Numerical Modelling and Simulation for Acoustics

Homework assignments reports

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1 Homework 1

1.1 Weak formulation of the problem

The aim of the following document is to solve the propagation problem that is described below:

$$\begin{cases} S(x)\psi_{tt} - \gamma^2[S(x)\psi_x(x,t)]_x = f(x,t) & (x,t) \in (0,1)\times(0,T], \\ \psi(x,0) = u_0(x) & x \in (0,1), \\ \psi_t(x,0) = v_0(x) & x \in (0,1), \\ \psi(0,t) = g(t) & t \in (0,T], \\ \psi_x(1,t) = 0 & t \in (0,T], \end{cases} \tag{1}$$

This equation is known as Webster's equation, where $\psi(x,t)$ is the so called *acoustic potential* f, u_0, v_0, g are given regular functions and $\gamma = c/L$ is a positive constant. c represents the wave propagation velocity and L the length of the medium (acoustic tube with cross-section $S(x)$). In the following sections we are going to solve the above mentioned problem using the Finite Element Method (FEM) employing different boundary conditions.

1.1.1 Weak Formulation

To start, we introduce a lifting function $R(x,t)$ that help us to satisfy the boundary conditions given by $\psi(0,t) = g(t)$ (non-homogeneous Dirichlet condition) and $\psi(1,t) = 0$ (homogeneous Neumann condition). So we obtain:

$$\begin{cases} R(0,t) = g(t) \\ R_x(L,t) = 0 \end{cases}$$

Where the acoustic potential $\psi(x,t)$ becomes:

$$\psi(x,t) = w(x,t) + R(x,t)$$

Our problem can be stated as:

$$\begin{cases} S(x)w_{tt} - \gamma^2[S(x)w_x(x,t)]_x = f(x,t) - S(x)R_{tt}(x,t) + \gamma^2[S(x)R(x,t)] & (x,t) \in (0,1)\times(0,T], \\ w(x,0) = w_0(x) & x \in (0,1), \\ w_t(x,0) = \dot{w}_0(x) & x \in (0,1), \\ w(0,t) = g(t) & t \in (0,T], \\ w_x(1,t) = 0 & t \in (0,T], \end{cases} \tag{2}$$

We search for a solution w in the Sobolev space defined as:

$$V \equiv H^1_\star(0,1) = \{\upsilon \in (0,1) \subseteq \mathbb{R} \to \mathbb{R}, \upsilon, \upsilon_x \in L^2(0,1), \upsilon(0) = 0\}$$

In the definition of the Sobolev space, $L^2(0,1)$ refers to the Lesbegue function space with norm:

$$||\upsilon||^2_{L^2(0,1)} = \sqrt{\int_0^1 \upsilon^2(x)dx}, \quad \forall \upsilon \in L^2(0,1)$$

So the norm of $H^1_\star(0,1)$ is:

$$||\upsilon||^2_{H^1_\star(0,1)} = ||\upsilon||^2_{L^2(0,1)} + ||\upsilon_x||^2_{L^2}(0,1), \quad \forall \upsilon \in H^1_\star(0,1)$$

To move forward with *weak formulation*, for the following steps we are multiplying the equation of the propagation problem by a test function $\upsilon \in V$ and then we should integrate over the space domain Ω, remembering that $L = 1$. This allows us to move from a second-order differential problem to a first-order integral one (search for solutions belonging to $C^1([0,1])$). The above mentioned integrals are performed as it follows:

$$\int_0^1 S(x)w_{tt}(x,t)\upsilon dx \underline{- \gamma^2 \int_0^1 (S(x)w_x(x,t))_x \upsilon dx} =$$

$$\int_0^1 f(x,t)\upsilon dx - \int_0^1 S(x)R_{tt}(x,t)\upsilon dx \underline{+ \gamma^2 \int_0^1 (S(x)R_x(x,t))_x \upsilon dx}$$

- Integrating by parts the underlined integral on the left side of the equation:

$$-\gamma^2 \int_0^1 (S(x)w_x(x,t))_x v dx = \gamma^2 \left[\int_0^L (S(x)w_x(x,t))v_x dx - [S(L)w_x(L,t)v(L,t) - S(0)w_x(0,t)v(0,t)] \right]$$

  The expression can be simplified recalling the Neumann boundary condition $w_x(L,t) = 0$, then $S(L)w_x(L,t)v(L,t) = 0$. In addition, $v(0,t) = 0$, so it makes $S(0)w_x(0,t)v(0,t) = 0$

- Integrating by parts the underlined integral on the right side of the equation:

$$\gamma^2 \int_0^L (S(x)R_x(x,t))_x v dx = \gamma^2 \left[[S(L)R_x(L,t)v(L,t) - S(0)R_x(0,t)v(0,t)] - \int_0^L S(x)R_x(x,t)v_x dx \right]$$

  This time the expression can be simplified recalling the definition of the lifting function $R_x(L,t) = 0$, then $S(L)R_x(L,t)v(L,t) = 0$. Again $v(0,t) = 0$, so it makes $S(0)R_x(0,t)v(0,t) = 0$

Taking into account what was developed, the *weak formulation* can be stated as:
Find $w \in V$ *such that* $\forall t \in (0,T]$ *and* $\forall v \in V$

$$\int_0^L S w_{tt} v dx + \gamma^2 \int_0^L S w_x v_x dx = \int_0^L f v dx - \int_0^L S R_{tt} v dx - \gamma^2 \int_0^L S R_x v_x dx$$

and such that:

$$\begin{cases} w(x,0) = w_0(x) & x \in (0,1), \\ w_t(x,0) = \dot{w}_0(x) & x \in (0,1) \end{cases} \tag{3}$$

Moreover, the linear functional F and bilinear forms m and a for the weak formulation can be done by introducing:

- m:

$$V(0,L) \times V(0,L) \to \mathbb{R} \quad s.t \quad m(u,v) = \int_0^1 S u v dx$$

- a:

$$V(0,L) \times V(0,L) \to \mathbb{R} \quad s.t \quad a(u,v) = \int_0^1 \gamma^2 S u_x v_x dx$$

- F:

$$V(0,L) \to \mathbb{R} \quad s.t \quad F(v) = \int_0^1 f v dx$$

The weak formulation can be expressed as:
Find $w \in V$ *such that* $\forall t \in (0,T]$ *and* $\forall v \in V$

$$m(w_{tt}, v) + a(w,v) = F(v) - m(R,v) - a(R,v)$$

and such that:

$$\begin{cases} w(0,t) = w(0,t) & x \in (0,1), \\ w_t(0,t) = \dot{w}_0(x) & x \in (0,1) \end{cases} \tag{4}$$

This weak formulation exists and it is unique if $a(\cdot,\cdot)$ is coercive and continuous and the linear functional F is continuous (Lax-Milgram Lemma)

1.2 Galerkin and Algebraic Formulation

For the derivation of the Galerkin formulation, we choose the trial subspace V_h consisting of continuous piece-wise linear functions (of first order $\mathbb{P}^1$) defined in the interval [0, 1] within the real numbers. The subspace is constructed based on a partition (mesh) $\mathcal{T}_h$ of [0, 1], and its dimension is represented by N_h. Let us define the said space:

$$V_h = \mathsf{X}^1_{h,\star} = \{v \in \mathsf{C}^0(0, L) : v|_{k_j} \in \mathbb{P}^1 \quad \forall k_j \in \mathcal{T}_h, \quad v(0) = 0\} \tag{5}$$

The trial subspace V_h is composed by the set of basis function $\varphi_j(x)$ with $j = 0, 1, 2, ..., N_h$. The said linear basis functions look as in the following graph:

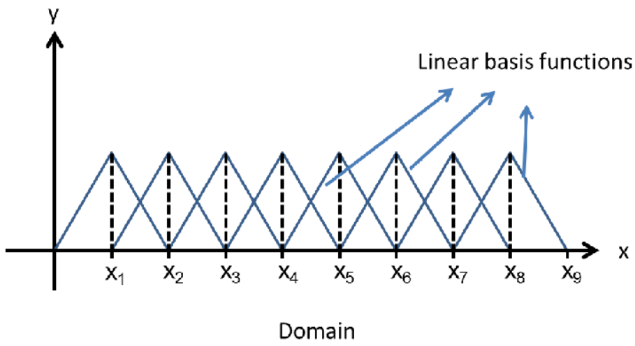


Figure 1: Linear basis functions $\varphi(x)$ of V_h associated to nodes x_1 to x_9

