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Factorization of operator-valued polynomials in several non-commuting variables ☆

Scott McCullough

Department of Mathematics, University of Florida, Gainesville, FL 32611-8105, USA

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Abstract

A version of Fejer–Riesz factorization and factorization of positive operator-valued polynomials in several non-commuting variables holds. The proofs use Arveson's extension theorem and matrix completions. © 2001 Elsevier Science Inc. All rights reserved.

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0. Introduction

Let $\mathscr{G}_m$ denote the free semi-group on m generators $g_1, \ldots, g_m$ with identity e. Let $\mathscr{U}_m$ denote the set of m-tuples $U = (U_1, \ldots, U_m)$ of of unitary operators on separable Hilbert space. Given a word $w = g_{j_1} g_{j_2} \cdots g_{j_k} \in \mathscr{G}_m$ and $U \in \mathscr{U}_m$, let

$$U^w = U_{j_1} U_{j_2} \cdots U_{j_k}.$$

Let $\mathscr{G}_m^n$ denote the words of length at most n, $\mathscr{F}_m \supset \mathscr{G}_m$ the free group on the m generators $g_1, \ldots, g_m$, and $\mathscr{H}_m^n$ the set of words of the form $v^{-1}w$, where $v, w \in \mathscr{G}_m^n$. ($\mathscr{H}_m^n$ are the hereditary words of order n.) For $h = v^{-1}w \in \mathscr{H}_m^n$ and $U \in \mathscr{U}_m$, define

$$U^h = (U^v)^* U^w,$$

and observe U^h depends only upon h, not v and w.

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E-mail address: sam@math.ufl.edu (S.A. McCullough).

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Given (complex) Hilbert spaces $\mathscr{C}$ and $\mathscr{E}$, let $\mathscr{L}(\mathscr{C})$ and $\mathscr{L}(\mathscr{C},\mathscr{E})$ denote the bounded operators on $\mathscr{C}$ and from $\mathscr{C}$ to $\mathscr{E}$, respectively.

Theorem 0.1. *Let positive integers m and n and operators* $A_h \in \mathscr{L}(\mathscr{C})$, $h \in \mathscr{H}^n_m$, *be given. If*

$$A(U) = \sum U^h \otimes A_h$$

is positive semidefinite for each $U \in \mathscr{U}_m$, *then there exists an auxiliary Hilbert space* $\mathscr{E}$ *(of dimension at most* $\dim(\mathscr{C}) \sum_0^n m^j$*) and operators* $B_w \in \mathscr{L}(\mathscr{C},\mathscr{E})$, $w \in \mathscr{G}^n_m$, *such that*

$$A(U) = B(U)^* B(U),$$

where

$$B(U) = \sum U^w \otimes B_w.$$

When $m = 1$, the spectral theorem implies that

$$A(U) = \sum_{-n}^{n} U^j \otimes A_j$$

is positive semidefinite for all unitary operators U if and only if

$$A(u) = \sum_{-n}^{n} u^j A_j$$

is positive semidefinite for all complex numbers u of modulus 1. Thus, as a corollary of Theorem 0.1, if $A(u)$ is positive semidefinite, then it factors as $B(u)^* B(u)$. This is somewhat weaker than the versions of Fejer–Riesz factorization of Rosenblatt, Gohberg, and the very general results of Rosenblum and Rovnyak [7,8], where $\mathscr{E} = \mathscr{C}$ and $B(u)$ is an outer function (no zeros in the unit disc in the scalar $\mathscr{C} = \mathbb{C}$ case). However, using Beurling's theorem, this stronger conclusion follows from the existence of a factorization. In the several variable case ($m > 1$) there is a version of Beurling's theorem, but it requires an auxiliary Hilbert space (usually countably many copies of $\mathscr{C}$) and so it seems unlikely that, in general, $\mathscr{E}$ can be chosen equal to $\mathscr{C}$ when $\mathscr{C}$ is finite dimensional.

