Musical Acoustics

Homework 2: 2D and 1D systems

Homework 2 Report

Students

Xinmeng Luan 10876787
Marco Bernasconi 10669941

POLITECNICO
MILANO 1863

Part 1 – Circular membrane characterization

It is given a circular membrane with radius $r = 0.15m$, and tension $T = 10N/m$. The unit surface weight is $\sigma = 0.07kg/m^2$.

a)

The propagation speed for transverse waves in the circular membrane is given by

$$c = \sqrt{T/\sigma} = 0.15\left[\frac{m}{s}\right]. \tag{1}$$

b)

The general solution of the displacement for circular membrane $w(r, \phi, t)$ is

$$w(r, \phi, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m(k_n r) e^{(m\phi)} e^{(\omega_m n t)} \tag{2}$$

where m corresponds to nodal diameters, and n is corresponding to nodal circles. The coefficients A_{mn} depends on the initial conditions, $J_m(k_n r)$ is a Bessel function and k_n is given by

$$k_n r = Z_n(J_m(kr)) \tag{3}$$

where $Z_n(J_m(kr))$ is the n^{th} zero of $J_m(k_n r)$. Thus the frequency of the membrane for each mode could be derived using the zeros of the first orders of the Bessel function, as shown in table.1.

n	$J_0(kr)$	$J_1(kr)$	$J_2(kr)$	$J_3(kr)$	$J_4(kr)$	$J_5(kr)$	$J_6(kr)$	$J_7(kr)$	$J_8(kr)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715	9.9361	11.0864	12.2251
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386	13.5893	14.8213	16.0378
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002	17.0038	18.2876	19.5545
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801	20.3208	21.6415	22.9452

Table 1: **Zeros of the first orders of the Bessel function.**

It can be concluded from the table that the first eighteen modes are (0,1), (1,1), (2,1), (0,2), (3,1), (1,2), (4,1), (2,2), (0,3), (1,6), (5,1), (3,2), (6,1), (1,3), (4,2), (7,1), (2,3), and (8,1). The eigenfrequency of these modes are shown in table.2.

Mode	(0,1)	(1,1)	(2,1)	(0,2)	(3,1)	(1,2)	(4,1)	(2,2)	(0,3)
Freq.(Hz)	191.62	305.32	409.21	439.85	508.39	559.02	604.65	670.70	689.54
Mode	(1,6)	(5,1)	(3,2)	(6,1)	(1,3)	(4,2)	(7,1)	(2,3)	(8,1)
Freq.(Hz)	698.93	777.78	791.73	810.64	881.66	883.39	925.89	939.57	974.12

Table 2: **Eigenfrequency of the first eighteen modes.**

From equation.2 we can take the real part and, by removing the constant A_{mn} and the dependancy on time it is possible to plot the modeshapes of the considered membrane

$$w_{mn}(r, \phi) = J_m(k_n r) e^{jm\phi} \tag{4}$$

Thus the first six mode shapes can be calculated by equation.4, as shown in Figure.1.

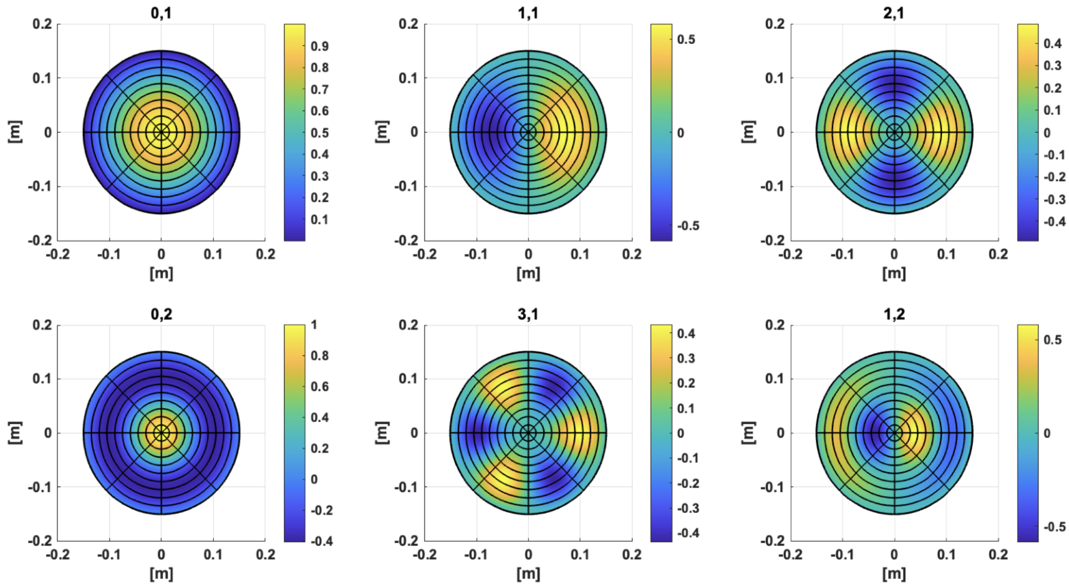


Figure 1: First six mode shapes of the membrane.

