Musical Acoustics

Homelab 3: Modeling of musical instruments

# Homework 3 Report

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# Ex 1

The finite difference method is used here to model the piano string considering the hammer interaction. In this exercise, we aim to model the piano string C1, whose fundamental frequency is $f_1 = 65.4$ Hz.

## Numerical modeling theory

A brief description of the numerical model is shown as follows. There are two domains need to be sampled: time and spatial domain. The sampling frequency of time is $Fs$, the sampling time is $T = 1/Fs$, the total modeling time is $L_t$, the time step is $N = L_t/T$. The length of the string is $L$, the spatial step of the length of the string is $M$, the length of each string segment is $X = L/M$.
The wave equation for the stiff and lossy string with the finite difference method in discrete-time with the hammer interaction is

$$\begin{aligned} y_m^{n+1} =& a_1(y_{m+2}^n + y_{m-2}^n) + a_2(y_{m+1}^n + y_{m-1}^n) + \\ & a_3 y_m^n + a_4 y_m^{n-1} + \\ & a_5(y_{m+1}^{n-1} + y_{m-1}^{n-1}) + a_F F_m^n, \end{aligned} \tag{1}$$

where $y_m^n$ means the transverse displacement of the $m$ segment along the string at time $n$. This equation is valid only in the interior of the string. The boundary conditions need to be considered at both of the end. Thus, the wave equations for $m = 1, M-1, 0, M$ have different expressions.
The coefficients of $a_i$ and $a_F$ are defined by:

$$a_1 = \frac{-\lambda\mu}{1 + b_1 T}$$

$$a_2 = \frac{\lambda^2 + 4\lambda\mu + \nu}{1 + b_1 T}$$

$$a_3 = \frac{2 - 2\lambda^2 - 6\lambda^2\mu - 2\nu}{1 + b_1 T}$$

$$a_4 = \frac{-1 + b_1 T + 2\nu}{1 + b_1 T}$$

$$a_5 = \frac{-\nu}{1 + b_1 T}$$

$$a_F = \frac{T^2/\rho}{1 + b_1 T},$$

where $\mu = \kappa^2/c^2X^2$ and $\nu = 2b_2T/X^2$. $b_1$ is the air damping coefficient, $b_2$ is the string internal friction coefficient, $\kappa$ is the string stiffness coefficient. The wave speed on the string is $c = \sqrt{T_e/\rho}$, with string tension $T_e$ and string linear density $\rho = M_S/L$, $M_S$ is the mass of the string. $\lambda = cT/X$ is Courant number, for ensuring that the discrete scheme works.
The wave equation for $m = 0$ is

$$\begin{aligned} y_m^{n+1} =& b_{L1} y_m^n + b_{L2} y_{m+1}^n + b_{L3} y_{m+2}^n + \\ & b_{L4} y_m^{n-1} + b_{LF} F_m^n, \end{aligned} \tag{2}$$

for $m = 1$ is

$$\begin{aligned} y_m^{n+1} =& a_1(y_{m+2}^n - y_m^n + 2y_{m-1}^n) + \\ & a_2(y_{m+1}^n + y_{m-1}^n) + a_3 y_m^n + \\ & a_4 y_m^{n-1} + a_5(y_{m+1}^{n-1} + y_{m-1}^{n-1}) + a_F F_m^n, \end{aligned} \tag{3}$$

for $m = M - 1$ is

$$\begin{aligned} y_m^{n+1} =& a_1(2y_{m+1}^n + y_{m-2}^n) + a_2(y_{m+1}^n + y_{m-1}^n) + \\ & a_3 y_m^n + a_4 y_m^{n-1} + a_5(y_{m+1}^{n-1} + y_{m-1}^{n-1}) + \\ & a_F F_m^n, \end{aligned} \tag{4}$$

and for $m = M$ is

$$\begin{aligned} y_m^{n+1} =& b_{R1} y_m^n + b_{R2} y_{m-1}^n + b_{R3} y_{m-2}^n + \\ & b_{R4} y_m^{n-1} + b_{RF} F_m^n. \end{aligned} \tag{5}$$

