The *Euclidean* Algorithm Generates Traditional Musical Rhythms

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Abstract

The *Euclidean* algorithm (which comes down to us from Euclid's *Elements*) computes the greatest common divisor of two given integers. It is shown here that the structure of the Euclidean algorithm may be used to generate, very efficiently, a large family of rhythms used as timelines (*ostinatos*), in sub-Saharan African music in particular, and world music in general. These rhythms, here dubbed *Euclidean* rhythms, have the property that their onset patterns are distributed as evenly as possible. *Euclidean* rhythms also find application in nuclear physics accelerators and in computer science, and are closely related to several families of words and sequences of interest in the study of the combinatorics of words, such as Euclidean strings, to which the *Euclidean* rhythms are compared.

1. Introduction

What do African rhythms, spallation neutron source (SNS) accelerators in nuclear physics, string theory (stringology) in computer science, and an ancient algorithm described by Euclid have in common? The short answer is: patterns distributed as evenly as possible. For the long answer please read on.

Mathematics and music have been intimately intertwined since the days of Pythagoras. However, most of this interaction has been in the domain of pitch and scales. For some historical snapshots of this interaction the reader is referred to H. S. M. Coxeter's delightful account [9]. Rhythm, on the other hand has been historically mostly ignored. Here we make some mathematical connections between musical rhythm and other areas of knowledge such as nuclear physics and computer science, as well as the work of another famous ancient Greek mathematician, Euclid of Alexandria.

2. Timing Systems in Neutron Accelerators

The following problem is considered by Bjorklund [5], [4] in connection with the operation of certain components (such as high voltage power supplies) of spallation neutron source (SNS) accelerators used in nuclear physics. Time is divided into intervals (in the case of SNS, 10 seconds). During some of these intervals a gate is to be enabled by a timing system that generates pulses that accomplish this task. The problem for a given number n of time intervals, and another given number $k < n$ of pulses, is to distribute the pulses as evenly as possible among these intervals. Bjorklund [5] represents this problem as a binary sequence of k one's and $n - k$ zero's, where each integer represents a time interval, and the one's represent the pulses. The problem then reduces to the following: construct a binary sequence of n bits with k one's, such that the k one's are distributed as evenly as possible among the zero's. If k divides evenly (without remainder) into n, then the solution is obvious. For example, if $n = 16$ and $k = 4$, the solution

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is [1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0]. The problem of primary interest is when k and n are relatively prime numbers [23], i.e., when k and n are evenly divisible only by 1.

Bjorklund's algorithm will be described simply by using one of his examples. Consider a sequence with $n = 13$ and $k = 5$. Since $13 - 5 = 8$, we start by considering a sequence consisting of 5 one's followed by 8 zero's which should be thought of as 13 sequences of one bit each:

[1 1 1 1 1 0 0 0 0 0 0 0 0]

We begin moving zero's by placing a zero after each one, to produce five sequences of two bits each, with three zero's remaining:

[10] [10] [10] [10] [10] [0] [0] [0]

Next we distribute the three remaining zeros in a similar manner, by placing a [0] sequence after each [10] sequence to obtain:

[100] [100] [100] [10] [10]

Now we have three sequences of three bits each, and a remainder of two sequences of two bits each. Therefore we continue in the same manner, by placing a [10] sequence after each [100] sequence to obtain:

[10010] [10010] [100]

The process stops when the remainder consists of only one sequence (in this case the sequence [100]), or we run out of zero's. The final sequence is thus the concatenation of [10010], [10010], and [100]:

[1 0 0 1 0 1 0 0 1 0 1 0 0]

Note that one could proceed one step further in this process by inserting [100] into [10010] [10010]. However, Bjorklund argues that since the sequence is cyclic it does not matter (hence his stopping rule). Bjorklund [5] shows that the final sequence may be computed from the initial sequence using $O(n)$ arithmetic operations in the worst case.