For the basis of V_h it is worth noting that the basis function $\varphi_0(x)$ may not be necessary to describe the elements of the trial subspace $\mathsf{X}^1_{h,*}$, and thus they can be omitted. However, all $N_h + 1$ basis elements are still considered. Therefore, any $u_h \in V_h$ can be expressed as a combination of these basis functions as can be seen here below.

$$u_h(x, t) = \sum_{j=0}^{N_h} u_j(t)\varphi_j(x) \tag{6}$$

$$\{\varphi_i\}_{i=1}^{N-1}$$

$$\begin{cases} u_h(x, t) = \sum_{j=0}^{N_h} u_j(t)\varphi(x), & \forall t(0, T] \\ u_{h,t}(x, t) = \frac{\partial u_h(x,t)}{\partial t} = \sum_{j=0}^{N_h} \dot{u}_j(t)\varphi_j(x), & \forall t \in (0, T] \\ u_{h,tt}(x, t) = \frac{\partial^2 u_h(x,t)}{\partial t^2} = \sum_{j=O}^{N_h} \ddot{u}_j(t)\varphi_j(x), & \forall t \in (0, T] \end{cases} \tag{7}$$

1.2.1 Garlerkin formulation:

It is possible to state the Garlekin formulation of this problem by limiting the *weak formulation* done in 4 to the trial subspace V_h.
For any $t \in (0, T]$ find $w_h(x, t) \in V_h$ such that:

$$m(w_{h,tt}, v_h) + a(w_h, v_h) = F(v_h) - m(R_{h,tt}, v) - a(R_h, v_h)\forall v_h \in V_h$$

and such that

$$\begin{cases} w_h(x, 0) = w_0(x) & x \in (0, 1), \\ w_{h,t}(x, 0) = \dot{w}_0(x) & x \in (0, 1) \end{cases} \tag{8}$$

Where w_h is the trial solution, v_h is any test function and the lifting function R is considered as a piece-wise expression of the lifting function. The above formulation is valid $\forall v_h \in V_h$. We can choose $v_h \equiv \varphi_i$ and by considering the superposition 6 we can write:

$$m(w_{w,tt}, v_h) = m\left(\sum_{i=0}^{N_h} \ddot{w}_j(t)\varphi_j(x), \varphi_i(x)\right)$$

$m(\cdot,\cdot)$ is a bilinear form

$$
\begin{aligned}
m(w_{w,tt}, v_h) &= \sum_{j=0}^{N_h} \ddot{w}_{j,t} \cdot m(\varphi_j(x), \varphi_i(x)) \\
&= \sum_{j=0}^{N_h} m_{i,j} \cdot \ddot{w}_j(t)
\end{aligned}
$$

with

$$m_{i,j} := m(\varphi_j(x), \varphi_i(x))$$

The same reasoning goes for $a(\cdot,\cdot)$

$$a(w_h, v_h) = \sum_{j=0}^{N_h} a_{i,j} \cdot w_j(t)$$

and for $F(\cdot)$

$$F(v_h) = F(\varphi_i) =: F_i(t)$$

Now for the lifting function R

$$m(R_{h,tt}, v_h) = \sum_{j=0}^{N_h} m_{i,j} \cdot \ddot{R}_j(t)$$

$$a(R_h, v_h) = \sum_{j=0}^{N_h} a_{i,j} \cdot R_j(t)$$

1.2.2 Algebraic formulation:

Introducing vectors $\underline{w}(t)$, $\underline{R}(t)$, $\underline{F}(t)$ and the respective time derivatives as:

$$\underline{w}(t) = \{w_0(t), ..., w_{N_h}(t)\}^T \qquad \underline{\dot{w}}(t) = \{\dot{w}_0(t), ..., \dot{w}_{N_h}(t)\}^T$$

$$\underline{\ddot{w}}(t) = \{\ddot{w}_0(t), ..., \ddot{w}_{N_h}(t)\}^T$$

$$\underline{R}(t) = \{R_0(t), ..., R_{N_h}(t)\}^T \qquad \underline{\dot{R}}(t) = \{\dot{R}_0(t), ..., \dot{R}_{N_h}(t)\}^T$$

$$\underline{\ddot{R}}(t) = \{\ddot{R}_0(t), ..., \ddot{R}_{N_h}(t)\}^T$$

$$\underline{F}(t) = \{F_0(t), ..., F_{N_h}(t)\}^T$$

And $(N_h+1) \times (N_h+1)$ matrices:

$$M = [m_{i,j}]_{i,j=0}^{N_h} = \left[\int_0^L S\varphi_i\varphi_j dx \right]_{i,j=0}^{N_h}$$

$$A = [a_{i,j}]_{i,j=0}^{N_h} = \left[\int_0^L \gamma^2 S \frac{\partial \varphi_i}{\partial x}\frac{\partial \varphi_j}{\partial x} dx \right]_{i,j=0}^{N_h}$$

The $Galerkin formulation$ can be expressed as an algebraic vectorial one or as a finite element formulation as it follows:
Find $\underline{w}(t) \in \mathbb{R}^{N_h+1}$ *such that* $\forall t \in (0,T]$

$$[M]\underline{\ddot{w}}(t) + [A]\underline{w}(t) = \underline{F}(t) - [M]\underline{\ddot{R}}(t) - [A]\underline{R}(t)$$

and such that

$$\begin{cases} \underline{w}(0) = \underline{w}_0 = \{w_0(0), ..., w_N(0)\}^T \\ \underline{\dot{w}}(0) = \underline{\dot{w}}_0 = \{w_1(0), ..., w_N(0)\}^T \end{cases} \tag{9}$$

Where $\underline{w}_0$ and $\underline{\dot{w}}_0$ are vectors containing the projections of the initial conditions $w_0(x)$ and $\dot{w}_0(t)$ into the trial subspace $\mathsf{X}^1_{h,*}$. Moreover, the mass matrix $[M]$ and the stiffness matrix $[A]$ are symmetric, defined positive and tridiagonal, i.e, they have nonzero elements only in the main diagonal and on the adjacent elements of the said diagonal thanks the properties of the basis functions.

1.3 Matlab finite element solver

The previous finite element formulation allow us to discretize the Webster's equation in space. Now what we need is to perform a time discretization. To do so, we choose a temporal step $\triangle t$ for the time interval $[0, T)$:

$$t_k = k\triangle t \qquad\qquad k = 0, ..., M$$

By doing so, the trial solution $\underline{w}(t)$ at time $t_k \in [0, T)$ is $\underline{w}_k = \underline{w}(t_k)$ and using the terms $\underline{w}_k$ and $\underline{w}_{k+1}$ it is possible to approximate $\underline{\ddot{w}}(t)$. The following integration scheme allows us to compute the finite element formulation 9

$$\begin{cases} \left(M + \frac{\triangle t^2}{2}A\right)\underline{w}_1 = \frac{\triangle t^2}{2}\underline{F}_1 + M\underline{w}_0 + \triangle t M\underline{\dot{w}}_0, & k = 0 \\ \left(M + \triangle t^2 A\right)\underline{w}_{k+1} = \triangle t^2\underline{F}_{k+1} + 2M\underline{w}_k - M\underline{w}_{k-1}, & k \geq 1 \end{cases} \tag{10}$$

This scheme is different for $k = 0$ to make possible to apply correctly the initial conditions. On regards the integration scheme stability we should refer to the Courant-Friedrichs-Lewy (CFL) conditions which tell us that the scheme is conditionally stable if and only if:

$$\triangle t \leq C\frac{h}{c}$$

Where h refers again to the mesh dimension, c is the propagation speed and C is a constant greater than 0.