To state a factorization for non-negative operator valued polynomials in several non-commuting variables, let $\mathscr{S}_m$ denote the set of m-tuples $S = (S_1, \dots, S_m)$ of self-adjoint operators on separable Hilbert space. Given $S \in \mathscr{S}_m$ and a word $w = g_{j_1} g_{j_2} \cdots g_{j_k}$, let

$$S^w = S_{j_1} S_{j_2} \cdots S_{j_k}.$$

Theorem 0.2. *Let m and n positive integers, and operators* $A_w \in \mathscr{L}(\mathscr{C})$, $w \in \mathscr{G}^{2n}_m$, *be given. If*

$$A(S) = \sum S^w \otimes A_w$$

is positive semidefinite for each $S \in \mathscr{S}_m$, *then there exists an auxiliary Hilbert space* (*of dimension at most* $\dim(\mathscr{C}) \sum_0^n m^j$) *and operators* $B_w \in \mathscr{L}(\mathscr{C}, \mathscr{E})$, $w \in \mathscr{G}_m^n$, *such that*

$$A(S) = B(S)^* B(S),$$

where

$$B(S) = \sum S^w \otimes B_w.$$

The remainder of the paper has four sections. Section 1 collects some preliminaries about completely positive maps and matrix completions. Hankel and Toeplitz matrices in several non-commuting variables are the topics of Sections 2 and 3, respectively. The results in these sections are closely related to recent results of Arias and Popescu [1], and Davidson and Pitts [4]. Section 4 contains the proofs of Theorems 0.1 and 0.2.

1. Preliminaries

1.1. Completely positive maps

A subset $\mathscr{S}$ of the C^*-algebra M_ℓ of $\ell \times \ell$ matrices is self-adjoint if $S^* \in \mathscr{S}$ whenever $S \in \mathscr{S}$. A linear map $\phi : \mathscr{S} \mapsto \mathscr{L}(\mathscr{C})$ is completely positive if the mapping $1_k \otimes \phi : M_k \otimes \mathscr{S} \mapsto M_k \otimes \mathscr{L}(\mathscr{C})$ is positive for each k.

Theorem 1.1 (Special case of Arveson's extension theorem). *If* $\mathscr{S} \subset M_\ell$ *is self-adjoint*, *if* $\mathscr{S}$ *contains a positive definite invertible matrix*, *and if* $\phi : \mathscr{S} \mapsto \mathscr{L}(\mathscr{C})$ *is completely positive*, *then there exists a completely positive map* $\bar{\phi} : M_\ell \mapsto \mathscr{L}(\mathscr{C})$ *which extends* ϕ.

Proof (*Sketch*). The usual hypothesis is that $\mathscr{S}$ contains the identity, rather than an invertible positive element [6].

If $\mathscr{S}$ does not contain the identity, but does contain a positive invertible matrix P, choose X so that $X^*X = P$, let $\mathscr{T} = (X^{-1})^* \mathscr{S} X^{-1}$ and define $\psi : \mathscr{T} \mapsto \mathscr{L}(\mathscr{C})$ by

$$\psi(T) = \phi(X^*TX).$$

ψ is completely positive and thus extends to a completely positive map $\bar{\psi} : M_\ell \mapsto \mathscr{L}(\mathscr{C})$. Define $\bar{\phi} : M_\ell \mapsto \mathscr{L}(\mathscr{C})$ by $\bar{\phi}(M) = \bar{\psi}((X^{-1})^* M X^{-1})$. Then $\bar{\phi}$ is a completely positive extension of ϕ. □

There is a simple characterization due to Choi [6] of completely positive maps $\phi : M_\ell \mapsto \mathscr{L}(\mathscr{C})$. Let $E_{\alpha,\beta} \in M_\ell$ denote the matrix with a one in the (α, β) position and zero elsewhere.

