5.8 Theorem. *Let $\mathcal{X}$ be a normed vector space.*

a. *If $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ and $x \in \mathcal{X} \setminus \mathcal{M}$, there exists $f \in \mathcal{X}^*$ such that $f(x) \neq 0$ and $f|\mathcal{M} = 0$. In fact, if $\delta = \inf_{y \in \mathcal{M}} \|x - y\|$, f can be taken to satisfy $\|f\| = 1$ and $f(x) = \delta$.*

b. *If $x \neq 0 \in \mathcal{X}$, there exists $f \in \mathcal{X}^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.*

c. *The bounded linear functionals on $\mathcal{X}$ separate points.*

d. *If $x \in \mathcal{X}$, define $\widehat{x} : \mathcal{X}^* \to \mathbb{C}$ by $\widehat{x}(f) = f(x)$. Then the map $x \mapsto \widehat{x}$ is a linear isometry from $\mathcal{X}$ into $\mathcal{X}^{**}$ (the dual of $\mathcal{X}^*$).*

Proof. To prove (a), define f on $\mathcal{M} + \mathbb{C}x$ by $f(y + \lambda x) = \lambda\delta$ ($y \in \mathcal{M}$, $\lambda \in \mathbb{C}$). Then $f(x) = \delta$, $f|\mathcal{M} = 0$, and for $\lambda \neq 0$, $|f(y + \lambda x)| = |\lambda|\delta \leq |\lambda| \|\lambda^{-1}y + x\| = \|y + \lambda x\|$. Thus the Hahn-Banach theorem can be applied, with $p(x) = \|x\|$ and $\mathcal{M}$ replaced by $\mathcal{M} + \mathbb{C}x$. (b) is the special case of (a) with $\mathcal{M} = \{0\}$, and (c) follows immediately: if $x \neq y$, there exists $f \in \mathcal{X}^*$ with $f(x - y) \neq 0$, i.e., $f(x) \neq f(y)$. As for (d), obviously $\widehat{x}$ is a linear functional on $\mathcal{X}^*$ and the map $x \mapsto \widehat{x}$ is linear. Moroever, $|\widehat{x}(f)| = |f(x)| \leq \|f\| \|x\|$, so $\|\widehat{x}\| \leq \|x\|$. On the other hand, (b) implies that $\|\widehat{x}\| \geq \|x\|$. ∎

1°) Can compute $\|x\|$ by looking at $\hat{x} \in X^{**}$'s action on $\Sigma \triangleq \{f \in X^*, \|f\| = 1\}$

$\to$ $\forall x \in X$ $\exists f \in X^*$, $\|f\| = 1$, $f(x) = \|x\|$, and

$\to$ $$\|x\| = \sup_{f \in \Sigma} |f(x)|$$

very similar to L^p spaces dual.

2°) Can separate closed subspaces from points using $\Sigma \triangleq \{f \in X^*, \|f\| = 1\}$

$\to$ If M is a closed subspace, and $x \in X \setminus M$, $\exists f \in \Sigma$, $f|_M = 0$, and $f(x) = \inf_{y \in M} \|x - y\|$.

Folland Exercises 5.22–5.25

22. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T \in L(\mathcal{X}, \mathcal{Y})$.

a. Define $T^\dagger : \mathcal{Y}^* \to \mathcal{X}^*$ by $T^\dagger f = f \circ T$. Then $T^\dagger \in L(\mathcal{Y}^*, \mathcal{X}^*)$ and $\|T^\dagger\| = \|T\|$. $T^\dagger$ is called the **adjoint** or **transpose** of T.

b. Applying the construction in (a) twice, one obtains $T^{\dagger\dagger} \in L(\mathcal{X}^{**}, \mathcal{Y}^{**})$. If $\mathcal{X}$ and $\mathcal{Y}$ are identified with their natural images $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ in $\mathcal{X}^{**}$ and $\mathcal{Y}^{**}$, then $T^{\dagger\dagger}|\mathcal{X} = T$.

c. $T^\dagger$ is injective iff the range of T is dense in $\mathcal{Y}$.

d. If the range of $T^\dagger$ is dense in $\mathcal{X}^*$, then T is injective; the converse is true if $\mathcal{X}$ is reflexive.

23. Suppose that $\mathcal{X}$ is a Banach space. If $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ and $\mathcal{N}$ is a closed subspace of $\mathcal{X}^*$, let $\mathcal{M}^0 = \{f \in \mathcal{X}^* : f|\mathcal{M} = 0\}$ and $\mathcal{N}^\perp = \{x \in \mathcal{X} : f(x) = 0$ for all $f \in \mathcal{N}\}$. (Thus, if we identify $\mathcal{X}$ with its image in $\mathcal{X}^{**}$, $\mathcal{N}^\perp = \mathcal{N}^0 \cap \mathcal{X}$.)

a. $\mathcal{M}^0$ and $\mathcal{N}^\perp$ are closed subspaces of $\mathcal{X}^*$ and $\mathcal{X}$, respectively.

b. $(\mathcal{M}^0)^\perp = \mathcal{M}$ and $(\mathcal{N}^\perp)^0 \supset \mathcal{N}$. If $\mathcal{X}$ is reflexive, $(\mathcal{N}^\perp)^0 = \mathcal{N}$.

c. Let $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ be the natural projection, and define $\alpha : (\mathcal{X}/\mathcal{M})^* \to \mathcal{X}^*$ by $\alpha(f) = f \circ \pi$. Then α is an isometric isomorphism from $(\mathcal{X}/\mathcal{M})^*$ onto $\mathcal{M}^0$, where $\mathcal{X}/\mathcal{M}$ has the quotient norm.

d. Define $\beta : \mathcal{X}^* \to \mathcal{M}^*$ by $\beta(f) = f|\mathcal{M}$; then β induces a map $\overline{\beta} : \mathcal{X}^*/\mathcal{M}^0 \to \mathcal{M}^*$ as in Exercise 15, and $\overline{\beta}$ is an isometric isomorphism.

24. Suppose that $\mathcal{X}$ is a Banach space.

a. Let $\widehat{\mathcal{X}}$, $(\mathcal{X}^*)\widehat{}$ be the natural images of $\mathcal{X}, \mathcal{X}^*$ in $\mathcal{X}^{**}, \mathcal{X}^{***}$, and let $\widehat{\mathcal{X}}^0 = \{F \in \mathcal{X}^{***} : F|\widehat{\mathcal{X}} = 0\}$. Then $(\mathcal{X}^*)\widehat{} \cap \widehat{\mathcal{X}}^0 = \{0\}$ and $(\mathcal{X}^*)\widehat{} + \widehat{\mathcal{X}}^0 = \mathcal{X}^{***}$.

b. $\mathcal{X}$ is reflexive iff $\mathcal{X}^*$ is reflexive.