1.3.1 Exact solution for different cases

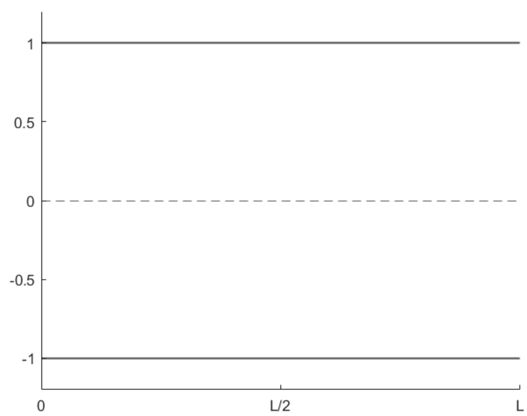
We need to set proper values for f, u_0 and υ_0. We do so by starting from the exact solution $\psi_{EX}(x, t)$ and then derive the other expressions:

$$\psi_{EX}(x, t) = cos(2\pi x)sin(2\pi t) \qquad (x, t) \in (0, L) \times (0, T]$$

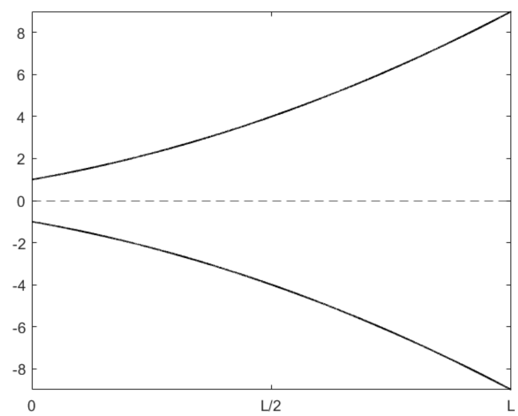
$$\frac{\partial \psi_{EX}}{\partial t} = 2\pi cos(2\pi x)cos(2\pi t) = \psi_{EX,t}$$

$$\begin{cases} \psi_{EX}(x, 0) = 0 = u_0(x) \\ \psi_{EX,t}(x, 0) = 2\pi cos(2\pi x) = \upsilon_0(x) \\ \psi_{EX}(0, t) = sin(2\pi t) = g(t) \end{cases} \tag{11}$$

The profiles we are going to test can be seen here below:



(a) Case a: $S(x) = 1$



(b) Case b: $S(x) = (1 + 2x)^2$

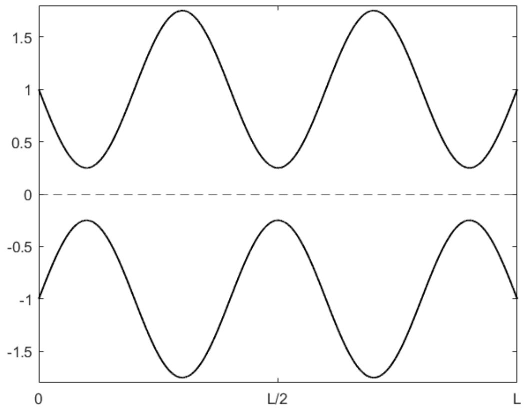


Figure 3: Case c: $S(x) = 1 - \frac{3}{4}sin(5\pi x)$

The accuracy of the implementation for the three different profiles is tested by means of the norm $||\psi - \psi_{EX}||_{L^2(0,1)}$ and $||\psi - \psi_{EX}||_{H^1_\star(0,1)}$ with respect to the mesh dimension (h) and different time integration steps ($\triangle t = 1 \times 10^{-3}$ and $\triangle t = 5 \times 10^{-4}$).

- Case a: $S(x) = 1$

  For this case $f(x,t) = 0$

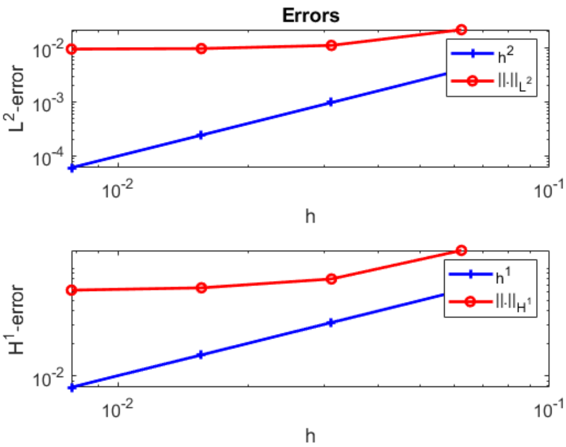


Figure 4: Case a: $\triangle t = 1x10^{-4}$

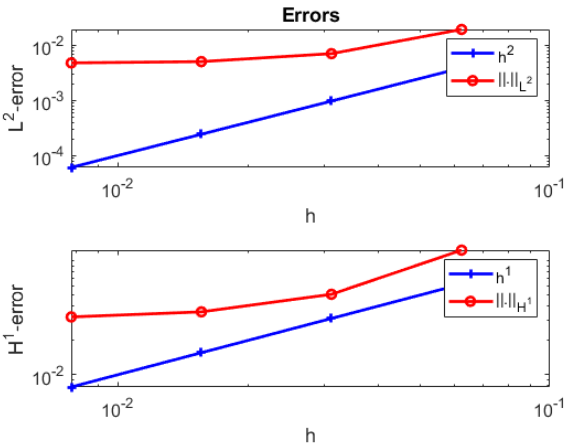


Figure 5: Case a: $\triangle t = 5x10^{-4}$

- Case b: $S(x) = (1 + 2x)^2$

  For this case $f(x,t) = (8x + 4)(2\pi sin(2\pi x)sin(2\pi t))$

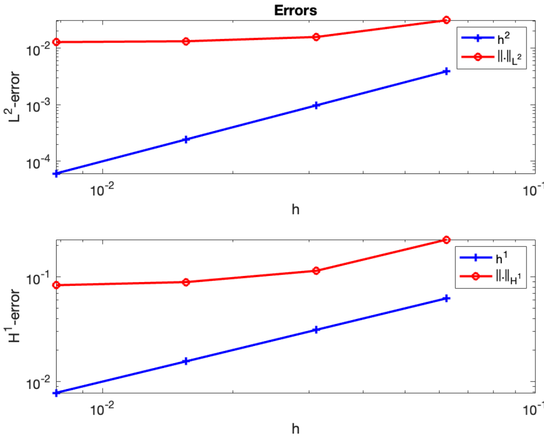


Figure 6: Case b: $\triangle t = 1x10^{-4}$

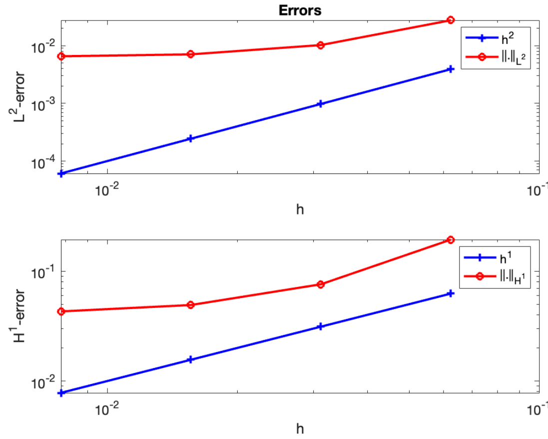


Figure 7: Case b: $\triangle t = 5x10^{-4}$

- Case c: $S(x) = 1 - \frac{3}{4}sin(5\pi x)$

  For this case $f(x,t) = -\frac{15}{4}\pi cos(5\pi x)(2\pi sin(2\pi x)sin(2\pi t))$

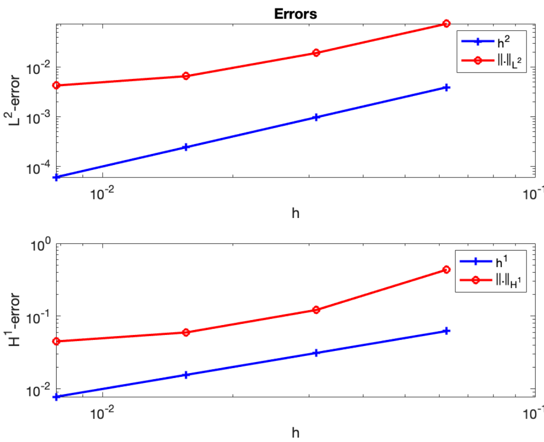


Figure 8: Case c: $\triangle t = 1x10^{-4}$

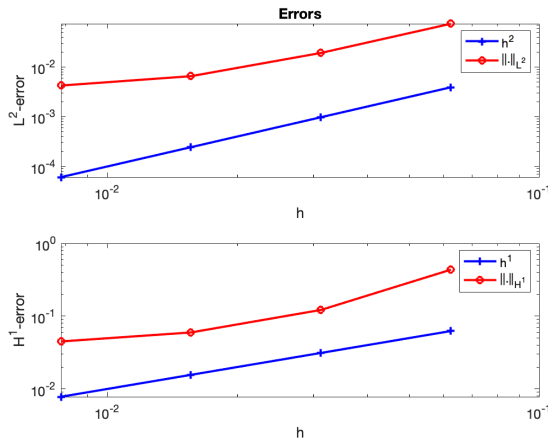


Figure 9: Case c: $\triangle t = 5x10^{-4}$

1.3.2 Comments on verification tests

Overall, the trends we observed in all the tests for each section profile confirms the validity of the performed system's discretization, since they converge in a good extent to the exact solution. However, we also noticed that for lower mesh size h the error fails to converge at the same rate as for higher sizes, in that cases the suggestion is to change the integration step $\triangle t$ for a lower value. This would make the models to keep convergence rate even for lower h values, yet increasing the computation time.