Theorem 1.2. *$\phi : M_\ell \mapsto \mathscr{L}(\mathscr{C})$ is completely positive if and only if*

$$(\phi(E_{\alpha,\beta})) \in \mathscr{L}(\bigoplus^\ell \mathscr{C})$$

is positive semidefinite.

1.2. Chordal graphs and matrix completions

A graph (undirected graph) G consists of a finite set V, the vertices, and a subset E of $V \times V$, the edges, such that $(v, v) \in E$ for each $v \in V$ and $(v, w) \in E$ if and only if $(w, v) \in E$. A subset C of V is a clique (of G) if $(v, w) \in E$ for each $v, w \in C$. A subset L_k of V is a loop of length k provided there is an enumeration $L_k = \{v_1, v_2, \ldots, v_k\}$ such that $(v_j, v_\ell) \in E$ if and only if $|j - \ell| \leqslant 1$ or $|j - \ell| = k - 1$. The graph G is chordal if it contains no loops of length 4 or more.

A partially positive matrix subordinate to the graph G is a set of $k \times k$ matrices $\{P_x : x \in E\}$ such that for each clique $C \subset V$ the matrix (with matrix entries)

$$(P_{(v,w)})_{v,w\in C}$$

is positive semidefinite.

Grone et al. [5] show that if G is chordal, then a partially positive matrix subordinate to G can be extended to a positive semi-definite matrix.

Theorem 1.3. *If G is a chordal graph and if $\{P_x : x \in E\}$ is a partial positive matrix subordinate to G, then there exists $k \times k$ matrices P_x for $x \notin E$ such that*

$$(P_{(v,w)})_{v,w\in V}$$

is positive semidefinite.

2. Hankel matrices in several non-commuting variables

Denote by $\mathfrak{G}^n_m$ the Hilbert space with orthonormal basis $\mathscr{G}^n_m$. Given a word $w = g_{j_1} g_{j_2} \cdots g_{j_k} \in \mathscr{G}_m$, let w^{t} denote the transpose of w,

$$w^{\mathrm{t}} = g_{j_k} \cdots g_{j_2} g_{j_1}.$$

A matrix $H \in \mathscr{L}(\mathfrak{G}^n_m)$ is a Hankel matrix if $H_{v,w}$ depends only upon $v^{\mathrm{t}} w$. Let $\mathbb{H}^n_m$ denote the collection of all such Hankel matrices. Thus $M_k \otimes \mathbb{H}^n_m$ is the set of Hankel matrices with $k \times k$ matrix entries. In this case $H_{v,w} \in M_k$. Observe that $\mathbb{H}^n_m$ is self-adjoint.

In systems theory, the transfer function corresponds to a Hankel matrix which then has a number of state space realizations. A similar representation holds for Hankel matrices in several non-commuting variables. The representation below for positive Hankel operators generalizes a result of Curto and Fialkow [2].

Theorem 2.1. *If $H \in M_k \otimes \mathbb{H}_m^n$ and if H is positive definite, then there exists an m-tuple $S = (S_1, \ldots, S_m)$ of self-adjoint operators on a Hilbert space $\mathscr{K}$ of dimension at most $k \sum_0^n m^j$ and an operator $V : \mathbb{C}^k \mapsto \mathscr{K}$ such that*

$$H_{v,w} = V^* S^{v^{\mathrm{t}}w} V$$

for all $v, w \in \mathscr{G}_m^n$.

The first step in the proof of Theorem 2.1 is to extend the positive definite Hankel matrix H. The same proof that positive definite Hankel matrices have positive completions [2] works for Hankel matrices in several non-commuting variables. Let $|w|$ denote the length of the word $w \in \mathscr{G}_m$.

