Interest with Monthly Payment Loan

A fixed monthly payment is allocated, whereby a percentage of it is used to pay the monthly interest while the remainder is set aside for the main loan. The monthly interest is calculated from the net balance at that point in time. Since the net balance decreases with time, the monthly interest also decreases in tandem. If a graph of the net balance against the duration is plotted, an exponential pattern can be deduced. (Note: The derivation below assumes that the interest rates are in their original multiplicative forms, not in percentages.)

Let
$P = Principal$
$n = Duration\ in\ months$
$a = Annual\ interest\ rate$
$i = Monthly\ interest\ rate = \frac{a}{12}$
$I_1 = Initial\ monthly\ interest = Pi$
$M = Monthly\ payment$
$T = Net\ value\ or\ total$

Month	Principal part, X	Interest part, I	Monthly payment
1	X_1	I_1	M_1
2	X_2	I_2	M_2
…	…	…	…
n	X_n	I_n	M_n

Before proceeding, we must acknowledge the assumption that:

$$X_n + I_n = M$$

Since every interest part is the net value at the point in time multiplied by the monthly interest rate, i, the following progression is a consequence.

$$\begin{aligned}
I_1 &= Pi \\
I_2 &= i(P - X_1) \\
&= Pi - X_1 i \\
&= I_1 - X_1 i \\
I_3 &= i(P - X_1 - X_2) \\
&= Pi - X_1 i - X_2 i \\
&= I_1 - X_1 i - X_2 i \\
I_n &= i\left(P - \sum_{k=1}^{n} X_k\right) \\
&= Pi - i\sum_{k=1}^{n} X_k \\
&= I_1 - i\sum_{k=1}^{n} X_k
\end{aligned}$$

Next, each monthly payment should be equal and so we equate the first monthly payment to the second:

$$M_1 = M_2$$
$$I_1 + X_1 = I_2 + X_2$$
$$I_1 + X_1 = I_1 - X_1 i + X_2$$
$$X_2 = X_1(1 + i)$$

Equating the second and the third monthly payments:

$$M_2 = M_3$$
$$I_2 + X_2 = I_3 + X_3$$
$$(I_1 - X_1 i) + X_2 = (I_1 - X_1 i - X_2 i) + X_3$$
$$X_3 = X_2(1 + i)$$
$$X_3 = X_1(1 + i)(1 + i)$$
$$X_3 = X_1(1 + i)^2$$

A pattern emerges when the method is repeated for further payments:

$$X_n = X_1(1 + i)^{\mathrm{n}-1}$$

Moving on, the sum of all the monthly payments is equal to the principal, P:

$$\sum_{k=1}^{n} X_k = P$$
$$X_1 + X_2 + \cdots X_n = P$$
$$X_1 + X_1(1 + i) + X_1(1 + i)^2 \cdots X_1(1 + i)^{\mathrm{n}-1} = \mathrm{P}$$
$$X_1(1 + (1 + i) + (1 + i)^2 \cdots (1 + i)^{\mathrm{n}-1}) = P$$

Using the geometric sum formula, the formula can be condensed into:

$$\frac{X_1((1 + i)^n - 1)}{(1 + i) - 1} = P$$
$$X_1 = \frac{Pi}{(1 + i)^n - 1}$$

Now that X_1 can be expressed in terms of P and i, the monthly payment, M, can finally be derived:

$$X_1 + I_1 = M$$
$$\frac{Pi}{(1 + i)^n - 1} + Pi = M$$
$$\frac{Pi + Pi((1 + i)^n - 1)}{(1 + i)^n - 1} = M$$
$$M = \frac{Pi(1 + i)^n}{(1 + i)^n - 1}$$

Obtaining the Monthly Interest Rate

Unlike the interest-only loan, the monthly interest rate, i, cannot be obtained purely through algebraic manipulation. Instead, it must be somehow approximated. For our program, we resorted to the Newton-Raphson approximation algorithm to approximate the value. Firstly, the equation is rearranged for the algorithm. We let y be the function which represents the equation:

$$M = \frac{Pi(1+i)^n}{(1+i)^n - 1}$$
$$0 = \frac{M}{P} - \frac{i(1+i)^n}{(1+i)^n - 1}$$
$$y = \frac{M}{P} - \frac{i(1+i)^n}{(1+i)^n - 1}$$

Next, we obtain the differential of y with respect to i:

$$y' = \frac{dy}{di} = -\frac{((1+i)^n - 1)((1+i)^n + ni(1+i)^{n-1}) - ni(1+i)^n(1+i)^{n-1}}{((1+i)^n - 1)^2}$$

The Newton-Raphson's algorithm utilises the following approach:

$$y' = \frac{y_2 - y_1}{i_2 - i_1}$$
$$y' = \frac{0 - y_1}{i_2 - i_1}, where\ y_2 = 0$$
$$i_2 = -\frac{y_1}{y'} + i_1$$
$$or, in\ general,$$
$$i_{n+1} = -\frac{y_n}{y'} + i_n$$

The program starts with $i_1 = 0$ and uses 50 iterations.
More info on the algorithm: https://bit.ly/2BiWESA

Obtaining the Duration

$Let\ A = (1+i)^n$

$$M = \frac{Pi(1+i)^n}{(1+i)^n - 1}$$
$$M = \frac{PiA}{A-1}$$
$$MA - M = PiA$$
$$MA - PiA = M$$
$$A(M - Pi) = M$$
$$A = \frac{M}{M - Pi}$$
$$(1+i)^n = \frac{M}{M - Pi}$$
$$n = \frac{\log \mathrm{M} - \log(\mathrm{M} - \mathrm{Pi})}{\log(1+i)}$$