Limit Inferior of a sequence of numbers.

$$\liminf_{n\to\infty} x_n = \sup_{n\geq 1} \inf_{m\geq n} x_m.$$

Limit Inferior of a sequence of sets.

$$\liminf_{n\to\infty} E_n = \bigcup_1^\infty \bigcap_n^\infty E_m.$$

<u>wts</u> $\mu(\liminf E_j) \leq \liminf \mu(E_j)$.

$\mu(F_n) = \mu(\bigcap_n^\infty E_j) \leq \mu(E_j) \quad \forall j \geq n.$

the sequence $F_n = \bigcap_n^\infty E_j$ is increasing.

$$\Rightarrow \quad \mu(F_n) \leq \inf_{j\geq n} \mu(E_j)$$

Both sides are increasing in n, so.

$$\sup_n \mu(F_n) \leq \sup_n \inf_{j\geq n} \mu(E_j).$$

By continuity from below. on the left

$$\sup_n \mu(F_n) = \mu(\bigcup_{n=1}^\infty F_n) \leq \liminf_{j\to\infty} \mu(E_j)$$

1.8 Theorem.

$\{E_j\}_1^\infty \subseteq \mathcal{M}$, and $E_1 \supseteq E_2 \supseteq \cdots$ and $\mu(E_1) < +\infty$.

then $\mu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.

<u>proof</u> The key is that we can do some subtraction magic.

1.8d We get a sequence ~~st~~

$(E_1 \setminus E_j) = F_j$, with all $\mu(E_j) \leq \mu(E_1) < +\infty$.

and $E_1 \setminus E_j \subseteq E_1 \setminus E_{j+1}$.

$E_j \supseteq E_{j+1}$. (Removing less $\Rightarrow$ set is larger),

then ① $\bigcup (E_1 \setminus E_j) = E_1 \setminus (\bigcap E_j)$.

② $\mu(\bigcup E_1 \setminus E_j) = \lim_{j\to\infty} \mu(E_1 \setminus E_j) = \mu(E_1 \setminus (\bigcap E_j))$.

~~and~~. Finite measure $\Rightarrow$ allows subtraction.

$\mu(E_1) - \mu(\bigcap E_j) = \lim_{j\to\infty} \mu(E_1 \setminus E_j) = \lim \mu(E_1) - \mu(E_j)$!

$\boxed{\Rightarrow \mu(\bigcap E_j) = \lim_{j\to\infty} \mu(E_j)}$

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wts $\limsup \mu(E_j) \leq \mu(\limsup E_j)$.

proof $\sup_{j\geq n} \mu(E_j) \leq \mu(\bigcup_n^\infty E_j) \quad \forall n \geq 1$

set $F_n = \bigcup_n^\infty E_j$, then $\{F_n\}_1^\infty$ is a decreasing sequence of sets with $\mu(F_n) \downarrow \mu(\limsup E_j)$.
(By 1.8d).

Both sides of the equation are decreasing in n, so

$\limsup_{j\to\infty} \mu(E_j) \leq \inf_{n\geq 1} \mu(F_n) = \mu(\limsup E_j)$.
