CS 61A Summer 2022

Recursion, Tree Recursion

Discussion 3: June 30, 2022

Recursion

A *recursive* function is a function that is defined in terms of itself.

Consider this recursive `factorial` function:

```
def factorial(n):
    """Return the factorial of N, a positive integer."""
    if n == 1:
        return 1
    else:
        return n * factorial(n - 1)
```

Inside of the body of `factorial`, we are able to call `factorial` itself, since the function body is not evaluated until the function is called.

When `n` is 1, we can directly return the factorial of 1, which is 1. This is known as the *base case* of this recursive function, which is the case where we can return from the function call directly, without having to first recurse (i.e. call `factorial`) and then returning. The base case is what prevents `factorial` from recursing infinitely.

Since we know that our base case `factorial(1)` will return, we can compute `factorial(2)` in terms of `factorial(1)`, then `factorial(3)` in terms of `factorial(2)`, and so on.

There are three main steps in a recursive definition:

1. **Base case.** You can think of the base case as the case of the simplest function input, or as the stopping condition for the recursion.

   In our example, `factorial(1)` is our base case for the `factorial` function.

2. **Recursive call on a smaller problem.** You can think of this step as calling the function on a smaller problem that our current problem depends on. We assume that a recursive call on this smaller problem will give us the expected result; we call this idea the "recursive leap of faith".

   In our example, `factorial(n)` depends on the smaller problem of `factorial(n-1)`.

3. **Solve the larger problem.** In step 2, we found the result of a smaller problem. We want to now use that result to figure out what the result of our current problem should be, which is what we want to return from our current function call.

   In our example, we can compute `factorial(n)` by multiplying the result of our smaller problem `factorial(n-1)` (which represents `(n-1)!`) by `n` (the reasoning being that `n! = n * (n-1)!`).

① If one of the inputs is one, you simply return the other input.

② The first call will calculate a value that is n less than the total, while the second will calculate a value that is m less. Either recursive call will work, but only multiply(m,n-1) is used in this solution.

Q1: Warm Up: Recursive Multiplication

These exercises are meant to help refresh your memory of the topics covered in lecture.

Write a function that takes two numbers `m` and `n` and returns their product. Assume `m` and `n` are positive integers. Use **recursion**, not `mul` or `*`.

Hint: `5 * 3 = 5 + (5 * 2) = 5 + 5 + (5 * 1)`.

For the base case, what is the simplest possible input for `multiply`?

For the recursive case, what does calling `multiply(m - 1, n)` do? What does calling `multiply(m, n - 1)` do? Do we prefer one over the other?

```
def multiply(m, n):
    """ Takes two positive integers and returns their product using
    recursion.
    >>> multiply(5, 3)
    15
    """
    "*** YOUR CODE HERE ***"
    if n==1:
        return m
    else:
        return m + multiply(m,n-1)


# You can use more space on the back if you want
```

Q2: Recursion Environment Diagram

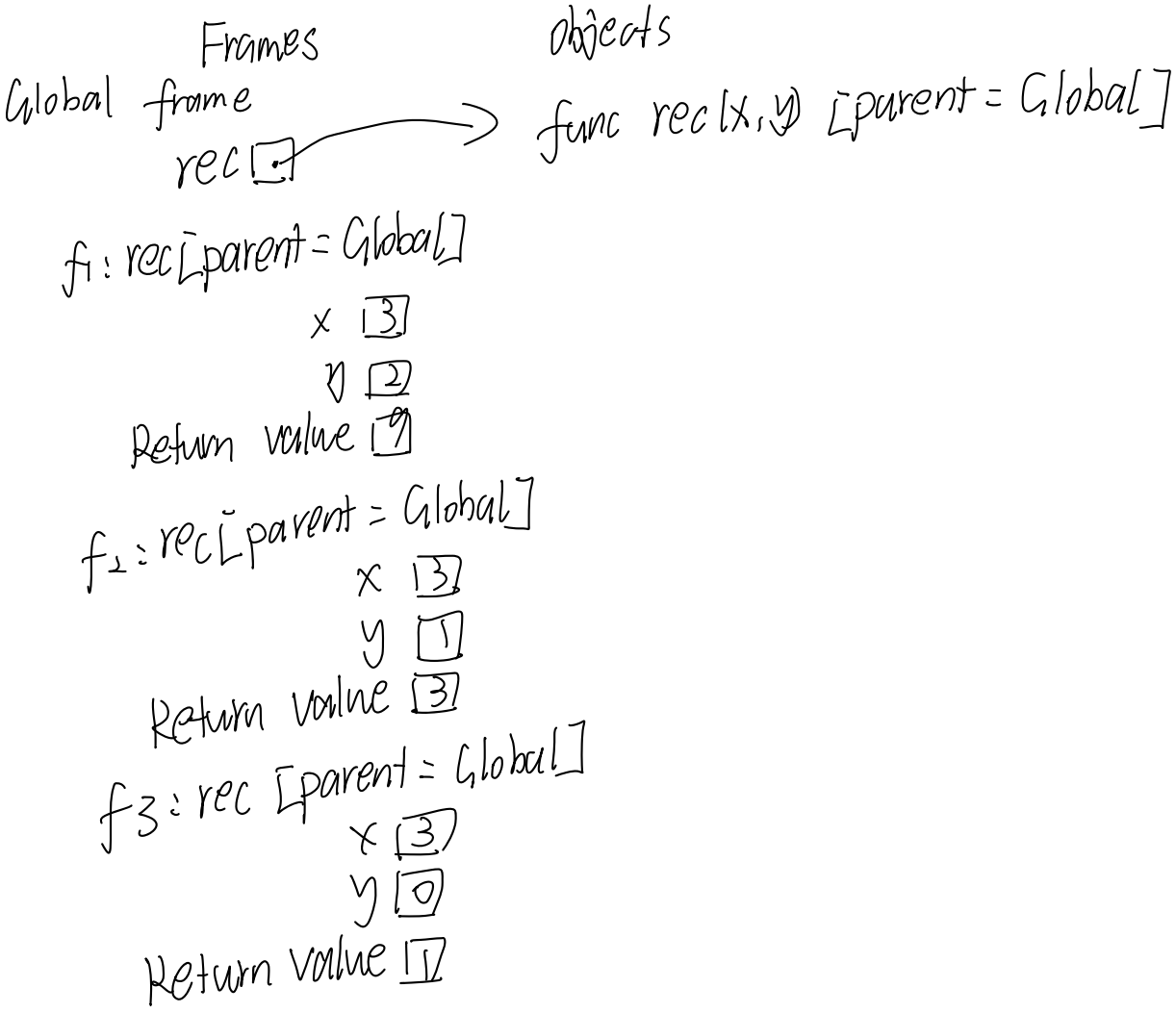
Draw an environment diagram for the following code:

Note: If you can't move elements around, make sure you're logged in!

```
def rec(x, y):
    if y > 0:
        return x * rec(x, y - 1)
    return 1

rec(3, 2)
```

Note: This problem is meant to help you understand what really goes on when we make the "recursive leap of faith". However, when approaching or debugging recursive functions, you should avoid visualizing them in this way for large or complicated inputs, since the large number of frames can be quite unwieldy and confusing. Instead, think in terms of the three steps: base case, recursive call, and solving the full problem.



Q3: Find the Bug

Find the bug with this recursive function.

```
def skip_mul(n):
    """Return the product of n * (n - 2) * (n - 4) * ...

    >>> skip_mul(5) # 5 * 3 * 1
    15
    >>> skip_mul(8) # 8 * 6 * 4 * 2
    384
    """
    if n == 2:
        return 2
    else:
        return n * skip_mul(n - 2)
```

if n==2 or n==1:
    return n

Q4: Is Prime

Write a function `is_prime` that takes a single argument `n` and returns `True` if `n` is a prime number and `False` otherwise. Assume `n > 1`. We implemented this in Discussion 1 iteratively, now time to do it recursively!

> *Hint*: You will need a helper function! Remember helper functions are nested functions that are useful if you need to keep track of more variables than the given parameters, or if you need to change the value of the input.

```
def is_prime(n):
    """Returns True if n is a prime number and False otherwise.

    >>> is_prime(2)
    True
    >>> is_prime(16)
    False
    >>> is_prime(521)
    True
    """
    "*** YOUR CODE HERE ***"
        def help(i):
            if i > n**0.5:
                return True
            elif n % i == 0:
                return False
            return help(i+1)
        return help(2)

# You can use more space on the back if you want
```

Q5: Recursive Hailstone

Recall the `hailstone` function from Homework 1. First, pick a positive integer `n` as the start. If `n` is even, divide it by 2. If `n` is odd, multiply it by 3 and add 1. Repeat this process until `n` is 1. Write a recursive version of `hailstone` that prints out the values of the sequence and returns the number of steps.

> Hint: When taking the recursive leap of faith, consider both the return value and side effect of this function.

```
def hailstone(n):
    """Print out the hailstone sequence starting at n, and return
    the number of elements in the sequence.
    >>> a = hailstone(10)
    10
    5
    16
    8
    4
    2
    1
    >>> a
    7
    >>> b = hailstone(1)
    1
    >>> b
    1
    """
    "*** YOUR CODE HERE ***"
    print(n)
    if n==1:
        return 1
    elif n%2==0:
        return 1+ hailstone(n//2)
    elif n%2==1:
        return 1+hailstone(3*n+1)


# You can use more space on the back if you want
```

Q6: Merge Numbers

Write a procedure `merge(n1, n2)` which takes numbers with digits in decreasing order and returns a single number with all of the digits of the two, in decreasing order. Any number merged with 0 will be that number (treat 0 as having no digits). Use recursion.

