CS 70 Fall 2022 — Discrete Mathematics and Probability Theory — Satish Rao and Babak Ayazifar — HW 01

Due: Saturday, 9/3, 4:00 PM
Grace period until Saturday, 9/3, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Solving a System of Equations

Alice wants to buy apples, beets, and carrots. An apple, a beet, and a carrot cost 16 dollars, two apples and three beets cost 23 dollars, and one apple, two beets, and three carrots cost 35 dollars. What are the prices for an apple, for a beet, and for a carrot, respectively? Set up a system of equations and show your work.

2 Calculus Review

(a) Compute the following integral:

$$\int_0^\infty \sin(t)e^{-t}\,\mathrm{d}t\,.$$

(b) Find the minimum value of the following function over the reals and determine where it occurs.

$$f(x) = \int_0^{x^2} e^{-t^2}\,\mathrm{d}t\,.$$

Show your work.

(c) Compute the double integral

$$\iint_R 2x + y\,\mathrm{d}A\,,$$

where R is the region bounded by the lines $x = 1$, $y = 0$, and $y = x$.

3 Implication

Which of the following assertions are true no matter what proposition Q represents? For any false assertion, state a counterexample (i.e. come up with a statement $Q(x,y)$ that would make the implication false). For any true assertion, give a brief explanation for why it is true.

(a) $\exists x \exists y Q(x,y) \implies \exists y \exists x Q(x,y)$.

(b) $\forall x \exists y Q(x,y) \implies \exists y \forall x Q(x,y)$.

(c) $\exists x \forall y Q(x,y) \implies \forall y \exists x Q(x,y)$.

(d) $\exists x \exists y Q(x,y) \implies \forall y \exists x Q(x,y)$.

4 Logical Equivalence?

Decide whether each of the following logical equivalences is correct and justify your answer.

(a) $\forall x\ \big(P(x) \wedge Q(x)\big) \equiv \forall x\ P(x) \wedge \forall x\ Q(x)$

(b) $\forall x\ \big(P(x) \vee Q(x)\big) \equiv \forall x\ P(x) \vee \forall x\ Q(x)$

(c) $\exists x\ \big(P(x) \vee Q(x)\big) \equiv \exists x\ P(x) \vee \exists x\ Q(x)$

(d) $\exists x\ \big(P(x) \wedge Q(x)\big) \equiv \exists x\ P(x) \wedge \exists x\ Q(x)$

5 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(b) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

(c) $f(A \cap B) \subseteq f(A) \cap f(B)$, and give an example where equality does not hold.

(d) $f(A \setminus B) \supseteq f(A) \setminus f(B)$, and give an example where equality does not hold.

6 Prove or Disprove

For each of the following, either prove the statement, or disprove by finding a counterexample.

(a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 4n$ is odd.

(b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$ then $a \leq 11$ or $b \leq 4$.

(c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational.

(d) $(\forall n \in \mathbb{Z}^+)$ $5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers)

7 Rationals and Irrationals

Prove that the product of a non-zero rational number and an irrational number is irrational.

8 Twin Primes

(a) Let $p > 3$ be a prime. Prove that p is of the form $3k+1$ or $3k-1$ for some integer k.

(b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

1.

Assume that an apple's price is \$x, a beets \$y and a carrot \$z. According to the desciption, we can get three equations.

$$\begin{cases} x+y+z=16 \\ 2x+3y=23 \\ x+2y+3z=35 \end{cases} \Rightarrow \begin{cases} x=16-y-z \\ 2(16-y-z)+3y=23 \\ 16-y-z+2y+3z=35 \end{cases}$$

$$\begin{cases} x=16-y-z \\ y-2z=-9 \\ y+2z=19 \end{cases} \Rightarrow \begin{cases} x=4 \\ y=5 \\ z=7 \end{cases}$$

Thus, an apple cost \$4, a beet cost \$5 and a carrot cost \$7.

