Folland Exercise 3.5

Let ν_1, ν_2 signed measures, s.t. $\nu = \nu_1 + \nu_2$ is again a signed measure, then

$$|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$$

(this is an equality of positive measures)

proof The total variation of a signed meaure

$$|\nu_1| = (\nu_1^+ + \nu_1^-) \geq 0.$$

Let $P \oplus N = X$ be a HJ decomp. for $(\nu_1 + \nu_2) = \nu$, then, $\forall E \in \mathcal{M}$,

$$|\nu_1 + \nu_2|(E) = (\nu_1 + \nu_2)^+(E \cap P) + (\nu_1 + \nu_2)^-(E \cap N)$$

$$= (\nu_1 + \nu_2)(E \cap P) - (\nu_1 + \nu_2)(E \cap N)$$

$$= (\nu_1^+ + \nu_2^+)(E \cap P) - (\nu_1^- + \nu_2^-)(E \cap P)$$

$$+ (-1)\left((\nu_1^+ + \nu_2^+)(E \cap N) - (\nu_1^- + \nu_2^-)(E \cap N) \right)$$

$$\leq (\nu_1^+ + \nu_2^+)(E) + (\nu_1^- + \nu_2^-)(E)$$

$$\leq |\nu_1|(E) + |\nu_2|(E).$$

Note, if $\nu_1 \perp \nu_2$, then we have equality. $\square$
see Scanned Notes on Radon Nikodym Theorem.

3.1 ✓
3.2 ✓
3.3 ✗
3.4 ✓
3.5 ✓
3.6 ✓
3.7 ✗
3.8 ✓
3.9 ✓

μ, ν are positive measures, $\mu \leq \nu$,
if ν is σ-finite, so is μ.

Decompositions of positive measures.

Let $\{\mu_j\}_1^\infty$, $\{\nu_j\}_1^\infty$ positive measures, and

$$\mu = \sum_1^\infty \mu_j, \quad \nu = \sum_1^\infty \nu_j,$$

Suppose further there is $\{E_j\}_1^\infty \subseteq \mathcal{M}$, $\bigcup_1^\infty E_j = X$, disjoint, such that

$$\mu(A \cap E_j) = \mu_j(A) \qquad \forall j \geq 1, \ \forall A \in \mathcal{M}$$

$$\nu(A \cap E_j) = \nu_j(A)$$

Then: $\mu \ll \nu$ iff $\mu_j \ll \nu_j \ \forall j \geq 1$

$\mu \perp \nu$ iff $\mu_j \perp \nu_j \ \forall j \geq 1$.

proof of Decompositions of positive measures

Note $\forall j \geq 1$, $\mu_j(E_j^c) = \mu(E_j \cap E_j^c) = 0$.

$\nu_j(E_j^c) = 0$ as well, so

$$E_j^c = X \setminus E_j = \bigcup_{i \neq j} E_i \in Ker(\mu_j) \cap Ker(\nu_j)$$

So that $\forall i \neq j$, $E_j \subseteq E_i^c \in Ker(\mu_i) \cap Ker(\nu_i)$
and because $E_j^c \in Ker(\mu_j) \cap Ker(\nu_j)$,
we see that: the cross terms are mutually singular (in the summation)

$$\mu_i \perp \mu_j, \quad \nu_i \perp \nu_j, \quad \mu_i \perp \nu_j.$$

We will prove the second claim in the proposition.
~~first~~, consider the characterisation of $Ker(\mu)$, and $Ker(\nu)$,

By some theorem in the Overleaf document.

$$Ker(\gamma) = \bigcap_1^\infty Ker(\gamma_j), \quad \text{and} \quad Ker(\alpha) = \bigcap_1^\infty Ker(\alpha_j)$$

and

Equivalence of mutual singularity

$$(\Sigma_1^\infty \gamma_j) \perp \alpha \iff (\forall j \geq 1,\ \gamma_j \perp \Sigma_1^\infty \alpha_i)$$

symmetric relation.

$$\iff (\forall i,j \geq 1,\ \gamma_j \perp \alpha_i)$$

$\boxed{\text{Because } \gamma_i \perp \alpha_j\ \forall i \neq j}$

$$\iff (\forall j \geq 1,\ \gamma_j \perp \alpha_j)$$

This proves the second claim of the proposition.

For the first claim

→ Suppose $\gamma << \alpha$, then fix $j \geq 1$ and $A \in Ker(\alpha_j)$,

so that $\alpha_j(A) = \alpha(E_j \cap A) = 0$

Implies the j^{th} slice of A, $E_j \cap A \in Ker(\alpha)$.

But $\alpha >> \gamma$, so $Ker(\alpha) \subseteq Ker(\gamma) \subseteq Ker(\gamma_j)$,

Hence $\gamma_j(E_j \cap A) = \gamma_j(A) = 0 \Rightarrow \gamma_j << \alpha_j$

→ Conversely, if $\forall j \geq 1$, $\gamma_j << \alpha_j$, ~~we~~ then:

$$\forall j \geq 1, \quad Ker(\alpha_j) \subseteq Ker(\gamma_j)$$

Taking intersections on both sides gives

$$Ker(\alpha) = \bigcap_1^\infty Ker(\alpha_j) \subseteq \bigcap_1^\infty Ker(\gamma_j) = Ker(\gamma).$$

~~ccc~~ If α is a complex measure

$L^1(\alpha) \triangleq L^1(\alpha_r) \cap L^1(\alpha_i)$, st $\forall f \in L^1(\alpha)$

$$\int_X f d\alpha \triangleq \int_X f d\alpha_r + i \int_X f d\alpha_i$$

all integrals above converge absolutely.