Lemma 2.2. *If $H \in M_k \otimes \mathbb{H}_m^n$ is positive definite, then there exist $G, G' \in M_k \otimes \mathbb{H}_m^{n+1}$ such that $G_{v,w} = G'_{v,w} = H_{v,w}$ for $v, w \in \mathscr{G}_m^n$ and*
1. *G' is positive definite, and*
2. *G is positive semidefinite and for each $x = \sum_{|w|=n+1} x_w \otimes w \in \mathbb{C}^k \otimes \mathfrak{G}_m^{n+1}$, there exists $y = \sum_{|w| \leqslant n} y_w \otimes w$ such that $G(x - y) = 0$.*

It is possible to parameterize all the choices for G, but to keep the exposition concise this is not done.

Proof of Lemma 2.2. Define $G_{v,w} = 0 = G'_{v,w}$ for $|v^{\mathrm{t}}w| = 2n + 1$. View $\mathfrak{G}_m^n$ as a subspace of $\mathfrak{G}_m^{n+1}$. Thus $\mathfrak{G}_m^n$ is the span of the words of length at most n and the orthogonal complement $(\mathfrak{G}_m^n)^\perp$ is the span of the words of length exactly $n + 1$. Define $X : \mathbb{C}^k \otimes (\mathfrak{G}_m^n)^\perp \mapsto \mathbb{C}^k \otimes \mathfrak{G}_m^n$ by

$$\langle X(x \otimes w), y \otimes v\rangle = \langle G_{v,w}x, y\rangle.$$

Observe that

$$\langle X^*(y \otimes v), x \otimes w\rangle = \langle G_{w,v}y, x\rangle,$$

since for $|v| < n$, $G_{v,w} = H_{v,w} = H^*_{w,v} = G^*_{w,v}$, and if $|v| = n$, $G_{v,w} = 0 = G_{w,v}$. Let $Y = X^* H^{-1} X$. Define, for $|v|, |w| = n + 1$,

$$\langle G_{v,w}x, y\rangle = \langle Y(x \otimes w), y \otimes v\rangle.$$

Then the Hankel operator

$$G = \begin{pmatrix} H & X \\ X^* & Y \end{pmatrix} = \begin{pmatrix} H^{1/2} \\ X^* H^{-1/2} \end{pmatrix} \begin{pmatrix} H^{1/2} & H^{-1/2}X \end{pmatrix}$$

is positive semidefinite. Moreover, given $x = \sum_{|w|=n+1} x_w \otimes w$, choose $y = H^{-1}Xx$, then $G(x - y) = 0$. This proves item (2). To prove item (1), define G' the same as G, except when $|v|, |w| = n + 1$ let

$$G'_{v^{\mathrm{t}}w} = G_{v^{\mathrm{t}}w} + \delta_{v,w} I. \qquad \square$$

Lemma 2.3. $\mathbb{M}_k \otimes H_m^n$ *contains a positive definite matrix.*

Proof. The proof proceeds by induction on n. For $n = 0$ there is little to prove. If the result holds for n, then it holds for $n + 1$ by an application of item (1) of Lemma 2.2. □

Proof of Theorem 2.1. By Lemma 2.2(2), there exists a positive semidefinite $G \in \mathbb{C}^k \otimes \mathbb{H}_m^{n+1}$ such that $G_{v,w} = H_{v,w}$ for $v, w \in \mathscr{G}_m^n$ and such that for each $x = \sum_{|w|=n+1} x_w \otimes w \in \mathscr{C} \otimes \mathfrak{G}_{n+1}$, there exists $y = \sum_{|w|\leqslant n} y_w \otimes w$ with

$$G(x - y) = 0. \tag{2.1}$$

Let $\mathscr{L}$ denote the vector space with basis $\mathscr{G}_m^{n+1}$ and let $H(G)$ denote the Hilbert space obtained from $\mathbb{C}^k \otimes \mathscr{L}$ by moding out null vectors using the positive semi-definite form

$$\begin{aligned}\left[\sum x_w \otimes w, \sum y_v \otimes v\right] &= \left\langle \left(\sum x_w \otimes w\right), \sum y_v \otimes v\right\rangle \\ &= \sum \langle G_{v,w} x_w, y_v\rangle.\end{aligned}$$

