

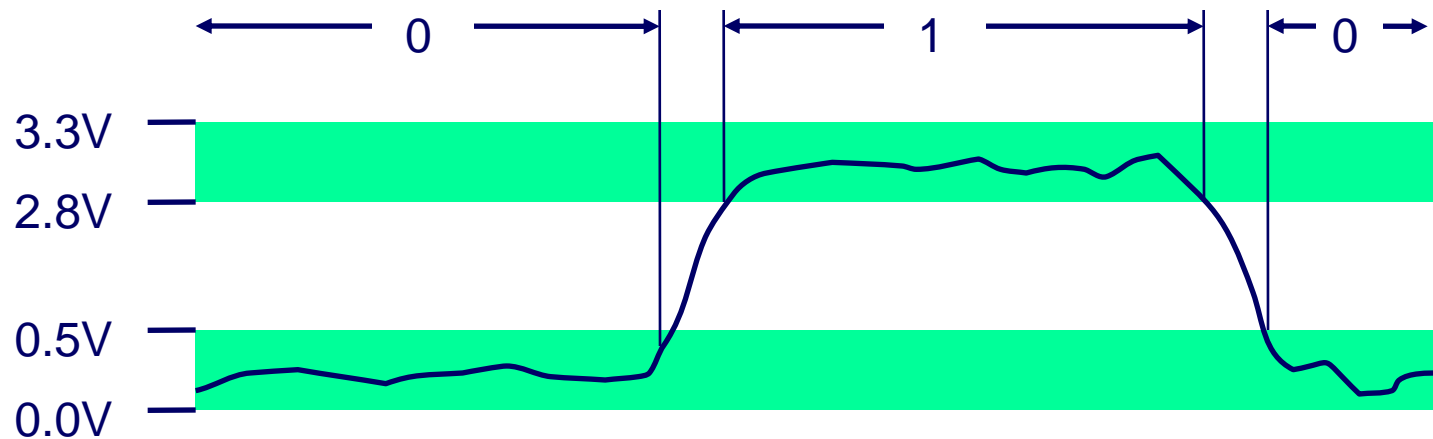
Week 02

Bits, Bytes, and Integers

Today: Bits, Bytes, and Integers

- **Representing information as bits**
- **Bit-level manipulations**
- **Integers**
 - Representation: unsigned and signed
 - Conversion, casting
 - Expanding, truncating
 - Addition, negation, multiplication, shifting
- **Summary**

Binary Representations



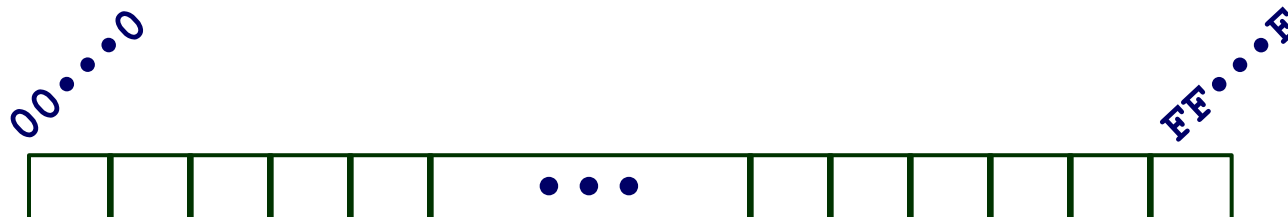
Encoding Byte Values

■ Byte = 8 bits

- Binary 00000000_2 to 11111111_2
- Decimal: 0_{10} to 255_{10}
- Hexadecimal 00_{16} to FF_{16}
 - Base 16 number representation
 - Use characters '0' to '9' and 'A' to 'F'
 - Write $FA1D37B_{16}$ in C as
 - `0xFA1D37B`
 - `0xfa1d37b`

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Byte-Oriented Memory Organization



■ Programs Refer to Virtual Addresses

- Conceptually very large array of bytes
- Actually implemented with hierarchy of different memory types
- System provides address space private to particular “process”
 - Program being executed
 - Program can clobber its own data, but not that of others

■ Compiler + Run-Time System Control Allocation

- Where different program objects should be stored
- All allocation within single virtual address space

Machine Words

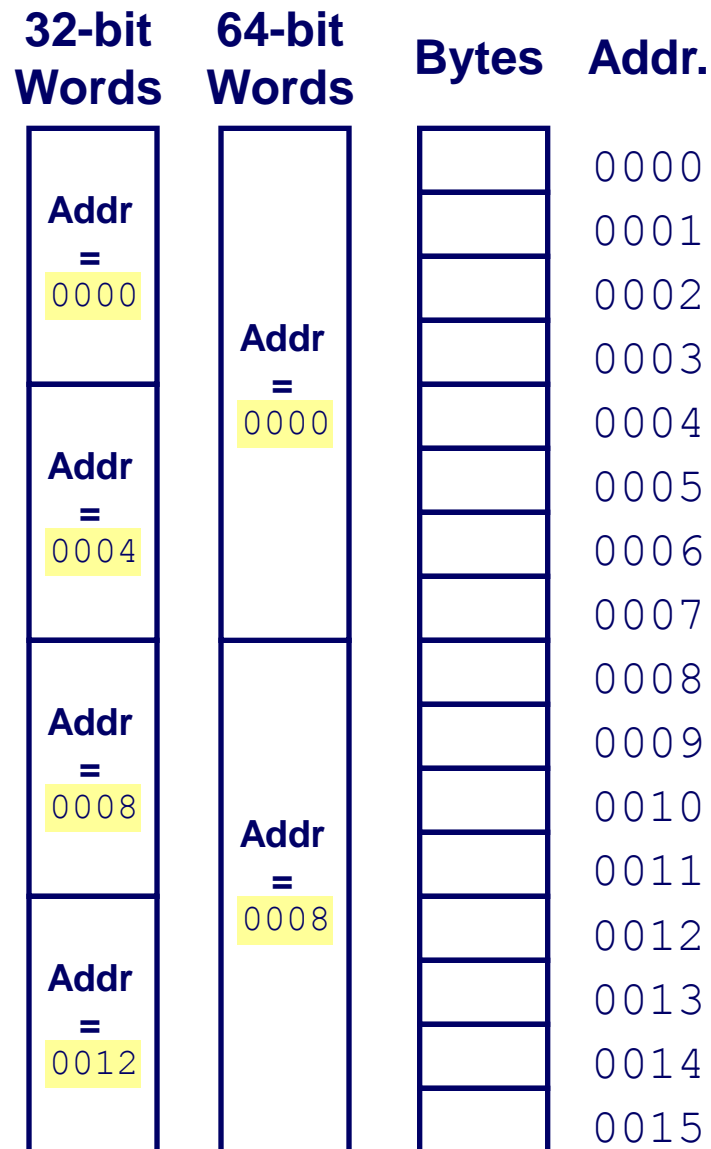
■ Machine Has “Word Size”