The coefficients of $b_{Li}$ and $b_{LF}$ is

$$b_{L1} = \frac{2 - 2\lambda^2\mu - 2\lambda^2}{1 + b_1 T + \zeta_l \lambda}$$

$$b_{L2} = \frac{4\lambda^2\mu + 2\lambda^2}{1 + b_1 T + \zeta_l \lambda}$$

$$b_{L3} = \frac{-2\lambda^2\mu}{1 + b_1 T + \zeta_l \lambda}$$

$$b_{L4} = \frac{-1 + b_1 T + \zeta_l \lambda}{1 + b_1 T + \zeta_l \lambda}$$

$$b_{LF} = \frac{T^2/\rho}{1 + b_1 T + \zeta_l \lambda},$$

where $\zeta_l$ is the left end normalized impedance.
The coefficients of $b_{Ri}$ and $b_{RF}$ is

$$b_{R1} = \frac{2 - 2\lambda^2\mu - 2\lambda^2}{1 + b_1 T + \zeta_b \lambda}$$

$$b_{R2} = \frac{4\lambda^2\mu + 2\lambda^2}{1 + b_1 T + \zeta_b \lambda}$$

$$b_{R3} = \frac{-2\lambda^2\mu}{1 + b_1 T + \zeta_b \lambda}$$

$$b_{R4} = \frac{-1 + b_1 T + \zeta_b \lambda}{1 + b_1 T + \zeta_b \lambda}$$

$$b_{RF} = \frac{T^2/\rho}{1 + b_1 T + \zeta_b \lambda},$$

where $\zeta_b$ is the bridge normalized impedance.
The initial condition of the string is rest, as

$$y_m^0 = 0.$$

The force density term is defined as

$$F_m^n = F_H(n)g(m, m_0), \tag{6}$$

with

$$F_H(n) = K|\eta^n - y_{m0}^n|^p, \tag{7}$$

where $K$ is the hammer felt stiffness, $p$ is the hammer felt stiffness exponent, $m0$ is the striking point along the string, $\eta$ is the hammer displacement and $g(m, m_0)$ is the spatial window (details are given later). For $n = 1, 2$, the force expressions are

$$F_H^1 = K|\eta^1 - y_{m_0}^1|^p, \tag{8}$$

$$F_H^2 = K|\eta^1 - y_{m_0}^2|^p, \tag{9}$$

respectively.
The hammer displacement is given by

$$\eta^{n+1} = d_1\eta^n + d_3\eta^{n-1} + d_F F_H(n). \tag{10}$$

The coefficients of $d_i$ and $d_F$ are

$$d_1 = \frac{2}{1 + b_H T/2M_H}$$

$$d_2 = \frac{-1 + b_H T/2M_H}{1 + b_H T/2M_H}$$

$$d_F = \frac{-T^2/M_H}{1 + b_H T/2M_H},$$

where $M_H$ is the hammer mass, and $b_H$ is the fluid damping coefficient. When considering the initial condition, the hammer displacement for $n = 1, 2$ is

$$\eta^1 = V_{H0}ts, \tag{11}$$

$$\eta^2 = 2\eta^1 - \eta^0 - [T^2 F_H^1]/M_H, \tag{12}$$

where $V_{H0}$ is the initial hammer velocity.

When considering the spatial window $g(m, m_0)$, the spatial step of the striking position along the string is defined by

$$m_0 = aM, \tag{13}$$

where $a$ is the relative striking position. The number of the spatial steps of the string that are contacted by the hammer is

$$m_w = w/X, \tag{14}$$

where $w$ is the width of the hammer spatial window. Then a hanning window with the length of $m_w$, centered at $m_0$ is defined as $g(m, m_0)$.

In order to satisfy the numerical stability, the spatial step should be limited. Here are two kinds of stability condition:

$$X_{max1} = \sqrt{\frac{1}{2}\left(c^2T^2 + 4b_2T + \sqrt{(c^2T^2 + 4b_2T)^2 + 16\kappa^2T^2}\right)}, \tag{15}$$

which is from [1], and

$$X_{max2} = \left[(-1 + (1 + 16\kappa\gamma^2)^{1/2})8\kappa\right]^{1/2} \tag{16}$$

which is from [2]. In the following modeling, we choose the spatial steps that smaller both of $X_{max1}$ and $X_{max2}$. The Courant-Friedrichs-Lewy (CFL) condition is given by

$$\lambda \leq 1, \tag{17}$$

which is used to satisfy Von Neumann stability analysis.

