Musical Acoustics

Homework 5: Design of a recorder

Homework 5 Report

Students

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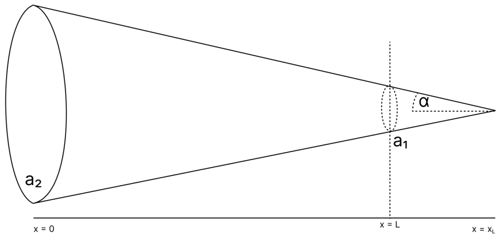

Figure 1: Cone Geometry

The goal of this homework is to design a recorder flute by dimensioning the bore, the last two finger holes, the flue channel and the instrument mouth.

First component: Resonator

The resonator is shaped like a cone with a semi-angle $\alpha = 0.75$ and a length $L = 0.45cm$

Question1: Bore Radius

The geometry of the recorder is shown in Fig.1: a_1 is the radius of the foot of the recorder, a_2 is the radius of the head of the recorder and x_L is the total length of the uncut cone.

We aim to obtain an E4 note ($329.63Hz$) when all the finger holes are closed. To get this result we need to choose a suitable radius for the mouth of the cone. The desired frequency needs to happen at a minimum of the input impedance curve of the truncated cone (also considering the mouth correction). Since the wall thickness of a recorder is not negligible, the flanged radiation type is used, which makes the effective length of the truncated cone as

$$L_{tot} = L + 0.85a_1, \tag{1}$$

The impedance of a truncated cone can be given by

$$Z_{cone} = \frac{\rho c}{S_1}\left[\frac{jZ_L\frac{\sin(kL_{tot}-\theta_2)}{\sin\theta_2} + \frac{\rho c}{S_2}\sin kL_{tot}}{Z_L\frac{\sin(kL_{tot}+\theta_1-\theta_2)}{\sin\theta_1\sin\theta_2} - \frac{j\rho c}{S_2}\frac{\sin(kL_{tot}+\theta_1)}{\sin\theta_1}}\right], \tag{2}$$

where ρ is the air density, c is the speed of sound in air, $k = \omega/c$ is the wave number, Z_L is the load impedance at output, $\theta_1 = \tan^{-1} kx_1$ and $\theta_2 = \tan^{-1} kx_2$. For our case, the radiation effect has been already taken into considered in the equivalent length, so Z_L is set as 0.

The mass of air contained in the mouth as an additional impedance can be modeled by the an impedance as [1]

$$M = \frac{\Delta L\rho}{S} \tag{3}$$

where ΔL is considered around $40mm$ for the low frequencies, and S is the cross section of the recorder at the mouth. The mouth impedance can then be considered as

$$Z_{mouth} = jM\omega_0 \tag{4}$$

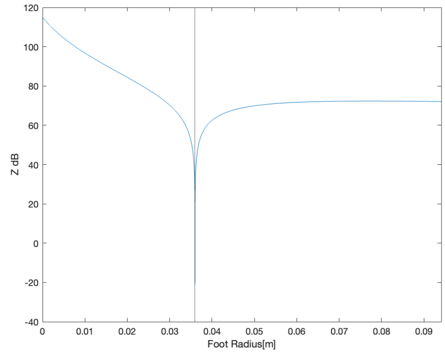

Figure 2: Cone impedance as function of its radii

Z_{mouth} is added in series to the impedance of the cone from Eq.2. To find the radius that will give the lowest impedance we define a_2 as linearly spaced elements from radius $L\tan(\alpha)$ which is the minimum radius possible to $0.1[m]$ and $a1 = a2 - L\tan(\alpha)$. By iterating the impedance computation we get the the best result is given at $a1 = 0.0359[m]$ ($d_1 = 0.0718[m]$) and $a2 = 0.0418[m]$ ($d_2 = 0.0836[m]$). The impedance plot is shown in Fig.2

Question 2: First Finger Hole

The next step is to find the position of the finger hole nearest to the resonator foot so that the frequency produced by the recorder is $369.99[Hz]$. The geometry is shown in Fig.3