Together, the hypothesis that H is a positive definite matrix and (2.1) imply that the cosets represented by $\{e_q \otimes w\colon |w| \leqslant n, 1 \leqslant q \leqslant k\}$ is a basis of $H(G)$, where $\{e_q\}$ is the standard basis of $\mathbb{C}^k$. On this basis, define the shift operators S_j by

$$S_j(e_q \otimes w) = e_q \otimes g_j w.$$

Compute

$$\begin{aligned}&\left[S_j \sum_{|w|\leqslant n} x_w \otimes w, \sum_{|v|\leqslant n} y_v \otimes v\right] \\ &\quad = \left[\sum_{|w|\leqslant n} x_w \otimes g_j w, \sum_{|v|\leqslant n} y_v \otimes v\right] \\ &\quad = \sum \langle G_{v,g_j w} x_w, y_v\rangle \\ &\quad = \sum \langle G_{g_j v,w} x_w, y_v\rangle \\ &\quad = \left[\sum_{|w|\leqslant n} x_w \otimes w, \sum_{|v|\leqslant n} y_v \otimes g_j v\right] \\ &\quad = \left[\sum_{|w|\leqslant n} x_w \otimes w, S_j \sum_{|v|\leqslant n} y_v \otimes v\right].\end{aligned}$$

Thus S_j is self-adjoint.

Define $V : \mathbb{C}^k \mapsto H(G)$ by $Vx = x \otimes e$ and compute, for $|v|, |w| \leqslant n$,

$$\begin{aligned}[V^* S^{v^{\mathrm{t}}w} Vx, y] &= [S^w Vx, S^v Vy] \\ &= [x \otimes w, y \otimes v] \\ &= \langle G_{v,w} x, y\rangle.\end{aligned}$$

Thus, for any $v, w \in \mathscr{G}_m^n$,

$$V^* S^{v^{\mathrm{t}}w} V = G_{v,w} = H_{v,w}. \qquad \square$$

3. Toeplitz matrices in several non-commuting variables

Using the notations of the previous section, $T \in \mathscr{L}(\mathfrak{G}_m^n)$ is a Toeplitz matrix if $T_{v,w}$ depends only upon $v^{-1}w$. Let $\mathbb{T}_m^n$ denote the set of all such Toeplitz matrices. Thus, $M_k \otimes \mathbb{T}_m^n$ is the set of Toeplitz operators—the Toeplitz matrices with $k \times k$ matrix entries. In particular, $t(h) \in M_k$ for $h \in \mathscr{H}_m^n$ determines a Toeplitz operator by requiring

$$\langle T(x \otimes w), y \otimes v\rangle = \langle t(v^{-1}w)x, y\rangle$$

for $v, w \in \mathscr{G}_m^n$ and $x, y \in \mathbb{C}^k$. Observe that $\mathbb{T}_m^n$ is self-adjoint. Choosing, $t(e) = 1$ and $t(h) = 0$ if $h \neq e$ shows $\mathbb{T}_m^n$ contains the identity matrix.

Compare the following representation for positive semi-definite Toeplitz matrices with the Naimark Dilation Theorem [6].

Theorem 3.1. *If $T \in M_k \otimes \mathbb{T}_m^n$ is positive semidefinite, then there exists an m-tuple $U = (U_1, \dots, U_m)$ of unitary operators on a Hilbert space $\mathscr{K}$ and an operator $V : \mathbb{C}^k \mapsto \mathscr{K}$ such that*

$$T_{v,w} = V^* U^{v^{-1}w} V,$$

for all $|v|, |w| \leqslant n$.

The proof of Theorem 3.1 depends on a positive, rather than contractive, version of Caratheodory interpolation in several non-commuting variables. Recent work of Arias and Popescu [1] and Davidson and Pitts [4] addresses Nevanlinna–Pick and Caratheodory interpolation in several non-commuting variables.

