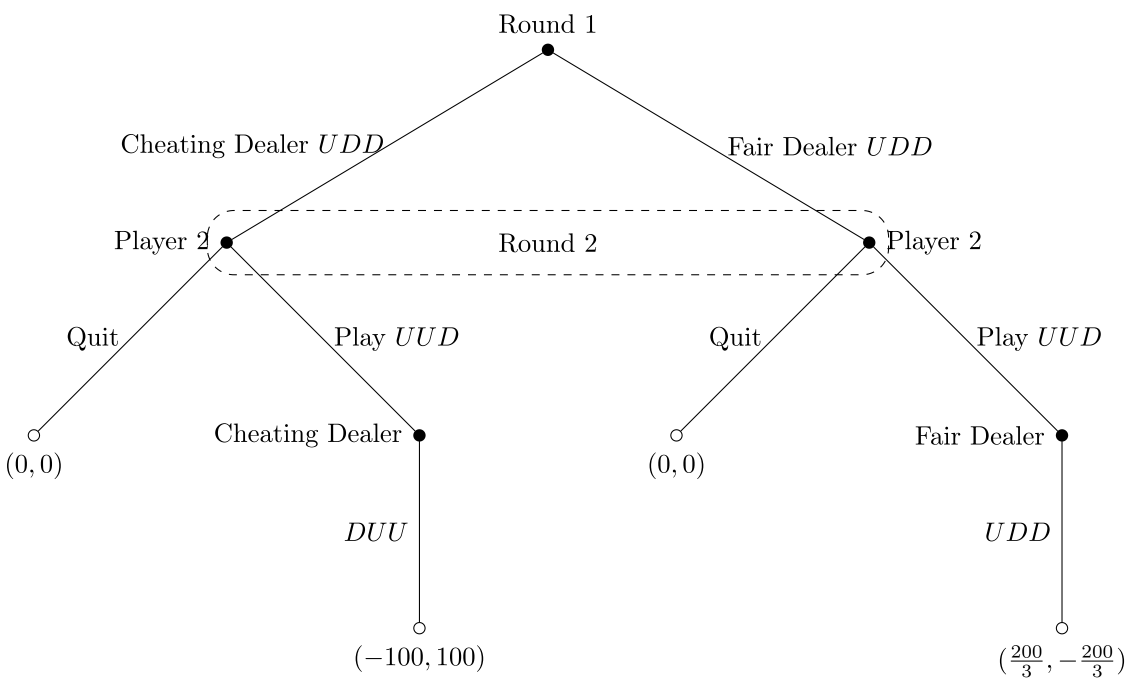
$$q(-50) + (1-q)\frac{100}{3} = 0$$

which delivers the value of $q = 2/5$. □

Conclusion 4.1. The key conclusion in a one-period game is: if $q > 2/5$, a rational Player 2 will not play. This threshold is quite low. Player 2 doesn't want to bear the risk even though cheating may not prevail because first, his expected payoff of certain winning is lower than the expected payoff of certain loss and second, he doesn't have the chance to update his belief and turn defeat into victory in the following rounds. In the following section, however, we will show that, by using a mixed strategy, a Cheating Dealer can persuade and keep Player 2 in the game even if the initial probability is higher than this threshold.

4.3 Two-period Game

We extend the one-period game to a two-period game and stipulate the following: in the second stage, the betting chips for each player should double, from 100 to 200. In this framework, the Cheating Dealer might have the incentive to disguise his type and pretend to be 'fair', aiming to keep Player 2 in the game and gain a higher payoff in the second round. For this reason, under some circumstances, both the Cheating Dealer and Player 2 will prefer to randomize and play mixed strategies in a PBE.



Proposition 4.2. *In the case where $q < 2/5$*[3]*, there is a unique Perfect Bayesian Nash Equilibrium in which Player 2 plays in both period 1 and period 2 (unless he saw DUU in period 1); Player 1 plays UDD and keeps silent in both periods, and the Cheating Dealer plays UDD in period 1 and plays DUU in period 2.*