3. The Euclidean Algorithm

One of the oldest algorithms known, described in Euclid's *Elements* (*circa* 300 B.C.) in Proposition 2 of *Book VII*, today referred to as the Euclidean algorithm, computes the greatest common divisor of two given integers [12], [14]. The idea is very simple. The smaller number is repeatedly subtracted from the greater until the greater is zero or becomes smaller than the smaller, in which case it is called the remainder. This remainder is then repeatedly subtracted from the smaller number to obtain a new remainder. This process is continued until the remainder is zero. To be more precise, consider as an example the numbers 5 and 8 as before. First 5 divides into 8 once with a remainder of 3. Then 3 divides into 5 once with a remainder of 2. Then 2 divides into 3 once with a remainder of 1. Finally, 2 divides into 2 once with a remainder of 0. The greatest common divisor is therefore 1. Although Euclid's original algorithm used repeated subtraction in this manner, standard division will work just as well, and is even faster. The steps of this process can be summarized by the following sequence of equations:

$$
\begin{aligned}
8 &= (1)(5) + 3 \\
5 &= (1)(3) + 2 \\
3 &= (1)(2) + 1 \\
2 &= (1)(2) + 0
\end{aligned}
$$

The algorithm may be described succinctly in a recursive manner as done in [8]. Let m and k be the input integers with $m > k$.

EUCLID(m, k)
1. **if** $k = 0$
2. **then return** m
3. **else return** EUCLID(k,$m \bmod k$)

Running this algorithm with $m = 8$ and $k = 5$ we obtain:
EUCLID(8,5) = EUCLID(5,3) = EUCLID(3,2) = EUCLID(2,1) = EUCLID(1,0) = 1

It is clear from the description of the Euclidean algorithm that if m and k are equal to the number of zero's and one's, respectively, in a binary sequence (with $n = m + k$) then the structure of the Euclidean algorithm has the same structure as that of Bjorklund's algorithm described in the preceding. Indeed, Bjorklund's algorithm uses the repeated subtraction form of division, just as Euclid did in his *Elements* [12]. It is also well known that if algorithm EUCLID(m, k) is applied to two $O(n)$ bit numbers (binary sequences of length n) it will perform $O(n)$ arithmetic operations in the worst case [8].

4. Euclidean Rhythms in Traditional World Music

A common method of representing musical rhythms is as binary sequences, where each bit is considered as one unit of time (for example a 16*th* note), and a zero bit represents a silence (or an unaccented note) whereas a one bit represents an attack (or onset) of a note [31]. Therefore, the binary sequences generated by Bjorklund's algorithm, as described in the preceding, may be considered as one family of rhythms. Furthermore, since Bjorklund's algorithm has the same structure as the Euclidean algorithm, these rhythms will be called Euclidean rhythms and denoted by $E(k, n)$, where k denotes the number of one's and n is the length of the sequence (zero's plus one's). For example $E(5, 13) = [1001010010100]$. The zero-one notation is not ideal for representing binary rhythms because it is difficult to visualize the locations of the onsets as well as the duration of the inter-onset intervals. In the musicology literature it is common to use the symbol 'x' for for the one bit and the symbol '.' for the zero bit. In this more iconic notation the preceding rhythm is written as E(5,13)=[x . . x . x . . x . x . .].

The rhythm E(5,13) is a cyclic rhythm with a time span (measure) of 13 units. This is not a common measure in world music. Let us consider for contrast two common values of k and n; in particular, what is $E(3, 8)$? Applying the Euclidean algorithm to the corresponding sequence [1 1 1 0 0 0 0 0], the reader may easily verify that the resulting Euclidean rhythm is E(3,8)=[x . . x . . x .]. This rhythm is illustrated as a polygon (triangle) in Figure 1 (a), another useful and common way to represent cyclic rhythms [31], where the rhythm is assumed to start at the location labelled 'zero', time flows in a clockwise direction, and the numbers by the sides of the triangle indicate the inter-onset duration intervals. Indeed, an even more compact representation of the rhythm is as the adjacent-inter-onset-interval vector (332).

The Euclidean rhythm $E(3, 8)$ pictured in Figure 1 (a) is none other than one of the most famous on the planet. In Cuba it goes by the name of the *tresillo* and in the USA is often called the *Habanera* rhythm used in hundreds of *rockabilly* songs during the 1950's. It can often be heard in early rock-and-roll hits in the left-hand patterns of the piano, or played on the string bass or saxophone [7], [15], [22]. A good example is the bass rhythm in Elvis Presley's *Hound Dog*. The tresillo pattern is also found widely in West African traditional music. For example, it is played on the *atoke* bell in the *Sohu*, an *Ewe* dance from Ghana [16]. The tresillo can also be recognized as the first bar of the ubiquitous two-bar *clave Son* given by [x . . x . . x . . . x . x . . .].