2.

(a)

$\sin(t)=u \qquad dv=e^{-t}$

$\qquad\qquad\qquad\quad v=e^{-t}$

Let $I=\int \sin(t)e^{-t}dt$

$$\begin{aligned} I &= \int -\sin(t)\, de^{-t} \\ &= [-\sin(t)\cdot e^{-t}] - \int e^{-t} d[\sin t] \\ &= -e^{-t}\sin(t) + \underbrace{\int e^{-t}[\cos(t)]dt} \end{aligned}$$

$$\begin{aligned} \int e^{-t}[\cos t]dt &= \int [-\cos(t)]\, de^{-t} \\ &= [-\cos(t)\cdot e^{-t}] - \int e^{-t} d[-\cos(t)] \\ &= -e^{-t}\cos(t) - \int e^{-t}\sin t\, dt \\ &= -e^{-t}\cos(t) - I \end{aligned}$$

Thus, $I = -e^{-t}\sin t - e^{-t}\cos t - I$

$$\begin{aligned} 2I &= -e^{t}\sin t - e^{-t}\cos t \\ I &= \frac{-e^{t}\sin t - e^{-t}\cos t}{2} \end{aligned}$$

$$\begin{aligned} \int_0^\infty \sin(t)e^{-t}dt &= I\Big|_0^\infty \\ &= 0-(-\tfrac{1}{2}) \\ &= \tfrac{1}{2} \end{aligned}$$

(b)

$f(x) = \int_0^{x^2} e^{-t^2} dt$

let $x^2 = y$

$$\frac{df}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx}$$

$$= e^{-y^2} \cdot 2x$$

$$= 2 \cdot x \cdot e^{-x^4}$$

Since e^{-x^4} is always larger than 0, then we only need to analyze $2x$

The derivative is 0 only at $x^* = 0$, with corresponding value of the function $f(x^*) = 0$ Hence, $f(x^*) = f(0) = 0$

Double check the found point is a minima by checking the second derivative:

$$\frac{d^2 f}{dx^2} = \frac{d\, 2x e^{-x^4}}{dx}$$

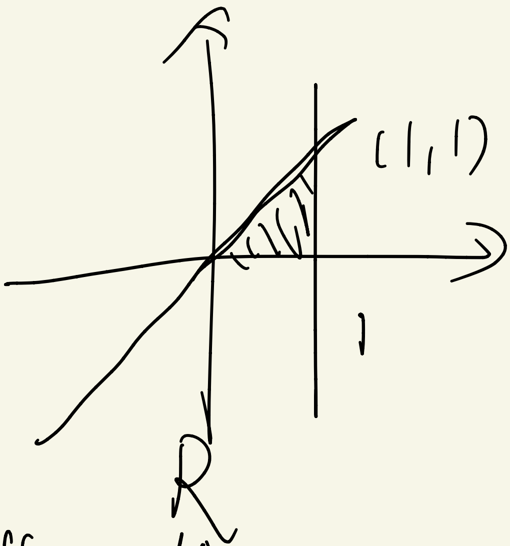
$$= 2 \cdot e^{-x^4} + 2x \cdot e^{-x^4} \cdot (-4x^3)$$

$$= 2e^{-x^4} - 8e^{-x^4} x^4$$

$$= e^{-x^4}(2 - 8x^4)$$

The second derivative evaluated at $x^* = 0$ is $2e^{-0^4} - 80^4 \cdot e^{-0^4} = 2$, which is positive, found point is a minima, as desired.

(c)



$$\iint_R 2x + y \, dA$$

$$\iint_R 2x + y \, dA$$

$$= \int_0^1 \int_0^x 2x + y \, dy \, dx$$

$$= \int_0^1 \left(\int_0^x d\, 2xy + \frac{y^2}{2} \right) dx$$

$$= \int_0^1 \left(2x^2 + \frac{x^2}{2} \right) dx$$

$$= \int_0^1 \frac{5}{2} x^2 \, dx$$

$$= \frac{5x^3}{2 \cdot 3} \Big|_0^1$$

$$= \frac{5}{6}$$

3.