1.4 Matlab implementation with different data

Considering the following data: $L = 1$, $c = 2$, $f = u_0 = v_0 = 0$ while

$$g = \begin{cases} \frac{1}{2}(1 + cos(\pi(t-0.2)/0.1)) & |t-0.2| \leq 0.1, \\ 0 & otherwise \end{cases} \tag{12}$$

The expression above corresponds to a half cosine pulse with unitary amplitude and width 0.2. The following figures depict the computed solution for different profiles presented in the previous section in terms is acoustic potential $\psi(x,t)$ and the pressure field p_h when the system is excited by an impulse-like function $g(t)$ applied to the left-hand side of its space domain. For all tests, the values for mesh size N_h and the temporal step $\triangle t$ were fixed to 2^7 and $1 \times 10^{-3}s$ respectively.

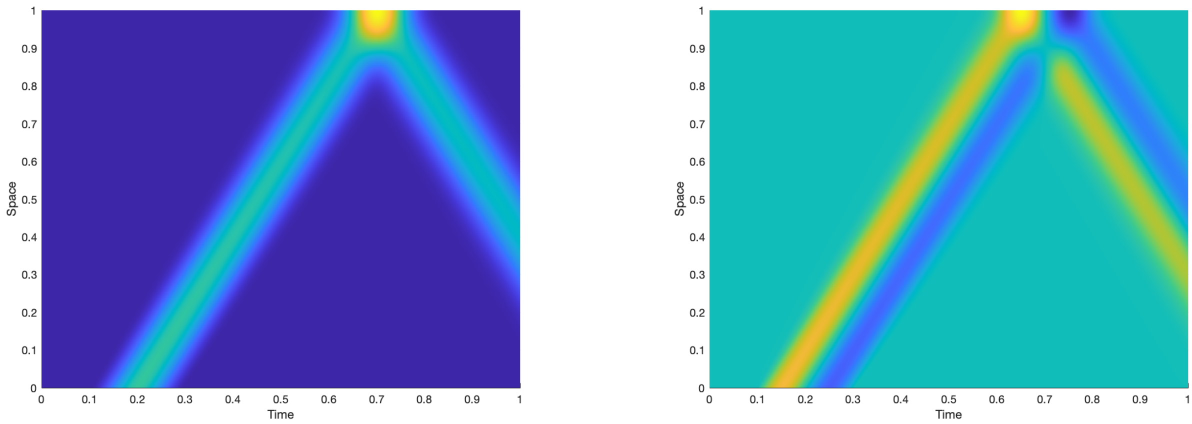


Figure 10: Case a: $S(x) = 1$

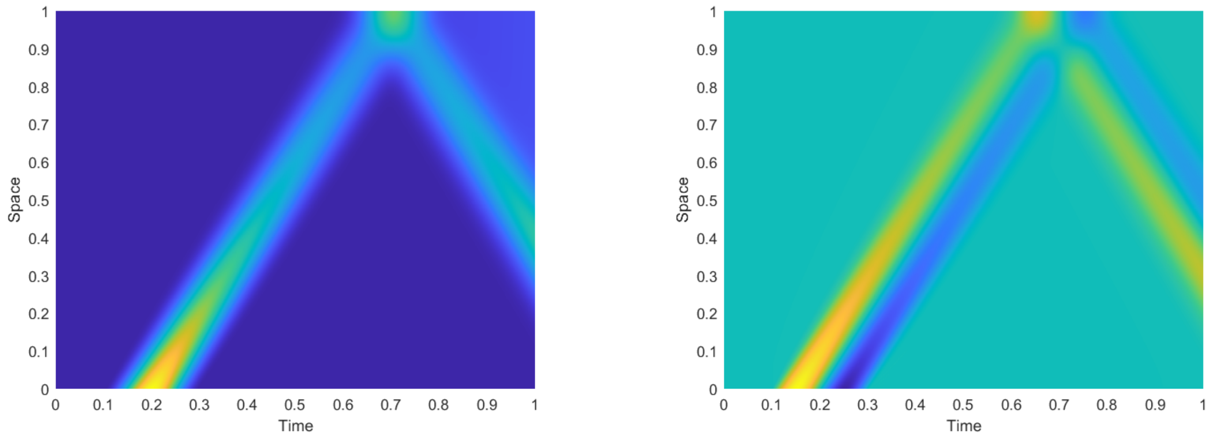


Figure 11: Case b: $S(x) = (1 + 2x)^2$

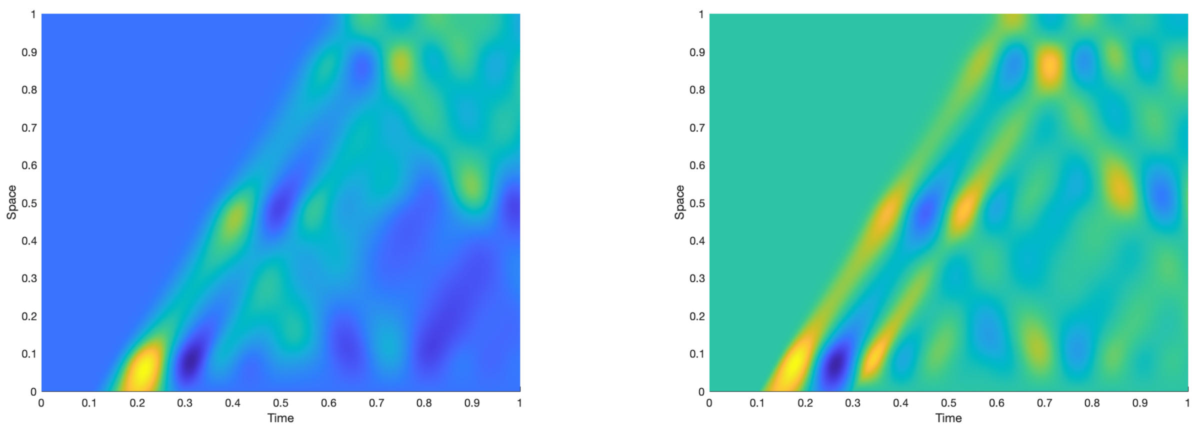


Figure 12: Case c: $S(x) = 1 - \frac{3}{4}sin(5\pi x)$

In the figures above, the ones on the left corresponds to the behaviour of the acoustic potential $\psi(x, t)$ and on the right it is possible to appreciate the behaviour of the pressure $p(x, t)$. On regards the first profile we can appreciate that the wave propagates without disturbance until it reaches the boundary on the right-most side of the domain and it is reflected without losses since there is no energy transmitted to the outside of the domain. For the second one, since the cross section of this profile changes as a function of the space (it gets wider) it is possible to appreciate an attenuation in the progressive wave when it is reflected, since for the regressive wave the cross-section decreases meaning that some energy is stored in the horn (as it can be seen on the right top corner of the figure 11) . For the last case, the particular shape of this domain causes reflection at each minima of the section, so what we get is that the energy is not concentrated at a spatial point for each time instant, but it is distributed in the domain as it is diffused at each reflection.