# Implementation

## Parameters set up

The first step to implement the finite difference scheme is to define all the string parameters and the simulation variables.
Since the physics is illustrated in section , table 1 shows the relation between the names of the parameters and variables used above and in Matlab file, and the value of the parameters used in this modeling. Only set up values are listed here.

## Sampling

In our case, the Courant number is $\lambda = 0.3552$, which is smaller than 1 and satisfies the CFL condition. The spatial step we used here is $M = 479$, which is smaller than both $X_{max1} = 1.3486e3$ and $X_{max2} = 479.7089$, to ensure the numerical modeling stability.

## Finite Difference scheme

To implement the finite difference model, we create a loop, with a particular consideration of the time step ($n = 0, 1, 2$) and the spatial step ($m = 0, 1, M-1, M$). The sound of the C1 piano string can be generated from the above finite difference model, approximated by averaging the string displacement over a small portion of the string. The structure diagram of a string on the grand piano is shown in Fig.1. After exciting the string, it transfers the vibration into the bridge and then makes the soundboard vibrate and radiate sound. Therefore, we choose the displacement of the string portion near to the bridge to generate the sound, by considering the last 12 spatial steps from $M - 11$ to $M$.

# Results

Figure 2 shows an example of the solution for the whole string displacement at time 0.0878741 s.
The estimated sound signal is shown in figure 5. Figure 3 and figure 4 shows the piano sound after FFT and its spectrogram, respectively.

[1] Saitis, C. (2008). Physical modelling of the piano: An investigation into the effect of string stiffness on the hammer-string interaction. Belfast: Sonic Arts Research Centre.

[2] Chaigne, A., & Askenfelt, A. (1994). Numerical simulations of piano strings. I. A physical model for a struck string using

| Parameter | Section 1 | Matlab | Value |
|---|---|---|---|
| Time sampling frequency | $Fs$ | fs | 176400[s$^{-1}$] |
| Sampling time | $T$ | ts | - |
| Total time | $L_t$ | signal_length | 8[s] |
| Time step | $N$ | N | - |
| Fundamental freq. | $f_1$ | f_0 | 65.4[Hz] |
| Left end normalized impedance | $\zeta_l$ | zeta_l | 1e20[Ω/(Kg·m $^2$·s)] |
| Bridge normalized impedance | $\zeta_l$ | zeta_l | 1000[Ω/(Kg·m $^2$·s)] |
| String length | $L$ | l | 1.92[m] |
| String mass | $M_S$ | Ms | 35e-3[Kg] |
| String linear density | $\rho$ | rho | - |
| String tension | $T_e$ | te | - |
| Air damping coefficient | $b_1$ | b1 | 0.5[1/s] |
| String internal friction coefficient | $b_2$ | b2 | 6.25e-9[s] |
| String stiffness coefficient | $\kappa$ | epsilon | 7.5e-6 |
| Wave speed | $c$ | c | - |
| Spatial step | $M$ | M | 479 |
| String segment length | $X$ | xm | - |
| Courant number | $\lambda$ | lambda | - |
| - | $\mu$ | mu | - |
| - | $\nu$ | nu | - |
| Wave equation coefficient 1 | $a_1$ | a1 | - |
| Wave equation coefficient 2 | $a_2$ | a2 | - |
| Wave equation coefficient 3 | $a_3$ | a3 | - |
| Wave equation coefficient 4 | $a_4$ | a4 | - |
| Wave equation coefficient 5 | $a_5$ | a5 | - |
| Wave equation-force coefficient | $a_F$ | af | - |
| Hammer mass | $M_H$ | Mh | 4.9e-3[kg] |
| Fluid damping coefficient | $b_H$ | bh | 1e-4[s$^{-1}$] |
| Hammer displacement coefficient 1 | $d_1$ | d1 | - |
| Hammer displacement coefficient 2 | $d_2$ | d2 | - |
| Hammer displacement-force coefficient | $d_F$ | df | - |
| Initial hammer velocity | $V_{H0}$ | v0h | 2.5[m/s] |
| Window width | $w$ | w | 0.2 |
| Relative striking point | $a$ | a | 0.12 |
| Center striking point segment | $m_0$ | xmh | - |
| Hammer contacted segments number | $m_w$ | window_lenght | - |
| Bridge boundary coefficient 1 | $b_{R1}$ | br1 | - |
| Bridge boundary coefficient 2 | $b_{R2}$ | br2 | - |
| Bridge boundary coefficient 3 | $b_{R3}$ | br3 | - |
| Bridge boundary coefficient 4 | $b_{R4}$ | br4 | - |
| Bridge boundary-force coefficient | $b_{RF}$ | brf | - |
| Left end boundary coefficient 1 | $b_{L1}$ | bl1 | - |
| Left end boundary coefficient 2 | $b_{L2}$ | bl2 | - |
| Left end boundary coefficient 3 | $b_{L3}$ | bl3 | - |
| Left end boundary coefficient 4 | $b_{L4}$ | bl4 | - |
| Left end-force boundary coefficient | $b_{LF}$ | blf | - |
| Felt stiffness exponent | $p$ | ph | 2.3 |
| Felt stiffness | $K$ | Kh | 4e8 |
| String displacement | $y$ | y | - |
| Hammer displacement | $\eta$ | eta | - |
| Force density term | $F_H$ | Fh | - |
| Force density | $F$ | F | - |
| - | $X_{max1}$ | M_max1 | - |
| - | $X_{max2}$ | M_max2 | - |

