NUMERICAL MODELING AND SIMULATION FOR ACOUSTICS
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HOMEWORK 1

Consider the following wave propagation problem (Webster's equation) in a longitudinal domain having cross section S, cf. Figure 1:

$$\begin{cases} S(x)\psi_{tt}(x,t) - \gamma^2(S(x)\psi_x(x,t))_x = f(x,t) & (x,t) \in (0,1) \times (0,T], \\ \psi(x,0) = u_0(x) & x \in (0,1), \\ \psi_t(x,0) = v_0(x) & x \in (0,1), \\ \psi(0,t) = g(t) & t \in (0,T], \\ \psi_x(1,t) = 0 & t \in (0,T], \end{cases} \tag{1}$$

where f, u_0, v_0, g are given regular functions and $\gamma = c/L$ is a positive constant, with c representing the wave velocity inside the medium and L its length.

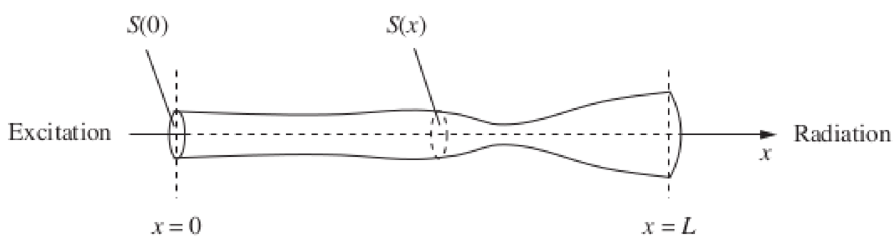

Figure 1: Example of a 1D acoustic tube, with excitation at the left end, and radiated sound produced at the right end.

1. Write the weak formulation for problem (1). State precisely the functional spaces where the problem is formulated.

2. Write the Galerkin formulation of the problem at the previous point and its algebraic formulation in the form
$$M\ddot{\boldsymbol{\psi}}(t) + A\boldsymbol{\psi}(t) = \mathbf{F}(t), \tag{2}$$
together with initial conditions $\boldsymbol{\psi}(0) = \boldsymbol{u}_0$ and $\dot{\boldsymbol{\psi}}(0) = \mathbf{v}_0$. Consider a linear finite element space discretization. Define precisely the entries of the matrices M and A and of the right hand side $\mathbf{F}$.

3. Implement in Matlab a finite element solver for problem (1) For the time integration of (2) consider the folling scheme:
$$\begin{cases} (M + \Delta t^2 A)\boldsymbol{\psi}(t^{n+1}) = \Delta t^2 \mathbf{F}(t^{n+1}) + 2M\boldsymbol{\psi}(t^n) - M\boldsymbol{\psi}(t^{n-1}), & n \geq 1 \\ (M + \frac{\Delta t^2}{2} A)\boldsymbol{\psi}(t^1) = \frac{\Delta t^2}{2}\mathbf{F}(t^1) + M\boldsymbol{u}_0 + \Delta t M\boldsymbol{v}_0. \end{cases}$$

   Consider the following section profiles:

a) $S(x) = 1$,

b) $S(x) = (1 + 2x)^2$,

c) $S(x) = 1 - \frac{3}{4}\sin(5\pi x)$.

Verify your implementation by considering $L = 1$, $T = 1$ and $c = 1$ and by computing the remaining data (i.e., f, u_0, v_0 and g) considering as exact solution $\psi_{ex}(x,t) = \cos(2\pi x)\sin(2\pi t)$. Report the behaviour of the norm $\|\psi - \psi_h\|_{L^2(0,L)}$ at the final observation time for different choiche of the discretization paramters h and Δt. Comment on the results.

4. Solve problem (1) by considering the following data: $L = 1$, $c = 2$, $f = u_0 = v_0 = 0$ while

$$g(t) = \begin{cases} \frac{1}{2}(1 + \cos(\pi(t - 0.2)/0.1)) & \text{if } |t - 0.2| \leq 0.1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

For the different parametrizations of S in $a) - c)$ report in a space-time plot the computed solution ψ_h and of the computed pressure field $p_h = \frac{1}{\gamma}(\psi_h)_t$. Choose properly the discretization parameters h and Δt. Comment on the results.

5. Consider now problem (1) where the last boundary condition is replaced by the following one, representing an open end with inertia and loss,

$$\psi_x(1,t) = -\alpha_1\psi_t(1,t) - \alpha_2\psi(1,t),$$

where

$$\alpha_1 = \frac{1}{2(0.8216)^2\gamma}, \quad \text{and} \quad \alpha_2 = \frac{L}{0.8216\sqrt{S(0)S(1)/\pi}}.$$

Repeat points 1,2 and 4. Comment on the results.

6. (Difficult). Consider the Webster's equation in the space-frequency domain

$$\begin{cases} -\omega^2 S(x)\widehat{\psi}(x) - \gamma^2(S(x)\widehat{\psi}_x(x))_x = 0 & x \in (0,1), \\ \widehat{\psi}(0) = 0, \\ \widehat{\psi}_x(1) = 0. \end{cases}$$

Compute the resonant modes by choosing $\omega \in (10^{-1}, 5 \cdot 10^3)$ for different definition of S in $a) - c)$. Report the maximum value of $|\widehat{\psi}|$ as a function of ω.