Box-Muller Method for Generating Normally Distributed Random Variables

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Introduction

The Box-Muller method is an algorithm that transforms two uniformly distributed random variables in the interval $[0, 1]$ into two independent normally distributed random variables with a mean of 0 and a variance of 1, i.e., random variables that follow the standard normal distribution $N(0, 1)$.

Problem Statement

Assume we have two independent random numbers u_1 and u_2, uniformly distributed in the interval $[0, 1]$. The task is to generate two independent random variables z_0 and z_1, which are normally distributed according to the standard normal distribution $N(0, 1)$.

Box-Muller Method

The method uses a transformation to polar coordinates. For two uniform random variables u_1 and u_2, the generation of standard normally distributed random variables is carried out by the following formulas:

$$z_0 = \sqrt{-2\ln(u_1)} \cdot \cos(2\pi u_2),$$
$$z_1 = \sqrt{-2\ln(u_1)} \cdot \sin(2\pi u_2),$$

where $u_1, u_2 \sim U[0, 1]$.

The generated random variables z_0 and z_1 are independent and follow the standard normal distribution $N(0, 1)$.

Correctness of the Method

The correctness of the Box-Muller method is based on converting uniform random variables into polar coordinates and the fact that the sum of squares of two independent standard normal random variables is exponentially distributed.

Consider two random variables X and Y, which are derived from the uniform numbers u_1 and u_2 as follows:

$$X = \cos(2\pi u_2), \quad Y = \sin(2\pi u_2).$$

Since the angle $\theta = 2\pi u_2$ is uniformly distributed in the interval $[0, 2\pi]$, the components X and Y are independent and normally distributed. The radial component is transformed using $r = \sqrt{-2\ln(u_1)}$, ensuring the correct distribution of distances from the origin, which corresponds to the normal distribution.

Transformation to General Normal Distribution $N(\mu, \sigma^2)$

To generate random variables from a normal distribution with a mean μ and variance σ^2, the transformation is straightforward. Given a standard normal variable $z \sim N(0, 1)$, the following equation will give a normally distributed variable with mean μ and variance σ^2:

$$x = \mu + \sigma z.$$

For example, if z_0 is a standard normal variable generated by the Box-Muller method, the corresponding normal variable x_0 will be:

$$x_0 = \mu + \sigma z_0.$$

Conclusion

The Box-Muller method is a reliable algorithm for transforming uniformly distributed random variables into normally distributed ones. It is computationally efficient and easy to implement in programming languages, as demonstrated in the Java example. With this method, one can generate random variables from any normal distribution $N(\mu, \sigma^2)$ by simply applying a linear transformation to the standard normal output.