Deposit Goals

With a constant monthly deposit and interest which is calculated from the balance at a point in time, the foundations for a formula which computes the future balance after a certain duration can be elementarily derived.

Let
$P = Principal$
$n = Duration\ in\ months$
$a = Annual\ interest\ rate$
$i = Monthly\ interest\ rate = \frac{a}{12}$
$I_1 = Initial\ monthly\ interest = Pi$
$M = Monthly\ deposit$
$T = Net\ value\ or\ total$

To simplify things, we let $A = (1 + i)$. The total at the end of the first month, T_1, would be:

$$\begin{aligned} T_1 &= P + I_1 \\ &= P + Pi \\ &= P(1 + i) \\ &= PA \end{aligned}$$

The monthly deposit, M, is added at the end of the first month or the start of the second month. The following, I_2, would be:

$$\begin{aligned} I_2 &= (T_1 + M)i \\ &= (P(1 + i) + M)i \\ &= (PA + M)i \end{aligned}$$

Thus, the second month's balance, T_2, would be:

$$\begin{aligned} T_2 &= T_1 + M + I_2 \\ &= P(1 + i) + M + (P(1 + i) + M)i \\ &= PA + M + (PA + M)i \\ &= (PA + M)(1 + i) \\ &= (PA + M)A \\ &= PA^2 + MA \end{aligned}$$

Next, we explore the value of third month's interest, I_3:

$$\begin{aligned} I_3 &= (T_2 + M)i \\ &= (PA^2 + MA + M)i \end{aligned}$$

Hence, the third month's balance, T_3, would be:

$$\begin{aligned} T_3 &= T_2 + M + I_3 \\ &= PA^2 + MA + M + (PA^2 + MA + M)i \\ &= (PA^2 + MA + M)(1 + i) \\ &= (PA^2 + MA + M)A \\ &= PA^3 + MA^2 + MA \end{aligned}$$

If we were to continue deriving the formulas for the subsequent balances, a pattern emerges. The balance after a specific duration, T_n, would be:

$$
\begin{aligned}
T_n &= PA^n + \mathrm{M}\sum_{\mathrm{k=1}}^{\mathrm{n-1}} \mathrm{A^k} \\
&= PA^n + \mathrm{M}(\mathrm{A} + A^2 + \cdots + A^{n-1})
\end{aligned}
$$

The formula can be condensed using the geometric sum formula:

$$
\begin{aligned}
T_n &= PA^n + \frac{\mathrm{MA(A^n - 1)}}{\mathrm{A - 1}} \\
&= PA^n + \frac{\mathrm{M(A^{n+1} - A)}}{\mathrm{A - 1}}
\end{aligned}
$$

Computing the Monthly Deposit

With some algebraic manipulation,

$$
M = \frac{(T_n - PA^n)(A - 1)}{\mathrm{A^{n+1} - A}}
$$

Computing the Interest Rate

Due to algebraic constraints, the monthly interest rate, $i, where\ i = A - 1$, cannot be obtained purely through algebraic manipulation. Instead, it must be somehow approximated. For our program, we resorted to the Newton-Raphson approximation algorithm to approximate the value. Firstly, the equation is rearranged for the algorithm. We let y be the function which represents the equation:

$$
\begin{aligned}
T_n &= PA^n + \frac{\mathrm{M(A^{n+1} - A)}}{\mathrm{A - 1}} \\
0 &= -T_n + PA^n + \frac{\mathrm{M(A^{n+1} - A)}}{\mathrm{A - 1}} \\
y &= -T_n + PA^n + \frac{\mathrm{M(A^{n+1} - A)}}{\mathrm{A - 1}}
\end{aligned}
$$

Next, we obtain the differential of y with respect to A:

$$
y' = \frac{dy}{dA} = nPA^{n-1} + \frac{M(A-1)\big((n+1)(A^n) - 1\big) - (\mathrm{A^{n+1} - A})}{(A-1)^2}
$$

The Newton-Raphson's algorithm utilises the following approach:

$$y' = \frac{y_2 - y_1}{A_2 - A_1}$$

$$y' = \frac{0 - y_1}{A_2 - A_1}, where\ y_2 = 0$$

$$A_2 = -\frac{y_1}{y'} + A$$

$$or, in\ general,$$

$$A_{n+1} = -\frac{y_n}{y'} + A_n$$

The program starts with $i_1 = 0$ and uses 50 iterations.
More info on the algorithm: https://bit.ly/2BiWESA

Obtaining the Duration

$Let\ B = A^n$

$$T_n = PA^n + \frac{\mathrm{MA}(\mathrm{A}^{\mathrm{n}} - 1)}{\mathrm{A} - 1}$$

$$T_n = PB + \frac{\mathrm{MA}(\mathrm{B} - 1)}{\mathrm{A} - 1}$$

$$T_n = \frac{PB(\mathrm{A} - 1) + \mathrm{MA}(\mathrm{B} - 1)}{\mathrm{A} - 1}$$

$$T_n(\mathrm{A} - 1) = PB(\mathrm{A} - 1) + MAB - MA$$

$$T_n(\mathrm{A} - 1) = B(P(\mathrm{A} - 1) + MA) - MA$$

$$B = \frac{T_n(\mathrm{A} - 1) + MA}{P(\mathrm{A} - 1) + MA}$$

$$A^n = \frac{T_n(\mathrm{A} - 1) + MA}{P(\mathrm{A} - 1) + MA}$$

$$n = \frac{\log \frac{T_n(\mathrm{A} - 1) + MA}{P(\mathrm{A} - 1) + MA}}{\log A}$$