(a) true, There exists can be switched if they are adjacent; $\exists x, \exists y$ and $\exists y, \exists x$ means there exists x and y in our universe.

(b) false, let $Q(x,y)$ be $x<y$, and the universe for x and y be the integers. Or let $Q(x,y)$ be $x=y$ and the universe be any set with at least two elements. In both cases, the antecedent is true and the consequence is false, thus the entire implication statement is false.

(c) true, the first statement says that there is an x, say x' where for every y, $Q(x,y)$ is true. Thus, one can choose $x=x'$ for the second statement and that statement will be true again for every y. Note: 4c and 4d are not logically equivalent. In fact, the converse of 4d is 4c, which we saw is false

(d) false. Suppose Q is the statement "y is 5, and x is any integer." The antecedent is true when $y=5$, but for $y \neq 5$, there is no x that will make it true.

4.

(a) Correct. Assume that the left hand side is true. Then we know for an arbitrary x $P(x) \wedge Q(x)$ is true. This means that both $\forall x P(x)$ and $\forall x Q(x)$. Therefore the right hand side is true. Now for the other direction assume that the right hand side is true. Since for any x $P(x)$ and for any y $Q(y)$ holds, then for an arbitrary x both $P(x)$ and $Q(x)$ must be true. Thus the left hand side is true.

(b) Incorrect. There are many possible counter examples — here we present only one. Suppose that the universe (i.e. the values that x can take on) is $\{1,2\}$ and that P and Q are truth functions defined on this universe. If we set $P(1)$ to be true, $Q(1)$ to be false $P(2)$ to be false and $Q(2)$ to be true. the left-hand side will be true, but the right-hand side will be false. Hence, we can find a universe and truth functions P and Q for which these two expressions have different values, so they must be different.

(c) Correct. Assuming that the left hand side is true, we know there exists some x such that one of $P(x)$ and $Q(x)$ is true. Thus $\exists x P(x)$ or $\exists x Q(x)$ and the right hand side is true. To prove the other direction, assume the left hand side is false. Then there does not exists an x for which $P(x) \vee Q(x)$ is true, which means there is no x for which $P(x)$ or $Q(x)$ is true. There fore the right hand side is false.

(d)

Incorrect, there are many possible counterexamples - here we present only one. Suppose that the universe (i.e. the values that x can take on) is the natural numbers $\mathbb{N}$, and that P and Q are truth functions defined on this universe. If we set $P(1)$ to be true and $P(x)$ to be false for all other x, and $Q(2)$ to be true and $Q(x)$ to be false for all other x, then the right hand side would be simultaneously true, so the left hand side would be no value of x at which both $P(x)$ and $Q(x)$ would be simultaneously true, so the left hand side would be false. Hence, we can find a universe and truth functions P and Q for which these two expressions have ~~different~~ values, so they must be different.

5.

(a)

Assume that $x \in f^{-1}(A \cap B)$, then $f(x) \in A \cap B$. So $f(x) \in A \cap f(x) \in B$

thus $x \in f^{-1}(A) \cap f^{-1}(B)$

Suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$, Then, x is in both $f^{-1}(A)$ and $f^{-1}(B)$, so $f(x) \in A$ and $f(x) \in B$, so $f(x) \in A \cap B$, so $x \in f^{-1}(A \cap B)$. So $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.