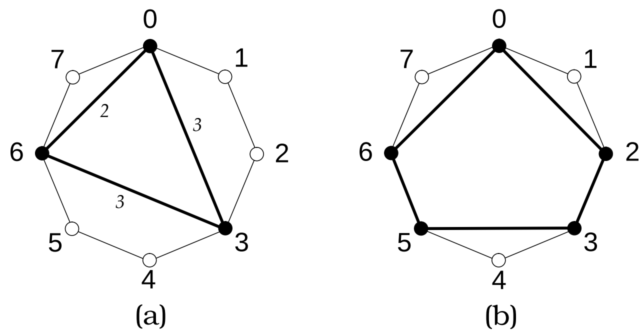

Figure 1: *(a) The Euclidean rhythm $E(3,8)$ is the Cuban* tresillo, *(b) The Euclidean rhythm $E(5,8)$ is the Cuban* cinquillo.

In the two examples in the preceding ($E(5,13)$ and $E(3,8)$) the number of one's is less than the number of zero's. If instead the number of one's is greater than the number of zero's, Bjorklund's algorithm yields the following steps with, for example $k = 5$ and $n = 8$.

[1 1 1 1 1 0 0 0]
[10] [10] [10] [1] [1]
[101] [101] [10]
[1 0 1 1 0 1 1 0]

The resulting Euclidean rhythm is E(5,8) = [x . x x . x x .]. This rhythm is illustrated as a polygon (pentagon) in Figure 1 (b). It is another famous rhythm on the world scene. In Cuba it goes by the name of the *cinquillo* and is intimately related to the tresillo [15]. It has been used in jazz throughout the 20*th* century [27], as well as in the *rockabilly* music of the 1950's. For example it is the hand-clapping pattern in Elvis Presley's *Hound Dog* [7]. The cinquillo pattern is also widely used in West African traditional music [26],[31].

In the remainder of this section we list some of the most common Euclidean rhythms found in world music. In some cases the Euclidean rhythm is a rotated version of a commonly used rhythm. If a rhythm is a rotated version of another we say that both belong to the same *necklace*. Thus a rhythm necklace is the inter-onset duration interval pattern that disregards the starting point in the cycle. An example of two rhythms that are instances of one and the same necklace is illustrated in Figure 2.

The simplest rhythms have a value of $k = 1$. This subfamily of Euclidean rhythms yields:
E(1,2) = [x .]
E(1,3) = [x . .]
E(1,4) = [x . . .], etc.

Note that since we are interested in cyclic non-periodic rhythms it is not necessary to enumerate these rhythms with multiples of k and n. For example, multiplying (1,3) by 4 gives (4,12) which yields:
E(4,12) = [x . . x . . x . . x . .],

which is periodic with four repetitions of E(1,3) = [x . .]. Incidentally, E(4,12) = [x . . x . . x . . x . .] is the (12/8)-time *Fandango* clapping pattern in the Flamenco music of southern Spain, where 'x' denotes a loud clap and '.' a soft clap [10].

E(2,3) = [x . x] is a common Afro-Cuban drum pattern. For example, it is the conga rhythm of the (6/8)-time *Swing Tumbao* [18]. It is also common in Latin American music, as for example in the *Cueca* [33].

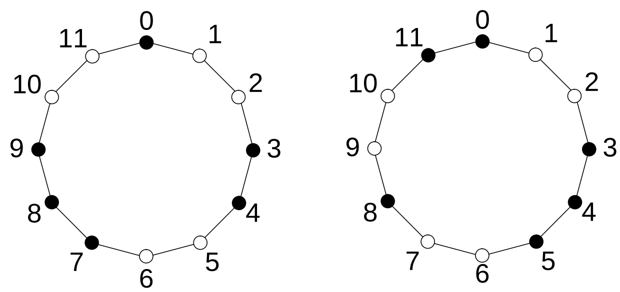

Figure 2: *These two rhythms are instances of one and the same rhythm* necklace.

