wts

1) For all $X, Y \geq 0$, $\binom{X}{0} = \binom{Y}{0} = 1$
$\binom{X}{X} = 1$

2) For all $0 \leq \gamma \leq X$, $\gamma_j = 0$, $(j = \underline{n})$, then $\binom{X}{\gamma} = \binom{X + e_j}{\gamma}$

3) $$\binom{\alpha + e_j}{\gamma} = \binom{\alpha}{\gamma - e_j} + \binom{\alpha}{\gamma}, \quad (\forall\, e_j \leq \gamma \leq \alpha, \quad j = \underline{n})$$

4) $\forall \alpha \geq 0$, $$\partial^\alpha (fg) = \sum_{\gamma=0}^{\alpha} \binom{\alpha}{\gamma} (\partial^\gamma f)(\partial^{\alpha - \gamma} g)$$

proof

1) Fix $X, Y \geq 0$, then

$$\binom{X}{0} = \prod_1^n \binom{X_i}{0} = 1 = \binom{Y}{0}$$

$$\binom{X}{X} = \prod_1^n \binom{X_i}{X_i} = 1$$

2) Fix $0 \leq \gamma < e_j$, $X \geq e_j$,

$$\binom{X}{\gamma} = \left(\prod_{i \neq j} \binom{X_i}{\gamma_i}\right) \binom{X_j}{0} = \binom{X + e_j}{\gamma}$$

3) $$\binom{\alpha + e_j}{\gamma} = \left(\prod_{i \neq j} \quad\right) \binom{\alpha_j + 1}{\gamma_j}$$

one-dimensional case

$$= (\qquad) \left[\binom{\alpha_j}{\gamma_j - 1} + \binom{\alpha_j}{\gamma_j} \right]$$

$$= \binom{\alpha}{\gamma - e_j} + \binom{\alpha}{\gamma}$$

4) $\alpha = 0$, nothing to prove, (the sum ranges from $0 \leq \gamma \leq \alpha$, so $\gamma \in \{0\}$),
Suppose holds for $|\alpha| \leq k$, then $\forall j = \underline{n}$,

$$\partial^{e_j}(\partial^\alpha fg) = \sum_{\gamma=0}^{\alpha} \binom{\alpha}{\gamma} \left[(\partial^{\gamma + e_j} f)(\partial^{\alpha - \gamma} g) + (\partial^\gamma f)(\partial^{\alpha + e_j - \gamma} g) \right].$$

Pay attention to the green exponent.
→ what is the range of the green exponent?.

$$= \sum_{\gamma = e_j}^{\alpha + e_j} \binom{\alpha}{\gamma - e_j} (\partial^\gamma f)(\partial^{\alpha + e_j - \gamma} g) + \sum_{\gamma=0}^{\alpha} \binom{\alpha}{\gamma} (\quad)(\quad)$$

By ①

$$= \left(\sum_{\gamma = e_j}^{\alpha} \binom{\alpha}{\gamma - e_j} (\quad)(\quad) \right) + \left(\sum_{\gamma=0}^{\alpha} \binom{\alpha}{\gamma} (\quad)(\quad) \right) + (1)(\gamma = \alpha + e_j)(\quad)$$

③

$$\sum_0^\alpha = \sum_{e_j}^{\alpha} + \sum_{(0 \leq \gamma \leq \alpha,\ \gamma_j = 0)}$$

$$= \sum_{\gamma = e_j}^{\alpha} \binom{\alpha + e_j}{\gamma} (\quad)(\quad) + \sum_{\substack{0 \leq \gamma \leq \alpha, \\ \gamma_j = 0}} \binom{\alpha}{\gamma} (\quad)(\quad) + (1)(\gamma = \alpha + e_j)(\quad)$$

②

$$= (\text{ same as above }) + \sum_{\substack{0 \leq \gamma \leq \alpha \\ \gamma_j = 0}} \binom{\alpha + e_j}{\gamma} (\quad)(\quad) + (1)(\partial^{\alpha + e_j} f)(\partial^0 g)$$

$$= \sum_{\gamma = 0}^{\alpha} \binom{\alpha + e_j}{\gamma} (\quad)(\quad) + \sum_{\gamma = \alpha + e_j} \binom{\alpha + e_j}{\gamma} (\quad)(\quad)$$