1 Truth Tables

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

(a) $P \land (Q \lor P) \equiv P \land Q$

(b) $(P \lor Q) \land R \equiv (P \land R) \lor (Q \land R)$

(c) $(P \land Q) \lor R \equiv (P \lor R) \land (Q \lor R)$

2 Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) $(\forall x \in \mathbb{Z})\ (x \in \mathbb{Q})$

(e) $(\forall x \in \mathbb{Z})\ (((2 \mid x) \lor (3 \mid x)) \implies (6 \mid x))$

(f) $(\forall x \in \mathbb{N})\ ((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$

3 Converse and Contrapositive

Consider the statement "if a natural number is divisible by 4, it is divisible by 2".

(a) Write the statement in propositional logic. Prove that it is true or give a counterexample.

(b) Write the inverse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The inverse of an implication $P \implies Q$ is $\neg P \implies \neg Q$.)

(c) Write the converse of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

(d) Write the contrapositive of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

4 Equivalences with Quantifiers

Evaluate whether the expressions on the left and right sides are equivalent in each part, and briefly justify your answers.

(a)	$\forall x\ \big((\exists y\ Q(x,y)) \Rightarrow P(x)\big)$	$\forall x\ \exists y\ \big(Q(x,y) \Rightarrow P(x)\big)$
(b)	$\neg\exists x\ \forall y\ \big(P(x,y) \Rightarrow \neg Q(x,y)\big)$	$\forall x\ \big((\exists y\ P(x,y)) \land (\exists y\ Q(x,y))\big)$
(c)	$\forall x\ \exists y\ \big(P(x) \Rightarrow Q(x,y)\big)$	$\forall x\ \big(P(x) \Rightarrow (\exists y\ Q(x,y))\big)$

1. (a) not equivalent

P	Q	R	$P \wedge (Q \vee P)$	$P \wedge Q$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(b) equivalent

P	Q	R	$(P \vee Q) \wedge R$	$(P \wedge R) \vee (Q \wedge R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

1(C). equivalent

P	Q	R	$(P \wedge Q) \vee R$	$(P \vee R) \wedge (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T
F	F	F	F	F

2.

(a) $(\exists x \in \mathbb{R})(x \notin \mathbb{Q})$

True, not all real numbers are rational, π

(b)

$(\forall x \in \mathbb{Z})[[(x \in \mathbb{N}) \vee (x < 0)] \wedge \neg[(x \in \mathbb{N}) \wedge (x < 0)]])$

True, since we define the naturals to contain all integers which are not integer.

(c)

$(\forall x \in \mathbb{N})[(6 | x) \Rightarrow ((2 | x) \vee (3 | x))]$

True, since any number divisible by 6 can be written as $6k = (2 \cdot 3k) = 2(3k)$, meaning it must also be divisible by 2.

(d)

All integers are rational.

True, it can be written as $\frac{n}{1}$

(e)

If a interger is divisible by 2 or 3, it is divisible by 6.

False, 14 can't be divisible by 6 but can be divisible by 2

(f)

If a natural number is larger than 7, there exists 2 natural number, their sum equals the number

True, $x = 0 + x$

3.

(a) $(\forall x \in N)(4 \mid x \Rightarrow 2 \mid x)$

True, $x = 4b = 2 \cdot 2b$, $2b$ is a integer, so it can be divisible by 2.

(b) If a natural number isn't divisible by 4, it isn't divisible by 2.

$(\forall x \in N)(4 \nmid x \Rightarrow 2 \nmid x)$

False, 18 isn't divisible by 4, but divisible by 2.

(c) If a natural number is divisible by 2, it is divisible by 4.

$(\forall x \in \mathbb{N})(2 \mid x \Rightarrow 4 \mid x)$

18 is divisible by 2, but not divisible by 4

(d) If a natural number isn't divisible by 2, it isn't divisible by 4

$(\forall x \in \mathbb{N})(2 \nmid x \Rightarrow 4 \nmid x)$

True

$(\forall x \in \mathbb{N})(2 \nmid x \Rightarrow 4 \nmid x)$

$\equiv (\forall x \in \mathbb{N})((2 \mid x) \vee (4 \nmid x))$

$\equiv (\forall x \in \mathbb{N})((4 \mid x) \Rightarrow (2 \mid x))$

To show that this true, first consider that saying that x is not divisible by 2 is equivalent to to saying that $x/2$ is not an integer. And if we divide a non-integer by an integer, we get back another non-integer - so $(x/2)/2 = x/4$ must also not be an integer.

4. (a) Not equivalent

left:

$$\forall x((\exists y\, Q(x,y)) \Rightarrow P(x))$$

$$\equiv \forall x(\neg(\exists y\, Q(x,y)) \vee P(x))$$

$$\equiv \forall x(\forall y \neg Q(x,y)) \vee P(x))$$

right:

$$\forall x \exists y(Q(x,y) \Rightarrow P(x))$$

$$\equiv \forall x \exists y(\neg Q(x,y) \vee P(x))$$

We can rewrite the left side as $\forall x((\neg(\exists y\, Q(x,y))) \vee P(x))$ and the right side as $\forall x \exists y(\neg Q(x,y) \vee P(x))$. Applying the negation on the left side of the equivalence $(\neg(\exists y Q(x,y)))$ changes the $\exists y$ to $\forall y$, and the two sides are clearly not the same.

(b) not equivalent

left:

$\neg \exists x \forall y (P(x,y) \Rightarrow \neg Q(x,y))$

$\equiv \forall x \exists y \neg (\neg P(x,y) \vee \neg Q(x,y))$

$\equiv \forall x \exists y (P(x,y) \wedge Q(x,y))$

right:

$\forall x ((\exists y P(x,y)) \wedge (\exists y\ Q(x,y)))$

Using De Morgan's Law to distribute the negation on the left side yields. $\forall x \exists y (P(x,y) \wedge Q(x,y))$

But $\exists$ does not distribute over $\wedge$, There could exist diffevent values of y such that $P(x,y)$ and $Q(x,y)$ for a given x, but not necessarily the same value

1(c)

left:

$\forall x \exists y (P(x) \Rightarrow Q(x, y))$

$\equiv \forall x \exists y (\neg P(x) \vee Q(x, y))$

right:

$\forall x (P(x) \Rightarrow (\exists y\ Q(x, y)))$

$\equiv \forall x (\neg P(x) \vee (\exists y\ Q(x, y)))$

$\equiv \forall x \exists y (\neg P(x) \vee Q(x, y))$

We can rewrite the left side as $\forall x \exists y (\neg P(x) \vee Q(x,y))$ and the right side as $\forall x (\neg P(x) \vee (\exists y\ Q(x, y)))$. Clearly, the two sides are the same if $\neg P(x)$ is True. If $\neg P(x)$ is false, then the two sides are still the same, because $\forall x \exists y (False \vee Q(x,y))$ $\equiv \forall x (False \vee \exists y\ Q(x, y))$