$E(2,5)$=[x . x . .] is a thirteenth century Persian rhythm called *Khafif-e-ramal* [34]. It is also the metric pattern of the second movement of Tchaikovsky's *Symphony No. 6* [17]. When it is started on the second onset ([x . . x .]) it is the metric pattern of Dave Brubeck's *Take Five* as well as *Mars* from *The Planets* by Gustav Holst [17].

$E(3,4)$=[x . x x] is the archetypal pattern of the *Cumbia* from Colombia [20], as well as a *Calypso* rhythm from Trinidad [13]. It is also a thirteenth century Persian rhythm called *Khalif-e-saghil* [34], as well as the *trochoid choreic* rhythmic pattern of ancient Greece [21].

$E(3,5)$=[x . x . x], when started on the second onset, is another thirteenth century Persian rhythm by the name of *Khafif-e-ramal* [34], as well as a Rumanian folk-dance rhythm [25].

$E(3,7)$=[x . x . x . .] is a *Ruchenitza* rhythm used in a Bulgarian folk-dance [24]. It is also the metric pattern of Pink Floyd's *Money* [17].

$E(3,8)$=[x . . x . . x .] is the Cuban *tresillo* pattern discussed in the preceding [15].

$E(4,7)$=[x . x . x . x] is another *Ruchenitza* Bulgarian folk-dance rhythm [24].

$E(4,9)$ = [x . x . x . x . .] is the *Aksak* rhythm of Turkey [6]. It is also the metric pattern used by Dave Brubeck in his piece *Rondo a la Turk* [17].

$E(4,11)$ = [x . . x . . x . . x .] is the metric pattern used by Frank Zappa in his piece titled *Outside Now* [17].

$E(5,6)$=[x . x x x x] yields the *York-Samai* pattern, a popular Arab rhythm, when started on the second onset [30].

$E(5,7)$=[x . x x . x x] is the *Nawakhat* pattern, another popular Arab rhythm [30].

$E(5,8)$=[x . x x . x x .] is the Cuban *cinquillo* pattern discussed in the preceding [15]. When it is started on the second onset it is also the *Spanish Tango* [13] and a thirteenth century Persian rhythm, the *Al-saghil-al-sani* [34].

$E(5,9)$=[x . x . x . x . x] is a popular Arab rhythm called *Agsag-Samai* [30]. When started on the second onset, it is a drum pattern used by the *Venda* in South Africa [26], as well as a Rumanian folk-dance rhythm [25].

$E(5,11)$=[x . x . x . x . x . .] is the metric pattern used by Moussorgsky in *Pictures at an Exhibition* [17].

$E(5,12)$ = [x . . x . x . . x . x .] is the *Venda* clapping pattern of a South African children's song [24].

$E(5,16)$ = [x . . x . . x . . x . . x . . . .] is the *Bossa-Nova* rhythm necklace of Brazil. The actual Bossa-Nova rhythm usually starts on the third onset as follows: [x . . x . . x . . . x . . x . .] [31]. However, there are other starting places as well, as for example [x . . x . . x . . x . . . x . .] [3].

$E(7,8)$ = [x . x x x x x x] is a typical rhythm played on the *Bendir* (frame drum), and used in the accompaniment of songs of the *Tuareg* people of Libya [30].

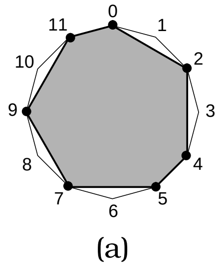

Figure 3: *Two right-rotations of the* Bemb´e *string: (a) the* Bemb´e, *(b) rotation by one unit, (c) rotation by seven units.*

E(7,12) = [x . x x . x . x x . x .] is a common West African bell pattern. For example, it is used in the *Mpre* rhythm of the *Ashanti* people of Ghana [32].

E(7,16) = [x . . x . x . x . . x . x . x .] is a *Samba* rhythm necklace from Brazil. The actual Samba rhythm is [x . x . . x . x . x . . x . x .] obtained by starting E(7,16) on the last onset. When E(7,16) is started on the fifth onset it is a clapping pattern from Ghana [24].