Lemma 3.2. *If $T \in M_k \otimes \mathbb{T}_m^n$ is positive semidefinite, then there exists $R \in M_k \otimes \mathbb{T}_m^{n+1}$ such that R is positive semidefinite and $R_{v,w} = T_{v,w}$ for $v, w \in \mathscr{G}_m^n$.*

Proof. Let G denote the graph with vertices $V = \mathscr{G}_m^{n+1}$ and edges $E = \{(v, w) : v^{-1}w \in \mathscr{H}_m^n\}$. Thus $(v, w) \in E$ if and only if either $|v|, |w| \leqslant n$, or there exists j such that $v = g_j v'$ and $w = g_j w'$. In particular, the maximal cliques of G are $\mathscr{G}_m^n$

and $g_j\mathscr{G}_m^n$, $j=1,2,\dots,m$ and there are no edges from $\{g_jw : |w|=n\}$ to $\{g_\ell w : |w|\leqslant n\}$ when $j\neq\ell$.

To see that G is chordal, suppose $L=\{v_1,\dots,v_k\}$ is a subset of V and (v_i,v_j) is an edge if either $|i-j|=1$ or $|i-j|=k$ with $k\geqslant 4$. If all of the v_j are in $\mathscr{G}_m^n$, then, as $\mathscr{G}_m^n$ is a clique, L is not a loop. Now suppose not all of the v_j are in $\mathscr{G}_m^n$. Without loss of generality, we may assume $v_1=g_1w_1$ for some $|w_1|=n$. Consequently, $v_2=g_1w_2$ and $g_k=g_1w_k$ for some $|w_2|,|w_k|\leqslant n$. But then $(v_2,v_k)\in E$, and thus L is not a loop. It follows that G contains no loops of length 4 or more. Thus G is chordal.

Let $t(h)=T_{v,w}$ for $h=v^{-1}w\in\mathscr{H}_m^n$. Since T is toeplitz, $t(h)$ is well defined. The data $t(h)$, $h\in\mathscr{H}_m^n$, determines a partially defined matrix R subordinate to the graph G by

$$R_{v,w}=t(v^{-1}w),\quad (v,w)\in E.$$

On the maximal cliques $g_j\mathscr{G}_m^n$,

$$\begin{aligned}(R_{v,w})_{v,w\in g_j\mathscr{G}_m^n} &= (R_{g_jv',g_jw'})_{v',w'\in\mathscr{G}_m^n}\\ &= (T_{v',w'})_{v',w'\in\mathscr{G}_m^n}.\end{aligned}$$

Thus, R defines a partially positive matrix subordinate to the graph G. It follows from Theorem 1.2 that there exists $R_{g_jv,g_\ell w}$ for $|w|,|v|=n$ and $j\neq\ell$ such that

$$(R_{v,w})_{v,w\in\mathscr{G}_m^{n+1}}$$

is positive semidefinite.

Proof of Theorem 3.1. From Lemma 3.2 and induction, there exists $Q_{v,w}\in M_k$, $v,w\in\mathscr{G}_m$, such that $Q_{v,w}$ depends only upon $v^{-1}w$, for each N the matrix

$$(Q_{v,w})_{|v|,|w|\leqslant N}$$

is positive semidefinite, and if $|v|,|w|\leqslant n$, then $Q_{v,w}=T_{v,w}$. Let $\mathscr{V}$ denote the vector space with basis $\mathscr{G}_m$. Let $H(Q)$ denote the Hilbert space obtained by moding out null vectors and completing $\mathbb{C}^k\otimes\mathscr{V}$ in the form

$$\left[\sum x_w\otimes w,\sum y_v\otimes v\right]=\sum\langle Q_{v,w}x_w,y_v\rangle.$$

