5.25

Suppose X^* separable, $\Rightarrow X$ separable

proof let $\{f_n\} \subseteq X^*$ be a countable dense subset of X^*, for each $n \geq 1$

By pulling the supremum downwards, we find

$$2^{-1}\|f_n\| \leq |f_n(x_n)|, \text{ with } \|x_n\| = 1.$$

Want to show that $M = \text{span}(x_n)_1^\infty$ is dense in X. It will be useful to remember the following technique,

If $\overline{M}$ is a closed proper subspace, one can find a non-zero continuous functional that vanishes on $\overline{M}$,

Suppose for contradiction that $y_0 \in X \setminus \overline{M}$, one finds:

$$|f_0(y_0)| = c > 0, \quad f_0|_{\overline{M}} \equiv 0,$$

Which implies, $\|f_0\| > \frac{c}{\|y_0\|} > 0$.

Let $\varepsilon > 0$ be so small that

$$\|f_0\| - 3\varepsilon > 0, \quad \text{then}$$

by density of $\{f_n\}_1^\infty$, we find $k \in \mathbb{N}^+$ with $\|f_0 - f_k\|_{X^*} \geq \big|\|f_0\| - \|f_k\|\big|$,

which allows us to approximate,

$$2^{-1}\|f_0\| - 2^{-1}\|f_0 - f_k\| \leq 2^{-1}\|f_k\| \leq |f_k(x_k)|,$$

the rightmost member is Bounded above by

$$|f_k(x_k)| \leq |f_0(x_k)| + \|f_0 - f_k\|,$$

so that $0 < 2^{-1}(\|f_0\| - 3\|f_0 - f_k\|) \leq |f_0(x_k)|$

this shows that no non-zero functional can vanish on $\overline{M}$, i.e: $\overline{M} = X$ $\square$

5.25 Commentary

the main idea is that, if a subspace is not dense, then it allows for HB separation on the closure,

5.29

Let $Y = \ell^1(\mathbb{N}^+)$ with the counting measure, and $X = \{f \in Y, \sum_1^\infty j|f(j)| < +\infty\}$.

a) X is a proper dense subspace of Y.

proof $\forall f, g \in X$, $a, b \in \mathbb{C}$.

$$\sum_1^\infty j|af(j) + bg(j)| \leq |a| \cdot \sum_1^\infty j|f| + |b|\sum_1^\infty j|g|.$$

X is dense, Because it contains the dense subspace of finitely supported sequences; $\Sigma \subseteq X$.

$$\Sigma = \{f \in Y, \ f(j) = 0 \text{ eventually}\}$$

It is clear that X is proper, subspace, because not every summable sequence is summable against $\{j\}$, Take $x_n = n^{-2}$,

b) $T: X \to Y$, $T(\{x_n\})(n) = n x_n$,

Then: T is linear, closed, But discontinuous.

Linearity is a triviality,

First we verify unboundedness, so we find sequences indexed by j:

$$\varphi(j,k) = \delta_{j,k} \in X,$$

$\| \varphi(\cdot,k)\| = 1 \ \forall k \geq 1$, But $\|T\varphi(\cdot,k)\| = k$

so that T is not bounded on the sphere.

We show that T is closed. This will require some effort. Suppose $(A,B) \in X \times Y$ is an adherent point of the graph of T, meaning we can find $\varphi : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{C}$,

$$\sum_j |\varphi(j,k) - A(j)| + \sum_j |j\varphi(j,k) - B(j)| = \omega_k$$

where $\omega_k \to 0$,

we have used the product metric.

Now, we first show that $A(\cdot) \in X$, then we prove $jA(j) = B(j)$.

Fatou's lemma implies the first claim, with $|\varphi(j,k)| \to |A(j)|$ pointwise as $k \nearrow +\infty$,

Bound $|j\varphi(j,k)| \leq |j\varphi(j,k) - B(j)| + |B(j)|$.

then

$$\sum_1^{\infty} |jA(j)| \leq \liminf_{k\to\infty} \omega_k + \sum_j^{\infty} |B(j)| < +\infty.$$

this shows $A \in X$. now, to show that

$$jA(j) = B(j),$$

It suffices to show that $\|jA(j) - j\varphi(j,k)\|_{\ell^1} \to 0$ by Ex 2.21, because $\{jA(j)\}, (j\varphi(j,k)) \in L^1$, and $j\varphi(j,k) \to jA(j)$ pwise,

$\|jA(j) - j\varphi(j,k)\|_{\ell^1} \to 0$ iff $\big| \|jA(j)\|_{\ell^1} - \|j\varphi(j,k)\|\big| \to 0$.

we show the later by the generalized Dominated convergence theorem.

$$|j\varphi(j,k)| \leq |j\varphi(j,k) - B(j)| + |B(j)|.$$

the right hand side converges pw to $|B(j)|$, and its sum converges to $\sum_1^{\infty} |B(j)|$ as $k \to +\infty$,

Then $\sum_{j=1}^{\infty} |j\varphi(j,k)| \to \sum_{j=1}^{\infty} |jA(j)|$.

and $\|jA(j) - B(j)\|_{\ell^1} \leq \omega_k + \|jA(j) - j\varphi(j,k)\|_{\ell^1}$
$\leq 2\omega_k$

So that $T(A(j)) = B(j)$ and the graph of T must be closed. $\square$

c) Define $S = T^{-1} : Y \to X$, then S is bounded, surjective but not open

proof S is bounded because it takes

$$\{x_n\} \xrightarrow{S} \{n^{-1} x_n\},$$ by Hölder's inequality.

and is clearly surjective. Now,

suppose that S is open, meaning we can find $C>0$ such that

$$B_X(C,0) \subseteq S(B_Y(1,0))$$

elements in the left member are precisely those: $\|\varphi(j)\|_{\ell^1} < C$, and $\|j\varphi(j)\|_{\ell^1} < +\infty$.

let $\frac{C}{2}m > 1$, and $\varphi(j) = 2^{-1}C\,\delta_{m,j}$. then $\|\varphi(j)\| = 2^{-1}C < C$, and become finitely supported, $\varphi \in X$. Now suppose that $\exists \psi \in Y$, $\varphi(j) = (S\psi)(j) = (j^{-1})\psi(j)$

Which means,

$$\psi(j) = (2^{-1}C\,j)\,\delta_{m,j}, \quad \text{and}$$

$$\|\psi(j)\|_{\ell^1} = 2^{-1}Cm > 1$$

so that S is not open. $\square$

Signed measures

$$\nu : \mathcal{M} \to [-\infty, +\infty]$$

1) $\nu(\emptyset) = 0$

2) ν assumes at most one $\pm\infty$

3) If $\{E_n\}$ disjoint,

$$\nu(\cup_1^\infty E_n) = \lim_{n\to\infty} \sum_1^n \nu(E_j).$$

the sum converges absolutely if $|\nu(\cup_1^\infty E_n)| < +\infty$

1°) If $E \in \mathcal{M}$, $\nu(E) = \pm\infty$, then $\nu(F) = \pm\infty$ $\forall F \supseteq E$, $F \in \mathcal{M}$.

Def A measurable subset $E \in \mathcal{M}$ is positive. if $\nu_E : \mathcal{M} \to [0, +\infty]$, where $\nu_E(A) = \nu(A \cap E)$ defines a positive measure.

Def A measurable subset $E \in \mathcal{M}$ is negate, if $(-1)\nu_E : \mathcal{M} \to [0, +\infty]$ defines a pos. measure

Def $E \in \mathcal{M}$ is null if it is Both positive and negative.