(b)

Suppose x is such that $f(x) \in A \setminus B$. Then, $f(x) \in A$ and $f(x) \notin B$, which means that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, which means that $x \in f^{-1}(A) \setminus f^{-1}(B)$. So $f^{-1}(A \setminus B) \subseteq f^{-1}(A) \setminus f^{-1}(B)$

Now, suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, So $f(x) \in A$ and $f(x) \notin B$, So $f(x) \in A \setminus B$, so $x \in f^{-1}(A \setminus B)$. So $f^{-1}(A) \setminus f^{-1}(B) \subseteq f^{-1}(A \setminus B)$

(c)

Suppose $x \in A \cap B$. Then, x lies in both A and B, so $f(x)$ lies in both $f(A)$ and $f(B)$, so $f(x) \in f(A) \cap f(B)$. Hence, $f(A \cap B) \subseteq f(A) \cap f(B)$.

Consider when there are elements $a \in A$ and $b \in B$ with $f(a) = f(b)$, but A and B are disjoint. Here, $f(a) = f(b) \in f(A) \cap f(B)$, but $f(A \cap B)$ is empty (Since $A \cap B$ is empty).

(d)

Suppose $y \in f(A) \setminus f(B)$. Since y is not in $f(B)$, there are no elements in B which map to y. Let x be any element of A that maps to y; by the previous sentence, x cannot lie in B. Hence, $x \in A \setminus B$, So $y \in f(A \setminus B)$. Hence, $f(A) \setminus f(B) \subseteq f(A \setminus B)$.

Consider when $B = \{0\}$ and $A = \{0,1\}$, with $f(0) = f(1) = 0$. One has $A \setminus B = \{1\}$, so $f(A \setminus B) = \{0\}$. However, $f(A) = f(B) = \{0\}$, so $f(A) \setminus f(B) = \emptyset$

6.

(a) True

We could use a direct proof.

Since n is odd, let $n = 2k+1$, $k \in N$

$n^2 + 4n = 4k^2 + 4k + 1 + 8k + 4$

$= 4k^2 + 12k + 5$

$= 2(2k^2 + 6k) + 5$

$(2k^2+6k)$ is integer, $2(2k^2+6k)$ is even while 5 is odd.

It apparent that n^2+4n is odd.

(b) true

Sin $a+b \le 15$, $b \le 15-a$.

if $a \le 11$, the the statement is true

if $a > 12$ and $b \le 15-a$

$-a < -12$ $b \le 3 < 4$

$15-a < 3$ the claim is right.

(c). True

We could use a proof by contraposition. Assume that r is rational. Since r is rational. let $r = \frac{a}{b}$, $b \neq 0$, $r^2 = \frac{a^2}{b^2}$, r^2 is a rational number, a^2 and b^2 are integers $b^2 \neq 0$

By the contraposition, the statement is right.

(d) if $n = 7$, $7 \in Z^+$

the left side equals 245

the right side equals 120

It's a contradiction to the claim.

7.

We prove the statement by contradiction

Suppose that a is a non-zero rational number and b is an irrational number, the product of a and b is ab which is a rational number.

a can be represented as $\frac{x}{y}$, $x \neq 0, y \neq 0$

c can be represented as $\frac{q}{p}$, $q \neq 0, p \neq 0$

$b = \frac{c}{a} = \frac{q}{p} \frac{x}{y} = \frac{q \cdot x}{py}$, $qx \neq 0, py \neq 0$

This form shows b is a rational number while it contradicts the initial assumption, thus our assumption is false and it means the statement is true.

8.

(a) We can prove it by contraposition. $3k-1$ can written as $3k+2$

Assume that $p > 3$, if $p = 3k+3$ which is divisible by 3, this contradicts the the claim that "p is a prime".

By contraposition, let $p > 3$ be a prime, p is of the form $3k+1$ or $3k-1$ for some integer k.

(b) Assume $x > 5$, and x is prime.

According to part (a), $x = 3k+1$ or $3k-1$

If $x = 3k+1$, $x+2 = 3+3 = 3(k+1)$, $x+2$ is not prime.

If $x = 3k-1$, $x-2 = 3k-3 = 3(k-1)$, $x-2$ is not prime.

when $x \leq 5$, $(5,7)$ are prime

$(3,5)$ are prime