- Nominal size of integer-valued data
 - Including addresses
- Most current machines use 32 bits (4 bytes) words
 - Limits addresses to 4GB
 - Becoming too small for memory-intensive applications
- High-end systems use 64 bits (8 bytes) words
 - Potential address space $\approx 1.8 \times 10^{19}$ bytes
 - x86-64 machines support 48-bit addresses: 256 Terabytes
- Machines support multiple data formats
 - Fractions or multiples of word size
 - Always integral number of bytes

Word-Oriented Memory Organization

■ Addresses Specify Byte Locations

- Address of first byte in word
- Addresses of successive words differ by 4 (32-bit) or 8 (64-bit)



Data Representations

C Data Type	Typical 32-bit	Intel IA32	x86-64
char	1	1	1
short	2	2	2
int	4	4	4
long	4	4	8
long long	8	8	8
float	4	4	4
double	8	8	8
long double	8	10/12	10/16
pointer	4	4	8

Byte Ordering

- **How should bytes within a multi-byte word be ordered in memory?**
- **Conventions**
 - Big Endian: Sun, PPC Mac, Internet
 - Least significant byte has highest address
 - Little Endian: x86
 - Least significant byte has lowest address

Byte Ordering Example

■ Big Endian

- Least significant byte has highest address

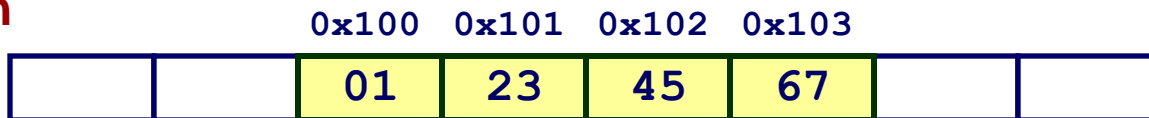
■ Little Endian

- Least significant byte has lowest address

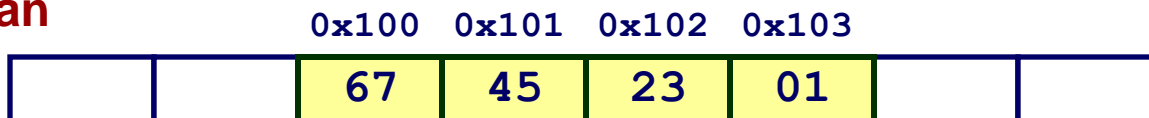
■ Example

- Variable x has 4-byte representation 0x01234567
- Address given by &x is 0x100

Big Endian



Little Endian



Reading Byte-Reversed Listings

■ Disassembly

- Text representation of binary machine code
- Generated by program that reads the machine code

■ Example Fragment

Address	Instruction Code	Assembly Rendition
8048365:	5b	pop %ebx
8048366:	81 c3 ab 12 00 00	add \$0x12ab,%ebx
804836c:	83 bb 28 00 00 00 00	cmpl \$0x0,0x28(%ebx)

■ Deciphering Numbers

- Value: 0x12ab
- Pad to 32 bits: 0x000012ab
- Split into bytes: 00 00 12 ab
- Reverse: ab 12 00 00

Examining Data Representations

■ Code to Print Byte Representation of Data

- Casting pointer to unsigned char * creates byte array

```
typedef unsigned char *pointer;

void show_bytes(pointer start, int len){
    int i;
    for (i = 0; i < len; i++)
        printf("%p\t0x%.2x\n", start+i, start[i]);
    printf("\n");
}
```

Printf directives:

%p: Print pointer

%x: Print Hexadecimal

show_bytes Execution Example

```
int a = 15213;  
printf("int a = 15213;\n");  
show_bytes((pointer) &a, sizeof(int));
```

Result (Linux):

```
int a = 15213;  
0x11ffffffcb8 0x6d  
0x11ffffffcb9 0x3b  
0x11ffffffcba 0x00  
0x11ffffffcbb 0x00
```

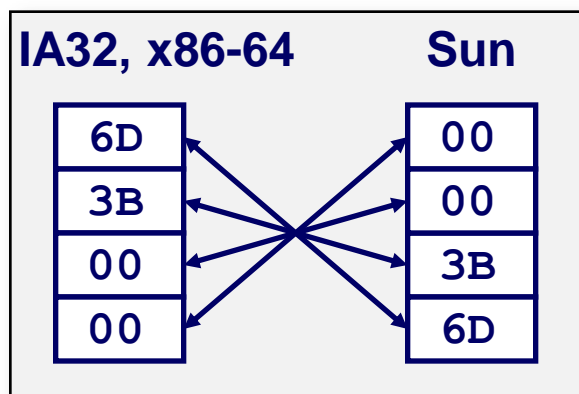
Representing Integers

Decimal: 15213

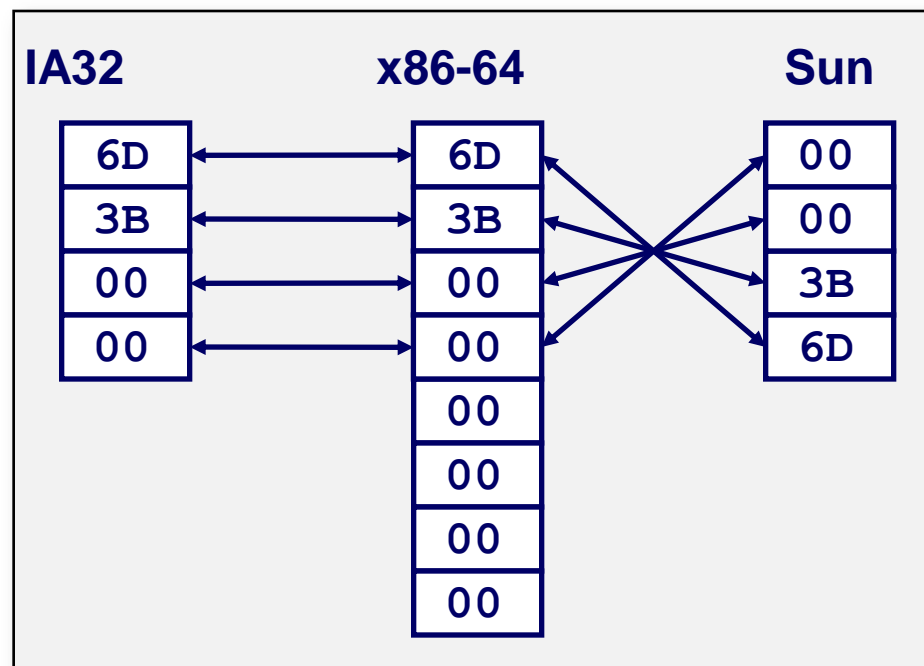
Binary: 0011 1011 0110 1101

Hex: 3 B 6 D

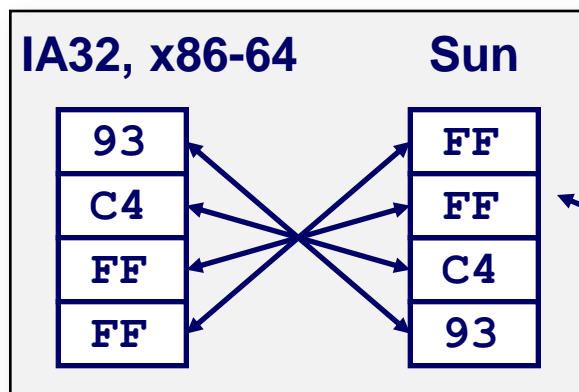
int A = 15213;



long int C = 15213;



int B = -15213;