Table 1: Modeling parameters.



finite difference methods. The Journal of the Acoustical Society of America, 95(2), 1112-1118.

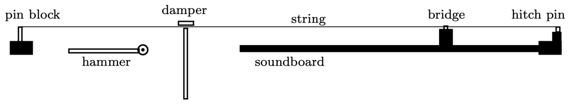



Figure 1: Simplified diagram of the piano.

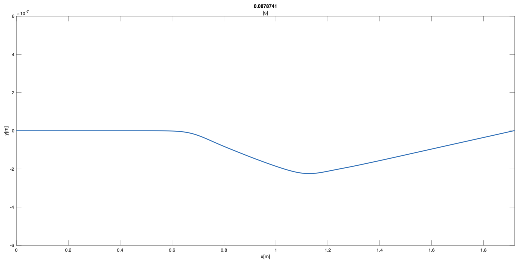


Figure 2: Piano C1 string displacement at time 0.0878741s.

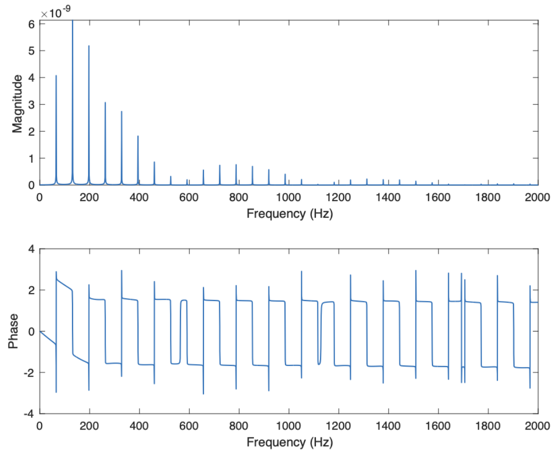



Figure 3: Amplitude and phase of the piano C1 sound.

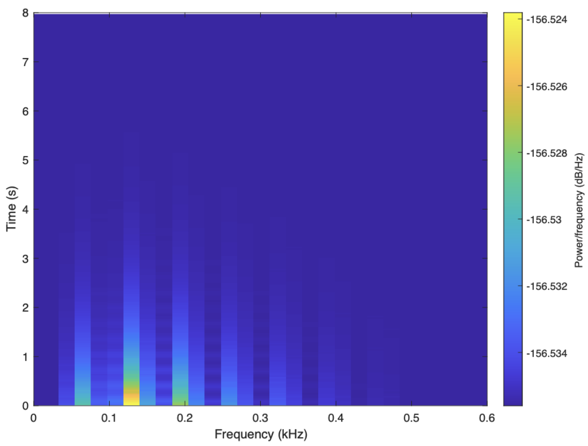



Figure 4: Spectrogram of the piano C1 sound.