1.5 Propagation problem with absorbing boundary condition

The propagation problem stated in the first section 1 becomes:

$$\begin{cases} S(x)\psi_{tt} - \gamma^2[S(x)\psi_x(x,t)]_x = f(x,t) & (x,t) \in (0,1)\times(0,T], \\ \psi(x,0) = u_0(x) & x \in (0,1), \\ \psi_t(x,0) = v_0(x) & x \in (0,1), \\ \psi(0,t) = g(t) & t \in (0,T], \\ \psi_x(1,t) = -\alpha_1\psi_t(1,t) - \alpha_2\psi(1,t) & t \in (0,T] \end{cases} \tag{13}$$

The expressions for α_1 and α_2 are:

$$\alpha_1 = \frac{1}{2(0.8216)^2\gamma} \qquad \alpha_2 = \frac{1}{0.8216\sqrt{S(0)S(1)/\pi}}$$

These two terms indicate us that the domain at right-most side ($x = 1$) presents Robin and absorption conditions. With this knowledge, in the present section we are aiming to solve the problem with the new boudary conditions by means of the finite element method.

1.5.1 Weak formulation

Again, as it was done in section 1.1, we should take the differential equation of our problem and multiply it by a test function υ and integrate the resulting expression by parts over the space domain. In that way, we search a solution u in the trial space V that would guaranteed the resulting equality for any υ belonging to a test space V_0. Once more, we introduce a lifting function $R(x,t)$ such that $w = \psi - R$ solves our problem:

$$\begin{cases} S(x)w_{tt} - \gamma^2[S(x)w_x(x,t)]_x = f(x,t) - S(x)R_{tt}(x,t) + \gamma^2[S(x)R(x,t)] & (x,t) \in (0,1)\times(0,T], \\ w(x,0) = w_0(x) & x \in (0,1), \\ w_t(x,0) = \dot{w}_0(x) & x \in (0,1), \\ w(0,t) = 0 & t \in (0,T], \\ \psi_x(1,t) = -\alpha_1 w_t(1,t) - \alpha_2 w(1,t) & t \in (0,T] \end{cases} \tag{14}$$

We have the conditions that $w(0,t) = \psi(0,t) - R(0,t)$ and $w_x = \psi_x(1,t) - R_x(1,t)$. Meaning the said lifting function R must be such that:

$$\begin{cases} R(0,t) = g(t) \\ R_x(1,t) = -\alpha_1 R_t(1,t) - \alpha_2 R(1,t) \end{cases}$$

For this case we also search the solution w within the Sobolev space mentioned in section 1.1 ($V = H^1_\star$). Putting all together, the weak formulation is:
Find $w \in V$ such that $\forall t \in (0,T]$ and $\forall \upsilon \in V$

$$\int_0^1 Sw_{tt}\upsilon dx - \underline{\gamma^2\int_0^1 (Sw_x)_x\upsilon dx} = \int_0^1 f\upsilon dx - \int_0^1 SR_{tt}\upsilon dx + \underline{\gamma^2\int_0^1 (SR_x)_x\upsilon dx}$$

$$\begin{cases} w(x,0) = w_0(x) & x \in (0,1), \\ w_t(x,0) = \dot{w}_0(x) & x \in (0,1) \end{cases} \tag{15}$$

- Integrating by parts the underlined integral on the left side of the equation:

$$-\gamma^2\int_0^1 (Sw_x)_x\upsilon dx = -[\gamma^2 Sw_x\upsilon]_0^1 + \int_0^1 \gamma Sw_x\upsilon_x dx =$$

$$-[\gamma^2 S(1)w_x(1)\upsilon(1) - \gamma^2 S(0)w_x(0)\upsilon(0)] + \int_0^1 \gamma Sw_x\upsilon_x dx$$

  The expression can be simplified recalling $\upsilon(0,t) = 0 \in V$, so it makes $S(0)w_x(0,t)\upsilon(0,t) = 0$ and the expression we get is:

$$\gamma^2 S(1)\alpha_1 w_t(1)\upsilon(1) + \gamma^2 S(1)\alpha_2 w(1)\upsilon(1) + \int_0^1 \gamma^2 Sw_x\upsilon_x dx$$

- Integrating by parts the underlined integral on the right side of the equation:

$$\gamma^2 \int_0^L (SR_x)_x \upsilon dx = [\gamma^2 SR_x \upsilon]_0^1 - \int_0^1 \gamma SR_x \upsilon_x dx =$$

$$[\gamma^2 S(1)R_x(1)\upsilon(1) - \gamma^2 S(0)R_x(0)\upsilon(0)] - \int_0^1 \gamma^2 SR_x dx$$

Again $\upsilon(0,t) = 0 \in V$, so it makes $S(0)R_x(0,t)\upsilon(0,t) = 0$. The expression is then:

$$-\gamma^2 S(1)\alpha_1 R_x(1)\upsilon(1) - \gamma^2 S(1)\alpha_2 R_x(1)\upsilon(1) - \int_0^1 \gamma SR_x \upsilon_x dx$$

Considering the integration results, we can rewrite the *weak formulation* in the following way:
Find $w \in V$ *such that* $\forall t \in (0,T]$ *and* $\forall \upsilon \in V$

$$\int_0^1 Sw_{tt}\upsilon dx + \gamma^2 \int_0^1 Sw_x \upsilon_x dx + \gamma^2 S(1)\alpha_1 w_t(1)\upsilon(1) + \gamma^2 S(1)\alpha_2 w(1)\upsilon(1) =$$

$$\int_0^L f\upsilon dx - \int_0^L SR_{tt}\upsilon dx - \gamma^2 \int_0^L SR_x \upsilon_x dx - \gamma^2 S(1)\alpha_1 R_x(1)\upsilon(1) - \gamma^2 S(1)\alpha_2 R_x(1)\upsilon(1)$$

and such that:

$$\begin{cases} w(x,0) = w_0(x) & x \in (0,1), \\ w_t(x,0) = \dot{w}_0(x) & x \in (0,1) \end{cases} \tag{16}$$

Recalling the bilinear forms and linear functional, the *weak formulation* is rewritten as:
Find $w \in V$ *such that* $\forall t \in (0,T]$ *and* $\forall \upsilon \in V$

$$m(w_{tt},\upsilon) + a(w,\upsilon) + \gamma^2 S(1)\alpha_1 w_t(1)\upsilon(1) + \gamma^2 S(1)\alpha_2 w(1)\upsilon(1) =$$

$$F(\upsilon) - m(R_{tt},\upsilon) - a(R,\upsilon) - \gamma^2 S(1)\alpha_1 R_t(1)\upsilon(1) - \gamma^2 S(1)\alpha_2 R(1)\upsilon(1)$$

and such that:

$$\begin{cases} w(x,0) = w_0(x) & x \in (0,1), \\ w_t(x,0) = \dot{w}_0(x) & x \in (0,1) \end{cases} \tag{17}$$

1.5.2 Galerkin formulation

Defining a space $V_h = \mathsf{X}^1_{h,*}$ which is an approximation of the trial space defined before as $V = H^1_\star$. It is also worth to remark that the subspace $V_h \subset V$ and it is of finite dimension N_h.
So reducing the previous *weak formulation* 17 to the trial subspace $V_h = \mathsf{X}^1_{h,*}$, the subspace of continuous functions in which the elements can be defined with a linear superposition of the basis functions $\{\varphi_j(x)\}_{j=0}^{N_h}$ as described in section 2. So the *Galerkin formulation* can be state by naming w_h as the trial solution, υ_h is any test function and the lifting function R is considered as a piece-wise expression of the lifting function:
Find $w_h \in V_h$ *such that* $\forall t \in (0,T]$ *and* $\forall \upsilon_h \in V_h$

$$m(w_{h,tt},\upsilon_h) + a(w_h,\upsilon_h) + \gamma^2 S(1)\alpha_1 w_{h,t}(1)\upsilon_h(1) + \gamma^2 S(1)\alpha_2 w_h(1)\upsilon_h(1) =$$

$$F(\upsilon_h) - m(R_{h,tt},\upsilon_h) - a(R_h,\upsilon_h) - \gamma^2 S(1)\alpha_1 R_{h,t}(1)\upsilon_h(1) - \gamma^2 S(1)\alpha_2 R_h(1)\upsilon_h(1)$$

and such that:

$$\begin{cases} w_h(x,0) = w_0(x) & x \in (0,1), \\ w_{h,t}(x,0) = \dot{w}_0(x) & x \in (0,1) \end{cases} \tag{18}$$