Two's complement representation
(Covered later)

Representing Pointers

```
int B = -15213;  
int *P = &B;
```

Sun

EF
FF
FB
2C

IA32

D4
F8
FF
BF

x86-64

0C
89
EC
FF
FF
7F
00
00

Different compilers & machines assign different locations to objects

Representing Strings

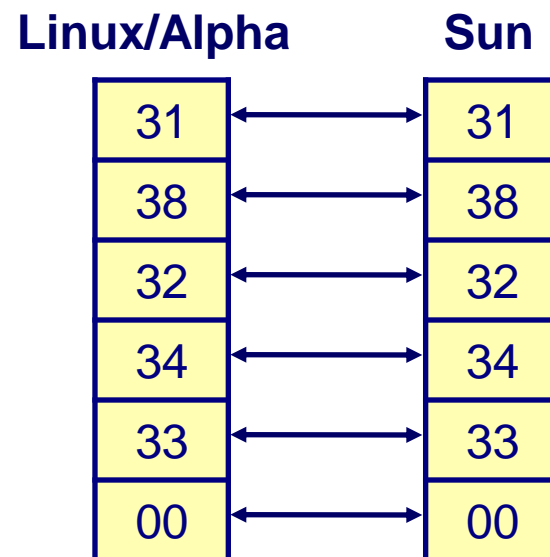
```
char S[6] = "18243";
```

■ Strings in C

- Represented by array of characters
- Each character encoded in ASCII format
 - Standard 7-bit encoding of character set
 - Character "0" has code 0x30
 - Digit i has code $0x30+i$
- String should be null-terminated
 - Final character = 0

■ Compatibility

- Byte ordering not an issue



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Boolean Algebra

■ Developed by George Boole in 19th Century

- Algebraic representation of logic
 - Encode “True” as 1 and “False” as 0

And

- $A \& B = 1$ when both $A=1$ and $B=1$

$\&$	0	1
0	0	0
1	0	1

Or

- $A | B = 1$ when either $A=1$ or $B=1$

$ $	0	1
0	0	1
1	1	1

Not

- $\sim A = 1$ when $A=0$

\sim	
0	1
1	0

Exclusive-Or (Xor)

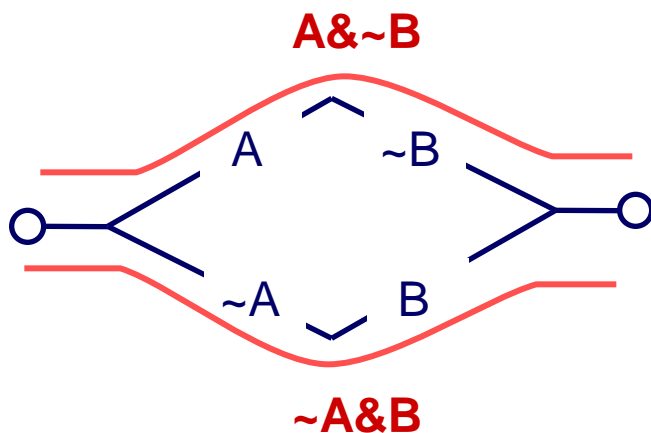
- $A \wedge B = 1$ when either $A=1$ or $B=1$, but not both

\wedge	0	1
0	0	1
1	1	0

Application of Boolean Algebra

■ Applied to Digital Systems by Claude Shannon

- 1937 MIT Master's Thesis
- Reason about networks of relay switches
 - Encode closed switch as 1, open switch as 0



Connection when

$$A \& \sim B \mid \sim A \& B$$

$$= A \wedge B$$

General Boolean Algebras

■ Operate on Bit Vectors

- Operations applied bitwise

01101001	01101001	01101001	01101001
& 01010101	01010101	^ 01010101	~ 01010101
<u> </u>	<u> </u>	<u> </u>	<u> </u>
01000001	01111101	00111100	10101010

■ All of the Properties of Boolean Algebra Apply

Representing & Manipulating Sets

■ Representation

- Width w bit vector represents subsets of $\{0, \dots, w-1\}$
- $a_j = 1$ if $j \in A$

■ 01101001 $\{0, 3, 5, 6\}$

■ 76543210

■ 01010101 $\{0, 2, 4, 6\}$

■ 76543210

■ Operations

- | | | |
|----------------------------|----------|------------------------|
| ■ & Intersection | 01000001 | $\{0, 6\}$ |
| ■ Union | 01111101 | $\{0, 2, 3, 4, 5, 6\}$ |
| ■ ^ Symmetric difference | 00111100 | $\{2, 3, 4, 5\}$ |
| ■ ~ Complement | 10101010 | $\{1, 3, 5, 7\}$ |

Bit-Level Operations in C

■ Operations $\&$, $|$, \sim , \wedge Available in C

- Apply to any “integral” data type
 - long, int, short, char, unsigned
- View arguments as bit vectors
- Arguments applied bit-wise

■ Examples (Char data type)

- $\sim 0x41 \rightarrow 0xBE$
 - $\sim 01000001_2 \rightarrow 10111110_2$
- $\sim 0x00 \rightarrow 0xFF$
 - $\sim 00000000_2 \rightarrow 11111111_2$
- $0x69 \& 0x55 \rightarrow 0x41$
 - $01101001_2 \& 01010101_2 \rightarrow 01000001_2$
- $0x69 | 0x55 \rightarrow 0x7D$
 - $01101001_2 | 01010101_2 \rightarrow 01111101_2$

Contrast: Logic Operations in C

■ Contrast to Logical Operators

- `&&`, `||`, `!`
 - View 0 as “False”
 - Anything nonzero as “True”
 - Always return 0 or 1
 - Early termination

■ Examples (char data type)

- `!0x41 → 0x00`
- `!0x00 → 0x01`
- `!!0x41 → 0x01`
- `0x69 && 0x55 → 0x01`
- `0x69 || 0x55 → 0x01`
- `p&& *p` (avoids null pointer access)

Shift Operations

■ Left Shift: $x \ll y$

- Shift bit-vector x left y positions
 - Throw away extra bits on left
 - Fill with 0's on right

■ Right Shift: $x \gg y$

- Shift bit-vector x right y positions
 - Throw away extra bits on right
- Logical shift
 - Fill with 0's on left
- Arithmetic shift
 - Replicate most significant bit on right

■ Undefined Behavior

- Shift amount < 0 or \geq word size

Argument x	01100010
$\ll 3$	00010000
Log. $\gg 2$	00011000
Arith. $\gg 2$	00011000

Argument x	10100010
$\ll 3$	00010000
Log. $\gg 2$	00101000
Arith. $\gg 2$	11101000

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Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

```
short int x = 15213;
short int y = -15213;
```

Sign
Bit



■ C short 2 bytes long

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
y	-15213	C4 93	11000100 10010011

■ Sign Bit

- For 2's complement, most significant bit indicates sign
 - 0 for nonnegative
 - 1 for negative

Encoding Example (Cont.)