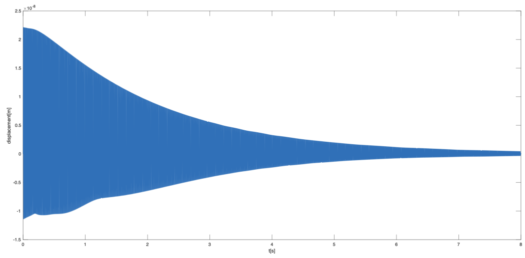


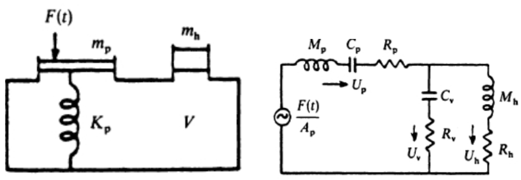
Figure 5: The simulated C1 piano sound.

The audio of the synthesis piano sound is saved in the file '10669941_Bernasconi_10876787_Luan_Piano.wav'.

# Ex 2

## Guitar Modeling

The electric model is built from the acoustic equivalent of a guitar that models the top plate and the volume of air contained inside the body of the guitar itself. Fig.6a shows that the top plate is represented as a mass-spring system using $m_p$ and $K_p$ while the mass $m_h$ of the sound hole and volume $V$ model the air inside the body of the guitar. This gets translated in an electric equivalent as shown in Fig.6b using RLC circuits to model the same behaviour in the electric domain.




(a) (b)

Figure 6: Equivalents of the Guitar.

From this it is possible to further expand the model. For example, we can simulate the excitation that comes from a string. This will be modeled in two ways. One is shown in Fig.7 and it is a Controlled Voltage Generator that creates a signal in the form:

$$f(t) = sign(sin(2\pi \cdot 300 \cdot t))e^{-3t}u(t) \qquad (18)$$

This kind of simulation is very rudimental and gives a rough approximation of a string. A more refined one can be see on the left side of Fig.8. Here the string is modeled using two symmetrical generators and a delay line. The two generators with the switches that short-circuit the delay line after the excitation is over model the reflection of the wave when it comes in contact with either the bridge or the nut of the guitar.
Another improvement in the model is shown on the right side of Fig.8. 20 more RLC circuits have been added in parallel, each one of them modeling one of the resonances of top plate, this enables the model to better

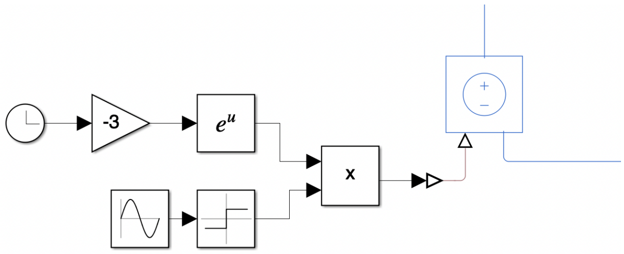



Figure 7: Damped Rectangle excitation circuit.

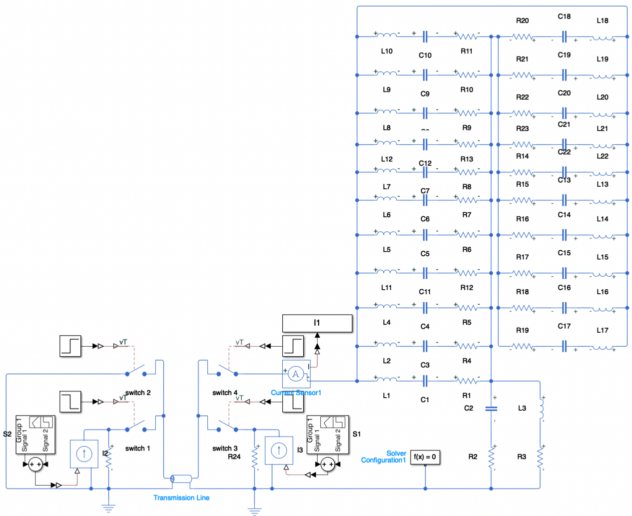



Figure 8: Electric Equivalent of the guitar with string simulation.