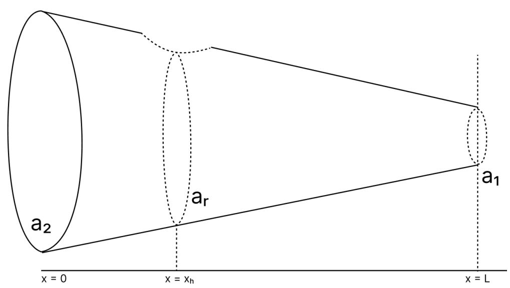

Figure 3: Cone Geometry with the first hole

A finger hole can be taken into consideration as an impedance which is put in parallel with the impedance that we get from the section between x_h to L. This impedance is the used as the radiation impedance of the section from $x = 0$ and x_h and finally the mouth impedance is added. The value of the impedance of the hole can be seen in Eq.5

$$Z_{hole} = -\frac{1}{j\frac{S_1\cot(kL_h)}{\rho c}} \tag{5}$$

Where L_h is the acoustic length of the hole which is computed as $a_r + 0.85a_1$, taking the height of the hole as the radius of the bore in correspondence of the hole plus the flanged end elongation of the hole. By iterating the computation of the impedance along the x axis we obtain a graph of Z as function of the position of the hole. The graph is shown in Fig.4

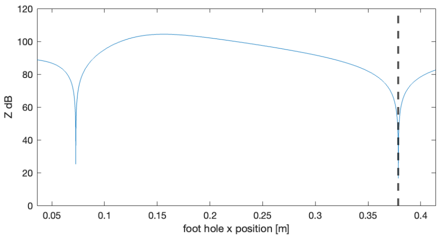

Figure 4: Impedance of the cone as function of the position of the foot hole

From the graph we get that $x_h = 0.378[m]$ The x axis is chosen by excluding a length a_1 from both the start and the end so that the hole would be fully inside the recorder and not to get results that would be impossible to achieve in a real recorder.

Question 3: Second Finger Hole

Now we repeat the same process but with a second finger hole. The only difference now is that we have two impedances which are in parallel so now we first put the first hole impedance in parallel with the impedance of the last section of the recorder, then we use this impedance as the radiation impedance for the middle section, this whole section is then put in parallel with the second hole and finally the result is used as the radiation impedance for the mouth section. In the end we add the mouth air impedance. Again we iterate the computation along the x axis, this time from a distance a_1 from the mouth to $x_h - 2a_1$ so that the two holes will not overlap. The resulting graph is whown in Fig.5

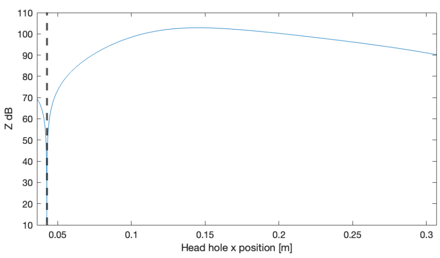

Figure 5: Impedance of the cone as function of the position of the head hole

From the graph we get that $x_{h2} = 0.0426[m]$ Now it is possible to give a visualization of the recorder. The resulting geometry is shown in Fig.6

Considerations

Since the results are not convincing in terms of the positioning of the holes we tried another approach to the problem which is given by evaluating the effect

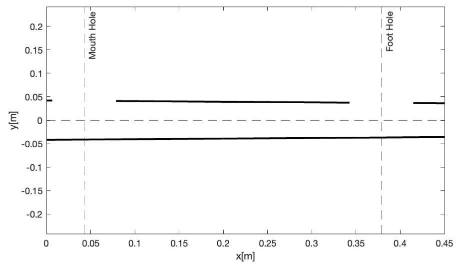

Figure 6: Plot of the shape of the recorder

that the holes have on the total acoustic length of the instrument. In particular we know[1] that for the specific case in which the diameter of the radius at the foot of the recorder is the same as the one for the holes the expression for the reduction of acoustic length is

$$\delta = D + \frac{\Delta^2}{D + 2\Delta} \tag{6}$$

Where D is the distance of the hole from the end of the pipe and Δ is the flanged end correction for the hole. We also know that

$$\omega = \frac{\pi c}{L_{acoustic}} \tag{7}$$

Where

$$L_{acoustic} = L + 0.85a_1 - \delta \tag{8}$$

So in the end we get

$$\delta = L + 0.85a_1 - \frac{\pi c}{\omega} \tag{9}$$

Now from Eq.6 we get that

$$D^2 + D(2\Delta - \delta) + \Delta^2 - 2\delta\Delta = 0 \tag{10}$$

In the end by solving Eq.10 we get the distance of the hole from the open end is $D_1 = 0.0028[m]$ so at $x = 0.4471[m]$

When considering the last hole open we need reconsider the acoustic length of the recorder now as

$$L_{acoustic} = L + 0.85a_1 - \delta \tag{11}$$

And the effective end correction for the finger hole is

$$\Delta' = D + \Delta - \delta \tag{12}$$

Reusing Eq.9 with the new data gives us a new value of δ to compute the position of the second hole which is at $x = 0.4028$. This method gives us a recorder which is impossible to build as the first hole would go outside the bore. Using this same method if we increase the length of the bore by $3[cm]$ so obtaining a $48[cm]$ bore we would get the following data:

- $a_1 = 0.0145[m]$ ($d_1 = 0.0290[m]$)
- $a_2 = 0.0207[m]$ ($d_2 = 0.0414$)
- $x_{h1} = 0.4550[m]$

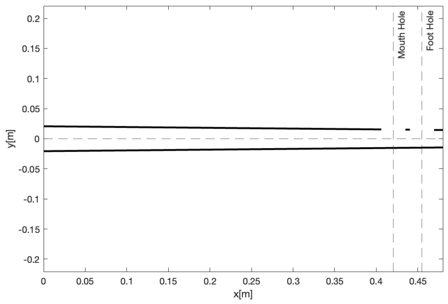

Figure 7: Plot of the shape of the recorder when L = 48 [cm]

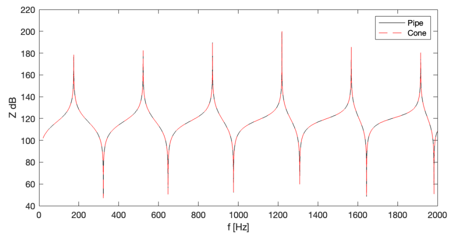

Figure 8: Plots of the cone and pipe impedance

- $x_{h2} = 0.4205[m]$

The resulting geometry can be seen in Fig.7
It can easily be seen that both the position and the size of the holes become much more reasonable just by increasing the length of the bore by a small amount. This approximation is however based on the assumption that the bore is cylindrical while ours is conical so we need to check that this assumption is reasonable. To do this we compute the impedance of a pipe of length $0.48[m]$ and the radius $a = 0.0145$ as in our last results and a cone with the results given in the list above. The equation for the impedance of the cone is given in Eq.2 while the one for the cylindrical pipe is given in Eq.113

$$Z_{pipe} = Z_0 \frac{Z_L \cos(kL) + jZ_0 \sin(kL)}{jZ_L \sin(kL) + Z_0 \cos(kL)} \tag{13}$$

Where Z_L is assumed zero as we are already taking it into consideration by adding the flanged end elongation to the length of the pipe, and

$$Z_0 = \frac{\rho c}{S} \tag{14}$$

with $S = \pi a^2$.
In both cases we add the mouth impedance. The results are shown in Fig.8 in a frequency range around the one that we take into consideration for the notes that we want to achieve on the recorder.
The plot clearly shows that the two are very similar and, in fact, overlap well over the frequency spectrum of the notes. This means that our approximation will not give us results that are too far from the case of the cylindrical pipe and so our computations are valid for this assumption.

Second component: Flue Channel

The target of the recorder design is to produce a spectrum whose centroid is at $f_{centroid} = 1.7$ kHz when the pressure difference between the player mouth and the flue channel entrance is $\Delta p = 55Pa$. The schematic diagram for the recorder is shown in 9.

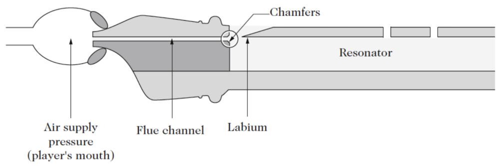

Figure 9: The schematic diagram for the recorder

Question 4

The estimate of the central velocity of the jet can be calculated by the over pressure Δp of the player's mouth with respect to the atmospheric pressure, as

$$U_j = \sqrt{\frac{2\Delta p}{\rho}} = 9.4761[m/s]. \tag{15}$$

The spectrum centroid of the recorder is related to the strongest amplification in flue oscillation as

$$f_{centroid} = 0.3U_j/h, \tag{16}$$

where h is the thickness of the flue channel, which returns the result of 1.7 mm. The Reynolds number Re is used to characterize the structure of the jet in terms of the ratio between inertial and viscous forces, given by

$$Re = \frac{U_j h}{\nu} = 1056.4[Hz], \tag{17}$$

with the kinematic viscosity as $\nu = 1.5 \times 10^{-5}[m^2/s]$. The Reynolds number helps to predict fluid flow patterns in different situations. When the air jet velocity increases, the structure becomes chaotic. The value of Re in our case is lower than 2000, which means that the jet could remain laminar for a short distance.

Question 5

When considering the air near the walls of the flue channel we can see that the viscosity of the air determines a layer of air in correspondence of the wall. Its thickness is given by

$$\delta(x) = \sqrt{\frac{\nu x}{U_j}} \tag{18}$$

The resulting thickness is $1.989e-04[m]$.

Bibliography

[1] Fletcher, N. H., & Rossing, T. D. (2012). The physics of musical instruments. Springer Science & Business Media.