25. If $\mathcal{X}$ is a Banach space and $\mathcal{X}^*$ is separable, then $\mathcal{X}$ is separable. (Let $\{f_n\}_1^\infty$ be a countable dense subset of $\mathcal{X}^*$. For each n choose $x_n \in \mathcal{X}$ with $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$. Then the linear combinations of $\{x_n\}_1^\infty$ are dense in $\mathcal{X}$.) *Note:* Separability of $\mathcal{X}$ does not imply separability of $\mathcal{X}^*$.

22a) fix $f \in Y^*$, and $\forall x \in X$

$$\langle T^\dagger f, x \rangle = \langle f, Tx \rangle$$

means

$$|\langle T^\dagger f, x\rangle| \leq |f| \cdot |T| \cdot |x|$$

$\Rightarrow$ $|T^\dagger f| \leq |T| \cdot |f|$ where $T^\dagger f \in X^*$ $\forall f \in Y^*$

At the same time, $T^\dagger : Y^* \to X^*$ is linear, so the last estimate also proves

$T^\dagger$ is a top.linear mapping with

$$\|T^\dagger\| \leq \|T\|.$$

We will show that $\|T^\dagger\| = \|T\|$.

$\forall x \in X$, Tx is an element in Y, so by existence of functionals, we can find $f \in Y^*$, $\|f\| = 1$, such that

$$f(Tx) = (T^\dagger f)(x) = \|Tx\|.$$

Hence: $\|Tx\| \leq \|T^\dagger\| \cdot \|f\| \cdot \|x\| = \|T^\dagger\| \cdot \|x\|$. $(\forall x \in X)$.

The number $\|T^\dagger\|$ satisfies some upperbound condition, for which $\|T\|$ is the infimum of ($\|T\| = \inf\{C > 0,\ \|Tx\| \leq C\|x\|\ \forall x \in X\}$)

Which means $\|T\| \leq \|T^\dagger\|$.

22b) Want to show, $\forall x \in X$, $T^{\dagger\dagger}(\hat{x})$ can be uniquely identified with $\hat{y} \in \hat{Y} = \{\hat{y}, y \in Y\}$ where $\hat{x} \in X^{**}$ is the natural embedding.

$$\begin{tikzcd} X \arrow[r, "\wedge"] \arrow[d, "T"'] & X^{**} \arrow[d, "T^{\dagger\dagger}"] \\ Y \arrow[r, "\wedge"] & Y^{**} \end{tikzcd}$$

Fix any $f \in Y^*$, because $T^{\dagger\dagger}(\hat{x})$ as an element in Y^{**}, is uniquely identified by its action on Y^*,

It suffices to show

$$\langle f, Tx \rangle = \langle T^{\dagger\dagger}(\hat{x}), f \rangle$$

To proceed,

$$\text{LHS} = \langle T^\dagger f, x \rangle = \langle \hat{x}, T^\dagger f \rangle_{(X^{**}, X^*)}$$

which is by definition of $T^{\dagger\dagger}$, equal to $\langle T^{\dagger\dagger}\hat{x}, f \rangle$.

and the claim is proven.

Lemma

Characterisation of Dense Subspaces

Let X be a Banach space and M a subspace (not necessarily closed), and define:

$$M^{\perp} \triangleq \{ f \in X^*,\ f(x) = 0\ \forall x \in M \},$$

Then $M^{\perp} = \{0\}$ iff $\overline{M} = X$ (M is dense).

proof For every subset $M \subseteq X$, $\{0\} \subseteq M^{\perp}$ trivially. It suffices to show

$$M^{\perp} \subseteq \{0\} \text{ iff } \overline{M} = X .$$

If M is not dense, we find $f_0 \in M^{\perp} \setminus \{0\}$. This can be done by selecting $x_0 \in X \setminus \overline{M}$ and $f_0 \in \Sigma = \{ f \in X^*, \|f\| = 1 \}$ as in Theorem 5.8a. with $f_0|_M = 0$, and $f_0(x_0) = \delta = \inf_{y \in M} |x_0 - y|$.

If M is dense, every $f \in M^{\perp}$ extends continuously to be the 0 functional on X, So the proposition is proven. ∎

What I mean is $f|_M \triangleq g$, and $\bar{g}$ be its bounded extension, $f = \bar{g} = 0$.

22c) wts: T^+ is injective iff $T(X)$ is dense in Y.

This can be easily proven using the lemma displayed on the left.

Note that $\operatorname{Ker}(T^+) = T(X)^{\perp}$. Indeed:

$$f \in \operatorname{Ker}(T^+) \text{ iff } \forall x \in X,\ (T^+ f)(x) = f(Tx) = 0.$$

So:

$$T^+ \text{ is injective} \iff T(X)^{\perp} = \{0\}$$
$$\iff T(X) \text{ is dense in } Y. \quad ∎$$

22d) $T^+(Y^*)$ dense in $X^* \iff T^{++}: X^{**} \to Y^{**}$ is injective.

So, if $T^+(Y^*)$ is dense in X^*, $T^{++}|_{\hat{X}} \approx T$ is an injection.

Assume the converse & X is reflexive, then,

T is an injection $\Rightarrow$ $\Psi_Y \circ T \circ \Psi_X^{-1}$ is an injection.

where

$\underbrace{\Psi_X : X \to X^{**}}_{\text{toplinear isomorphism}}$, $\underbrace{\Psi_Y : Y \to Y^{**}}_{\text{toplinear injection}}$.

But $T^{++} = \Psi_Y \circ T \circ \Psi_X^{-1}$, so T^{++} is an injection, and $T^+(Y^*)$ must be dense in X^*.

22. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T \in L(\mathcal{X}, \mathcal{Y})$.

a. Define $T^\dagger : \mathcal{Y}^* \to \mathcal{X}^*$ by $T^\dagger f = f \circ T$. Then $T^\dagger \in L(\mathcal{Y}^*, \mathcal{X}^*)$ and $\|T^\dagger\| = \|T\|$. $T^\dagger$ is called the **adjoint** or **transpose** of T.

b. Applying the construction in (a) twice, one obtains $T^{\dagger\dagger} \in L(\mathcal{X}^{**}, \mathcal{Y}^{**})$. If $\mathcal{X}$ and $\mathcal{Y}$ are identified with their natural images $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ in $\mathcal{X}^{**}$ and $\mathcal{Y}^{**}$, then $T^{\dagger\dagger}|\mathcal{X} = T$.

c. $T^\dagger$ is injective iff the range of T is dense in $\mathcal{Y}$.

d. If the range of $T^\dagger$ is dense in $\mathcal{X}^*$, then T is injective; the converse is true if $\mathcal{X}$ is reflexive.