Define the shift operators S_j densely on $H(Q)$ by

$$S_j\sum x_w\otimes w=\sum x_w\otimes g_jw.$$

The equality,

$$\begin{aligned}&\left[S_j\sum x_w\otimes w, S_j\sum y_v\otimes v\right]\\ &\quad=\left[\sum x_w\otimes g_jw,\sum y_v\otimes g_jv\right]\\ &\quad=\left\langle\sum Q_{g_jv,g_jw}x_w,y_v\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= \left\langle \sum Q_{v,w} x_w, y_v \right\rangle \\
&= \left[\sum x_w \otimes w, \sum y_v \otimes v \right]
\end{aligned}
$$

shows that S_j is (well defined and) an isometry and so extends to an isometry on $H(G)$.

There exists a Hilbert space $K(G)$ containing $H(G)$ and unitary operators U_j on $K(G)$ such that $H(G)$ is invariant for U_j and U_j restricted to $H(G)$ is S_j. Define $V : \mathbb{C}^k \mapsto K(G)$ by $Vx = x \otimes e$ and compute, for $|v|, |w| \leqslant n$,

$$
\begin{aligned}
\langle V^* U^{v^{-1}w} Vx, y \rangle &= \langle U^w x \otimes e, U^v y \otimes e \rangle \\
&= \langle S^w x \otimes e, S^v y \otimes e \rangle \\
&= \langle x \otimes w, y \otimes v \rangle \\
&= \langle Q_{v,w} x, y \rangle \\
&= \langle T_{v,w} x, y \rangle
\end{aligned}
$$

Thus, $V^* U^{v^{-1}w} V = T_{v,w}$. $\square$

4. Main results

This section contains the proofs of Theorems 0.1 and 0.2.

Proof of Theorem 0.2. Let $\ell = \sum_0^n m^j$ (the cardinality of $\mathscr{G}_m^n$). As before, let $\mathfrak{G}_m^n$ denote the Hilbert space with orthonormal basis $\mathscr{G}_m^n$ and identify M_ℓ, the $\ell \times \ell$ matrices, with $\mathscr{L}(\mathfrak{G}_m^n)$ in the natural way.

Let $E_{v,w} \in M_\ell$ denote the matrix with a one in the (v, w) position and zeros elsewhere. Given $u \in \mathscr{G}_m^{2n}$, let $e(u) = \sum\{E_{v,w} : u = v^{\mathrm{t}} w\}$. The set $\{e(u) : u \in \mathscr{G}_m^{2n}\}$ is a basis of $\mathbb{H}_m^n$. Define $\phi : \mathbb{H}_m^n \mapsto \mathscr{L}(\mathscr{C})$ by

$$\phi(e(u)) = A_u.$$

To show that ϕ is completely positive, suppose $H \in M_k \otimes \mathbb{H}_m^n$ is positive definite and let $h(u) = H_{v^{\mathrm{t}}w}$, where $v, w \in \mathscr{G}_m^n$ and $u = v^{\mathrm{t}} w$. Since H is a Hankel operator, $h(u)$ is well defined. Note also that $H = \sum h(u) \otimes e(u)$. By Theorem 2.1, there exists an m-tuple $S = (S_1, \ldots, S_m)$ of self-adjoint operators on a Hilbert space $\mathscr{K}$ and an operator $V : \mathbb{C}^k \mapsto \mathscr{K}$ such that $h(u) = V^* S^u V$ for $|u| \leqslant 2n$. Thus,

$$
\begin{aligned}
(1_k \otimes \phi)(H) &= (1_k \otimes \phi) \left(\sum h(u) \otimes e(u) \right) \\
&= \sum h(u) \otimes A_u \\
&= (V \otimes 1_{\mathscr{C}})^* \left(\sum S^u \otimes A_u \right) (V \otimes 1_{\mathscr{C}}).
\end{aligned}
$$