Proof 4.2. For $q < 2/5$, we first consider an equilibrium in which the Cheating Dealer plays UDD with probability 1 in period 1 if Player 2 plays, meaning:

$$Pr(UDD|Cheating\ Dealer(CD)) = 1 \tag{1}$$

$$Pr(UDD|Fair\ Dealer(FD)) = Pr(UDD|Player\ 1) = 1 \tag{2}$$

So the belief at the start of the second stage game:

$$\begin{aligned} Pr(CD|UDD) =& \frac{Pr(UDD|CD) * Pr(CD)}{Pr(UDD|CD) * Pr(CD) + Pr(UDD|FD) * Pr(FD)} \\ =& \frac{1 * q}{1 * q + 1 * (1-q)} \\ =& q \end{aligned} \tag{3}$$

is not updated. In period 2, given $q < 2/5$, Player 2 will play and the Cheating Dealer will play DUU so the expected payoff for the Cheating Dealer is given by:

$$U_{d,\theta_1}(UDD \text{ with } Pr = 1) = -\frac{100}{3} + 100 = \frac{200}{3}$$

If the Cheating Dealer deviates to play DUU somehow in period 1, with the belief that a fair game never lets it happen, Player 2 will quit the second round, thus the expected payoff for the Cheating Dealer is given by:

$$U_{d,\theta_1}(Deviation) = 50 + 0 < \frac{200}{3}$$

Consequently, there is no profitable deviation for the Cheating Dealer. For Player 2, his actions should be consistent with his belief in the equilibrium, so given the equilibrium strategies of the Cheating Dealer, he doesn't want to deviate as well. □

Proposition 4.3. *In the case where $q > 2/5$, there is a unique Perfect Bayesian Nash Equilibrium in which*

1. *In period 1, Player 2 plays if $q < 16/25$ and quits at the beginning if $q > 16/25$. Player 1 plays UDD and keeps silent if Player 2 plays. The Cheating Dealer plays UDD with probability $2(1-q)/3q$ if Player 2 plays.*

[3] Even though in the second round bets double, the condition for stage game indifference is $q(-100) + (1-q)200/3 = 0$, which gives the same threshold $q = 2/5$.

2. *In period 2, Player 2 quits if DUU was played in period 1. If UDD was played in period 1, Player 2 mixes and plays the second round with a probability of 5/6. Player 1 plays UDD and keeps silent if Player 2 plays the second round. The Cheating Dealer plays DUU if Player 2 plays the second round.*

Proof 4.3.1. For $q > 2/5$, first of all, we rule out any pure strategy equilibrium.

1. *There cannot be a PBE where the Cheating Dealer plays UDD with probability 1 in period 1 if player 2 plays.*

   As shown in formulas from 1 to 3, if the Cheating Dealer plays UDD with probability 1 in period 1, the belief is not updated at the start of the second stage. Given $q > 2/5$ here, Player 2 will quit the second round for sure, so the expected payoff for the Cheating Dealer is given by:

$$U_{d,\theta_1}(UDD \text{ with } Pr = 1) = -\frac{100}{3} + 0 = -\frac{100}{3}$$

   If the Cheating Dealer deviates to DUU in period 1 when Player 2 Plays, he will gain:

$$U_{d,\theta_1}(Deviation) = 50 + 0 > -\frac{100}{3}$$

   Thus, we proved *1*.

2. *There cannot be a PBE where the Cheating Dealer plays DUU with probability 1 in period 1 if player 2 plays.*

   In this case, the belief of player 2 in such an equilibrium and the expected payoff for the Cheating Dealer are separately given by:

$$Pr(UDD|CD) = 0$$
$$U_{d,\theta_1}(DUU \text{ with } Pr = 1) = 50 + 0 = 50$$

   We consider a deviation in which the Cheating Dealer chooses UDD in period 1, and the belief is given by:

$$\begin{aligned} Pr(CD|UDD) &= \frac{Pr(UDD|CD) * Pr(CD)}{Pr(UDD|CD) * Pr(CD) + Pr(UDD|FD) * Pr(FD)} \\ &= \frac{0 * q}{0 * q + 1 * (1-q)} \\ &= 0 \end{aligned}$$

   This means that Player 2 now regards the Dealer as a fair one, and he will play the second round. Then the expected payoff for the Cheating Dealer is:

$$U_{d,\theta_1}(Deviation) = -\frac{100}{3} + 100 = \frac{200}{3} > 50$$

Thus, we proved 2.

$\square$

Proof 4.3.2.

1. *In period 2, equilibrium strategies are consistent with Proposition 3.2 (1).*

   The Cheating Dealer's strategy is obvious: DUU if Player 2 plays since this is the last period. If player 2 saw DUU in the first period, apparently he quits the second round in equilibrium.