23. Suppose that $\mathcal{X}$ is a Banach space. If $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ and $\mathcal{N}$ is a closed subspace of $\mathcal{X}^*$, let $\mathcal{M}^0 = \{f \in \mathcal{X}^* : f|\mathcal{M} = 0\}$ and $\mathcal{N}^\perp = \{x \in \mathcal{X} : f(x) = 0$ for all $f \in \mathcal{N}\}$. (Thus, if we identify $\mathcal{X}$ with its image in $\mathcal{X}^{**}$, $\mathcal{N}^\perp = \mathcal{N}^0 \cap \mathcal{X}$.)

a. $\mathcal{M}^0$ and $\mathcal{N}^\perp$ are closed subspaces of $\mathcal{X}^*$ and $\mathcal{X}$, respectively.

b. $(\mathcal{M}^0)^\perp = \mathcal{M}$ and $(\mathcal{N}^\perp)^0 \supset \mathcal{N}$. If $\mathcal{X}$ is reflexive, $(\mathcal{N}^\perp)^0 = \mathcal{N}$.

c. Let $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ be the natural projection, and define $\alpha : (\mathcal{X}/\mathcal{M})^* \to \mathcal{X}^*$ by $\alpha(f) = f \circ \pi$. Then α is an isometric isomorphism from $(\mathcal{X}/\mathcal{M})^*$ onto $\mathcal{M}^0$, where $\mathcal{X}/\mathcal{M}$ has the quotient norm.

d. Define $\beta : \mathcal{X}^* \to \mathcal{M}^*$ by $\beta(f) = f|\mathcal{M}$; then β induces a map $\overline{\beta} : \mathcal{X}^*/\mathcal{M}^0 \to \mathcal{M}^*$ as in Exercise 15, and $\overline{\beta}$ is an isometric isomorphism.

24. Suppose that $\mathcal{X}$ is a Banach space.

a. Let $\widehat{\mathcal{X}}$, $(\mathcal{X}^*)\widehat{}$ be the natural images of $\mathcal{X}, \mathcal{X}^*$ in $\mathcal{X}^{**}, \mathcal{X}^{***}$, and let $\widehat{\mathcal{X}}^0 = \{F \in \mathcal{X}^{***} : F|\widehat{\mathcal{X}} = 0\}$. Then $(\mathcal{X}^*)\widehat{} \cap \widehat{\mathcal{X}}^0 = \{0\}$ and $(\mathcal{X}^*)\widehat{} + \widehat{\mathcal{X}}^0 = \mathcal{X}^{***}$.

b. $\mathcal{X}$ is reflexive iff $\mathcal{X}^*$ is reflexive.

25. If $\mathcal{X}$ is a Banach space and $\mathcal{X}^*$ is separable, then $\mathcal{X}$ is separable. (Let $\{f_n\}_1^\infty$ be a countable dense subset of $\mathcal{X}^*$. For each n choose $x_n \in \mathcal{X}$ with $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$. Then the linear combinations of $\{x_n\}_1^\infty$ are dense in $\mathcal{X}$.) *Note:* Separability of $\mathcal{X}$ does not imply separability of $\mathcal{X}^*$.

5.23a) $\forall x \in X$ $\forall f \in X^*$, $Ker(\hat{x}) \subseteq X^*$ and $Ker(f) \subseteq X$ are closed subspaces. Hence

1°) $M^\circ = \{f \in X^*, \hat{x}(f) = 0 \ \forall x \in M\} = \bigcap_{x \in M} Ker(\hat{x})$

2°) $N^\perp = \Psi_X^{-1}(N^\circ)$, where $\Psi_X : X \to X^{**}$ is the top.linear embedding (which is continuous).

Both are closed subspaces of X^*, X respectively.

5.23b) $(M^\circ)^\perp = \overline{M}$ (for any subspace M).

$\forall x \in M$, $\forall f \in M^\circ = \bigcap_{x \in M} Ker(\hat{x})$, $f(x) = \hat{x}(f) = \hat{f}(\hat{x}) = 0$. (where $\hat{f} \in X^{***}$ injected through natural map)

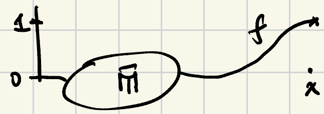
Hence: $x \in \Psi_X^{-1}\left(\bigcap_{f \in M^\circ} Ker(\hat{f})\right)$

and $M \subseteq (M^\circ)^\perp$, since $(M^\circ)^\perp$ is closed, it contains the closure of M as well.

Conversely, if $x \in (M^\circ)^\perp$, then $\Psi_X(x) \in M^{\circ\circ}$ meaning every functional f that vanishes on M, must vanish on x.

$\forall f \in X^*$ st $f \in \bigcap_{z \in M} Ker(\hat{z})$, $\hat{x}(f) = f(x) = 0$.

Then x has to be in $\overline{M}$, because if $x \notin \overline{M}$,



we can find a functional $f \in X^*$, $f|_{\overline{M}} = 0$, and $f(x) \neq 0 \Rightarrow x \notin (M^\circ)^\perp$.

Idea of subordinates and superordinates, but applied to Banach Spaces?

↗ $M \subseteq X$ closed subset of a T_4 space.

$M^\circ = \{f \in C(X, [0,1]), f(x) = 0 \ \forall x \in M\}$

$\mathcal{J}_1(M) \triangleq M^\circ$

what makes a good definition for $\mathcal{J}_2$?

22. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T \in L(\mathcal{X},\mathcal{Y})$.

a. Define $T^\dagger : \mathcal{Y}^* \to \mathcal{X}^*$ by $T^\dagger f = f \circ T$. Then $T^\dagger \in L(\mathcal{Y}^*,\mathcal{X}^*)$ and $\|T^\dagger\| = \|T\|$. $T^\dagger$ is called the **adjoint** or **transpose** of T.

b. Applying the construction in (a) twice, one obtains $T^{\dagger\dagger} \in L(\mathcal{X}^{**},\mathcal{Y}^{**})$. If $\mathcal{X}$ and $\mathcal{Y}$ are identified with their natural images $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ in $\mathcal{X}^{**}$ and $\mathcal{Y}^{**}$, then $T^{\dagger\dagger}|\mathcal{X} = T$.

c. $T^\dagger$ is injective iff the range of T is dense in $\mathcal{Y}$.

d. If the range of $T^\dagger$ is dense in $\mathcal{X}^*$, then T is injective; the converse is true if $\mathcal{X}$ is reflexive.