Thus $(1_k \otimes \phi)(H)$ is positive semidefinite. If H is merely positive semidefinite, rather than positive definite, choose, by Lemma 2.3, a positive definite $G \in M_k \otimes \mathbb{H}^n_m$. Since, for $\delta > 0$, $H + \delta G$ is positive definite, $(1_k \otimes \phi)(H + \delta G)$ is positive semidefinite. Letting δ tend to zero shows $(1_k \otimes \phi)(H)$ is positive semidefinite. Thus ϕ is completely positive. $\square$

From Theorem 1.1, there exists a completely positive extension $\bar{\phi} : M_\ell \mapsto \mathscr{L}(\mathscr{C})$ of ϕ. From Theorem 1.2,

$$(\bar{\phi}(E_{v,w})) \in \mathscr{L}\left(\bigoplus^\ell \mathscr{C}\right)$$

is positive semidefinite. Thus, there exists operators $B_w : \mathscr{C} \mapsto \oplus^\ell \mathscr{C}$ such that

$$B_v^* B_w = \bar{\phi}(E_{v,w}).$$

In particular,

$$\begin{aligned} A_u &= \phi(e(u)) \\ &= \phi\left(\sum\{E_{v,w} : u = v^{\mathrm{t}} w\}\right) \\ &= \sum\{\bar{\phi}(E_{v,w}) : u = v^{\mathrm{t}} w\} \\ &= \sum\{B_v^* B_w : u = v^{\mathrm{t}} w\}. \end{aligned}$$

Proof of Theorem 0.1. Let ℓ, M_ℓ and let $E_{v,w}$ be as in the proof above, but now let $e(h) = \sum\{E_{v,w} : h = v^{-1} w\}$, for $h \in \mathscr{H}^n_m$. The set $\{e(h)\}$ is a basis of $\mathbb{T}^n_m$. Define $\phi : \mathbb{T}^n_m \mapsto \mathscr{L}(\mathscr{C})$ by

$$\phi(e(h)) = A_h.$$

Given a positive semidefinite $T \in \mathbb{M}_k \otimes \mathbb{T}^n_m$, let $t(h) = T_{v,w}$, where $h = v^{-1} w$. In particular $T = \sum t(h) \otimes e(h)$. By Theorem 3.1, there exists an m-tuple $U = (U_1, \ldots, U_m)$ of unitary operators on a Hilbert space $\mathscr{K}$ and a bounded operator $V : \mathbb{C}^k \mapsto \mathscr{K}$ such that

$$t(v^{-1} w) = V^* U^{v^{-1} w} V.$$

Consequently,

$$\begin{aligned} (1_k \otimes \phi)(T) &= 1_k \otimes \phi\left(\sum_h t(h) \otimes e(h)\right) \\ &= \sum t(h) \otimes A_h \\ &= (V \otimes 1_{\mathscr{C}})^* \left(\sum U^h \otimes A_h\right) (V \otimes 1_{\mathscr{C}}). \end{aligned}$$

It follows that ϕ is completely positive.

Since ϕ is completely positive, there exists a completely positive $\bar{\phi} : M_\ell \mapsto \mathscr{L}(\mathscr{C})$ extending ϕ. The operator

$$(\bar{\phi}(E_{v,w})) \in \mathscr{L}\left(\bigoplus^\ell \mathscr{C}\right)$$

is positive semidefinite. Thus, there exists $B_w : \mathscr{C} \mapsto \bigoplus^\ell \mathscr{C}$ such that

$$B_v^* B_w = \bar{\phi}(E_{v,w}).$$

Consequently,

$$\begin{aligned} A_h &= \phi(e(h)) \\ &= \phi\left(\sum\{E_{v,w} : h = v^{-1}w\}\right) \\ &= \sum\{\bar{\phi}(E_{v,w}) : h = v^{-1}w\} \\ &= \sum\{B_v^* B_w : h = v^{-1}w\}. \qquad \square \end{aligned}$$

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