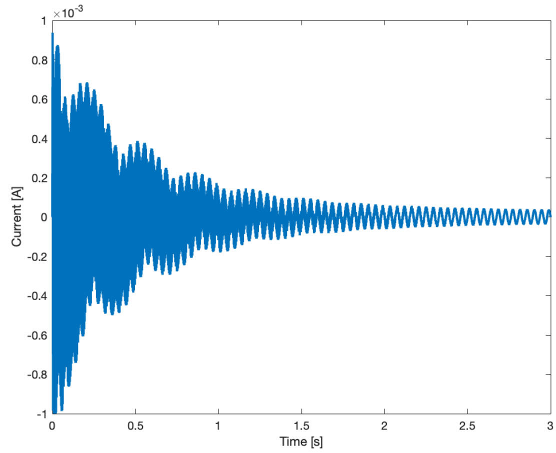



Figure 9: Resulting signal with String simulation.

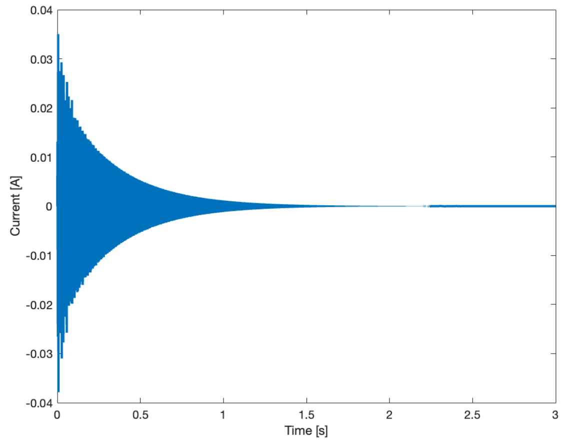



Figure 10: Resulting signal without String simulation.

represent the real guitar also for higher frequencies. The values of the components are chosen to specifically introduce resonances exactly at the frequencies where measures on the top plate show a resonance in the real guitar.

## Simulation

The simulation of the sound produced by the guitar model takes place over a span of 3 seconds and it was done using as input both the basic string simulation and the more complex one. The output signal is taken as the current that flows in the guitar plate and sound box model. The resulting signals are shown in Fig.9 and in Fig.10
We can clearly see in Fig.9 that the signal hits the body many times while, in Fig.10 the absence of the string modeling results in an output signal which is overall a less accurate sound, closer to a square wave than to the sound of a guitar. Here in Fig.11 and Fig.12 we have the frequency and phase analysis of both signals.

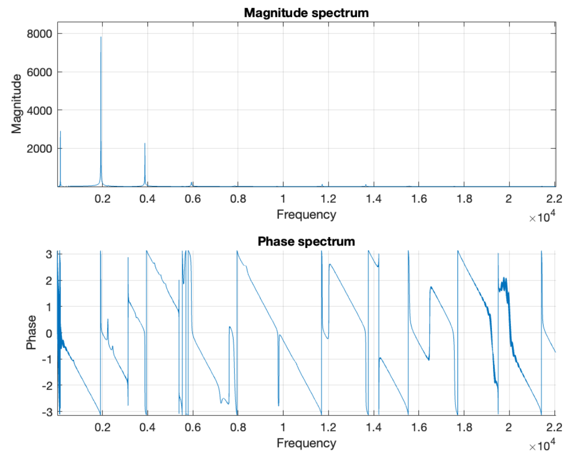



Figure 11: Amplitude and phase of the guitar sound with string simulation.