23. Suppose that $\mathcal{X}$ is a Banach space. If $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ and $\mathcal{N}$ is a closed subspace of $\mathcal{X}^*$, let $\mathcal{M}^0 = \{f \in \mathcal{X}^* : f|\mathcal{M} = 0\}$ and $\mathcal{N}^\perp = \{x \in \mathcal{X} : f(x) = 0$ for all $f \in \mathcal{N}\}$. (Thus, if we identify $\mathcal{X}$ with its image in $\mathcal{X}^{**}$, $\mathcal{N}^\perp = \mathcal{N}^0 \cap \mathcal{X}$.)

a. $\mathcal{M}^0$ and $\mathcal{N}^\perp$ are closed subspaces of $\mathcal{X}^*$ and $\mathcal{X}$, respectively.

b. $(\mathcal{M}^0)^\perp = \mathcal{M}$ and $(\mathcal{N}^\perp)^0 \supset \mathcal{N}$. If $\mathcal{X}$ is reflexive, $(\mathcal{N}^\perp)^0 = \mathcal{N}$.

c. Let $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ be the natural projection, and define $\alpha : (\mathcal{X}/\mathcal{M})^* \to \mathcal{X}^*$ by $\alpha(f) = f \circ \pi$. Then α is an isometric isomorphism from $(\mathcal{X}/\mathcal{M})^*$ onto $\mathcal{M}^0$, where $\mathcal{X}/\mathcal{M}$ has the quotient norm.

d. Define $\beta : \mathcal{X}^* \to \mathcal{M}^*$ by $\beta(f) = f|\mathcal{M}$; then β induces a map $\overline{\beta} : \mathcal{X}^*/\mathcal{M}^0 \to \mathcal{M}^*$ as in Exercise 15, and $\overline{\beta}$ is an isometric isomorphism.

24. Suppose that $\mathcal{X}$ is a Banach space.

a. Let $\widehat{\mathcal{X}}$, $(\mathcal{X}^*)\widehat{}$ be the natural images of $\mathcal{X}$, $\mathcal{X}^*$ in $\mathcal{X}^{**}$, $\mathcal{X}^{***}$, and let $\widehat{\mathcal{X}}^0 = \{F \in \mathcal{X}^{***} : F|\widehat{\mathcal{X}} = 0\}$. Then $(\mathcal{X}^*)\widehat{} \cap \widehat{\mathcal{X}}^0 = \{0\}$ and $(\mathcal{X}^*)\widehat{} + \widehat{\mathcal{X}}^0 = \mathcal{X}^{***}$.

b. $\mathcal{X}$ is reflexive iff $\mathcal{X}^*$ is reflexive.

25. If $\mathcal{X}$ is a Banach space and $\mathcal{X}^*$ is separable, then $\mathcal{X}$ is separable. (Let $\{f_n\}_1^\infty$ be a countable dense subset of $\mathcal{X}^*$. For each n choose $x_n \in \mathcal{X}$ with $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$. Then the linear combinations of $\{x_n\}_1^\infty$ are dense in $\mathcal{X}$.) *Note:* Separability of $\mathcal{X}$ does not imply separability of $\mathcal{X}^*$.

12. Let $\mathcal{X}$ be a normed vector space and $\mathcal{M}$ a proper closed subspace of $\mathcal{X}$.

a. $\|x + \mathcal{M}\| = \inf\{\|x + y\| : y \in \mathcal{M}\}$ is a norm on $\mathcal{X}/\mathcal{M}$.

b. For any $\epsilon > 0$ there exists $x \in \mathcal{X}$ such that $\|x\| = 1$ and $\|x + \mathcal{M}\| \geq 1 - \epsilon$.

c. The projection map $\pi(x) = x + \mathcal{M}$ from $\mathcal{X}$ to $\mathcal{X}/\mathcal{M}$ has norm 1.

d. If $\mathcal{X}$ is complete, so is $\mathcal{X}/\mathcal{M}$. (Use Theorem 5.1.)

e. The topology defined by the quotient norm is the quotient topology as defined in Exercise 28 in §4.2.

13. If $\|\cdot\|$ is a seminorm on the vector space $\mathcal{X}$, let $\mathcal{M} = \{x \in \mathcal{X} : \|x\| = 0\}$. Then $\mathcal{M}$ is a subspace, and the map $x + \mathcal{M} \mapsto \|x\|$ is a norm on $\mathcal{X}/\mathcal{M}$.

14. If $\mathcal{X}$ is a normed vector space and $\mathcal{M}$ is a nonclosed subspace, then $\|x + \mathcal{M}\|$, as defined in Exercise 12, is a seminorm on $\mathcal{X}/\mathcal{M}$. If one divides by its nullspace as in Exercise 13, the resulting quotient space is isometrically isomorphic to $\mathcal{X}/\overline{\mathcal{M}}$. (Cf. Exercise 5.)

15. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T \in L(\mathcal{X},\mathcal{Y})$. Let $\mathcal{N}(T) = \{x \in \mathcal{X} : Tx = 0\}$.

a. $\mathcal{N}(T)$ is a closed subspace of $\mathcal{X}$.

b. There is a unique $S \in L(\mathcal{X}/\mathcal{N}(T), \mathcal{Y})$ such that $T = S \circ \pi$ where $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ is the projection (see Exercise 12). Moreover, $\|S\| = \|T\|$.

5.23c) Let $M \subseteq X$ closed subspace.

$$\|x+M\|_{X/M} = \inf_{y \in M} \|x-y\|$$

If X is Banach, so is X/M. (Ex 5.12d)

For every $f \in (X/M)^*$, define

$$\alpha(f) \triangleq (\pi^* f) \in X^*.$$

$$\begin{array}{c} X \\ \downarrow \pi \\ X/M \end{array}$$

We claim that ① $\alpha : (X/M)^* \to M^\circ$ is an ② isometric isomorphism

① Fix $f \in (X/M)^*$ and $x \in M$, then,

$$\alpha(f)(x) = f(x+M) = f(0+M) = 0$$

So $\alpha((X/M)^*) \subseteq M^\circ$, which is a closed subspace of X^*, and is again a Banach Space.

② Suffices to show that α is linear, continuous, isometry, surjection, which will imply α is an isometric isomorphism.

α is clearly toplinear (as a mapping into X^*),

Moreover, $\|\alpha\| = \|\pi^*\| = \|\pi\| = 1$, so $\forall f \in (X/M)^*$,

$$\|\alpha(f)\| \le \|\alpha\| \cdot \|f\| = \|f\|.$$

We wish to show the reverse equality.

→ Approach $\|f\|_{(X/M)^*}$ with $\{z_n + M\} \subseteq (X/M)$, such that

$$\lim |f(z_n+M)| = \|f\|, \text{ and } \|z_n + M\| = 1.$$

By using the definition of quotient norm, fix $\{\epsilon_n\}_1^\infty \in c_0 \cap \ell^{++}$, and $\forall n \ge 1$, pick $x_n \in X$ such that.

$$\pi(x_n) = z_n + M, \text{ and } 1 \le \|x_n\| \le 1 + \epsilon_n.$$

(With $\liminf \|x_n\| = 1$.)

→ Now, we bound $|f(z_n+M)| = |\alpha(f)(x_n)|$,

$$|f(z_n+M)| \le \|\alpha(f)\| \cdot \|x_n\| \le \|\alpha(f)\| \cdot (1+\epsilon_n).$$

Hence: $\|f\| \le \|\alpha(f)\|$ upon liminf ≤ liminf.

and $\alpha : (X/M)^* \to X^*$ is a toplinear isometry,

→ Therefore α is a toplinear isomorphism onto its range.

