

wts

- 1) For all $X, Y \geq 0$, $\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} Y \\ 0 \end{pmatrix} = 1$
 $\begin{pmatrix} X \\ X \end{pmatrix} = 1$
- 2) For all $0 \leq r \leq X$, $r_j = 0$, ($j=n$), then $\begin{pmatrix} X \\ r \end{pmatrix} = \begin{pmatrix} X+e_j \\ r \end{pmatrix}$
- 3) $\begin{pmatrix} \alpha+e_j \\ r \end{pmatrix} = \begin{pmatrix} \alpha \\ r-e_j \end{pmatrix} + \begin{pmatrix} \alpha \\ r \end{pmatrix}$, ($\forall e_j \leq r \leq \alpha$, $j=n$)
- 4) $\forall \alpha \geq 0$, $\partial^\alpha(fg) = \sum_{r=0}^{\alpha} \begin{pmatrix} \alpha \\ r \end{pmatrix} (\partial^r f) (\partial^{\alpha-r} g)$

proof

- 1) Fix $X, Y \geq 0$, then
 $\begin{pmatrix} X \\ 0 \end{pmatrix} = \prod_i \begin{pmatrix} X_i \\ 0 \end{pmatrix} = 1 = \begin{pmatrix} Y \\ 0 \end{pmatrix}$
 $\begin{pmatrix} X \\ X \end{pmatrix} = \prod_i \begin{pmatrix} X_i \\ X_i \end{pmatrix} = 1$
- 2) Fix $0 \leq r < e_j$, $X \geq e_j$,
 $\begin{pmatrix} X \\ r \end{pmatrix} = \left(\prod_{i \neq j} \begin{pmatrix} X_i \\ r_i \end{pmatrix} \right) \begin{pmatrix} X_j \\ 0 \end{pmatrix} = \begin{pmatrix} X+e_j \\ r \end{pmatrix}$
- 3) $\begin{pmatrix} \alpha+e_j \\ r \end{pmatrix} = \left(\prod_{i \neq j} \begin{pmatrix} \alpha_i \\ r_i \end{pmatrix} \right) \begin{pmatrix} \alpha_j+1 \\ r_j \end{pmatrix}$
 $= \left(\prod_{i \neq j} \begin{pmatrix} \alpha_i \\ r_i \end{pmatrix} \right) \left[\begin{pmatrix} \alpha_j \\ r_j-1 \end{pmatrix} + \begin{pmatrix} \alpha_j \\ r_j \end{pmatrix} \right]$ } one-dimensional case
 $= \begin{pmatrix} \alpha \\ r-e_j \end{pmatrix} + \begin{pmatrix} \alpha \\ r \end{pmatrix}$
- 4) $\alpha=0$, nothing to prove, (the sum ranges from $0 \leq r \leq \alpha$, so $r \in \{0\}$),
 Suppose holds for $|k| \leq k$, then $\forall j=n$,
 $\partial^{e_j}(\partial^\alpha f g) = \sum_{r=0}^{\alpha} \begin{pmatrix} \alpha \\ r \end{pmatrix} [(\partial^{r+e_j} f)(\partial^{\alpha-r} g) + (\partial^r f)(\partial^{\alpha+e_j-r} g)].$
 Pay attention to the green exponent.
 \rightarrow What is the range of the green exponent?
 $= \sum_{r=e_j}^{\alpha+e_j} \begin{pmatrix} \alpha \\ r-e_j \end{pmatrix} (\partial^r f) (\partial^{\alpha+e_j-r} g) + \sum_{r=0}^{\alpha} \begin{pmatrix} \alpha \\ r \end{pmatrix} (\partial^r f) (\partial^{\alpha+e_j-r} g)$
 $= \left(\sum_{r=e_j}^{\alpha} \begin{pmatrix} \alpha \\ r-e_j \end{pmatrix} (\partial^r f) (\partial^{\alpha+e_j-r} g) \right) + \left(\sum_{r=0}^{\alpha} \begin{pmatrix} \alpha \\ r \end{pmatrix} (\partial^r f) (\partial^{\alpha+e_j-r} g) \right) + (1) (\partial^{\alpha+e_j} f) (\partial^0 g)$
 $= \sum_{r=e_j}^{\alpha} \begin{pmatrix} \alpha+e_j \\ r \end{pmatrix} (\partial^r f) (\partial^{\alpha+e_j-r} g) + \sum_{\substack{0 \leq r \leq \alpha \\ r_j=0}} \begin{pmatrix} \alpha \\ r \end{pmatrix} (\partial^r f) (\partial^{\alpha+e_j-r} g) + (1) (\partial^{\alpha+e_j} f) (\partial^0 g)$
 $= \left(\text{same as above} \right) + \sum_{\substack{0 \leq r \leq \alpha \\ r_j=0}} \begin{pmatrix} \alpha+e_j \\ r \end{pmatrix} (\partial^r f) (\partial^{\alpha+e_j-r} g) + (1) (\partial^{\alpha+e_j} f) (\partial^0 g)$
 $= \sum_{r=0}^{\alpha} \begin{pmatrix} \alpha+e_j \\ r \end{pmatrix} (\partial^r f) (\partial^{\alpha+e_j-r} g) + \sum_{r=\alpha+e_j}^{\alpha+e_j} \begin{pmatrix} \alpha+e_j \\ r \end{pmatrix} (\partial^r f) (\partial^{\alpha+e_j-r} g)$