$x =$ 15213: 00111011 01101101
 $y =$ -15213: 11000100 10010011

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum	15213		-15213	

Numeric Ranges

■ Unsigned Values

- $UMin = 0$
000...0
- $UMax = 2^w - 1$
111...1

■ Two's Complement Values

- $TMin = -2^{w-1}$
100...0
- $TMax = 2^{w-1} - 1$
011...1

■ Other Values

- Minus 1
111...1

Values for $W = 16$

	Decimal	Hex	Binary
UMax	65535	FF FF	11111111 11111111
TMax	32767	7F FF	01111111 11111111
TMin	-32768	80 00	10000000 00000000
-1	-1	FF FF	11111111 11111111
0	0	00 00	00000000 00000000

Values for Different Word Sizes

	W			
	8	16	32	64
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

■ Observations

- $|TMin| = TMax + 1$
 - Asymmetric range
- $UMax = 2 * TMax + 1$

■ C Programming

- `#include<limits.h>`
- Declares constants, e.g.,
 - `ULONG_MAX`
 - `LONG_MAX`
 - `LONG_MIN`
- Values platform specific

Unsigned & Signed Numeric Values

X	$B2U(X)$	$B2T(X)$
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	-8
1001	9	-7
1010	10	-6
1011	11	-5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15	-1

■ Equivalence

- Same encodings for nonnegative values

■ Uniqueness

- Every bit pattern represents unique integer value
- Each representable integer has unique bit encoding

■ \Rightarrow Can Invert Mappings

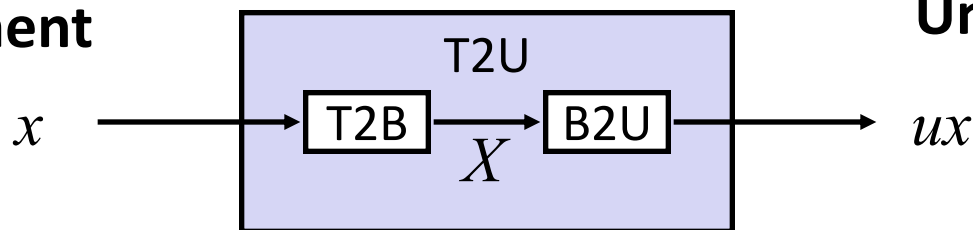
- $U2B(x) = B2U^{-1}(x)$
 - Bit pattern for unsigned integer
- $T2B(x) = B2T^{-1}(x)$
 - Bit pattern for two's comp integer

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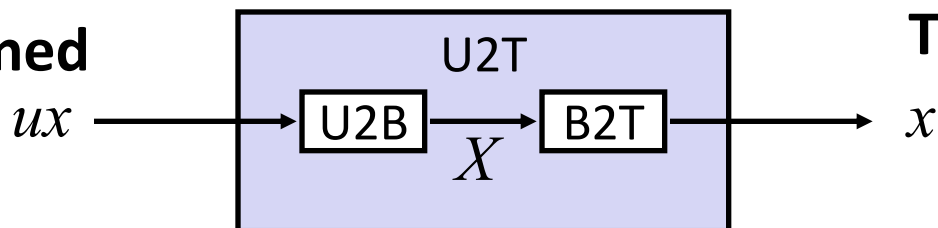
Mapping Between Signed & Unsigned

Two's Complement



Maintain Same Bit Pattern

Unsigned



Maintain Same Bit Pattern

- Mappings between unsigned and two's complement numbers:
keep bit representations and reinterpret

Mapping Signed \leftrightarrow Unsigned

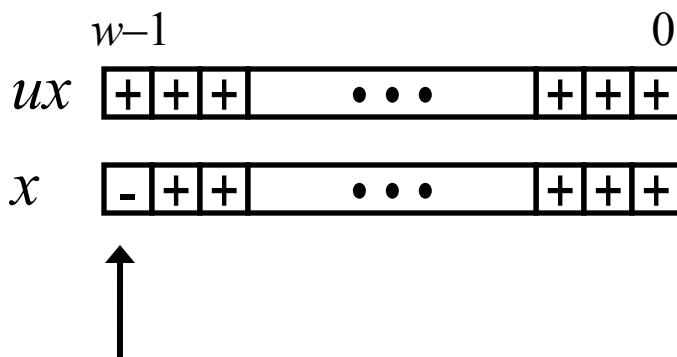
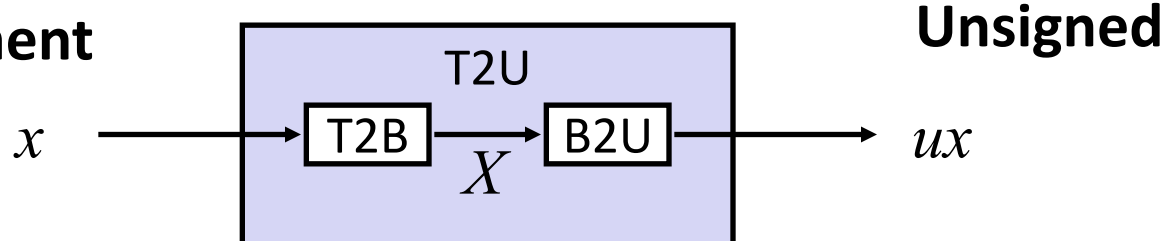
Bits	Signed		Unsigned
0000	0	\rightarrow T2U \rightarrow U2T \leftarrow	0
0001	1		1
0010	2		2
0011	3		3
0100	4		4
0101	5		5
0110	6		6
0111	7		7
1000	-8		8
1001	-7		9
1010	-6		10
1011	-5		11
1100	-4		12
1101	-3		13
1110	-2		14
1111	-1		15

Mapping Signed \leftrightarrow Unsigned

Bits	Signed		Unsigned
0000	0	\longleftrightarrow =	0
0001	1		1
0010	2		2
0011	3		3
0100	4		4
0101	5		5
0110	6		6
0111	7		7
1000	-8	\longleftrightarrow +/- 16	8
1001	-7		9
1010	-6		10
1011	-5		11
1100	-4		12
1101	-3		13
1110	-2		14
1111	-1		15