We compute its range. We know $\alpha((X/M)^*) \subseteq M^\circ$.
Conversely, $\forall g \in M^\circ$, define

$$f : (X/M) \to \mathbb{R}, \quad \text{by} \quad f(x+M) \triangleq g(x).$$

This is independent of representative since $g \in M^\circ$, and produces a linear mapping. (Not continuous)

proof for continuity.

Moreover, $\forall \varepsilon > 0$ $\forall (x+M) \in X/M$:
there exists $y \in X$:
1°) $y + M = x + M$
2°) $\|x+M\| \le \|y\| \le \|x+M\| + \varepsilon$.

So we may take $f(x+M) = g(y)$. with y above.
And $\forall (x+M) \in X/M$, $\|x+M\| = 1$,

$$|f(x+M)| \le \|g\|(1+\varepsilon)\|x+M\|$$

This defines a continuous linear functional on (X/M), and almost by definition, $\alpha(f) = g$,
Therefore α is a surjection onto M° $\square$

5.23d) The restriction map is clearly toplinear, the Kernel of $\beta : X^* \to M^*$, is $M^\circ \subseteq X^*$

→ $\forall f \in \mathrm{Ker}(\beta)$, $\forall x \in M$, $\beta(f)(x) = f(x) = 0$
so $f \in M^\circ$. Conversely if $f \in M^\circ$ clearly $f|_M = \beta(f) = 0$.

(Also: β is clearly a surjection, fix $f \in M^*$, extend by Hahn-Banach to $\tilde{f} \in X^*$, and $\beta(\tilde{f}) = f$)

→ Repeating Ex 5.15, if $\pi : X^* \to X^*/M^\circ$ is the Quotient projection, there is a unique $\bar{\beta} \in L(X^*/M^\circ, M^*)$ such that the diagram commutes.

$$\begin{tikzcd} X^* \arrow[d, "\pi"'] \arrow[dr, "\beta"] & \\ X^*/M^\circ \arrow[r, "\bar{\beta}"'] & M^* \end{tikzcd}$$

Also: $\|\bar{\beta}\| = \|\beta\|$ by (5.15), and if $i_M : M \to X$ is the toplinear inclusion, then $\|i_M\| = 1$,
Notice: $\beta = i_M^*$, so $\|\beta\| = \|i_M^*\| = \|i_M\| = 1$.
$\Rightarrow$ $\|\bar{\beta}\| = 1$, and

$$\|\bar{\beta}(f + M^\circ)\| \le \|\bar{\beta}\|\|f + M^\circ\| \le \|f + M^\circ\|.$$

Hahn-Banach Extensions (same norm)

If $M \leq X$ subspace, and $f: M \to \mathbb{C}$ linear, with

$$\|f\| \triangleq \sup\{|f(x)|,\ x \in M,\ \|x\|=1\}$$

then there exists $\bar{f} \in X^*$, not unique, with $\|\bar{f}\| = \|f\|$, and $\bar{f}|_M = f$.

proof Apply Hahn-Banach with the semi-norm Analytic (Extension form)

$$p(x) = \|f\| \cdot \|x\| \quad \text{where}$$

$$|f(x)| \leq p(x) \qquad \forall x \in M$$

$\square$

→ Show isometry: Fix $f + M^\circ \in X^*/M^\circ$ and $g \triangleq \beta(f)$ (this is independent of representative)

We know that, by Hahn-Banach, there is an extension of $g: M \to \mathbb{R}$ to X, with the same operator norm.

$$\to \bar{g}: X \to \mathbb{R}, \quad \|\bar{g}\| = \|g\|, \qquad (\bar{g} - f)|_M = 0$$

$$\to \beta(\bar{g}) = g = \bar{\beta}(f + M^\circ),$$

So that

Quotient Projections are Norm decreasing.

$$\|\bar{\beta}(f+M^\circ)\| = \|\bar{g}\| \geq \|\pi(\bar{g})\| = \|f + M^\circ\|$$

Therefore

$$\|\bar{\beta}(f+M^\circ)\| = \|f + M^\circ\|,$$ and $\bar{\beta}$ is a surjective isometry, hence an isometric isomorphism.

$\square$

22. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T \in L(\mathcal{X}, \mathcal{Y})$.

a. Define $T^\dagger : \mathcal{Y}^* \to \mathcal{X}^*$ by $T^\dagger f = f \circ T$. Then $T^\dagger \in L(\mathcal{Y}^*, \mathcal{X}^*)$ and $\|T^\dagger\| = \|T\|$. $T^\dagger$ is called the **adjoint** or **transpose** of T.

b. Applying the construction in (a) twice, one obtains $T^{\dagger\dagger} \in L(\mathcal{X}^{**}, \mathcal{Y}^{**})$. If $\mathcal{X}$ and $\mathcal{Y}$ are identified with their natural images $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ in $\mathcal{X}^{**}$ and $\mathcal{Y}^{**}$, then $T^{\dagger\dagger}|\mathcal{X} = T$.

c. $T^\dagger$ is injective iff the range of T is dense in $\mathcal{Y}$.

d. If the range of $T^\dagger$ is dense in $\mathcal{X}^*$, then T is injective; the converse is true if $\mathcal{X}$ is reflexive.

23. Suppose that $\mathcal{X}$ is a Banach space. If $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ and $\mathcal{N}$ is a closed subspace of $\mathcal{X}^*$, let $\mathcal{M}^0 = \{f \in \mathcal{X}^* : f|\mathcal{M} = 0\}$ and $\mathcal{N}^\perp = \{x \in \mathcal{X} : f(x) = 0 \text{ for all } f \in \mathcal{N}\}$. (Thus, if we identify $\mathcal{X}$ with its image in $\mathcal{X}^{**}$, $\mathcal{N}^\perp = \mathcal{N}^0 \cap \mathcal{X}$.)

a. $\mathcal{M}^0$ and $\mathcal{N}^\perp$ are closed subspaces of $\mathcal{X}^*$ and $\mathcal{X}$, respectively.

b. $(\mathcal{M}^0)^\perp = \mathcal{M}$ and $(\mathcal{N}^\perp)^0 \supset \mathcal{N}$. If $\mathcal{X}$ is reflexive, $(\mathcal{N}^\perp)^0 = \mathcal{N}$.

c. Let $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ be the natural projection, and define $\alpha : (\mathcal{X}/\mathcal{M})^* \to \mathcal{X}^*$ by $\alpha(f) = f \circ \pi$. Then α is an isometric isomorphism from $(\mathcal{X}/\mathcal{M})^*$ onto $\mathcal{M}^0$, where $\mathcal{X}/\mathcal{M}$ has the quotient norm.

d. Define $\beta : \mathcal{X}^* \to \mathcal{M}^*$ by $\beta(f) = f|\mathcal{M}$; then β induces a map $\overline{\beta} : \mathcal{X}^*/\mathcal{M}^0 \to \mathcal{M}^*$ as in Exercise 15, and $\overline{\beta}$ is an isometric isomorphism.