c)

The membrane is struck at coordinates $r_k = 0.075m$, $\Phi_k = 15°$, and a displacement sensor is mounted at coordinates $r_j = 0.075m$, $\Phi_j = 195°$. The modal superposition approach can be used when finite modes are considered, then a n-dof system can be thought as n 1-dof systems. We want the modal vectors to reflect the motion of the struck membrane at that point. We can obtain this by rotating the eigenmodes in such a way that one of the minima for each of the modes always lays under the striking point. Since the system is thought in radial coordinates, and the dependency on the angle is only given by a complex exponential, it is possible to compute the necessary rotation of the eigenmodes by taking the angle of rotation θ as the minima of the real part of the exponential function at the considered mode.

$$\theta_m = \min_{\phi \in [0,2\pi]} \mathbb{R}\left[e^{jm\phi}\right] = \pi/m \tag{5}$$

where m is the first modal coordinate of the currently considered modeshape.
Limited to the frequency range where the first eighteen modes lie, the transfer receptance can be computed as:

$$G(\omega) = \frac{w_{j0}}{F_{k0}} = \sum_{i=1}^{18} \frac{\Phi_i(x_j)\Phi_i(x_k)}{-\omega^2 + j\frac{\omega\Omega_{mn}}{Q} + \Omega_{mn}^2} \tag{6}$$

where subscript k is for input and j is for output. Ω_{mn} is the resonant frequency for the given mode. The general Φ is:

$$\Phi_i = J_m(k_n r)e^{jm\phi_i n} \tag{7}$$

The results of the receptance are shown in Figure.2.
The impact force signal of the hammer is

$$f(t) = 0.1e^{-\frac{(t-0.03)^2}{0.01^2}} \tag{8}$$

Then the displacement response can be derived from the convolution between the force signal and the inverse Fourier Transform of the receptance

$$x(t) = f(t) * F^{-1}[G(j\omega)] \tag{9}$$

as shown in Figure.3.

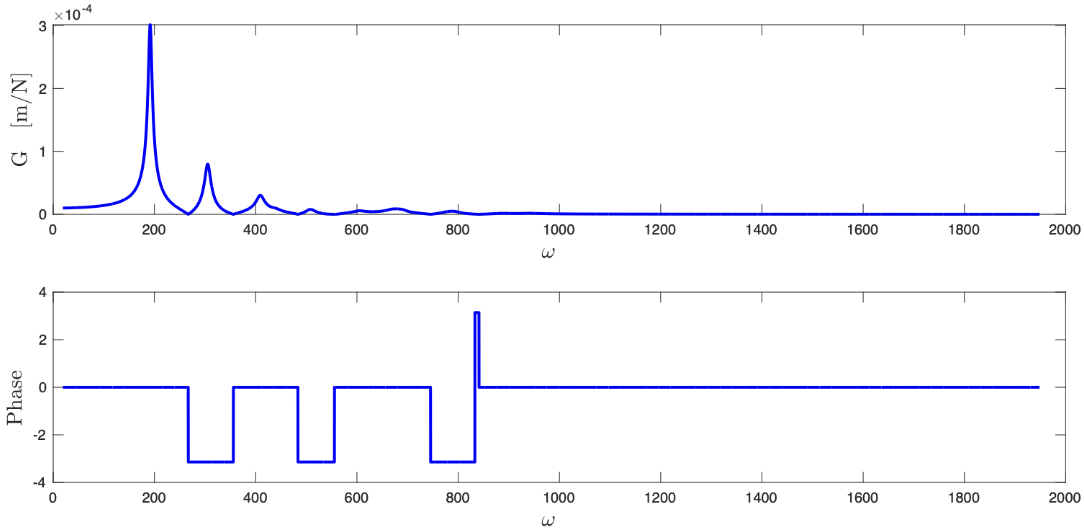


Figure 2: Transfer receptance of the membrane. The top is for the absolute value and the bottom is for the phase.

