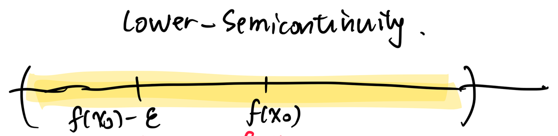
Lower Semi-continuity

" $\Phi(\bar{A}) \subseteq \overline{\Phi(A)}$ ".

"pushforward of accumulation points."

Let $\Phi : \prod_1^k X_j \to Y$, continuous mapping and if $z_j \in X_j$ is an adherent point,

Lower-Semicontinuity.

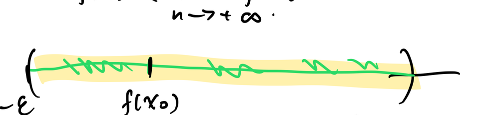


→ inverse image of this is open

→ Reflects the fact that the inverse image of some open sets is again open.

→ If $x_n \to x$ in X, then

$$f(x) \leq \liminf_{n\to+\infty} f(x_n).$$



Green: eventually where $f(x_n)$ are.

→ Where does LSC come from?

X = topological space, and

$$f : X \to (-\infty, +\infty],$$

f is lower-semicontinuous iff $\forall a \in \mathbb{R}$

$f^{-1}((a, +\infty])$ is open in X

→ Suppose $x_n \to x$, then $\forall \varepsilon > 0$.

$f(x) - \varepsilon < f(x_n)$ eventually.

so $f(x) - \varepsilon \leq \inf_{n\geq m} f(x_n)$ $\exists m \in \mathbb{N}^+$.

$f(x) - \varepsilon \leq \liminf f(x_n)$ $\forall \varepsilon > 0$.

Therefore $f(x) \leq \liminf f(x_n)$.

→ The limit inferior is formed by increasing tail infima.

or: The Limit Inferior is the supremum of tail infima.

→ Fatou's Lemma, " Limit Inferior Inside is Less Than Limit Inferior Outside (the integral."

$$\int \liminf (f_n)\, dx \leq \liminf \int f_n\, dx.$$

For every non-negative measurable sequence of functions.

Upper and lower approximations.

Semi-norms Behave like measures.

5.13 Semi-norm $\|\cdot\|$, on a vector space, and $M = \{x \in X, \|x\| = 0\}$.

then X/M has norm

$$\|x + M\| = \inf\{\|x+y\|, y \in M\}.$$

proof First, notice that for all $x \in X$, $y \in M$,

$$\|x\| - \|y\| \leq \|x + y\| \leq \|x\| + \|y\|$$

Triangle Inequality and homogeneity is easy to show that is a norm,

Suppose $\|x + M\| = 0$, we find $(-y_n) \subseteq M$, such that $\|x - y_n\| \to \|x + M\| = 0$.

and $x \in M$ and $x + M = 0$ (mod M),

so that $\|\cdot\|_{X/M}$ must be positive definite. □

12. Let $\mathcal{X}$ be a normed vector space and $\mathcal{M}$ a proper closed subspace of $\mathcal{X}$.
a. $\|x + \mathcal{M}\| = \inf\{\|x + y\| : y \in \mathcal{M}\}$ is a norm on $\mathcal{X}/\mathcal{M}$.
b. For any $\epsilon > 0$ there exists $x \in \mathcal{X}$ such that $\|x\| = 1$ and $\|x + \mathcal{M}\| \geq 1 - \epsilon$.
c. The projection map $\pi(x) = x + \mathcal{M}$ from $\mathcal{X}$ to $\mathcal{X}/\mathcal{M}$ has norm 1.
d. If $\mathcal{X}$ is complete, so is $\mathcal{X}/\mathcal{M}$. (Use Theorem 5.1.)
e. The topology defined by the quotient norm is the quotient topology as defined in Exercise 28 in §4.2.

13. If $\|\cdot\|$ is a seminorm on the vector space $\mathcal{X}$, let $\mathcal{M} = \{x \in \mathcal{X} : \|x\| = 0\}$. Then $\mathcal{M}$ is a subspace, and the map $x + \mathcal{M} \mapsto \|x\|$ is a norm on $\mathcal{X}/\mathcal{M}$.