Relation between Signed & Unsigned

Two's Complement



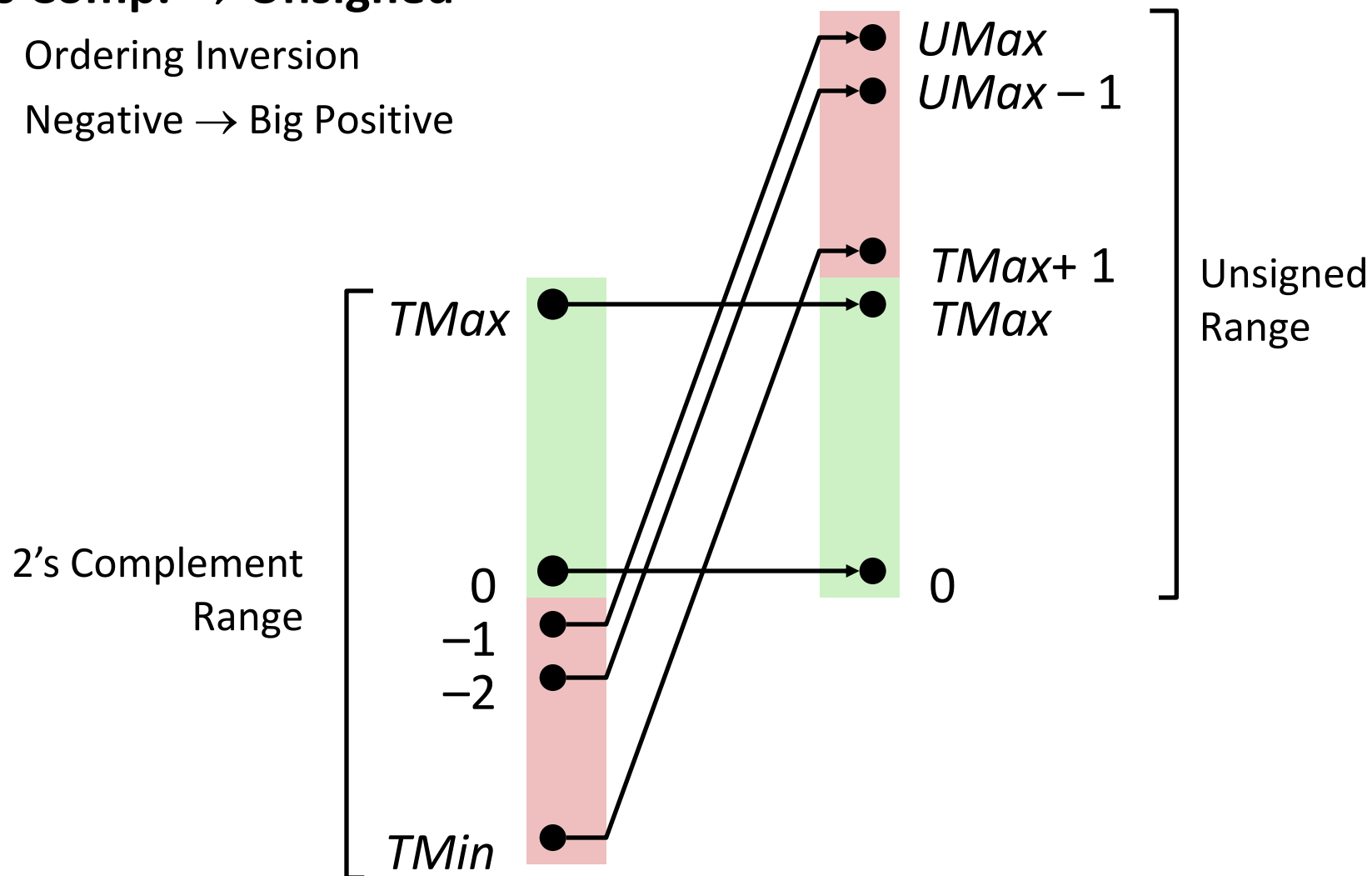
Large negative weight
becomes
Large positive weight

$$ux = \begin{cases} x & x \geq 0 \\ x + 2^w & x < 0 \end{cases}$$

Conversion Visualized

■ 2's Comp. → Unsigned

- Ordering Inversion
- Negative → Big Positive



Signed vs. Unsigned in C

■ Constants

- By default are considered to be signed integers
- Unsigned if have “U” as suffix

`0U, 4294967259U`

■ Casting

- Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;  
unsigned ux, uy;  
tx = (int) ux;  
uy = (unsigned) ty;
```

- Implicit casting also occurs via assignments and procedure calls

```
tx = ux;  
uy = ty;
```

Casting Surprises

■ Expression Evaluation

- If there is a mix of unsigned and signed in single expression,
signed values implicitly cast to unsigned
- Including comparison operations $<$, $>$, $==$, $<=$, $>=$
- Examples for $W = 32$: **TMIN = -2,147,483,648**, **TMAX = 2,147,483,647**

■ Constant ₁	Constant ₂	Relation	Evaluation
0	0U	==	unsigned
-1	0	<	signed
-1	0U	>	unsigned
2147483647	-2147483647-1	>	signed
2147483647U	-2147483647-1	<	unsigned
-1	-2	>	signed
(unsigned)-1	-2	>	unsigned
2147483647	2147483648U	<	unsigned
2147483647	(int) 2147483648U	>	signed

Summary

Casting Signed \leftrightarrow Unsigned: Basic Rules

- Bit pattern is maintained
- But reinterpreted
- Can have unexpected effects: adding or subtracting 2^w
- Expression containing signed and unsigned int
 - `int` is cast to `unsigned`!!

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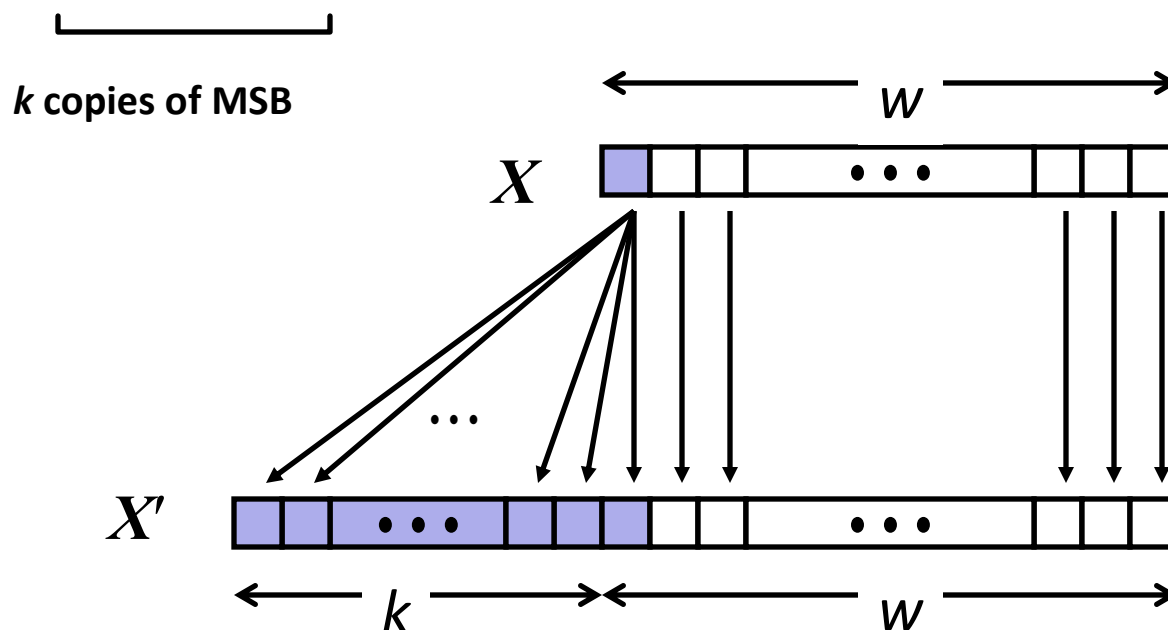
Sign Extension

■ Task:

- Given w -bit signed integer x
- Convert it to $w+k$ -bit integer with same value

■ Rule:

- Make k copies of sign bit:
- $X' = \underbrace{x_{w-1}, \dots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \dots, x_0$



Sign Extension Example