24. Suppose that $\mathcal{X}$ is a Banach space.

a. Let $\widehat{\mathcal{X}}$, $\widehat{(\mathcal{X}^*)}$ be the natural images of $\mathcal{X}$, $\mathcal{X}^*$ in $\mathcal{X}^{**}$, $\mathcal{X}^{***}$, and let $\widehat{\mathcal{X}}^0 = \{F \in \mathcal{X}^{***} : F|\widehat{\mathcal{X}} = 0\}$. Then $\widehat{(\mathcal{X}^*)} \cap \widehat{\mathcal{X}}^0 = \{0\}$ and $\widehat{(\mathcal{X}^*)} + \widehat{\mathcal{X}}^0 = \mathcal{X}^{***}$.

b. $\mathcal{X}$ is reflexive iff $\mathcal{X}^*$ is reflexive.

25. If $\mathcal{X}$ is a Banach space and $\mathcal{X}^*$ is separable, then $\mathcal{X}$ is separable. (Let $\{f_n\}_1^\infty$ be a countable dense subset of $\mathcal{X}^*$. For each n choose $x_n \in \mathcal{X}$ with $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$. Then the linear combinations of $\{x_n\}_1^\infty$ are dense in $\mathcal{X}$.) *Note:* Separability of $\mathcal{X}$ does not imply separability of $\mathcal{X}^*$.

5.24a) Consider the $(id - pr) + (pr) = id$ trick.

Because X is Banach, $\hat{X}$ is closed in X^{**}, and let $\beta: X^{***} \to (\hat{X})^*$ by restricting

$$\beta(f) = f|_{\hat{X}} \qquad \forall f \in X^{***}.$$

If $\psi_X: X \to \hat{X}$ is the isometric isomorphism, every $f \in X^{***}$, induces $\psi_X^*(\beta(f)) \in X^*$,

such that $\forall x \in X$,

$$\psi_X^*(\beta(f))(x) = \beta(f)(\psi_X(x))$$

$$= \beta(f)(\hat{x})$$

$$= f|_{\hat{X}}(\hat{x}) = f(\hat{x}).$$

Let $\psi_{X^*}: X^* \to \widehat{(X^*)} \leq X^{***}$ be the isometric isomorphism, consider

$$g \triangleq \psi_{X^*}(\psi_X^*(\beta(f))) \in \widehat{(X^*)}, \quad \text{and}$$

$$f = g + (f - g),$$

22. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T \in L(\mathcal{X}, \mathcal{Y})$.

a. Define $T^\dagger : \mathcal{Y}^* \to \mathcal{X}^*$ by $T^\dagger f = f \circ T$. Then $T^\dagger \in L(\mathcal{Y}^*, \mathcal{X}^*)$ and $\|T^\dagger\| = \|T\|$. $T^\dagger$ is called the **adjoint** or **transpose** of T.

b. Applying the construction in (a) twice, one obtains $T^{\dagger\dagger} \in L(\mathcal{X}^{**}, \mathcal{Y}^{**})$. If $\mathcal{X}$ and $\mathcal{Y}$ are identified with their natural images $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ in $\mathcal{X}^{**}$ and $\mathcal{Y}^{**}$, then $T^{\dagger\dagger}|\mathcal{X} = T$.

c. $T^\dagger$ is injective iff the range of T is dense in $\mathcal{Y}$.

d. If the range of $T^\dagger$ is dense in $\mathcal{X}^*$, then T is injective; the converse is true if $\mathcal{X}$ is reflexive.

23. Suppose that $\mathcal{X}$ is a Banach space. If $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ and $\mathcal{N}$ is a closed subspace of $\mathcal{X}^*$, let $\mathcal{M}^0 = \{f \in \mathcal{X}^* : f|\mathcal{M} = 0\}$ and $\mathcal{N}^\perp = \{x \in \mathcal{X} : f(x) = 0$ for all $f \in \mathcal{N}\}$. (Thus, if we identify $\mathcal{X}$ with its image in $\mathcal{X}^{**}$, $\mathcal{N}^\perp = \mathcal{N}^0 \cap \mathcal{X}$.)

a. $\mathcal{M}^0$ and $\mathcal{N}^\perp$ are closed subspaces of $\mathcal{X}^*$ and $\mathcal{X}$, respectively.

b. $(\mathcal{M}^0)^\perp = \mathcal{M}$ and $(\mathcal{N}^\perp)^0 \supset \mathcal{N}$. If $\mathcal{X}$ is reflexive, $(\mathcal{N}^\perp)^0 = \mathcal{N}$.

c. Let $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ be the natural projection, and define $\alpha : (\mathcal{X}/\mathcal{M})^* \to \mathcal{X}^*$ by $\alpha(f) = f \circ \pi$. Then α is an isometric isomorphism from $(\mathcal{X}/\mathcal{M})^*$ onto $\mathcal{M}^0$, where $\mathcal{X}/\mathcal{M}$ has the quotient norm.

d. Define $\beta : \mathcal{X}^* \to \mathcal{M}^*$ by $\beta(f) = f|\mathcal{M}$; then β induces a map $\overline{\beta} : \mathcal{X}^*/\mathcal{M}^0 \to \mathcal{M}^*$ as in Exercise 15, and $\overline{\beta}$ is an isometric isomorphism.

24. Suppose that $\mathcal{X}$ is a Banach space.

a. Let $\widehat{\mathcal{X}}$, $(\mathcal{X}^*)\widehat{}$ be the natural images of $\mathcal{X}, \mathcal{X}^*$ in $\mathcal{X}^{**}, \mathcal{X}^{***}$, and let $\widehat{\mathcal{X}}^0 = \{F \in \mathcal{X}^{***} : F|\widehat{\mathcal{X}} = 0\}$. Then $(\mathcal{X}^*)\widehat{} \cap \widehat{\mathcal{X}}^0 = \{0\}$ and $(\mathcal{X}^*)\widehat{} + \widehat{\mathcal{X}}^0 = \mathcal{X}^{***}$.

b. $\mathcal{X}$ is reflexive iff $\mathcal{X}^*$ is reflexive.

25. If $\mathcal{X}$ is a Banach space and $\mathcal{X}^*$ is separable, then $\mathcal{X}$ is separable. (Let $\{f_n\}_1^\infty$ be a countable dense subset of $\mathcal{X}^*$. For each n choose $x_n \in \mathcal{X}$ with $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$. Then the linear combinations of $\{x_n\}_1^\infty$ are dense in $\mathcal{X}$.) *Note:* Separability of $\mathcal{X}$ does not imply separability of $\mathcal{X}^*$.