   Proof for mixed strategies of Player 2 with equilibrium belief when he observes UDD is tricky. Still, we apply the Bayesian rule to solve:

$$\begin{aligned} Pr(CD|UDD) &= \frac{Pr(UDD|CD) * Pr(CD)}{Pr(UDD|CD) * Pr(CD) + Pr(UDD|FD) * Pr(FD)} \\ &= \frac{p * q}{p * q + 1 * (1 - q)} \end{aligned}$$

   where p is the probability of the Cheating Dealer playing UDD to make Player 2 indifferent between $Play$ and $Quit$ in the second round in the Equilibrium. Thus, we have:

$$\begin{aligned} \frac{p * q}{p * q + 1 * (1 - q)} &= \frac{2}{5} \\ p &= \frac{2(1 - q)}{3q} \end{aligned}$$

2. *In period 1, equilibrium strategies are consistent with Proposition 3.2 (2).*

   The mixing by Player 2 in period 2 in equilibrium should make the Cheating Dealer indifferent between choosing UDD and DUU in period 1. Thus, the condition follows:

$$\begin{aligned} 50 + 0 = &-\frac{100}{3} + 0 * Pr(Player\ 2\ quits\ second\ round|UDD) \\ &+ 100 * Pr(Player\ 2\ plays\ second\ round|UDD) \end{aligned}$$

   which then delivers:

$$Pr(Player\ 2\ plays\ second\ round|UDD) = \frac{5}{6}$$

   So far, we have proven mixed strategies used by all players in PBE. Finally, we need to find out the entry condition for Player 2 in the first period, which is given by:

$$\frac{100(1 - q)}{3} + q\left\{\frac{2(1 - q)}{3q} * \frac{100}{3} + \left[1 - \frac{2(1 - q)}{3q}\right] * (-50)\right\} > 0$$

Thus, we have a threshold value:

$$q < \frac{16}{25}$$

□

Conclusion 4.2. The key conclusion in a two-period game is:

1. If the prior belief for Player 2 of encountering cheating is smaller than 2/5, a Cheating Dealer is always inclined to mimic the behavior of the fair type to keep Player 2 in the game. Furthermore, with the Doubling Betting System, one can expect a Cheating Dealer to keep mimicking when an infinite period game is played.

   Assume now we have an n-period game, and let t represent the number of periods of mimicking the fair type[4], then the expected payoff is given by:

   $$U_{d,\theta_1}(UDD \text{ for } t \text{ period}) = 50 * 2^t - \sum_{i=1}^{t} \frac{100}{3} * 2^{i-1} = \frac{50}{3}(2^t + 2)$$

   Given this payoff is increasing in t, the Cheating Dealer prefers to keep mimicking for $t = n - 1$ periods and generously show his evil scheme in the last round to maximize the payoff. Therefore, if n goes to infinity, we can verify that the game will never stop. However, this mechanism is very constrained in real life. First, it requires the Cheating Dealer to have enormous wealth to sustain the loss that grows at an exponential rate in early rounds. Second, this mechanism can easily drive out normal players if they are professional enough to realize the trick. And even though the game itself is fair, given the zero-sum feature of this game, one risk-aversion player will unsurprisingly quit after several rounds because of the exponentially growing risk under the corresponding assumption that the win-loss outcomes of each bet are independent and identically distributed random variables (Ottaviani & Sørensen, 2010).

2. If the prior belief for Player 2 of encountering cheating is greater than 2/5, both the Cheating Dealer and Player 2 will randomize their strategy. One of the critical differences between a one-period game and a two-period game is the region of threshold. In a two-period game, Player 2 only quits at the very beginning if $q > 16/25$ instead of $q > 2/5$ in a one-period game, which implies the change of 'Cheating Tolerance' in a multi-period game. With potential double rewards in the second round, and mixed strategy played by the Dealer, Player 2 is reasonably willing to play the first round when q is relatively high and has a high probability of playing the second round if the Cheating Dealer does disguise himself in the first round. For the Cheating Dealer, he also needs to mix given the prior belief is relatively higher which may convince Player 2 not to play the game—always mimicking

[4]From $t + 1$, the Cheating Dealer will play DUU, and the game will terminate in this period.

the fair type in the first round cannot ensure entering and then profitable reimbursement in the second round. However, the Cheating Dealer plays a dominant role in both games.