## Results

Even without hearing the actual sound of these simulations it's clear that the Fourier analysis in Fig.11 is more accurate to a real one as is much less harmonically rich and the harmonics are better spread across the spectrum as integer multiples of the fundamental,

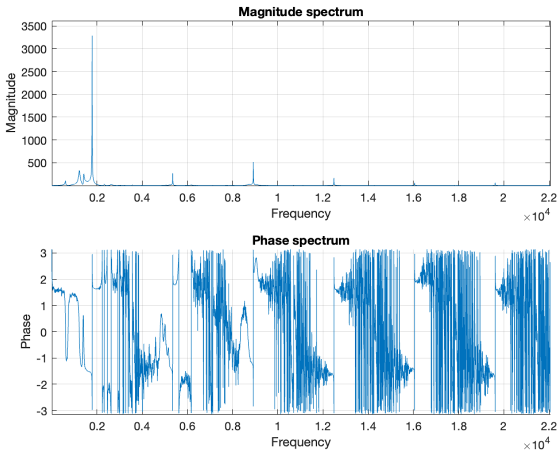



Figure 12: Amplitude and phase of the guitar sound without string simulation.

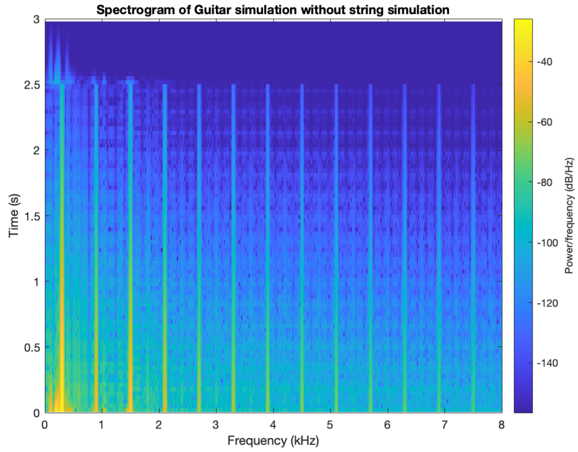



Figure 14: Spectrogram of Guitar without string simulation.

as expected in a musical instrument. This is further proved after taking a look at the spectrograms of both signals with respect to the one of a real guitar sample. It is clear that the harmonic content in Fig.14 stays for the whole length of the signal while the other two cases have different decay times for their harmonic contents. In particular the simulation with the string model more closely resembles the way in which lower harmonics stay longer while higher harmonics decay faster as in the real guitar example of Fig.15.

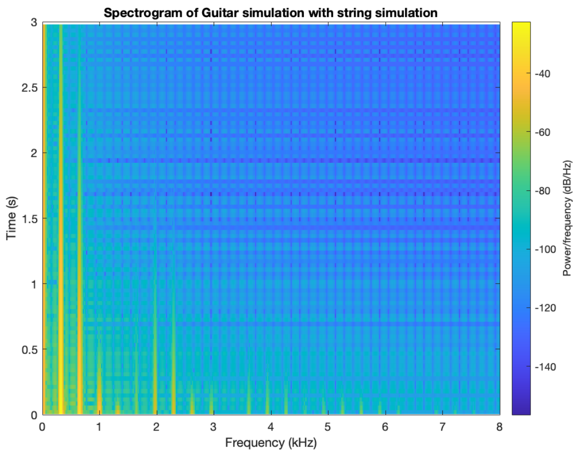



Figure 13: Spectrogram of Guitar with string simulation.

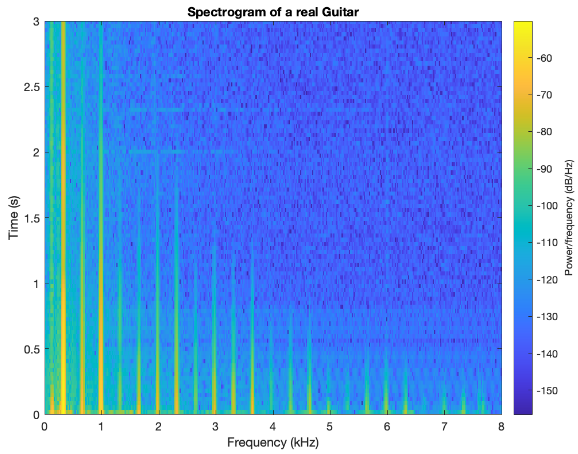



Figure 15: Spectrogram of a real Guitar sample.