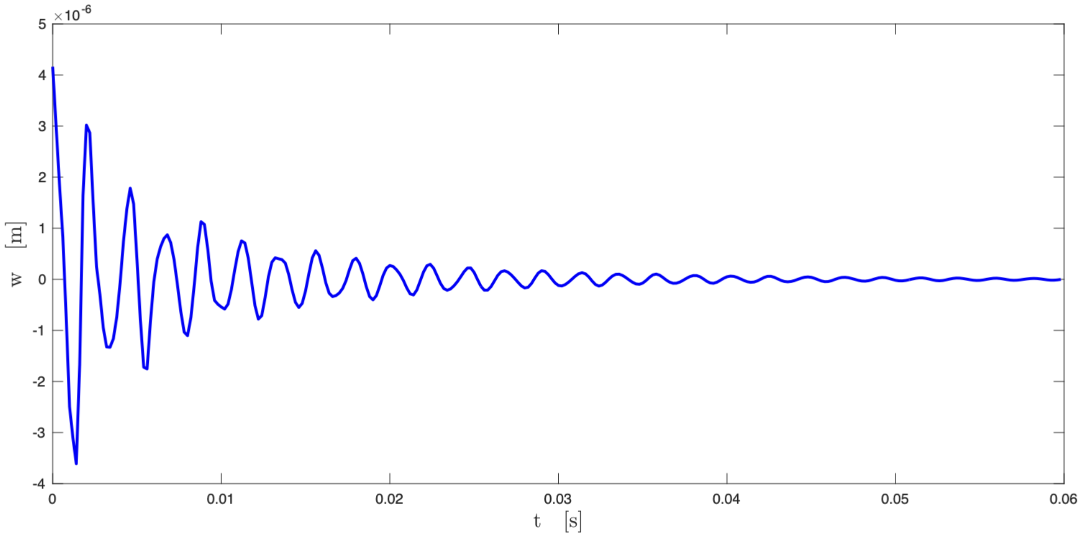


Figure 3: Real part of the displacement response of the membrane.

Part 2 – Circular plate characterization

Consider now a thin plate with clamped edges and with the same size of the membrane. The plate has a thickness of $h = 1mm$, and it is made with aluminum ($E = 69GPa$, $\rho = 2700kg/m^3$, $\nu = 0.334$).

d)

The speed of quasi-longitudinal and longitudinal waves propagating in this thin plate can be calculated by

$$C_L = \sqrt{\frac{E}{\rho(1-\nu^2)}} = 5.3632 \times 10^3 \tag{10}$$

and

$$C_L' = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} = 6.1992 \times 10^3 \tag{11}$$

separately.

e)

The propagation of bending waves in thin plate depend on the frequency of the wave itself. This phenomenon is called dispersion and the wave speed is given by

$$v(f) = \sqrt{1.8fhC_L} \tag{12}$$

The results are shown in Figure.4.

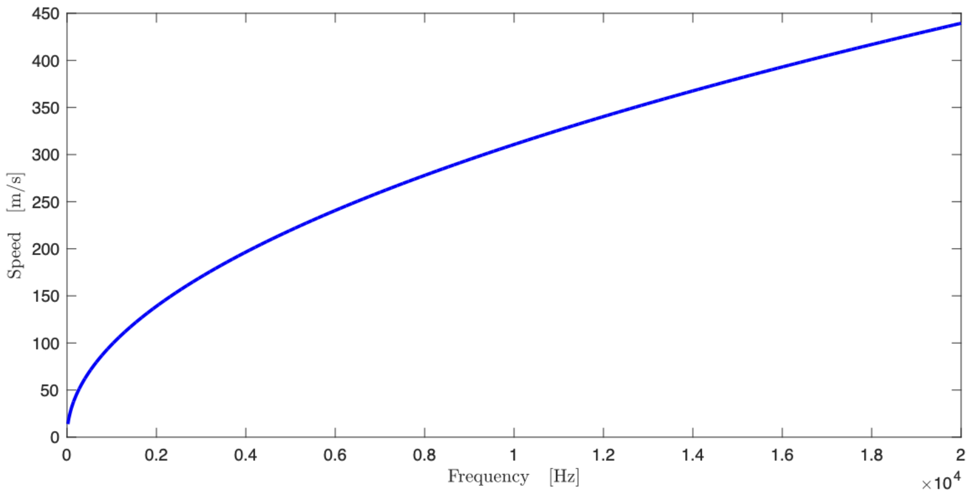


Figure 4: Propagation speed of the bending waves in thin plate.