```
short intx = 15213;
    int ix = (int) x;
short inty = -15213;
    int iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- Converting from smaller to larger integer data type
- C automatically performs sign extension

Summary:

Expanding, Truncating: Basic Rules

- **Expanding (e.g., short int to int)**
 - Unsigned: zeros added
 - Signed: sign extension
 - Both yield expected result

- **Truncating (e.g., unsigned to unsigned short)**
 - Unsigned/signed: bits are truncated
 - Result reinterpreted
 - Unsigned: mod operation
 - Signed: similar to mod
 - For small numbers yields expected behaviour

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Negation: Complement & Increment

■ Claim: Following Holds for 2's Complement

$$\sim x + 1 == -x$$

■ Complement

- Observation: $\sim x + x == 1111\dots111 == -1$

$$\begin{array}{r}
 x \quad \boxed{1} \boxed{0} \boxed{0} \boxed{1} \boxed{1} \boxed{1} \boxed{0} \boxed{1} \\
 + \quad \sim x \quad \boxed{0} \boxed{1} \boxed{1} \boxed{0} \boxed{0} \boxed{0} \boxed{1} \boxed{0} \\
 \hline
 -1 \quad \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{1}
 \end{array}$$

■ Complete Proof?

Complement & Increment Examples

x = 15213

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
~x	-15214	C4 92	11000100 10010010
~x+1	-15213	C4 93	11000100 10010011
y	-15213	C4 93	11000100 10010011

x = 0

	Decimal	Hex	Binary
0	0	00 00	00000000 00000000
~0	-1	FF FF	11111111 11111111
~0+1	0	00 00	00000000 00000000

Unsigned Addition

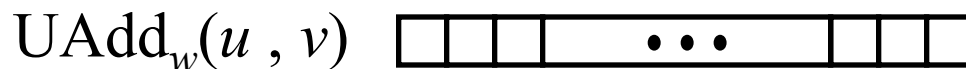
Operands: w bits



True Sum: $w+1$ bits



Discard Carry: w bits



■ Standard Addition Function

- Ignores carry output

■ Implements Modular Arithmetic

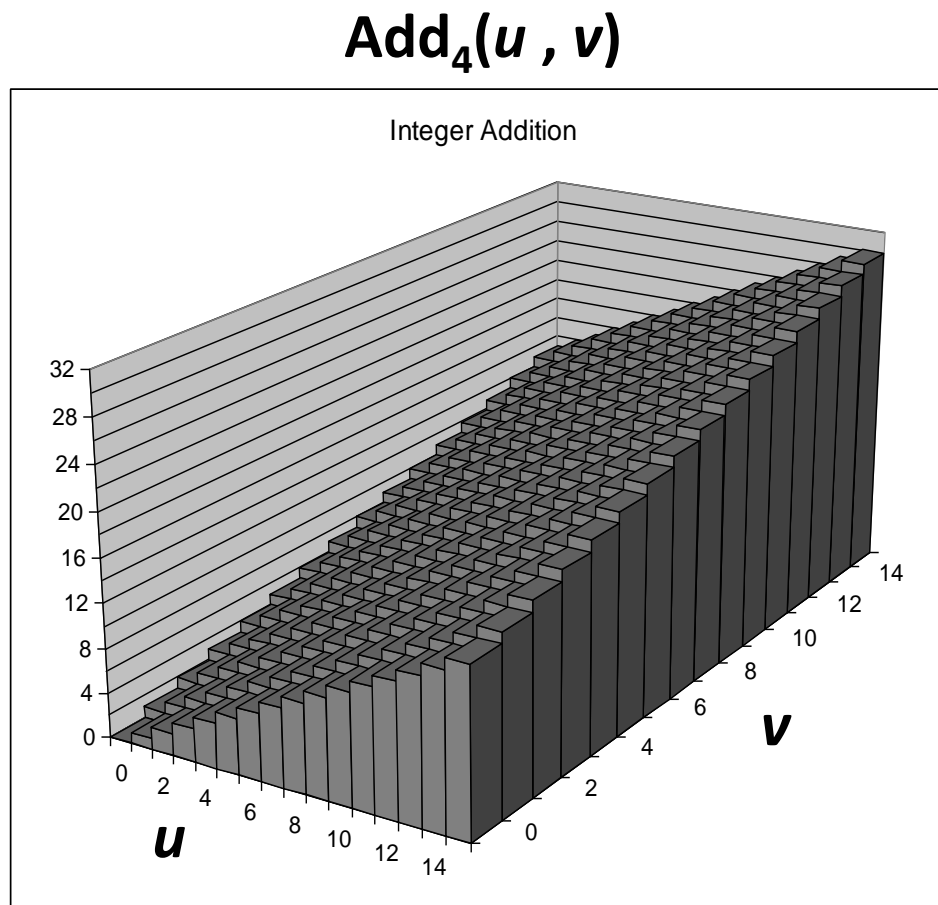
$$s = \text{UAdd}_w(u, v) = u + v \bmod 2^w$$

$$\text{UAdd}_w(u, v) = \begin{cases} u + v & u + v < 2^w \\ u + v - 2^w & u + v \geq 2^w \end{cases}$$