> Hint: If you can figure out which number has the smallest digit out of both, then we know that the resulting number will have that smallest digit, followed by the merge of the two numbers with the smallest digit removed.

```
def merge(n1, n2):
    """ Merges two numbers by digit in decreasing order
    >>> merge(31, 42)
    4321
    >>> merge(21, 0)
    21
    >>> merge (21, 31)
    3211
    """
    "*** YOUR CODE HERE ***"
    if n1 == 0:
        return n2
    elif n2 == 0:
        return n1
    elif n1 % 10 < n2 % 10:
        return n1 % 10 + 10 * merge(n1 // 10, n2)
    else:
        return n2 % 10 + 10 * merge(n1, n2 // 10)


# You can use more space on the back if you want
```

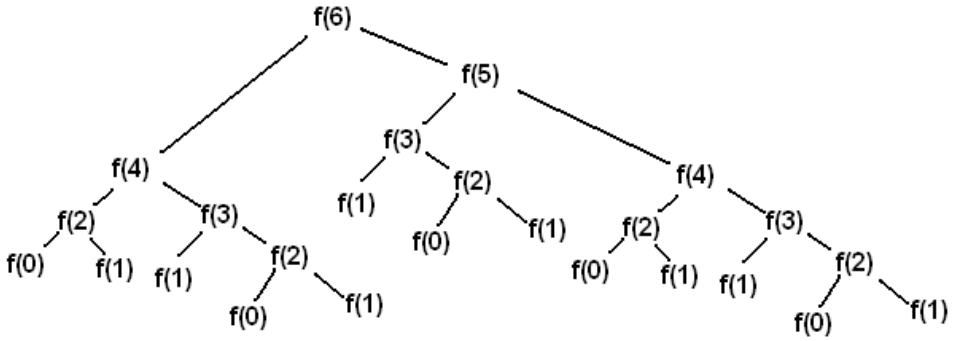
Tree Recursion

A tree recursive function is a recursive function that makes more than one call to itself, resulting in a tree-like series of calls.

For example, let's say we want to recursively calculate the `n`th Virahanka-Fibonacci number, defined as:

```
def virfib(n):
    if n == 0 or n == 1:
        return n
    return virfib(n - 1) + virfib(n - 2)
```

Calling `virfib(6)` results in the following call structure that looks like an upside-down tree (where `f` is `virfib`):



Virahanka-Fibonacci Tree

Each `f(i)` node represents a recursive call to `virfib`. Each recursive call `f(i)` makes another two recursive calls, which are to `f(i-1)` and `f(i-2)`. Whenever we reach a `f(0)` or `f(1)` node, we can directly return `0` or `1` rather than making more recursive calls, since these are our base cases.

In other words, base cases have the information needed to return an answer directly, without depending upon results from other recursive calls. Once we've reached a base case, we can then begin returning back from the recursive calls that led us to the base case in the first place.

Generally, tree recursion can be effective for problems where there are multiple possibilities or choices at a current state. In these types of problems, you make a recursive call for each choice or for a group of choices.

Q7: Count Stair Ways

Imagine that you want to go up a flight of stairs that has `n` steps, where `n` is a positive integer. You can either take 1 or 2 steps each time. How many different ways can you go up this flight of stairs? In this question, you'll write a function `count_stair_ways` that solves this problem. Before you code your approach, consider these questions.

How many different ways are there to go up a flight of stairs with `n = 1` step? How about `n = 2` steps? Try writing out some other examples and see if you notice any patterns.

What's the base case for this question? What is the simplest input?

What do `count_stair_ways(n - 1)` and `count_stair_ways(n - 2)` represent?

Fill in the code for `count_stair_ways`:

```
def count_stair_ways(n):
    """Returns the number of ways to climb up a flight of
    n stairs, moving either 1 step or 2 steps at a time.
    >>> count_stair_ways(4)
    5
    """
    "*** YOUR CODE HERE ***"
    if n == 1:
        return 1
    elif n == 2:
        return 2
    else:
        return count_stair_ways(n-1) +
                count_stair_ways(n-2)
# You can use more space on the back if you want
```

when there is only one step, there is only one way to go up the stairs. when there are two steps we can go up in two ways: take a single 2-step, or take two 1-steps

Our first base case is when there is one step left. This is, by definition, the smallest input since it is the smallest positive integer. Our second base case is when we have two steps left. we need this base case for a similar reason that fibonacci need 2 base cases: to cover both recursive calls

The former represents the number of different ways to go up the last n-1 stairs (this is the case where we take 1 step as our move). The latter represents the number of different ways to go up the last n-2 stairs (this is the case where we take 2 steps as our move

Q8: Count K

Consider a special version of the `count_stair_ways` problem, where instead of taking 1 or 2 steps, we are able to take up to and including `k` steps at a time. Write a function `count_k` that figures out the number of paths for this scenario. Assume `n` and `k` are positive.

> *Hint*: Your solution will follow a very similar logic to what you did for `count_stair_ways`.

```
def count_k(n, k):
    """ Counts the number of paths up a flight of n stairs
    when taking up to and including k steps at a time.
    >>> count_k(3, 3) # 3, 2 + 1, 1 + 2, 1 + 1 + 1
    4
    >>> count_k(4, 4)
    8
    >>> count_k(10, 3)
    274
    >>> count_k(300, 1) # Only one step at a time
    1
    """
    "*** YOUR CODE HERE ***"
    if n==0:
        return 1
    elif n<0:
        return 0
    else:
        total,i = 0, 1
        while i<=k:
            total += count_k(n-i,k)
            i += 1
        return total


# You can use more space on the back if you want
```