4.4 Extended Game

As we have clarified in *Conclusion 4.2*, the 'Doubling Betting System' (Martingale) will, in practice, negatively affect the continuity of the game. Here, we model one possible application that affects the dynamic motion of threshold value from the perspective of Player 2 as follows:

1. In a t-period game, we refine the last-period entry condition[5] as:

$$q(-50 * 2^{t-1}) + (1-q)\frac{100 * 2^{t-1}}{3} = G(t)$$

   where $G(1) = 0$ and $G(t)$ is increasing in t. Here, instead of setting the right-hand side to 0 for all periods, we employ an increasing function to represent the corresponding altitude at which players should anticipate higher earnings in this game to offset the exponential growth of risk as the game goes on. The indifference condition of entering is no longer applicable on 0. Then we define $H(t)$ as:

$$H(t) = \frac{G(t)}{2^{t-1}}$$

   Then we can obtain an expression for q as $q = -\delta H(t) + \varepsilon$. The dynamics of the threshold is thus given by:

$$\frac{dq_{threshold}}{dt} = -\delta h(t)$$

   in which, without loss of generality, δ is some positive constant and we set $h(t)$ as a non-negative increasing function in t. Together, they measure the time-relevant (bet-relevant) risk aversion for Player 2.

2. In a 2-period discrete-time game, the corresponding second-period entry condition is thus given by:

$$\frac{q^2_{threshold} - q^1_{threshold}}{2-1} = -\delta\frac{H(2) - H(1)}{2-1}$$
$$q^2_{threshold} = q^1_{threshold} - \delta H(2)$$

   Given $q^1_{threshold} = 2/5$, we have $q^2_{threshold} = 2/5 - \delta H(2)$. Then, we use the same method in section 4.3 to solve for PBE.

[5] We are checking the entry condition in period t without any belief updating according to actions from previous $t-1$ periods.

Proposition 4.4. *In the case where $q < 2/5 - \delta H(2)$, there is a unique Perfect Bayesian Nash Equilibrium in which Player 2 plays in both period 1 and period 2 (unless he saw DUU in period 1); Player 1 plays UDD and keeps silent in both periods, and the Cheating Dealer plays UDD in period 1 and plays DUU in period 2.*

Proposition 4.5. *In the case where $2/5 - \delta H(2) < q < 2/5$, there is a unique Perfect Bayesian Nash Equilibrium in which*

1. *In period 1, Player 2 plays[6]. Player 1 plays UDD and keeps silent if Player 2 plays. The Cheating Dealer plays UDD with probability $[(2/5 - \delta H(2))(1 - q)]/[(3/5 + \delta H(2))q]$ if Player 2 plays.*

2. *In period 2, Player 2 quits if DUU was played in period 1. If UDD was played in period 1, Player 2 mixes and plays the second round with a probability of $5/6$. Player 1 plays UDD and keeps silent if Player 2 plays the second round. The Cheating Dealer plays DUU if Player 2 plays the second round.*

Proposition 4.6. *In the case where $q > 2/5$, there is a unique Perfect Bayesian Nash Equilibrium in which*

1. *In period 1, Player 2 plays if $q < 16/25 - (3/5) * \delta H(2)$ and quits at the beginning if $q > 16/25 - (3/5) * \delta H(2)$. Player 1 plays UDD and keeps silent if Player 2 plays. The Cheating Dealer plays UDD with probability $[(2/5 - \delta H(2))(1 - q)]/[(3/5 + \delta H(2))q]$ if Player 2 plays.*

2. *In period 2, Player 2 quits if DUU was played in period 1. If UDD was played in period 1, Player 2 mixes and plays the second round with a probability of $5/6$. Player 1 plays UDD and keeps silent if Player 2 plays the second round. The Cheating Dealer plays DUU if Player 2 plays the second round.*

We set $\delta H(2) = 0$, $\delta H(2) = 1/20$, $\delta H(2) = 1/10$, $\delta H(2) = 1/5$ and $\delta H(2) = 2/5$ respectively and find the PBE expected payoff for the Cheating Dealer as follows:

[6] Player 2 plays if $q < 16/25 - (3/5) * \delta H(2)$ and quits at the beginning if $q > 16/25 - (3/5) * \delta H(2)$. Given $\delta H(2) \leq 2/5$, $16/25 - (3/5) * \delta H(2) \geq 2/5$, which makes $q < 16/25 - (3/5) * \delta H(2)$ always bind since here we assume $q < 2/5$.