E(9,16) = [x . x x . x . x . x x . x . x .] is a rhythm necklace used in the Central African Republic [2]. When it is started on the fourth onset it is a rhythm played in West and Central Africa [15], as well as a cow-bell pattern in the Brazilian *samba* [29]. When it is started on the penultimate onset it is the bell pattern of the *Ngbaka-Maibo* rhythms of the Central African Republic [2].

E(11,24) = [x . . x . x . x . x . x . . x . x . x . x . x .] is a rhythm necklace of the *Aka* Pygmies of Central Africa [2]. It is usually started on the seventh onset.

E(13,24) = [x . x x . x . x . x . x . x x . x . x . x . x .] is another rhythm necklace of the *Aka* Pygmies of the upper *Sangha* [2]. It is usually started on the fourth onset.

5. Euclidean Strings

In the study of the combinatorics of words and sequences, there exists a family of strings called Euclidean strings [11]. In this section we explore the relationship that exists between Euclidean strings and Euclidean rhythms. We use the same terminology and notation introduced in [11].

Let $P = (p_0, p_1, ..., p_{n-1})$ denote a string of non-negative integers. Let $\rho(P)$ denote the right rotation of P by one position, i.e., $\rho(P) = (p_{n-1}, p_0, p_1, ..., p_{n-2})$, and let $\rho^d(P)$ denote the right rotation of P by d positions. Figure 3 illustrates the $\rho(P)$ operator with P equal to the *Bembé* bell-pattern of West Africa [32]. Figure 3 (a) shows the *Bembé* bell-pattern, Figure 3 (b) shows $\rho(P)$, which is a hand-clapping pattern from West Africa [24], and Figure 3 (c) shows $\rho^7(P)$, which is the *Tambú* rhythm of Curaçao [28].

Ellis et al., [11] define a string $P = (p_0, p_1, ..., p_{n-1})$ as a *Euclidean string* if increasing p_0 by one, and decreasing p_{n-1} by one, yields a new string, denoted by $\tau(P)$, that is a rotation of P, i.e., P and $\tau(P)$ are instances of one and the same necklace. Therefore, if we represent rhythms as binary sequences, Euclidean rhythms cannot be Euclidean strings because by virtue of the Euclidean algorithm employed, all Euclidean rhythms begin with a 'one'. Increasing p_0 by one makes it a 'two', which is not a binary string. Therefore, to explore the relationship between Euclidean strings and Euclidean rhythms, here we will represent rhythms by their adjacent-inter-onset-duration-interval-vectors, (interval-vectors for short) which are also strings of non-negative integers. As an example, consider the *Aksak* rhythm of Turkey [6] given by E(4,9) = [x . x . x . x . .]. In interval-vector notation we have that E(4,9) = (2223). Now $\tau(2223) = (3222)$, which is a rotation of E(4,9), and is thus a Euclidean string. Indeed, for P=E(4,9), $\tau(P) = \rho^3(P)$. As a second example. consider the West African clapping-pattern shown in Figure 3 (b) given by P=(1221222). We have that $\tau(P) =$ (2221221) $= \rho^6(P)$, the pattern shown in Figure 3 (c), which also happens to be the mirror image of P about the $(0, 6)$ axis. Therefore P is a Euclidean string. However, note that P is not a Euclidean rhythm. Nevertheless, P is a rotation of the Euclidean rhythm E(7,12)=(2122122).

Ellis et al., [11] have many beautiful results about Euclidean strings. They show that Euclidean strings exist if, and only if, n and $(p_0 + p_1 + ... + p_{n-1})$ are relatively prime numbers, and that when they exist they are unique. They also show how to construct Euclidean strings using an algorithm that has the same structure as the Euclidean algorithm. In addition they relate Euclidean strings to many other families of sequences studied in the combinatorics of words [1], [19].