1.5.3 Algebraic formulation

Just as we do for the previous $Garlekin formulation$ 18 and knowing that it is valid $\forall \upsilon_h \in V_h$, we can choose $\upsilon_h \equiv \varphi_i$ and by considering the superposition 6 to describe the vectors $\underline{w}(t)$, $\underline{R}(t)$, $\underline{F}(t)$, their time derivatives and matrices $[M]$ and $[A]$ (refer to exercise 2). This time we need to consider matrices $[I]$ and $[D]$ that are in the following form:

$$[I] = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \gamma^2 S(1)\alpha_1 \end{bmatrix} \qquad [D] = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \gamma^2 S(1)\alpha_2 \end{bmatrix}$$

We can rewrite the previous formulation as an algebraic one:
Find $\underline{w_h} \in \mathbb{R}_h^{N+1}$ *such that* $\forall t \in (0,T]$

$$[M]\underline{\ddot{w}}(t) + [I]\underline{\dot{w}}(t) + ([D]+[A])\underline{w}(t) = \underline{F}(t) - [M]\underline{\ddot{R}}(t) - [I]\underline{R}(t) - ([D]+[A])\underline{R}(t)$$

and such that

$$\begin{cases} \underline{w}(0) = \underline{w}_0 = \{w_0(0), ..., w_N(0)\}^T \\ \underline{\dot{w}}(0) = \underline{\dot{w}}_0 = \{w_1(0), ..., w_N(0)\}^T \end{cases} \tag{19}$$

Where $\underline{w}_0$ and $\underline{\dot{w}}_0$ are vectors containing the projections of the initial conditions $w_0(x)$ and $\dot{w}_0(t)$ into the trial subspace $\mathsf{X}^1_{h,*}$. Naming the right-hand side of the equation as $\underline{F}_g$ in such a way that:

$$\underline{F}_g(t) = \underline{F}(t) - [M]\underline{\ddot{R}}(t) - [I]\underline{R}(t) - ([D]+[A])\underline{R}(t)$$

This can be done once the vector $\underline{F}(t)$ is computed and the lifting function vector $\underline{R}(t)$ is chosen, meaning that the right-hand side of the equation can be treated as a single $\mathbb{R}^{N_h+1}$ vector. So the final finite element formulation is this one:
Find $\underline{w_h} \in \mathbb{R}_h^{N+1}$ *such that* $\forall t \in (0,T]$

$$[M]\underline{\ddot{w}}(t) + [I]\underline{\dot{w}}(t) + ([D]+[A])\underline{w}(t) = \underline{F}_g(t)$$

and such that

$$\begin{cases} \underline{w}(0) = \underline{w}_0 \\ \underline{\dot{w}}(0) = \underline{\dot{w}}_0 \end{cases} \tag{20}$$

The actual solution is given by $\underline{\psi}(t) = \underline{w}(t) + \underline{R}(t)$*, where:*

$$\underline{\psi}(t) = \{g(t), w_1(t), ..., w_{N_h}\}^T \quad \forall t \in (0,T]$$

1.5.4 Impulsive excitation on one side of the system

Recalling the definition of $g(t)$ given in section 4, an impulse-like function made with a half cosine 12. We want to study the system's response for the different section profiles a, b and c when it its excited by $g(t)$ on one side and there are Robin and absorbing boundary conditions on the right-side. We are studying the response in terms of the acoustic potential $\psi(x,t)$ and the pressure field $p(x,t) = \frac{1}{\gamma}\psi(x,t)$. As it was done for exercise 4, for all tests the values for mesh size N_h and the temporal step $\triangle t$ were fixed to 2^{-7} and $1 \times 10^{-4}s$ respectively.

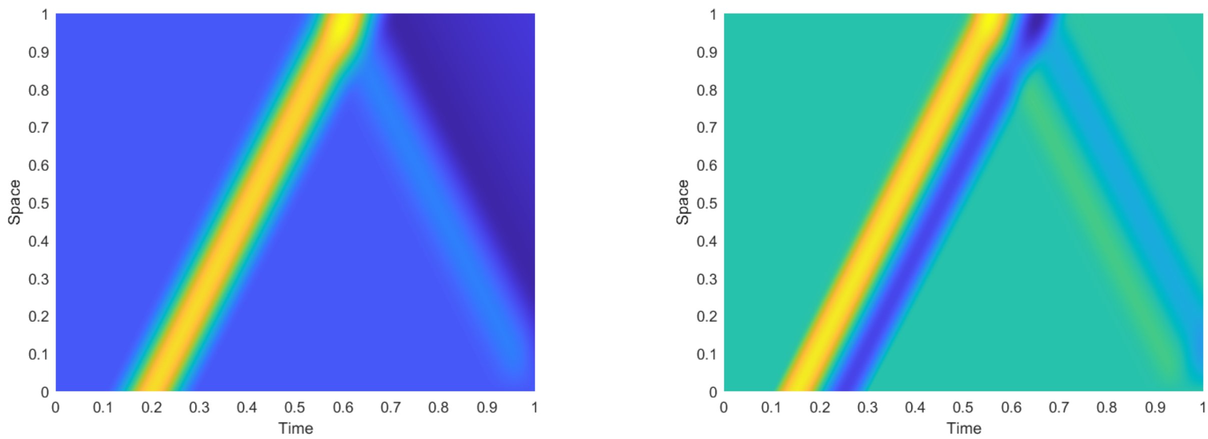


Figure 13: Case a: $S(x) = 1$

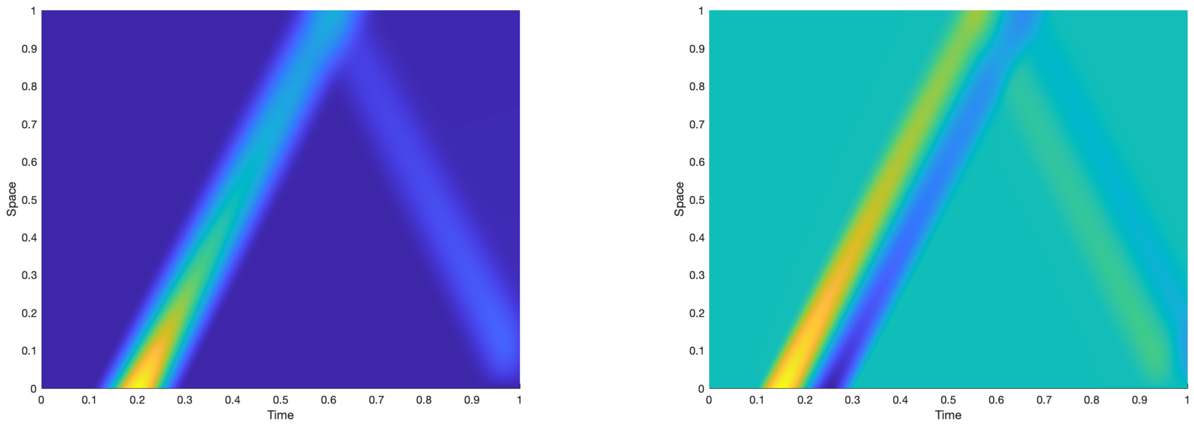


Figure 14: Case b: $S(x) = (1 + 2x)^2$

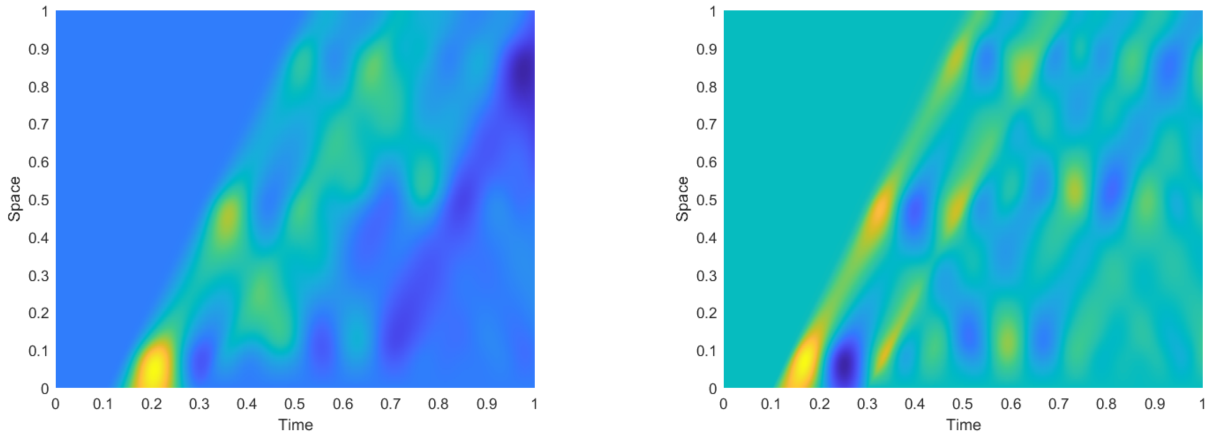


Figure 15: Case c: $S(x) = 1 - \frac{3}{4}sin(5\pi x)$

The comments we did for the results obtained in the previous section still hold true. However, this time we should also take into account the absorbing boundary present on the right side of the domain, thanks to it it is noticeable the attenuation in the regressive wave since part of the energy is transmitted outside the domain instead of being reflected back.