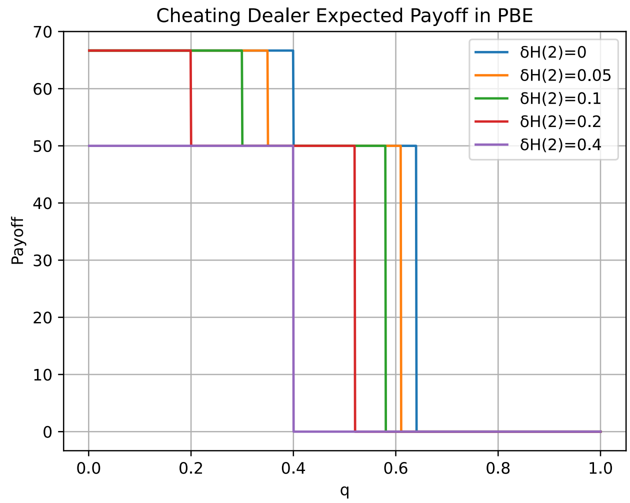


Figure 2: Result of PBE

Conclusion 4.3.

1. The initial draw of q is key. Prior belief directly determines the equilibrium strategies and thus the expected payoff. Given other factors unchanged, a Cheating Dealer can expect a high PBE payoff when the prior belief from Player 2 is low.

2. The shrinkage rate of the threshold plays a crucial role in shaping equilibrium and the expected payoff. Given other factors unchanged, a Cheating Dealer can expect a high PBE payoff when the shrinkage rate is low.

3. The expected payoff for the Cheating Dealer in the PBE is always non-negative. However, due to the zero-sum nature of the game, this means that the other player inevitably faces a non-positive expected payoff.

5. Conclusion

DO NOT GAMBLE!!!

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A. Codes

A.1 Simulation

```
import random

class UDGame(object):
    def __init__(self):
        self.check = {
            'DDD':[0,0,0],     #[pick_times, win_times, win_throws_times]
            'DDU':[0,0,0],
            'DUD':[0,0,0],
            'DUU':[0,0,0],
            'UUU':[0,0,0],
            'UUD':[0,0,0],
            'UDU':[0,0,0],
            'UDD':[0,0,0]
            }
        self.c = ['DDD','DUD','DDU','DUU', 'UUU','UUD', 'UDU','UDD']

    def game(self,c1,c2):
        string = ''
        count = 0
        while True:
            string += random.sample(['U','D'],1)[0] # throw dice 1 time
            count+=1
            # then determine if the game is over and who wins
            if string[-3:] == c1:
                return c1,count
            if string[-3:] == c2:
                return c2,count

    def play(self,times):
        for i in range(times):
            c1,c2 = random.sample(self.c,2)
            winner,count = self.game(c1,c2)
            self.check[c1][0]+=1
            self.check[c2][0]+=1
            self.check[winner][1]+=1
            self.check[winner][2]+=count

    def print_score(self):
        for poi in self.c:
            print(poi+' win_rate: {}  expected_value: {}'.format(\
                round(self.check[poi][1]/(self.check[poi][0]+1e-9),3),\
                   round(self.check[poi][2]/(self.check[poi][1]+1e-9),2)
                                                                  ))
```

```
    def reset(self):
        self.check = {
            'DDD':[0,0,0],
            'DUD':[0,0,0],
            'DDU':[0,0,0],
            'DUU':[0,0,0],
            'UUU':[0,0,0],
            'UUD':[0,0,0],
            'UDU':[0,0,0],
            'UDD':[0,0,0]
            }

def main(repeat_times = 100000):

    game = UDGame()
    game.play(repeat_times)
    game.print_score()
    game.reset()

if __name__ == '__main__':
    repeat_times = 1000000
    main(repeat_times)
```

A.2 PBE

```
import numpy as np
import matplotlib.pyplot as plt

def piecewise_function(x, c):
    if 0 < x < 2/5 - c:
        return 200/3
    elif 2/5 - c < x < 2/5:
        return 50
    elif 2/5 < x < 16/25 - 3*c/5:
        return 50
    elif x > 16/25 - 3*c/5:
        return 0


x_values = np.linspace(0, 1, 1000)

fig, ax = plt.subplots()

for c in [0, 1/20, 1/10, 1/5, 2/5]:
    y_values = [piecewise_function(x, c) for x in x_values]
    plt.plot(x_values, y_values, label=f"\u03B4H(2)={c}")
```

```
plt.xlabel('q')
plt.ylabel('Payoff')
plt.title('Cheating Dealer Expected Payoff in PBE')
plt.legend()
plt.grid(True)


plt.show()
fig.savefig('pbe.svg') # save the .svg picture
```