Let's $p = a+b < c+d$, $q = a<c$ or $b<d$
therefore $\neg q = a \geq c$ and $b \geq d$
we can get $a+b \geq c+d$ which is $\neg p$
$\neg q \Rightarrow \neg p$ means $p \Rightarrow q$, the claim is correct

Prove it by contradition. Suppose that everyone has different number of friends. For each person, he has i friends, which $i \in \{0, 1, \cdots, n-1\}$. We place them into $n-1$ containers. Each container has a different number from $\{0, 1, \cdots, n-1\}$. Let's analyze the 0 container and $(n-1)$ container, at least one of them must be empty, in other words, they can't contain person at the same time. For example, if the 0 contain isn't empty, that implies nobody can make friends with this guy. Based on the above instruction, we place n person into $n-1$ or less containers. According to the pigeonhole principle, at least one container must contain more than one person. There exists two share the same number of friends.

We give a proof by contraposition. Suppose there does not exist an all-red column. This means that, in each column, we can find a blue pebble. Therefore, if we take one blue pebble from each column, we have a way of choosing one pebble from each column without any red pebbles. This is the negation of the original hypothesis, so we are done.

(a) let $x = f^{-1}(A \cup B)$ which means that $f(x) \in A \cup B$, then either $f(x) \in A$ in which case $x \in f^{-1}(A)$, or $f(x) \in B$, in which case $x \in f^{-1}(B)$, so in either case we have $x \in f^{-1}(A) \cup f^{-1}(B)$

Now, Suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Suppose, without loss of generality, that $x \in f^{-1}(A)$. Then $f(x) \in A$, so $f(x) \in A \cup B$. The argument for $x \in f^{-1}(B)$ is the same.

Hence, $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$

(b) Suppose that $x \in A \cup B$. Then either $x \in A$, in which case $f(x) \in f(A)$, or $x \in B$ in which case $f(x) \in f(B)$. In either case, $f(x) \in f(A) \cup f(B)$, so $f(A \cup B) \subseteq f(A) \cup f(B)$

Now, suppose that $y \in f(A) \cup f(B)$, Then either $y \in f(A)$ or $y \in f(B)$. In the first case, there is an element $x \in A$ with $f(x)=y$; in the second case, there is an element $x \in B$ with $f(x)=y$. In either case, there is an element $x \in A \cup B$ with $f(x)=y$, which means that $y \in f(A \cup B)$. So $f(A) \cup f(B) \subseteq f(A \cup B)$