1.6 Webster's equation in the space-frequency domain

In the space-frequency domain the Webster's equation becomes:

$$\begin{cases} -\omega^2 S(x)\hat{\psi}(x) - \gamma^2(S(x)\hat{\psi}_x(x))_x = f(x,t) & x \in (0,1), \\ \hat{\psi}(0) = 0, \\ \hat{\psi}_x(1) = 0 \end{cases}$$

This formulation can be derived from the expression presented in 1 by considering the acoustical potential $\psi(x,t)$ and the driving force $f(x,t)$ as harmonic propagating waves with with angular frequency ω

$$\psi(x,t) = \hat{\psi}(x)e^{j\omega t} \qquad f(x,t) = \hat{f}(x)e^{j\omega t}$$

Rewriting the space-frequency equation including the complex functions for the external force and acoustic potential:

$$-\omega^2 S(x)\hat{\psi}(x)e^{j\omega t} - \gamma^2(S(x)\hat{\psi}_x(x))_x e^{j\omega t} = \hat{f}(x,t)e^{j\omega t}$$

We can neglect the time dependence, since $e^{j\omega t} \neq 0 \; \forall t$. Also, let us call the generalized wave number as $k = \omega/\gamma$, for the case of $\hat{f}(x) = 0$ we obtain:

$$\begin{cases} k^2 S(x)\hat{\psi}(x) + [S(x)\hat{\psi}_x(x)]_x = 0 & x \in (0,1) \\ \hat{\psi}(0) = 0 \\ \hat{\psi}_x(1) = 0 \end{cases} \tag{21}$$

1.6.1 Weak formulation

To derive the weak formulation and find solutions that belong to the function space V, we follow the same approach as in previous section. This involves multiplying the equation by a test function $\upsilon \in V$ and integrating over the spatial domain $(0,1)$. The goal is to seek a solution within the function space V, which has been explained in section 1. Taking that into account, the *weak formulation* is:
Find $\hat{\psi} \in H^1_*(0,1)$ *such that*

$$\int_0^1 k^2 S\hat{\psi}\upsilon dx + \underline{\int_0^1 (S\hat{\psi}_x)_x \upsilon dx} = 0 \qquad \forall \upsilon \in H^1_*(0,1)$$

The underline term in the previous expression contains a derivate in space, so we can develop it by integration by parts:

$$\int_0^1 (S\hat{\psi}_x)_x \upsilon dx = [S\hat{\psi}_x]_0^1 - \int_0^1 S\hat{\psi}_x \upsilon_x = [S(1)\hat{\psi}_x(1)\upsilon(1) - S(0)\hat{\psi}_x(0)\upsilon(0)] - \int_0^1 S\hat{\psi}_x \upsilon_x dx = -\int_0^1 S\hat{\psi}_x \upsilon_x dx$$

From the boundary conditions presented at the beginning of this section we know that $S(1)\hat{\psi}_x(1)\upsilon(1) = 0$, also $S(0)\hat{\psi}_x(0)\upsilon(0) = 0$ since $\upsilon \in H^1_*(0,1)$ This reduces the *weak formulation* to:
Find $\hat{\psi} \in H^1_*(0,1)$ *such that*

$$\int_0^1 k^2 S\hat{\psi}\upsilon dx - \int_0^1 S\hat{\psi}_x \upsilon_x dx = 0 \quad \forall \upsilon \in H^1_*(0,1) \tag{22}$$

The problem can be also formulated recalling the bilinear forms $m(\hat{\psi},\upsilon)$, $a(\hat{\psi},\upsilon)$ and the linear form $F(\upsilon)$ (already described in section 1).
Find $\hat{\psi} \in H^1_*(0,1)$ *such that*

$$k^2 m(\hat{\psi},\upsilon) - a(\hat{\psi},\upsilon) = 0 \quad \forall \upsilon \in H^1_*(0,1) \tag{23}$$

1.6.2 Galerkin formulation

To translate our problem into a finite-dimensional discrete domain, i.e. *Galerkin formulation*, we utilize the same set of basis functions $\{\varphi(x)\}_0^{N_h}$ (see eq. 7) and the same family of functional spaces $V_h \in V$ of dimension V_h. As a result, the *Galerkin formulation* can be expressed as follows:
Find $\hat{\psi} \in \mathsf{X}^1_{h,*}$ *such that*

$$k^2 m(\hat{\psi},\upsilon_h) - a(\hat{\psi}_h,\upsilon_h) = 0, \quad \forall \upsilon_h \in \mathsf{X}^1_{h,*} \tag{24}$$

1.6.3 Algebraic formulation

The problem can be expressed in matrix form, to do so, we choose $\upsilon_h = \varphi_i$ then the acoustic potential $\hat{\psi}_h$ can be written as a linear combination of the basis function.

$$\hat{\psi}_h(x) = \sum_{j=0}^{N_h} \psi_j \varphi_j(x) \tag{25}$$

$$m(\hat{\psi}_h,\upsilon_h) = m\left(\sum_{j=0}^{N} \psi_j \varphi_j(x), \varphi_i(x)\right)$$

$m(\cdot,\cdot)$ is a bilinear form (Integral in space - function in time is a constant)

$$m(\hat{\psi}_h,\upsilon_h) = \sum_{j=0}^{N} \psi_j \cdot m(\varphi_j(x),\varphi_i(x)) \qquad ; \qquad m(\varphi_j(x),\varphi_i(x)) =: m_{i,j}$$

$$= \sum_{j=0}^{N} m_{i,j} \cdot \psi_j$$

Same for a, $a(\cdot,\cdot)$ is bilinear

$$a(\hat{\psi}_h,\upsilon_h) = \sum_{j=0}^{N_h} a_{i,j}\psi_j$$

The $\mathbb{R}^{N_h+1}$ vector is:

$$\underline{\psi} = \{\psi, ..., \psi_{N_h}\}^T$$

And the $(N_h + 1) \times (N_h + 1)$ matrices:

$$[M] = [m_{i,j}]_{i,j=0}^{N_h} = \left[\int_0^L S\varphi_i\varphi_j dx\right]_{i,j=0}^{N_h}$$

$$[A] = [a_{i,j}]_{i,j=0}^{N_h} = \left[\int_0^L \gamma^2 S\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x}dx\right]_{i,j=0}^{N_h}$$

Now it is feasible to state the *Algebraic formulation*:
Find $\underline{\psi} \in \mathbb{R}^{N_h+1}$ *such that*

$$k^2[M]\underline{\psi} - A\underline{\psi} = 0$$

1.6.4 Resonant modes

To perform the calculation of the resonant modes of the system, we should make a modal analysis of the system . To do so, we should carry out the calculation of the modal mass matrix $[M_q]$ and the modal stiffness matrix $[A_q]$ in the following way:

$$[M_q] = \underline{\phi}^T[M]\underline{\phi} \qquad [A_q] = \underline{\phi}^T[A]\underline{\phi}$$

$[M_q]$ and $[A_q]$ are diagonal by definition and $\underline{\phi}$ is the modal vector that contains the mode shapes associated to each resonant mode. Where the *i-th* element of $\underline{\phi}$ will be obtained from:

$$\phi_i = eig_i([M]^{-1}[A])$$

We will get then a set of uncoupled equations with $m_{j,j}$ and $a_{j,j}$ related to each resonant mode, so the *Frequency Response Matrix (FRM)* can be obtained by means of the modal superposition approach:

$$H_\psi(\omega) = \sum_{j=0}^{N_h} \frac{1}{-\omega^2 m_{j,j} + a_{j,j}} \tag{26}$$

In eq. 26 each resonant mode corresponds to the poles of the transfer function, i.e, the values of ω that makes the denominator goes to zero.