14. If $\mathcal{X}$ is a normed vector space and $\mathcal{M}$ is a nonclosed subspace, then $\|x + \mathcal{M}\|$, as defined in Exercise 12, is a seminorm on $\mathcal{X}/\mathcal{M}$. If one divides by its nullspace as in Exercise 13, the resulting quotient space is isometrically isomorphic to $\mathcal{X}/\overline{\mathcal{M}}$. (Cf. Exercise 5.)

15. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T \in L(\mathcal{X}, \mathcal{Y})$. Let $\mathcal{N}(T) = \{x \in \mathcal{X} : Tx = 0\}$.
a. $\mathcal{N}(T)$ is a closed subspace of $\mathcal{X}$.
b. There is a unique $S \in L(\mathcal{X}/\mathcal{N}(T), \mathcal{Y})$ such that $T = S \circ \pi$ where $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ is the projection (see Exercise 12). Moreover, $\|S\| = \|T\|$.

5.14

5. If $\mathcal{X}$ is a normed vector space, the closure of any subspace of $\mathcal{X}$ is a subspace.

6. Suppose that $\mathcal{X}$ is a finite-dimensional vector space. Let $e_1, \ldots, e_n$ be a basis

For every NVS X, M any non-closed subspace.

$$(X/M)/\{z + M \in X/M, \|z + M\|_{X/M} = 0\} \simeq X/\bar{M}$$

none of these spaces are assumed to be complete.

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d. If $\mathcal{X}$ is complete, so is $\mathcal{X}/\mathcal{M}$. (Use Theorem 5.1.)
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15) First Isomorphism Theorem.

5.12 a) Let M proper closed subspace, then, up to isomorphism, let $N \subseteq X$ be the linear complement of M, then,

$$\forall x \in X, \quad \|x + M\| \triangleq \inf(\|x + y\|, y \in M)$$

Note that $x = 0$ mod M iff $x \in M$.

so that $\forall x \in M$,

$\|x + M\| = 0$ induces a sequence $\{-y_n = z_n\} \subseteq M$.

$\|x - z_n\| \to \|x + M\| = 0$, because $z_n \to x$.

$$\left(\begin{aligned} &\|x + y_n\| \to 0 \\ &\|x - (-y_n)\| \to 0 \end{aligned} \right)$$

so that $\|x - z_n\| \to 0$, and $x \in M$ because M is closed.

Fix $x_1, x_2 \in X$, and the numbers

$$\|x_1 + M\| = \lim_{n\to+\infty} \|x_1 + y_n\|, \quad \|x_2 + M\| = \lim_{k\to\infty} \|x_2 + z_k\|.$$

where $\forall (n,k) \in (\mathbb{N}^+)^2$, we have.

$$\|(x_1 + x_2) + M\| \leq \|(x_1 + x_2) + (y_n + z_k)\|$$

$$\leq \|x_1 + y_n\| + \|x_2 + z_k\|.$$

Taking limits on the right, because Addition is continuous, the right hand side converges to $\|x_1 + M\| + \|x_2 + M\|$. so the $\triangle$ inequality holds,

Let $(y_n) \subseteq M$ and $\alpha \in \mathbb{C}$, then if $\|x + y_n\| \to \|x + M\|$.

$$|\alpha| \cdot \|x + y_n\| = \|\alpha x + \alpha y_n\| \to |\alpha| \cdot \|x + M\|$$

so that $|\alpha| \cdot \|x + M\|$ is an adherent point of $\{\|\alpha x + z\|. z \in M\}$.

and $|\alpha| \cdot \|x + M\| \geq \|\alpha x + M\|$

Conversely fix $(\alpha z_n) \subseteq M$, if $\alpha \neq 0$. (if $\alpha = 0$ then $\|0 \cdot x + M\| = 0 = |0| \cdot \|x + M\|$).

where $\|\alpha x + \alpha z_n\| \to \|\alpha x + M\|$.

and $\|\alpha x + M\|$ is an adherent point of $\{|\alpha| \cdot \|x + y\|, y \in M\}$.

} the multiplication by a has been moved outside of the norm, here.