If α, γ are complex measures, then

we write $\alpha \perp \gamma$ iff ?

First we need to define Kernel of a complex measure.

we cannot take

$$Ker(\gamma) \triangleq \{E \in M,\ \gamma|_E \text{ is real}\} \cap \{E \in M,\ \gamma|_E \text{ is imaginary}\}?$$

because no Hahn Jordan decomposition?

A complex measure example

$$\gamma(\{0\}) = 1+i, \quad X = \{0\},$$

?

We define without motivation

$$Ker(\gamma) \triangleq Ker(\gamma_r) \cap Ker(\gamma_i).$$

and ~~corresponding~~ extend the notions of mutual singularity and absolute continuity.

If γ, α are complex measures, $\gamma \perp \alpha$ iff there exists $A \in M$, such that

$$A \in Ker(\gamma) \text{ and } A^c \in Ker(\alpha)$$

In this case, it means, which implies,

$$\begin{array}{l} A \in Ker(\gamma_r) \cap Ker(\gamma_i) \\ A^c \in Ker(\gamma_r) \cap Ker(\alpha_i) \end{array} ,\ \Rightarrow \begin{cases} \gamma_r \perp \alpha_r & \gamma_r \perp \alpha_i \\ \gamma_i \perp \alpha_i & \gamma_i \perp \alpha_r. \end{cases}$$

Conversely, if the 4 perpendicular relations hold, let $A_{rr}, A_{ri} \in Ker(\gamma_r)$ $\quad A_{ir}, A_{ii} \in Ker(\gamma_i)$

~~A_{rr}, A_{ii}~~

~~A_{rr}^c, A_{ii}^c~~

$A_{rr}^c, A_{ir}^c \in Ker(\alpha_r)$. $\quad A_{ri}^c, A_{ii}^c \in Ker(\alpha_i)$

For every

$F(x)$ increasing and right continuous

$\mu_F : \mathcal{E} \to [0, \infty]$, where $\mathcal{E}$ = elementary family of h-intervals

$F(\infty) = \sup F(x)$

$F(-\infty) = \inf F(x)$, for every finite a, b,

$$\mu_F((a,b]) = F(b) - F(a)$$

Every increasing and right continuous function induces a premeasure on the algebra of finite disjoint unions of h-intervals,

$$\underbrace{\{(a,b]\}}_{H_1} \cup \underbrace{\{(-\infty, b]\}}_{H_2} \cup \{\phi\} = H$$

I think we really have to use Dykin's lemma here, or else it gets too complicated.

If μ is a Borel measure that is finite on compacts,

Define
$$F(x) = \begin{cases} \mu((0,x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x,0]) & x < 0 \end{cases}$$

then F is ① increasing: if $x \leq y$, and $x \geq 0$, then $\mu((0,y]) \geq \mu((0,x])$.

if $x \leq 0 \leq y$, then clearly $F(x) \leq 0 \leq F(y)$

if $x \leq y \leq 0$, then $(y, 0] \supseteq (x, 0]$

implies $\mu_F(y, 0] \geq \mu_F(x, 0]$

implies $F(x) \leq F(y) \leq 0$.

F is right continuous,

$\forall x \geq 0$, clearly $F(x) = \mu((0,x]) = \inf\limits_{\varepsilon > 0} \mu((0, x+\varepsilon])$ [continuity from above]

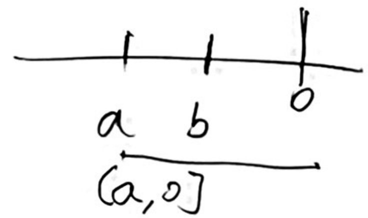
$\forall x < 0$, $F(x) = (-1)\mu((x, 0]) = (-1) \sup\limits_{\substack{\varepsilon > 0 \\ |\varepsilon| < |x|}} \mu((x+\varepsilon, 0])$

> in both cases,
>
> $F(x) = \inf\limits_{\varepsilon > 0} F(x+\varepsilon) \quad \forall x \in \mathbb{R}.$

therefore F is increasing and right continuous //

we verify $\mu_F|_A = \mu|_A$

$a < b \leq 0$



if $(a,b]$ finite, then $\mu_F((a,b]) = F(b) - F(a)$

we need to separate into cases,

if $0 \leq a < b$, then $F(b) - F(a) = \mu((0,b]) - \mu((0,a])$

$\mu_F((a,b]) = \mu((a,b])$

if $a < b \leq 0$, then $\mu_F((a,b]) = (-1)\mu((b,0]) + \mu((a,0])$

$= \mu((a,b])$

if $a \leq 0 \leq b$, then $\mu_F((a,b]) = \mu((0,b]) + \mu((a,0])$

$= \mu((a,b])$

$(a,b]$

For intervals that are unbounded above or below, we use σ-additivity, and generate

$$(a,\infty) = \bigcup_1^\infty (a,n]$$
$$(-\infty,b] = \bigcup_1^\infty (-n,b]$$

Because by the proof of Theorem 1.15, the induced premeasures of any increasing, right continuous F, is σ-additive, so it really does suffice to verify on bounded h-intervals.