pointwise convergence can be rewritten.

$f_j = f + g_j$, $g_j \to 0$ pw a.e.

Suppose $j : \mathbb{C} \to \mathbb{C}$, contin, $j(0) = 0$,

and $\forall \varepsilon > 0$, $\exists\, \varphi_\varepsilon, \psi_\varepsilon$, continuous, non-negative

$$|j(a+b) - j(a)| \le \varepsilon\, \varphi_\varepsilon(a) + \psi_\varepsilon(b).$$

Suppose further that,

$\to$ $j(f) \in L^1$;

(uniformly Bounded)

$\to$ $\sup_{n \ge 1, \varepsilon > 0} \int_X \varphi_\varepsilon(g_n(x))\, dx \le C$

$\to$ $\int_X \psi_\varepsilon(f(x))\, dx < +\infty$ $\forall \varepsilon > 0$.

then $\int_X |j(f_j) - j(f) - j(f_j - f)|\, dx \to 0$ as $j \to +\infty$.

① $a \le b \Leftrightarrow a^+ \le b^+$ and $b^- \le a^-$
② $a \le (b-a)_- + b$ $\forall a, b \in \mathbb{R}$
③ $a \le (a-b)_+ + b$ $\forall a, b \in \mathbb{R}$
④ $\max(o(1) - b_n, 0) \to 0$ (if $b_n \ge 0$, & converges)

Investigating the pointwise limit of $W_{n,\varepsilon}(x)$, is of the form:

first.

$S_n(x) = |j(f+g_n) - j(f) - j(g_n)|$

converges pointwise a.e to 0, because $g_n \to 0$ a.e, by continuity

and $W_{n,\varepsilon}(x) = \max(S_n(x) - \varepsilon\varphi_\varepsilon(g_n(x)), 0)$ converges pointwise to 0 by equation ④, as $n \to +\infty$, for all $\varepsilon > 0$.

The sequence of functions (with ε fixed), $\{W_{n,\varepsilon}(x)\}_{n \ge 1}$ uniformly dominated. by ⑥, so that, by DCT

⑦ $\lim_{n\to\infty} \int_X W_{n,\varepsilon}(x)\,dx = 0.$ $\forall \varepsilon > 0$.

Fix $\varepsilon > 0$

obtaining the first pointwise Bound:

$\forall x \in X$,

⑤ $|j(f_n) - j(f) - j(f_n - f)| \le \Big[|j(f_n) - j(f) - j(\underbrace{f_n - f}_{g_n})| - \varepsilon\varphi_\varepsilon(g_n)\Big]_+ + \varepsilon\varphi_\varepsilon(g_n)$.

another pointwise Bound, set $W_{n,\varepsilon}(x) = [|j(f_n(x)) - j(f(x)) - j(f_n(x) - f(x))| - \varepsilon\varphi_\varepsilon(g_n(x))]_{(+)}$

By the triangle inequality, then by the j property.

$$|j(\underbrace{f_n}_{f_n = f + g_n}) - j(f) - j(g_n)| - \varepsilon\varphi(g_n) \le \underbrace{|j(f)|}_{L^1} + |j(f+g_n) - j(g_n)| - \varepsilon\varphi_\varepsilon(g_n)$$

$$\le \underbrace{|j(f)|}_{L^1} + \varepsilon(\varphi_\varepsilon(g_n)) + \underbrace{\psi_\varepsilon(f)}_{L^1} - \varepsilon\varphi_\varepsilon(g_n)$$

so that

⑥ $W_{n,\varepsilon}(x) \le \big[|j(f)| + \psi_\varepsilon(f)\big]_{(+)} = |j(f)| + \psi_\varepsilon(f)$. $\forall x \in X$, $\forall n \ge 1$, $\forall \varepsilon > 0$.

now, estimate

$S_n(x) \le W_{n,\varepsilon}(x) + \varepsilon\varphi_\varepsilon(g_n)$.

and

$\limsup_{n\to\infty} \int_X S_n(x)\,dx \le$

with $S_n(x) = |j(f+g_n) - j(f) - j(g_n)|$,

and $S_n(x) \le W_{n,\varepsilon}(x) + \varepsilon \varphi_\varepsilon(g_n(x))$ pw a.e

$$\int_X S_n(x) \le \int_X W_{n,\varepsilon}(x)\,dx + \varepsilon \int_X \varphi_\varepsilon(g_n(x))\,dx \qquad \forall \varepsilon > 0\ \forall n \geq 1$$

the second term diasppears under the limit, and the third term is uniformly Bounded by ($\forall \varepsilon > 0\ \forall n \geq 1$)

$$\varepsilon \int_X \varphi_\varepsilon(g_n(x))\,dx \le \varepsilon \cdot C,$$

$$\limsup_{n\to\infty} \int_X S_n(x)\,dx \le \limsup_{n\to\infty} \left[\int_X W_{n,\varepsilon}(x) + \varepsilon \int_X \varphi_\varepsilon(g_n(x))\,dx\right].$$

$$\le \left(\limsup_{n\to\infty} \int_X W_{n,\varepsilon}(x)\,dx\right) + \left(\limsup_{n\to\infty} \varepsilon \int_X \varphi_\varepsilon(g_n(x))\,dx\right)$$

$\le 0 + \varepsilon C$.

this holds for all $\varepsilon > 0$, so $\limsup_{n\to\infty} \int_X S_n(x) = 0$.

(as $0 \le \int_X S_n(x)\,dx$)