Recall $\inf\{|\alpha| \cdot \|x + y\|, y \in M\} = |\alpha| \inf\{\|x + y\|, y \in M\}$

$$= |\alpha| \cdot \|x + M\|.$$

Therefore $|\alpha| \cdot \|x + M\| = \|\alpha x + M\|$, and $\|\cdot + M\|$ is a norm on X/M.

dsdasdasdas

5.12 b

Start the proof here

Fix an element $x + M \in X/M$ (non-trivial normed vector space). By dilations, we assume that

$$\|x + M\| = 1 - \varepsilon.$$

Now we can assume that, if $\frac{\varepsilon}{2} > 0$, this induces some $y_\varepsilon \in M$ such that

$$\|x + y_\varepsilon\| \in [1 - \varepsilon, 1 - \varepsilon/2].$$

We can assume $y_\varepsilon \neq 0$. by the reduction in the generating set of

By the intermediate value theorem.

$$\|x + t y_\varepsilon\| \geq |t| \cdot \|y_\varepsilon\| - \|x\|. \to +\infty.$$

there exists some $t_0 \in \mathbb{R}$ s

$$\|x + t_0 y_\varepsilon\| = 1.$$

Setting $z = x + t_0 y_\varepsilon$ finishes the proof □

12 c) $\forall x \in X$, $\|\pi x\| \leq \|x\|$?.

Yes, pick $y = 0$, $\|x + M\| \leq \|x + 0\| = \|x\|$.

Now $\forall \varepsilon > 0$, we find some $\|x_\varepsilon\| = 1$

$1 - \varepsilon \leq \|\pi x_\varepsilon\| \leq \|\pi\|_{op}$, so that

$$\limsup_{\varepsilon > 0} (1 - \varepsilon) \leq \|\pi\|_{op} = 1$$

12d) If X is complete, so is X/M.

Fix a absolutely convergent series in X/M.

$\{x_n + M\}_{n\geq 1} \subseteq X/M$. , let $(c_n) \in \ell^1 \cap \ell^{++}$ (Summable, strictly positive sequences)

$$\sum \|x_n + M\| + c_n \geq \sum \|x_n + y_n\|.$$

for some $\{y_n\} \subseteq M$, then the RHS is finite.

And set $z = \sum (x_n + y_n)$, because the projection map is continuous,

and the sums

$\{\sum_{j\leq N} (x_j + y_j)\}_{N\geq 1}$ converge to z

in the norm of X. So that

$\pi(\sum_{j\leq N} (x_j + y_j))$ converges to something.

and let $z^* \in X/M$ be the limit, of $\{\pi(\sum_1^N (x_j + y_j))\}$

$$\|\pi(\sum_1^N (x_j + y_j)) - z^*\|_{X/M} \to 0.$$

implies $\|[\sum_1^N (x_j + y_j) + M] - z^*\|_{X/M} \to 0.$

$$\Rightarrow \|(\sum_1^N x_j + M) - z^*\|_{X/M} \to 0.$$

therefore X/M is complete.

12e) Because the projection map is continuous, for every open $U \overset{\circ}{\subseteq} X/M$,

$\pi^{-1}(U)$ is open in X.

and $\tau_{X/M} \subseteq \{U \subseteq X/M, \pi^{-1}(U) \overset{\circ}{\subseteq} X\}$

Now, If $U \subseteq X/M$, and $\pi^{-1}(U) \overset{\circ}{\subseteq} X$, for all $x \in U$, we can find some ball that fits in U.

What about $\pi(x) + \varepsilon_x \pi(0)$? $\pi(x) \in U \subseteq X/M$.

Suppose $\exists \varepsilon_x > 0$ $\forall y \in X$, $\|y - x\| < \varepsilon_x$.

$\Rightarrow y \in \pi^{-1}(U)$. Because π is a surjection, $\pi(\pi^{-1}(U)) = U$.

Then, $\|\pi(x) - \pi(y)\| = \inf\{\|(x - y) + z\|, z \in M\}$

Alternate proof:

By the open-mapping theorem, every toplinear surjection is an open map.

Convexity of norms

$\|c_t(x,y)\| \leq c_t(\|x\|, \|y\|)$.

and by translation invariance

$\forall x_0 \in X$,

$$\|x_0 + t y_1 + (1-t) y_2\| \leq t\|x_0 + y_1\| + (1-t) \cdot \|x_0 + y_2\|.$$

Can we generalize this argument using limit inferior.

can we improve the "generating set" of the quotient norm?

"Generating set = a subset on which a function is determined."

$\forall x \in X$,

$$\|x + M\| = \inf\{\|x + y\|, y \in M \setminus \{0\}\} ?$$

The point that has been removed, can be approximated. g.

