NUMERICAL MODELING AND SIMULATION FOR ACOUSTICS
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HOMEWORK 2

Consider the following wave propagation problem:

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t) & (x,t) \in (0,L) \times (0,T], \\ u(x,0) = u_0(x) & x \in (0,L), \\ u_t(x,0) = v_0(x) & x \in (0,L), \\ u(0,t) = g_1(t) & t \in (0,T], \\ c^2 u_x(1,t) = g_2(t) & t \in (0,T], \end{cases} \tag{1}$$

where u_0, v_0, g_1 and g_2 are given regular functions, and c represents the (constant) wave velocity inside the medium.

1. Consider the spectral element formulation for problem (1). Show that it leads to the following system
$$M\ddot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) = \mathbf{F}(t), \tag{2}$$
together with initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and $\dot{\boldsymbol{u}}(0) = \mathbf{v}_0$. Define precisely the entries of the matrices M and A and of the right hand side $\mathbf{F}$.

2. Implement in Matlab a spectral element solver for problem (1), by considering the leap-frog scheme as time integrator for (2). Verify your implementation on a test problem for which you know the exact solution u. Report the behaviour of the norm $\|u - u_h\|_{L^2(0,L)}$ at the final observation time for different choiches of the discretization parameters, h, Δt and r (polynomial degree). Comment on the results.

3. Write problem (1) as a first order hyperbolic system. Discretize the latter by using different finite difference schemes such as the Lax-Friedrichs and the Lax-Wendroff methods applied to the systems of characteristic variables. State precisely the boundary conditions and their numerical treatment.

4. Verify your implementation at the previous point on a test problem for which you know the exact solution u. Report the behaviour of the norm $\|u - u_h\|_{\Delta,2}$ at the final observation time for different choiches of h and Δt. Comment on the results.

5. Compare the method at points 1-2 with the Lax-Wendroff method at point 3 for solving (1) with the following data: $L = 1$, $c = 2$, $T = 1$, $f = u_0 = v_0 = g_2 = 0$ while
$$g_1(t) = \begin{cases} \frac{1}{2}(1 + \cos(\pi(t - 0.1)/0.05)) & \text{if } |t - 0.1| \le 0.05, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$
Use the same choiche of the discretization parameters h and Δt for both methods. For the spectral element method consider polynomial degree $r = 2$ and $r = 3$. Report the results by varying h and Δt and comment on them.