Let $R(P)$ denote the reversal (or mirror image) of P, i.e., $R(P) = (p_{n-1}, p_{n-2}, ..., p_1, p_0)$. For example, for the *Aksak* rhythm where P = (2223), we obtain that $R(P)$ = (3222), i.e., $R(P)$ implies playing the rhythm P backwards by starting at the same onset. Now we may determine which of the Euclidean rhythms used in world music listed in the preceding, are Euclidean strings or *reverse* Euclidean strings. The length of a Euclidean string is defined as the number of integers it has. This translates in the rhythm domain to the number of onsets a rhythm contains. Furthermore, strings of length one are Euclidean strings, trivially. Therefore all the trivial Euclidean rhythms with only one onset, such as E(1,2) = [x .] = (2), E(1,3) = [x . .] = (3), and E(1,4) = [x . . .] =(4), etc., are both Euclidean strings as well as reverse Euclidean strings. In the lists that follow the Euclidean rhythms are shown in their box-notation format as well as in the interval-vector representation. The styles of music that use these rhythms is also included. Finally, if only a rotated version of the Euclidean rhythm is played, then it is still included in the list but referred to as a necklace.

The following Euclidean rhythms are Euclidean strings:

E(2,5)=[x . x . .] = (23) (classical, jazz, and Persian).
E(3,7)=[x . x . x . .] = (223) (Bulgarian folk).
E(4,9) = [x . x . x . x . .] = (2223) (Turkey).
E(5,11)=[x . x . x . x . x . .] = (22223) (classical).
E(5,16) = [x . . x . . x . . x . . x . . . .] = (33334) (Brazilian necklace).

The following Euclidean rhythms are reverse Euclidean strings:

E(2,3) = [x . x] = (21) (West Africa, Latin America).
E(3,4)=[x . x x] = (211) (Trinidad, Persia).
E(3,5)=[x . x . x] = (221) (Rumanian and Persian necklaces).
E(3,8)=[x . . x . . x .] = (332) (West Africa).
E(4,7)=[x . x . x . x] = (2221) (Bulgaria).
E(4,11) = [x . . x . . x . . x .] = (3332) (Frank Zappa).
E(5,6)=[x . x x x x] = (21111) (Arab).
E(5,7)=[x . x x . x x] = (21211) (Arab).
E(5,9)=[x . x . x . x . x] = (22221) (Arab rhythm, South African and Rumanian necklaces).
E(5,12) = [x . . x . x . . x . x .] = (32322) (South Africa).
E(7,8) = [x . x x x x x x] = (2111111) (Tuareg rhythm of Libya).
E(7,16) = [x . . x . x . x . . x . x . x .] = (3223222) (Brazilian necklace).
E(11,24) = [x . . x . x . x . x . x . . x . x . x . x . x .] = (32222322222) (Central Africa).

The following Euclidean rhythms are neither Euclidean nor reverse Euclidean strings:

E(5,8)=[x . x x . x x .] = (21212) (West Africa).
E(7,12) = [x . x x . x . x x . x .] = (2122122) (West Africa).
E(9,16) = [x . x x . x . x . x x . x . x .] = (212221222) (West and Central African, and Brazilian necklaces).
E(13,24) = [x . x x . x . x . x . x . x x . x . x . x . x .] = (2122222122222) (Central African necklace).

6. Concluding Remarks

A new family of musical rhythms has been described, called Euclidean rhythms, which are obtained by using Bjorklund's sequence generation algorithm, which has the same structure as the Euclidean algorithm. It was shown that many rhythms used in world music are Euclidean rhythms. Some of these Euclidean rhythms are also Euclidean strings [11].

The three groups of Euclidean rhythms listed in the preceding section reveal a tantalizing pattern. Those Euclidean rhythms that are also Euclidean strings (the first four of group one) are favoured in classical, jazz, Bulgarian, Turkish and Persian music, but are not popular in African music. The Euclidean rhythms that are neither Euclidean strings nor reverse Euclidean strings (the first two of group three) are used only in sub-Saharan African music. Finally, the Euclidean rhythms that are reverse Euclidean strings (the second group) appear to have a much wider appeal. Finding musicological explanations for the preferences apparent in these mathematical properties raizes an interesting ethnomusicological question.

The Euclidean strings defined in [11] determine another family of rhythms, many of which are also used in world music but are not necessarily Euclidean rhythms, as for example (1221222), an Afro-Cuban bell pattern. Therefore it would be interesting to explore empirically the relation between Euclidean strings and world music rhythms, and to determine formally the exact mathematical relation between Euclidean rhythms and Euclidean strings.

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