Schwartz Space

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Let $\{f_n\} \subseteq \mathcal{S}$ be a Cauchy sequence in $\mathcal{S}$, that is: for every (N,α), the difference

$$\|f_j - f_k\|_{(N,\alpha)} \to 0 \quad \text{as} \quad \max(j,k) \to \infty.$$

Because $\{f_n\} \subseteq BC(\mathbb{R}^n)$, and the sequence $\|f_j - f_k\|_{(0,\alpha)} \to 0$, there exists a unique $g_\alpha \in BC(\mathbb{R}^n)$ such that $\|\partial^\alpha f_j - g_\alpha\|_u \to 0$. (This comes from the completeness of $BC(X)$). We will show that $g_0 \in C^\infty(\mathbb{R}^n, \mathbb{C})$, and $\partial^\alpha g_0 = g_\alpha$. It is clear that $g_0 \in BC(\mathbb{R}^n, \mathbb{C})$.

Suppose for induction that $g_0 \in C^k(\mathbb{R}^n, \mathbb{C})$, and $\partial^\alpha g_0 = g_\alpha$ for $|\alpha| \le k$. By fixing e_j, the following equation holds at every x, and no matter how large $t \in \mathbb{R}$ is.

$$(\partial^\alpha f_n)(x + te_j) - (\partial^\alpha f_n)(x) = t\int_0^1 (\partial^{\alpha+e_j} f_n)(x + ste_j)\,ds$$

We can take limits as $n \to \infty$ on the left,

$$g_\alpha(x + te_j) - g_\alpha(x) = \lim_{n\to\infty} t\int_0^1 (\partial^{\alpha+e_j} f_n)(x + ste_j)\,ds$$

The sequence of integrands is uniformly dominated

$$|(\partial^{\alpha+e_j} f_n)(x + ste_j)| \le [\sup_{n\ge1} \|f_n\|_{(0,\alpha+e_j)}] \in L^1([0,1], ds),$$

and pointwise convergent to $g_{\alpha+e_j}(x+ste_j)$ for $s \in [0,1]$. By the dominated convergence theorem,

$$g_\alpha(x + te_j) - g_\alpha(x) = t\int_0^1 g_{\alpha+e_j}(x + ste_j)\,ds \quad \forall x \in \mathbb{R}^n, t \in \mathbb{R}.$$

Changing variables gives

$$g_\alpha(x + te_j) - g_\alpha(x) = \int_0^t g_{\alpha+e_j}(x + se_j)\,ds \quad \forall x \in \mathbb{R}^n, t \in \mathbb{R}.$$

By restricting $|t| \le 1$, we can consider the mapping

$$s \mapsto g_{\alpha+e_j}(x + se_j) \quad \forall s \in [-1, +1]$$

The integrand $s \mapsto g_{\alpha+e_j}(x+se_j)$ is continuous (and hence regulated mapping from a compact interval). Using FTC, because 0 is an interior point of $[-1,+1]$,

$$\partial^{e_j} g_\alpha(x) = g_{\alpha+e_j}(x) \quad \forall x \in \mathbb{R}^n.$$

Which shows that $g_\alpha \in C^1(\mathbb{R}^n, \mathbb{C})$, but $g_\alpha = (\partial^\alpha g_0)$. Therefore $g_0 \in C^{k+1}(\mathbb{R}^n, \mathbb{C})$. Repeating this process shows that $g_0 \in C^\infty(\mathbb{R}^n, \mathbb{C})$, and $\partial^\alpha g_0 = g_\alpha$ for every $\alpha \ge 0$.

It remains to show that $g_0 \in \mathcal{S}$. The space

$$\mathcal{S} = \begin{matrix}\text{smooth functions whose derivatives} \\ \text{decay faster than polynomials}\end{matrix}.$$

Decay estimates can be proven using uniform convergence. For every (N,α), the number

$$C_{(N,\alpha)} = \sup_{n\ge1} \|f_n\|_{(N,\alpha)} < \infty.$$

And

$$|\partial^\alpha g_0(x)| = \lim_n |\partial^\alpha f_n(x)| \le C_{(N,\alpha)}(1 + |x|)^{-N} \quad \forall x \in \mathbb{R}^n,$$

which is the same as saying $g_0 \in \mathcal{S}$. Finally, it is clear that $\|f_n - g_0\|_{(N,\alpha)} \to 0$.

There is another way to prove $g_0 \in C^\infty(\mathbb{R}^n, \mathbb{C})$ and that $\partial^\alpha g_0 = g_\alpha$ for every $\alpha \ge 0$. If $\alpha = 0$, it follows (again) from the completeness of $BC(\mathbb{R}^n)$. We can reuse some Frechet derivative machinery.

Given $f \in C^\infty(\mathbb{R}^n, \mathbb{C})$, we write $D^p f \in C^\infty(\mathbb{R}^n, L_p(\mathbb{R}^n, \mathbb{C}))$. Where $L_p(\mathbb{R}^n, \mathbb{C})$ is the space of p-multilinear complex-valued maps from $\mathbb{R}^n$. We know from Lang's text, if $\{f_n\} \subseteq C^\infty(\mathbb{R}^n, \mathbb{C})$, and $\|D^p f_n - D^p f_m\|_u \to 0$ for all $p \ge 1$. Then there exists $g_0 \in C^\infty(\mathbb{R}^n, \mathbb{C})$, such that

$$\|D^p f_n - D^p g_0\|_u \to 0.$$

I will just state an equation, whose proof comes from a tensor algebra argument,

$$\|D^p f(x)\|_{L_p(\mathbb{R}^n,\mathbb{C})} \le \sum_{|\alpha|=p} |\partial^\alpha f(x)| \le n^p \|D^p f(x)\|_{L_p(\mathbb{R}^n,\mathbb{C})}.$$

Now assume $\{f_n\} \subseteq \mathcal{S}$ is Cauchy, then

$$\sup_{x\in\mathbb{R}^n} \|D^p f_j(x) - D^p f_k(x)\|_{L_p(\mathbb{R}^n,\mathbb{C})} \le \sum_{|\alpha|=p} \|f_j - f_k\|_{(0,\alpha)} \to 0$$

as $\max(j,k) \to \infty$. We are in a position to use Lang's "pointwise + uniform = C^1 uniform converence" theorem. This means $g_0 \in C^\infty(\mathbb{R}^n, \mathbb{C})$, $D^p g_0 = g_p = \lim_{n\to\infty} D^p f_n$ for $p \geq 1$. Extracting the partials is easy, because

$$\partial^\alpha g_0(x) = D^p g_0(x)(e_\alpha),$$

where $e_\alpha = \sum_1^n \alpha_i e_i$