Brezis weak topology

$x_n \to x$ in E strongly whenever

① $\|x_n - x\| \to 0$

② $\|j(x_n) - j(x)\|_{E^{**}} = \sup_{\|f\| \le 1} \langle j(x_n) - j(x), f \rangle_{E^{**}}$

converges to 0.

this is precisely "convergence on Bounded subsets".

Alternatively, we can demand:

$j(x_n) \to j(x)$ in the pointwise topology rather than the bounded convergence topology.

We say that $x_n \to x$ weakly whenever $\forall f \in E^*$, $j(x_n)(f) \to j(x)(f)$.

→ The weak topology is generated by the family of functionals E^*.

More precisely, it is generated by the neighbourhood base, $\forall x \in E$,

$W_{x,f,n} = \{ y \in E, \ |f(x-y)| < n^{-1} \}_{n \ge 1, f \in E^*}$

The strong topology clearly contains every $W_{x,f,n}$, therefore every weakly open (resp. closed) set is strongly open (resp. closed)

Investigation into weak topology.

The weak topology ensures that the continuous functionals on E stay continuous,

→ The strict sublevel / superlevel sets are open, and hypersurfaces stay closed

→ The set $\{f \neq \alpha\}$ is weakly disconnected, and hence strongly disconnected

If C is convex, then strongly closed implies weakly closed. This is because of the "completely regular" property of $\sigma(E, E^*)$ with respect to the linear functionals.

Hahn-Banach - second geometric form

Strictly Separates a compact convex set from a closed convex set.

Let X be a set and $\{Y_\alpha\}_{\alpha\in A}$ family of topological spaces, and $\mathcal{F} = \{f_\alpha : X \to Y_\alpha\}$

→ The weak topology on X (Rel $\mathcal{F}$) is the weakest topology on X that makes all f_α continuous

Set $\mathcal{E} = \{ f_\alpha^{-1}(U_\alpha),\ \alpha \in A,\ U_\alpha \overset{\circ}{\subseteq} Y_\alpha . \}$

then $\mathcal{E}$ forms a sub-base of X. $\left(X = \bigcup_{U\in\mathcal{E}} U\right)$

→ A sub-base can be upgraded to a base by considering its finite intersections.

→ If X has a weak topology Rel $\mathcal{F}$. $\forall x \in X$, we find the neighbourhood base:

$$\mathcal{N}(x) = \left\{ \bigcap_1^N f_{\alpha_i}^{-1}(U_{\alpha_i}),\ N \geq 1 \right\}$$

→ Characterisation of weak-adherent point

In any topological space, $x \in \bar{E}$ iff there exists a net $\langle x_\alpha\rangle_A \subseteq E$ that converges to x.

We recap some of the properties of nets,

Def: Directed Set A is equipped with a Binary Relation $\lesssim$ such that

→ $\alpha \lesssim \alpha$ for all $\alpha \in A$,

→ if $\alpha \lesssim \beta$ and $\beta \lesssim \gamma \Rightarrow \alpha \lesssim \gamma$ (Transitive)

→ $\forall \alpha, \beta \in A$, $\exists \gamma \in A$, $\alpha \lesssim \gamma$, $\beta \lesssim \gamma$ (unBounded above)

Def A subnet of $\langle x_\alpha\rangle_A$ is another net $\langle y_\beta\rangle_B$. together with a map:

$$F : B \to A$$

such that

1)

$$\begin{tikzcd} \beta \arrow[r, "F"] \arrow[dr] & F(\beta) = \alpha \arrow[d] \\ & y_\beta = x_\alpha \end{tikzcd}$$

2) $\forall \alpha_0 \in A$, we find $F(\beta_0) \in F(B)$ such that F preserves the order:

$$\alpha_0 \lesssim F(\beta) \qquad \forall \beta \gtrsim \beta_0$$

Def A subset $B \subseteq A$ is cofinal if it is a slimmed down version of A. This means: $\forall \alpha_0 \in A$, $\exists \beta_0 \in B$, $\alpha_0 \lesssim \beta_0$.

Using the same order structure inherited from A, B is again a directed set.

4.30 a) The inclusion map $B \hookrightarrow A$ for cofinal B, makes $\langle x_\beta\rangle_B$ a subnet of $\langle x_\alpha\rangle_A$

proof First, we check that B is a directed set, $\forall \beta \in B$, clearly $\beta \lesssim \beta$, and the transitive relation holds too, Now fix $\beta_0, \beta_1 \in B$, we find $\alpha \in A$ such that $\max(\beta_0, \beta_1) \lesssim \alpha \lesssim \beta_2 \in B$. So that B is a directed set, and is easy to see, $\langle x_\beta\rangle_B$ is a subnet $\square$

b) If $\langle x_\alpha\rangle_A$ is a net $\langle x_\alpha\rangle \to x$ iff for all cofinal B, there exists a cofinal $C \subseteq B$ such that $\langle x_\gamma\rangle_{\gamma\in C} \to x$

proof Suppose $x_\alpha \to x$, then we claim that every subnet of $\langle x_\alpha \rangle$ also converges. Indeed, if $\langle y_\beta \rangle$ is a subnet of $\langle x_\alpha \rangle$. $\forall U \overset{\circ}{\subseteq} X$ we find $\alpha_0 \in A$ $\forall \alpha \gtrsim \alpha_0$, $x_\alpha \overset{\circ}{\in} U$.
Now, $\exists \beta_0 \in B$, $\forall \beta \gtrsim \beta_0$, $F(\beta) \gtrsim \alpha_0$, and

$$y_\beta = x_{F(\beta)} \overset{\circ}{\in} U$$

So that $y_\beta \longrightarrow x$ as well. And it suffices to take $C = B$.

Suppose $\langle x_\alpha \rangle_A$ does not converge to x. So that we can find an open set about x that separates the eventual tails of x_α from x.

$$\forall \alpha_0 \in A,\ \exists \alpha \gtrsim \alpha_0,\quad x_\alpha \notin U$$

set $B = \{\alpha \in A,\ x_\alpha \notin U\}$, it suffices to verify that B is a cofinal subset of A, this is clear because "the set of indices that $x_\alpha \notin U$ is unbounded above."
If $C \subseteq B$, then $\forall \gamma \in C$, $x_\gamma \notin U$, so x_γ cannot converge to x. $\square$