Suffices to show $(f-g) \in (\hat{X})^0$. Indeed, if $\hat{x} \in \hat{X}$, then,

$$f(\hat{x}) - g(\hat{x}) = f(\hat{x}) - \langle \Psi_{X^*} \circ \Psi_X^*(\beta(f)), \hat{x} \rangle_{(X^{***}, X^{**})}$$ Def of Ψ_{X^*}

$$= f(\hat{x}) - \langle \hat{x}, \Psi_X^*(\beta(f)) \rangle_{(X^{**}, X^*)}$$ Def of $\hat{x} = \Psi_X(x)$

$$= f(\hat{x}) - \langle \Psi_X^*(\beta(f)), x \rangle_{(X^*, X)}$$

$$= f(\hat{x}) - \langle \beta(f), \underbrace{\Psi_X(x)}_{= \hat{x}} \rangle_{(X^{***}, X^{**})}$$ (Def of Adjoint)

$$= f(\hat{x}) - f(\hat{x}) = 0$$

5.24b) Note that a Banach Space is reflexive, iff $\hat{X} = X^{**}$, since $\hat{X}$ is closed subspace in X^{**}, $\hat{X}$ is dense iff $(\hat{X})^0 = \{0\}$.
(this comes from the lemma proven in Exercise 5.22c, characterisation of dence subspaces).

So:

$$X \text{ is reflexive} \iff (\hat{X})^0 = \{0\}$$

$$\iff \widehat{(X^*)} = X^{***} \iff X^* \text{ is reflexive.}$$

27. There exist meager subsets of $\mathbb{R}$ whose complements have Lebesgue measure zero.

28. The Baire category theorem remains true if X is assumed to be an LCH space rather than a complete metric space. (The proof is similar; the substitute for completeness is Proposition 4.21.)

29. Let $\mathcal{Y} = L^1(\mu)$ where μ is counting measure on $\mathbb{N}$, and let $\mathcal{X} = \{f \in \mathcal{Y} : \sum_1^\infty n|f(n)| < \infty\}$, equipped with the L^1 norm.

a. $\mathcal{X}$ is a proper dense subspace of $\mathcal{Y}$; hence $\mathcal{X}$ is not complete.

b. Define $T : \mathcal{X} \to \mathcal{Y}$ by $Tf(n) = nf(n)$. Then T is closed but not bounded.

c. Let $S = T^{-1}$. Then $S : \mathcal{Y} \to \mathcal{X}$ is bounded and surjective but not open.

30. Let $\mathcal{Y} = C([0,1])$ and $\mathcal{X} = C^1([0,1])$, both equipped with the uniform norm.

a. $\mathcal{X}$ is not complete.

b. The map $(d/dx) : \mathcal{X} \to \mathcal{Y}$ is closed (see Exercise 9) but not bounded.

31. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and let $S : \mathcal{X} \to \mathcal{Y}$ be an unbounded linear map (for the existence of which, see §5.6). Let $\Gamma(S)$ be the graph of S, a subspace of $\mathcal{X} \times \mathcal{Y}$.

a. $\Gamma(S)$ is not complete.

b. Define $T : \mathcal{X} \to \Gamma(S)$ by $Tx = (x, Sx)$. Then T is closed but not bounded.

c. $T^{-1} : \Gamma(S) \to \mathcal{X}$ is bounded and surjective but not open.

32. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the vector space $\mathcal{X}$ such that $\|\cdot\|_1 \leq \|\cdot\|_2$. If $\mathcal{X}$ is complete with respect to both norms, then the norms are equivalent.

which is a subspace of $\mathcal{X} \times \mathcal{Y}$. (From a strict set-theoretic point of view, of course, T and $\Gamma(T)$ are identical; the distinction is a psychological one.) We say that T is **closed** if $\Gamma(T)$ is a closed subspace of $\mathcal{X} \times \mathcal{Y}$. Clearly, if T is continuous, then T is closed, and if $\mathcal{X}$ and $\mathcal{Y}$ are complete the converse is also true:

and T can be realized by projecting onto Y

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "T"'] & \Gamma(T) \arrow[d, "\pi_2"] \\ & Y \end{tikzcd}$$

5.12 The Closed Graph Theorem. *If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ is a closed linear map, then T is bounded.*

Closed Graph $\Rightarrow$ $\Gamma(T)$ is a closed subspace of a Banach Space, and $X \approx \Gamma(T)$

Proof. Let π_1 and π_2 be the projections of $\Gamma(T)$ onto $\mathcal{X}$ and $\mathcal{Y}$, that is, $\pi_1(x, Tx) = x$ and $\pi_2(x, Tx) = Tx$. Obviously $\pi_1 \in L(\Gamma(T), \mathcal{X})$ and $\pi_2 \in L(\Gamma(T), \mathcal{Y})$. Since $\mathcal{X}$ and $\mathcal{Y}$ are complete, so is $\mathcal{X} \times \mathcal{Y}$, and hence so is $\Gamma(T)$ since T is closed. The map π_1 is a bijection from $\Gamma(T)$ to $\mathcal{X}$, so by Corollary 5.11, π_1^{-1} is bounded. But then $T = \pi_2 \circ \pi_1^{-1}$ is bounded. ∎

Continuity of a linear map $T : \mathcal{X} \to \mathcal{Y}$ means that if $x_n \to x$ then $Tx_n \to Tx$, whereas closedness means that if $x_n \to x$ and $Tx_n \to y$ then $y = Tx$. Thus the significance of the closed graph theorem is that in verifying that $Tx_n \to Tx$ when $x_n \to x$, we may assume that Tx_n converges to *something,* and we need only to show that the limit is the right thing. This frequently saves a lot of trouble.

The completeness of $\mathcal{X}$ and $\mathcal{Y}$ was used in a crucial way in proving the open mapping theorem and hence also in proving the closed graph theorem. In fact, the conclusions of both of these theorems may fail if either $\mathcal{X}$ or $\mathcal{Y}$ is incomplete; see Exercises 29–31.

5.27)

The name of this theorem comes from Baire's terminology for sets: If X is a topological space, a set $E \subset X$ is **of the first category**, according to Baire, if E is a countable union of nowhere dense sets; otherwise E is **of the second category**. Thus Baire's theorem asserts that every complete metric space is of the second category in itself. A more modern and more descriptive synonym for "of the first category" is **meager**. The complement of a meager set is called **residual**.

30a) Can we think of an example using mollifiers. What are the approximation theorems with mollifiers?

5.25

Suppose X^* separable, $\Rightarrow X$ separable

proof let $\{f_n\} \subseteq X^*$ be a countable dense subset of X^*, for each $n \geq 1$

By pulling the supremum downwards, we find

$$2^{-1}\|f_n\| \leq |f_n(x_n)|, \text{ with } \|x_n\| = 1.$$

Want to show that $M = \text{span}(x_n)_1^\infty$ is dense in X. It will be useful to remember the following technique.

If $\overline{M}$ is a closed proper subspace. one can find a non-zero continuous functional that vanishes on $\overline{M}$.