Visualizing (Mathematical) Integer Addition

■ Integer Addition

- 4-bit integers u, v
- Compute true sum $\text{Add}_4(u, v)$
- Values increase linearly with u and v
- Forms planar surface

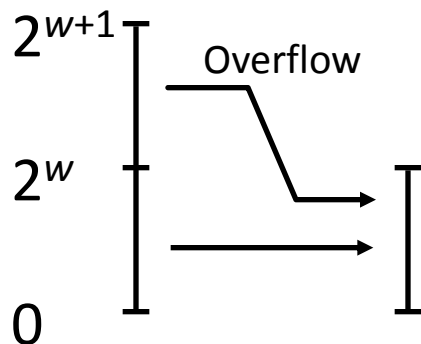


Visualizing Unsigned Addition

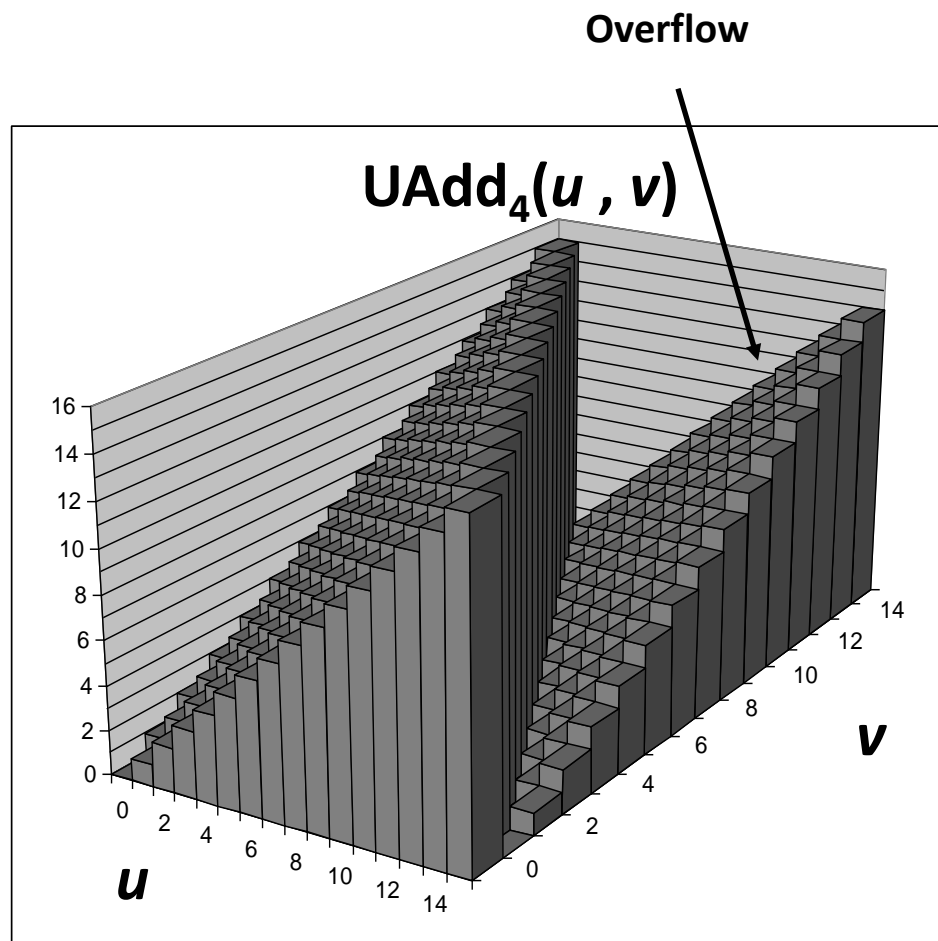
■ Wraps Around

- If true sum $\geq 2^w$
- At most once

True Sum



Modular Sum



Mathematical Properties

■ Modular Addition Forms an *Abelian Group*

- **Closed** under addition

$$0 \leq \text{UAdd}_w(u, v) \leq 2^w - 1$$

- **Commutative**

$$\text{UAdd}_w(u, v) = \text{UAdd}_w(v, u)$$

- **Associative**

$$\text{UAdd}_w(t, \text{UAdd}_w(u, v)) = \text{UAdd}_w(\text{UAdd}_w(t, u), v)$$

- **0** is additive identity

$$\text{UAdd}_w(u, 0) = u$$

- Every element has additive **inverse**

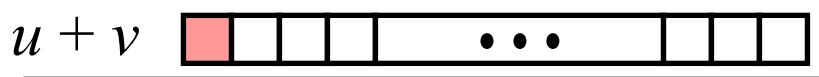
- Let $\text{UComp}_w(u) = 2^w - u$
 $\text{UAdd}_w(u, \text{UComp}_w(u)) = 0$

Two's Complement Addition

Operands: w bits



True Sum: $w+1$ bits



Discard Carry: w bits



■ TAdd and UAdd have Identical Bit-Level Behavior

- Signed vs. unsigned addition in C:

```
int s, t, u, v;
```

```
    s = (int) ((unsigned) u + (unsigned) v);
```

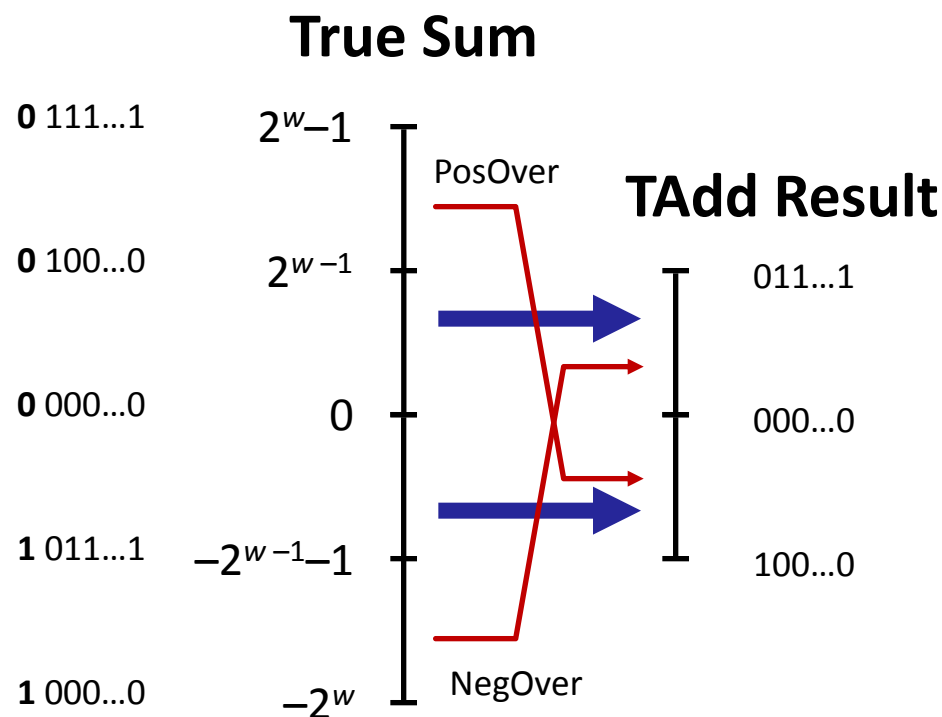
```
    t = u + v;
```

- Will give `s == t`

TAdd Overflow

■ Functionality

- True sum requires $w+1$ bits
- Drop off MSB
- Treat remaining bits as 2's comp. integer