f)

For a clamped edges thin plate, the frequency of the first five bending modes are

$$f_{01} = 0.4694C_Lh/r^2 = 111.8892[Hz]$$

$$f_{11} = 2.08f_{01} = 232.7296[Hz]$$

$$f_{21} = 3.41f_{01} = 381.5422[Hz]$$

$$f_{02} = 3.89f_{01} = 435.2490[Hz]$$

and

$$f_{12} = 5.95f_{01} = 559.4460[Hz]$$

Part 3 – Interaction between coupled systems

Consider now that a string is attached to the considered plate, and its fundamental mode is tuned to the frequency of the first mode of the plate. The string is made with iron ($\rho = 5000kg/m^3$), its cross section is circular with a radius of $r_s = 0.001m$, and its length is $L_s = 0.4m$. Due to internal losses and sound radiation, the plate at the frequency of the considered mode dissipates energy, and the merit factor is 50.

g)

The frequency of the fundamental mode (0,1) for the clamped thin plate has been given in point f). Since the fundamental mode is tuned with the first mode of the soundboard, the tension of the string is

$$T_s = (\frac{\omega L_s}{\pi})^2 \mu = (2fL)^2 \rho \pi r_s^2 = 125.8567[N] \tag{13}$$

h)

Now considering the stiffness of the string, whose Young modulus of the iron is $E = 200 \times 10^9 Pa$. Assume the string is hinged at both ends, the frequencies of the first five modes of the string considering its stiffness can be calculated by

$$f_n = n f_1^{\circ} \sqrt{1 + B_n^2} \left[1 + \frac{2}{\pi}\sqrt{B} + \frac{4}{\pi^2} B \right], \quad n = 1, 2, 3, 4, 5 \tag{14}$$

With $B = \frac{\pi E S K}{T_s L_s^2}$ and $K = \frac{r_s}{2}$ which results are

$$f_1 = 125.6632[Hz]$$

$$f_2 = 260.1878[Hz]$$

$$f_3 = 411.4821[Hz]$$

$$f_4 = 585.9301[Hz]$$

and

$$f_5 = 788.2997[Hz]$$

i)

Now considering the string without stiffness and the bending modes of the plate we need to determine what modes are strongly coupled. To do this we need to take into consideration the following inequality:

$$\frac{m}{n^2 M} > \frac{2\pi}{4Q_B^2} \tag{15}$$

where m is the mass of the string, M is the mass of the board, n is the mode number of the string and Q_B is the quality factor of the resonance of the board.
The disequality holds for $n \leq 5$, this means that we need to consider strong coupling at most up to the fifth mode of the string. It's easy to see that mode 1 and 5 have the same frequency both in the string and the board while modes 2,3 and 4 have different frequencies. To determine the coupled frequencies for modes 1 and 5 we use the graph in figure.5 and use an external code

to get the shift in frequency according to the factor $\frac{m}{n^2M}$. The code was made in Processing and takes the image of the graph as the input, then maps the left part of Equation.15 to the number of horizontal pixels in the image. From this the program finds the nearest black pixel with the same x value as computed just before and maps the y coordinate to the y coordinates of the graph, draws a point on the graph with the coordinates and also prints in the console the mapped y value. The code is made to be as general as possible so it computes points for the system given as input and the required number of modes. Because of this we ignore points 2,3 and 4 and see that the fifth mode doesn't lay inside the graph so we can consider that the frequency shift doesn't happen for that mode. By taking the ratio given on the y axis we can then compute the shifted frequencies

$$f_{n\pm} = f_n \pm \frac{y_{ratio} f_0}{2} \tag{16}$$

With $y_{ratio} = \frac{f_{0+} - f_{0-}}{f_0}$. The resulting coupled frequencies are:

$$f_{1+} = 112.4822[Hz]$$

$$f_{1-} = 111.2962[Hz]$$

For modes 2,3 and 4 we instead use the graph in figure.6 and, from that, it is possible to derive the coupled frequencies as a function of $\frac{\omega_s - \omega_b}{\omega_b}$ where ω_s and ω_b are the considered resonant frequencies for the n^{th} mode of the string and board respectively. The code used follows the same logic as the one for the coinciding frequencies case but iterates two times for each x value. From this we see that the third mode has frequencies which further apart than the span of the graph(the normalized difference between the two is -0.12) and so that one frequency will stay the same as the one of the string while the other will be equal to the one of the plate. The resulting frequencies are then computed from the ratios given by the graph in the following way:

$$f_{n\pm} = f_n(y_{ratio\pm} \cdot f_{board} + f_{board}) \tag{17}$$

With $y_{ratio\pm} = \frac{f_{\pm} - f_{board}}{f_{board}}$. The following results are then obtained from the graph:

$$f_{2+} = 235.0568[Hz]$$

$$f_{2-} = 222.0239[Hz]$$

$$f_{3+} = 381.5422[Hz]$$

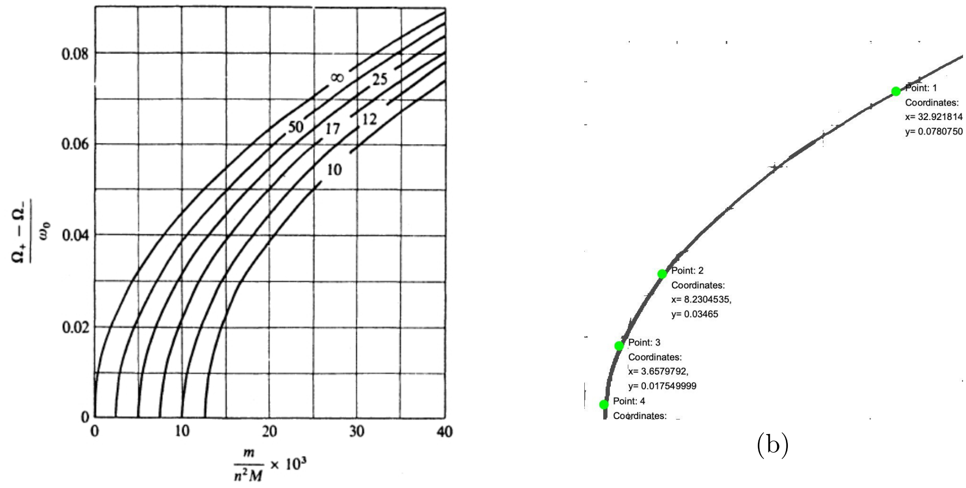
$$f_{3-} = 335.6676[Hz]$$

$$f_{4+} = 452.1366[Hz]$$

$$f_{4-} = 430.7224[Hz]$$

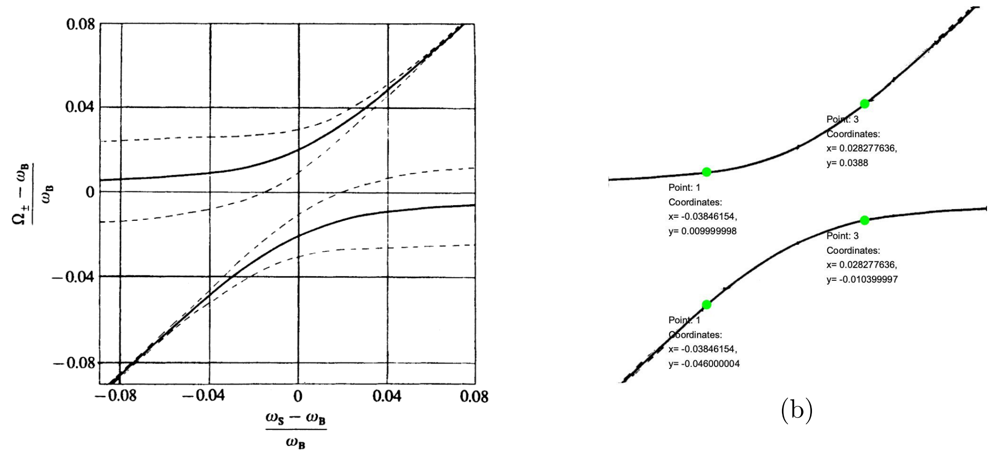
while the frequency for mode 5 remains unchanged:

$$f_5 = 559.4460[Hz]$$



(a) (b)

Figure 5: Normal mode splitting as a function of the ratio of string mass m to soundboard mass M times the mode number n on the string when the resonance frequencies of the string and soundboard mode are the same. Ω_+ and Ω_- are the mode angular frequencies in the coupled system, and Q values for the soundboard appear on each curve.



(a) (b)

Figure 6: Normal mode frequencies of a string coupled to a soundboard as a function of their uncoupled frequencies for strong coupling.