Suppose for contradiction that $y_0 \in X \setminus \overline{M}$, one finds:

$$|f_0(y_0)| = c > 0, \quad f_0|_{\overline{M}} \equiv 0.$$

Which implies, $\|f_0\| > \frac{c}{\|y_0\|} > 0$.

let $\varepsilon > 0$ be so small that

$$\|f_0\| - 3\varepsilon > 0, \quad \text{then}$$

by density of $\{f_n\}_1^\infty$, we find $k \in \mathbb{N}^+$

with $\|f_0 - f_k\|_{X^*} \geq \big|\|f_0\| - \|f_k\|\big|$,

which allows us to approximate,

$$2^{-1}\|f_0\| - 2^{-1}\|f_0 - f_k\| \leq 2^{-1}\|f_k\| \leq |f_k(x_k)|,$$

the rightmost member is Bounded above by

$$|f_k(x_k)| \leq |f_0(x_k)| + \|f_0 - f_k\|,$$

so that $0 < 2^{-1}(\|f_0\| - 3\|f_0 - f_k\|) \leq |f_0(x_k)|$

this shows that no non-zero functional can vanish on $\overline{M}$, i.e: $\overline{M} = X$ □

5.25 Commentary

The main idea is that, if a subspace is not dense, then it allows for HB separation on the closure,

5.29

Let $Y = \ell^1(\mathbb{N}^+)$ with the counting measure, and $X = \{f \in Y, \sum_1^\infty j|f(j)| < +\infty\}$.

a) X is a proper dense subspace, of Y.

proof $\forall f, g \in X$, $a, b \in \mathbb{C}$.

$$\sum_1^\infty j|af(j) + bg(j)| \leq |a| \cdot \sum_1^\infty j|f| + |b|\sum_1^\infty j|g|.$$

X is dense, because it contains the dense subspace of finitely supported sequences; $\Sigma \subseteq X$.

$$\Sigma = \{f \in Y, \ f(j) = 0 \text{ eventually}\}$$

It is clear that X is proper, subspace, because not every summable sequence is summable against $\{j\}$. Take $x_n = n^{-2}$.

b) $T: X \to Y$, $T(\{x_n\})(n) = n x_n$,

Then: T is linear, closed, But discontinuous.

Linearity is a triviality,
First we verify unboundedness, so
we find sequences indexed by j:

$$\varphi(j,k) = \delta_{j,k} \in X,$$

$\|\varphi(\cdot,k)\| = 1 \quad \forall k \geq 1$, But $\|T\varphi(\cdot,k)\| = k$
so that T is not bounded on the sphere.

We show that T is closed. This will require some effort. Suppose $(A,B) \in X \times Y$ is an adherent point of the graph of T, meaning we can find $\varphi : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{C}$,

$$\sum_j |\varphi(j,k) - A(j)| + \sum_j |j\varphi(j,k) - B(j)| = w_k$$

where $w_k \to 0$,
we have used the product metric.
Now, we first show that $A(\cdot) \in X$, then we prove $jA(j) = B(j)$.

Fatou's lemma implies the first claim, with
$|\varphi(j,k)| \to |A(j)|$ pointwise as $k \nearrow +\infty$,
Bound $|j\varphi(j,k)| \leq |j\varphi(j,k) - B(j)| + |B(j)|$.
then

$$\sum_1^{\infty} |jA(j)| \leq \liminf_{k\to\infty} w_k + \sum_j^{\infty} |B(j)| < +\infty.$$

this shows $A \in X$. now,
to show that

$$jA(j) = B(j),$$

It suffices to show that $\|jA(j) - j\varphi(j,k)\|_{\ell^1} \to 0$
by Ex 2.21, because $(jA(j))$, $(j\varphi(j,k)) \in L^1$,
and $j\varphi(j,k) \to jA(j)$ pwise,

$$\|jA(j) - j\varphi(j,k)\|_{\ell^1} \to 0 \quad \text{iff} \quad \big|\|jA(j)\|_{\ell^1} - \|j\varphi(j,k)\|\big| \to 0.$$

we show the later by the generalised Dominated convergence theorem.

$$|j\varphi(j,k)| \leq |j\varphi(j,k) - B(j)| + |B(j)|.$$

the right hand side converges pw to $|B(j)|$,
and its sum converges to $\sum_1^{\infty} |B(j)|$
as $k \to +\infty$,

Then $\sum_{j=1}^{\infty} |j\varphi(j,k)| \to \sum_{j=1}^{\infty} |jA(j)|$.

and $\|jA(j) - B(j)\|_{\ell^1} \leq w_k + \|jA(j) - j\varphi(j,k)\|_{\ell^1}$
$\leq 2w_k$

So that $T(A(j)) = B(j)$ and the graph of T must be closed. $\square$

c) Define $S = T^{-1} : Y \to X$, then
S is bounded, surjective but not open.
proof S is bounded because it takes

$$\{x_n\} \xrightarrow{S} \{n^{-1} x_n\},$$

by Hölder's inequality.
and is clearly surjective. Now,

suppose that S is open, meaning we can find $C>0$ such that

$$B_X(C,0) \subseteq S(B_Y(1,0))$$

elements in the left member are precisely those: $\|\varphi(j)\|_{\ell^1} < C$, and $\|j\varphi(j)\|_{\ell^1} < +\infty$.

let $\frac{C}{2}m > 1$, and $\varphi(j) = 2^{-1}C\,\delta_{m,j}$. then $\|\varphi(j)\| = 2^{-1}C < C$, and become finitely supported, $\varphi \in X$. Now suppose that

$$\exists \psi \in Y,\ \varphi(j) = (S\psi)(j) = (j^{-1})\psi(j)$$

Which means,

$$\psi(j) = (2^{-1}Cj)\,\delta_{m,j},$$ and

$$\|\psi(j)\|_{\ell^1} = 2^{-1}Cm > 1$$

so that S is not open. $\square$

Signed measures

$\nu : \mathcal{M} \to [-\infty, +\infty]$

1) $\nu(\phi) = 0$

2) ν assumes at most one $\pm\infty$

3) If $\{E_n\}$ disjoint,

$$\nu(\bigcup_1^\infty E_n) = \lim_{n\to\infty} \sum_1^n \nu(E_j).$$

the sum converges absolutely if $|\nu(\bigcup_1^\infty E_n)| < +\infty$

1°) If $E \in \mathcal{M}$, $\nu(E) = \pm\infty$, then $\nu(F) = \pm\infty$ $\forall F \supseteq E$, $F \in \mathcal{M}$.

Def A measurable subset $E \in \mathcal{M}$ is positive. if $\nu_E : \mathcal{M} \to [0, +\infty]$, where $\nu_E(A) = \nu(A \cap E)$ defines a positive measure.

Def A measurable subset $E \in \mathcal{M}$ is negative, if $(-1)\nu_E : \mathcal{M} \to [0, +\infty]$ defines a pos. measure

Def $E \in \mathcal{M}$ is <u>null</u> if it is Both positive and negative.