$\forall (y_n) \subseteq M$ that converges to 0.

$$\|x + 0\| = \liminf_{n\to\infty} \|x + y_n\|$$

Quotient Topology (Folland)

28. Let X be a topological space equipped with an equivalence relation, $\tilde{X}$ the set of equivalence classes, $\pi : X \to \tilde{X}$ the map taking each $x \in X$ to its equivalence class, and $\mathcal{T} = \{U \subset \tilde{X} : \pi^{-1}(U) \text{ is open in } X\}$.
a. $\mathcal{T}$ is a topology on $\tilde{X}$. (It is called the **quotient topology**.)
b. If Y is a topological space, $f : \tilde{X} \to Y$ is continuous iff $f \circ \pi$ is continuous.
c. $\tilde{X}$ is T_1 iff every equivalence class is closed.

$$X \xrightarrow{\pi} \hat{X}$$

a) Both $\phi, \hat{X}$ are open because $\pi^{-1}(\tilde{X}) = X$. Clearly closed under finite intersections and arbitrary unions. Therefore is a topological space.

b) A mapping f, is continuous iff the precomposed map through π is continuous.

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "f\circ\pi"'] & \hat{X} \arrow[d, "f"] \\ & Y \end{tikzcd}$$

If f is cont then the composition is continuous, because

$\pi : X \to \tilde{X}$ is cont almost by definition of the Quotient topology.

Conversely if $f \circ \pi$ is cont, fix open $V \overset{\circ}{\subseteq} Y$, then $\pi^{-1}(f^{-1}(V))$ is open in X means $f^{-1}(V)$ is open in $\tilde{X}$.

c) Suppose every equivalence class, as subsets of X with the original topology, is closed.

Let $[z]$ be any equivalence class identified as an element in $\tilde{X}$, then,

The set of points in X, which are not in this equivalence class (here identified as a subset of X) is open, and is precisely equal to

$$\pi^{-1}(\tilde{X} \setminus \{[z]\}) \overset{\circ}{\subseteq} X$$

the set of points in X, when one applies the projection map to it, maps to an equivalence class that is not equal to $[z]$.

So that $\tilde{X} \setminus \{[z]\}$ is open, for all $[z] \in \tilde{X}$.

The converse is the same argument: as.

$V \subseteq \tilde{X}$ is open iff $\pi^{-1}(V)$ is open in X.

→ Continuous, open mapping, surjection:

Definition. Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a ***quotient map*** provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

This condition is stronger than continuity; some mathematicians call it "strong continuity." An equivalent condition is to require that a subset A of Y be closed in Y if and only if $p^{-1}(A)$ is closed in X. Equivalence of the two conditions follows from equation

$$f^{-1}(Y - B) = X - f^{-1}(B).$$

Another way of describing a quotient map is as follows: We say that a subset C of X is ***saturated*** (with respect to the surjective map $p : X \to Y$) if C contains every set $p^{-1}(\{y\})$ that it intersects. Thus C is saturated if it equals the complete inverse image of a subset of Y. To say that p is a quotient map is equivalent to saying that p is continuous and p maps *saturated* open sets of X to open sets of Y (or saturated closed sets of X to closed sets of Y).

Two special kinds of quotient maps are the *open maps* and the *closed maps*. Recall that a map $f : X \to Y$ is said to be an ***open map*** if for each open set U of X, the set $f(U)$ is open in Y. It is said to be a ***closed map*** if for each closed set A of X, the set $f(A)$ is closed in Y. It follows immediately from the definition that if $p : X \to Y$ is a surjective continuous map that is either open or closed, then p is a quotient map. There are quotient maps that are neither open nor closed. (See Exercise 3.)

Intermediate Value Theorem:

$$\lim_{|t|\to+\infty} \|x + ty\| \to +\infty ?.$$

$$\geq \|ty\| - \|x\|.$$

$$\geq |t| \cdot \|y\| - \|x\|.$$

Suppose we find $y \in M$ such that

$\|x + y\| \leq 1 - \frac{3}{2}\varepsilon.$, $(y \neq 0)$.

$\|x + ty\| \geq |t| \cdot \|y\| - \|x\|$.

and there exists $t_0 \in \mathbb{R}$ such that.

$\|x + t_0 y\| = 1$, and.

$$\|x + t_0 y + M\| \geq \|x + M\| = 1 - \varepsilon.$$