Visualizing 2's Complement Addition

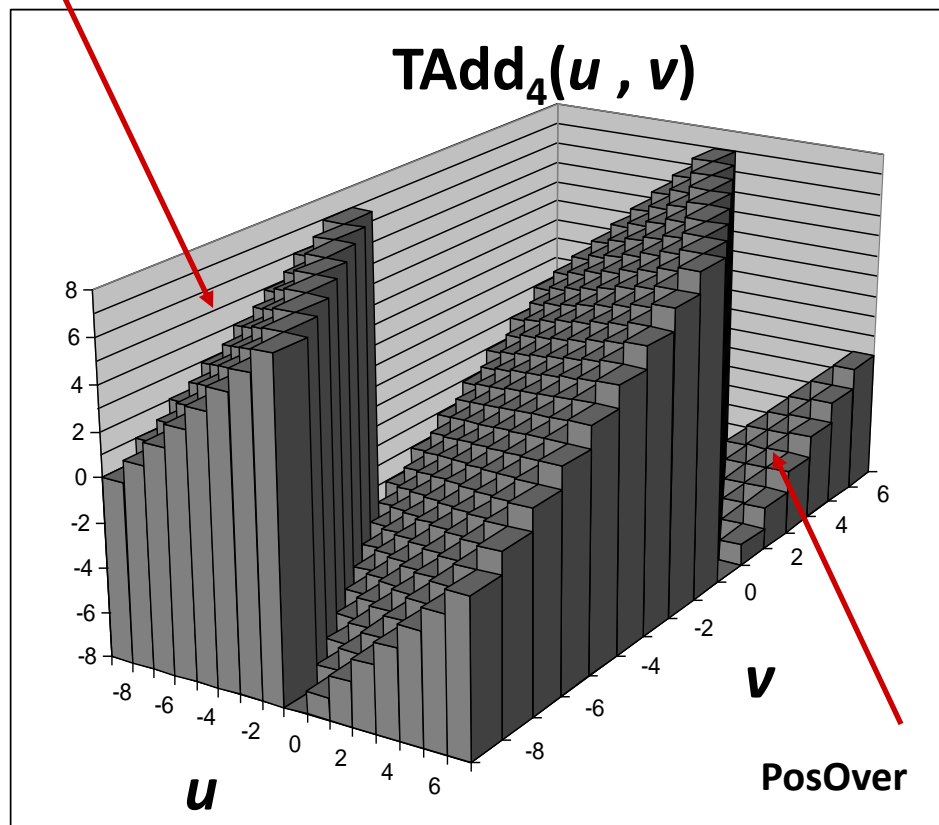
■ Values

- 4-bit two's comp.
- Range from -8 to +7

■ Wraps Around

- If $\text{sum} \geq 2^{w-1}$
 - Becomes negative
 - At most once
- If $\text{sum} < -2^{w-1}$
 - Becomes positive
 - At most once

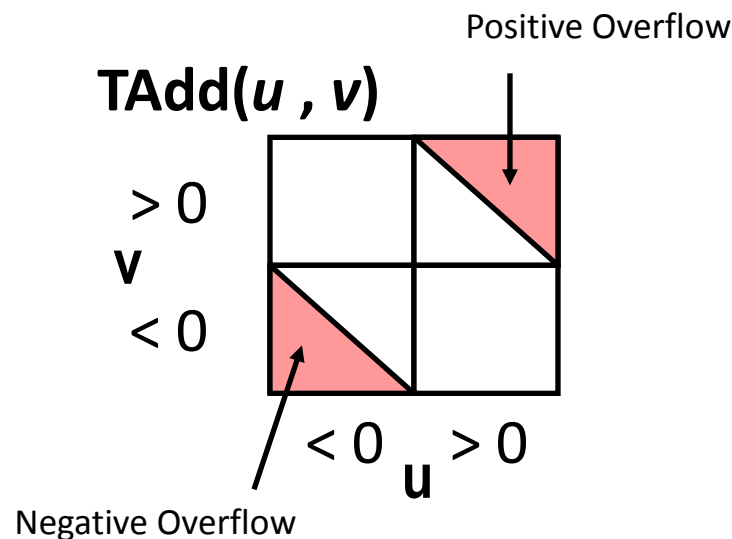
NegOver



Characterizing TAdd

■ Functionality

- True sum requires $w+1$ bits
- Drop off MSB
- Treat remaining bits as 2's comp. integer



$$TAdd_w(u, v) = \begin{cases} u + v + 2^w & u + v < TMin_w \text{ (NegOver)} \\ u + v & TMin_w \leq u + v \leq TMax_w \\ u + v - 2^w & TMax_w < u + v \text{ (PosOver)} \end{cases}$$

Quiz

Practice Problem 2.1

Perform the following number conversions:

- A. 0x39A7F8 to binary
- B. Binary 1100100101111011 to hexadecimal
- C. 0xD5E4C to binary
- D. Binary 1001101110011110110101 to hexadecimal

Quiz

Practice Problem 2.2

Fill in the blank entries in the following table, giving the decimal and hexadecimal representations of different powers of 2:

n	2^n (Decimal)	2^n (Hexadecimal)
9	512	0x200
19	_____	_____
_____	16,384	_____
_____	_____	0x10000
17	_____	_____
_____	32	_____
_____	_____	0x80

Quiz

Practice Problem 2.3

A single byte can be represented by two hexadecimal digits. Fill in the missing entries in the following table, giving the decimal, binary, and hexadecimal values of different byte patterns:

Decimal	Binary	Hexadecimal
0	0000 0000	0x00
167	_____	_____
62	_____	_____
188	_____	_____
_____	0011 0111	_____
_____	1000 1000	_____
_____	1111 0011	_____
_____	_____	0x52
_____	_____	0xAC
_____	_____	0xE7

Quiz

Practice Problem 2.4

Without converting the numbers to decimal or binary, try to solve the following arithmetic problems, giving the answers in hexadecimal. **Hint:** Just modify the methods you use for performing decimal addition and subtraction to use base 16.

- A. $0x503c + 0x8 = \underline{\hspace{2cm}}$
- B. $0x503c - 0x40 = \underline{\hspace{2cm}}$
- C. $0x503c + 64 = \underline{\hspace{2cm}}$
- D. $0x50ea - 0x503c = \underline{\hspace{2cm}}$