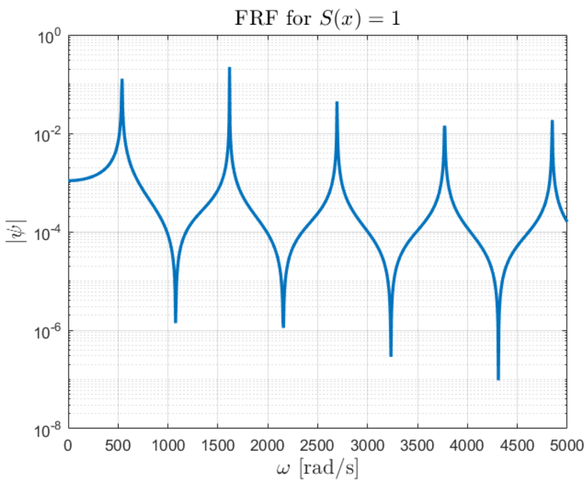


Figure 16: Case a: $S(x) = 1$

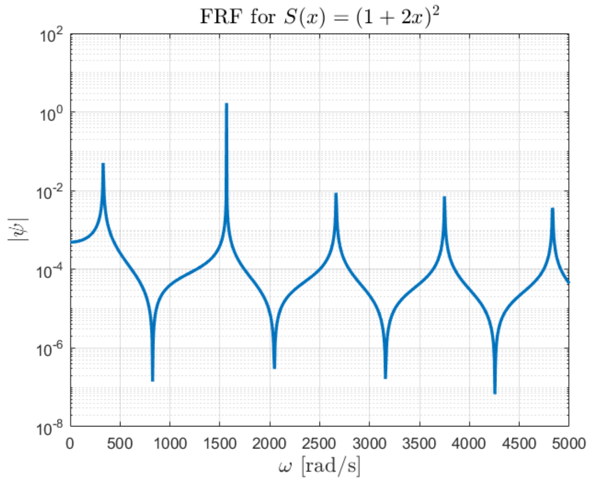


Figure 17: Case b: $S(x) = (1 + 2x)^2$

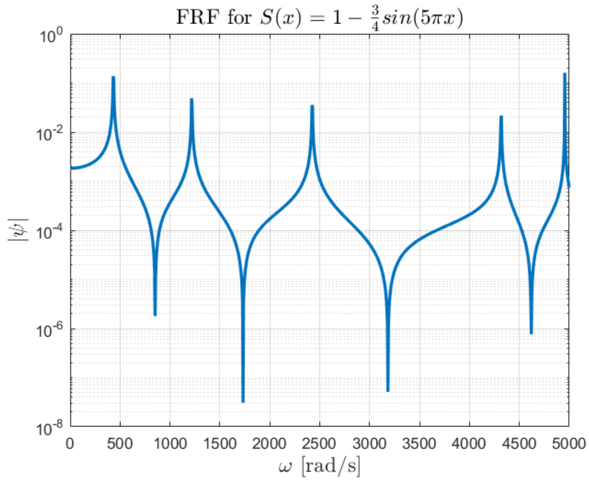


Figure 18: Case c: $S(x) = 1 - \frac{3}{4}sin(5\pi x)$

2 Homework 2

Introduction

The following wave propagation problem is proposed:

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t) & (x,t) \in (0,L) \times (0,T], \\ u(x,0) = u_0(x) & x \in (0,L), \\ u_t(x,0) = v_0(x) & x \in (0,L), \\ u(0,t) = g_1(t) & t \in (0,T], \\ c^2 u_x(1,t) = g_2(t) & t \in (0,T] \end{cases} \tag{1}$$

Where u_0, v_0, g_1 and g_2 are given regular functions, c represents the wave propagation velocity inside the medium and L is the length of the spatial domain. In the following sections we are going to solve the above mentioned problem using two methods: Spectral Element Method (SEM) and the Finite Difference Method (FDM).

2.1 Spectral Element Method formulation

The aim of this section is to show that the propagation problem introduced before 1 leads to the following system:

$$M\ddot{u}(t) + Au(t) = F(t) \tag{2}$$

Taking into account that the initial conditions are $u(0) = u_0$ and $\dot{u} = v_0$

2.1.1 Weak formulation

To start, we introduce a lifting function $R(x,t)$ that help us to satisfy the boundary conditions given by $u(0,t) = g_1(t)$ (non-homogeneous Dirichlet condition) and $c^2 u_x(L,t) = g_2(t)$ (non-homogeneous Neumann condition). So we obtain:

$$\begin{cases} R(0,t) = g_1(t) & t \in (0,T] \\ c^2 R_x(L,t) + c^2 w_x(L,T) = g_2(t) & t \in (0,T] \end{cases} \tag{3}$$

Where:

$$w(0,t) = u(0,t) - R(0,t)$$

And

$$w_x(L,t) = u_x(L,t) - R_x(L,t)$$

Our problem becomes:

$$\begin{cases} w_{tt}(x,t) - c^2 w_{xx}(x,t) = f(x,t) - R_{tt}(x,t) + c^2 R_{xx}(x,t) & (x,t) \in (0,L) \times (0,T], \\ w(x,0) = w_0(x) & x \in (0,L), \\ w_t(x,0) = \dot{w}_0(x) & x \in (0,L), \\ w(0,t) = 0 & t \in (0,T], \\ c^2 w_x(x,t) = c^2 u_x(x,t) - c^2 R_x(x,t) & t \in (0,T], \end{cases} \tag{4}$$

We search for a solution w in the Sobolev space defined as:

$$V \equiv H^1_*(0,L) = \{\upsilon \in (0,L) \subseteq \mathbb{R} \to \mathbb{R}, \upsilon, \upsilon_x \in L^2(0,L), \upsilon(0) = 0\}$$

To move forward with *weak formulation*, for the following steps we are multiplying the equation of the problem by a test function $\upsilon \in V$ and then we should integrate over the domain.

$$\int_0^L w_{tt}\upsilon dx \underline{- c^2 \int_0^L w_{xx}\upsilon dx} = \int_0^L f\upsilon dx - \int_0^L R_{tt}\upsilon dx \underline{+ c^2 \int_0^1 R_{xx}\upsilon dx}$$

- Integrating by parts the underlined integral on the left side of the equation:

$$-c^2 \int_0^L w_{xx}\upsilon dx = -c^2[w_x \upsilon]_0^L + c^2 \int_0^L w_x \upsilon_x dx$$

$$= -c^2[w_x(L)\upsilon(L) - w_x(0)\upsilon(0)] + c^2 \int_0^L w_x \upsilon_x dx$$

The expression can be simplified recalling $\upsilon(0,t) = 0 \in V$, so it makes $w_x(0,t)\upsilon(0,t) = 0$

- Integrating by parts the underlined integral on the right side of the equation:

$$c^2 \int_0^L R_{xx} \upsilon dx = -c^2 [R_x \upsilon]_0^L - c^2 \int_0^L R_x \upsilon_x dx$$

$$= -c^2 [R_x(L)\upsilon(L) - R_x(0)\upsilon(0)] + c^2 \int_0^L R_x \upsilon_x dx$$

Again $\upsilon(0,t) = 0 \in V$, so it makes $R_x(0,t)\upsilon(0,t) = 0$

Recalling the definition for $g_2(t)$, that is $g_2(t) = c^2 R_x(L,t) + c^2 w_x(L,t)$ we gave on eq.3 and taking into account what was developed, the *weak formulation* can be stated as:
Find $w \in H^1_*(0,L)$ *such that* $\forall t \in (0,T]$

$$\int_0^L w_{tt} \upsilon dx + c^2 \int_0^L w_x \upsilon_x dx = \int_0^L f \upsilon dx - \int_0^L R_{tt} \upsilon dx - c^2 \int_0^L R_x \upsilon_x dx + g_2(t)\upsilon(L) \quad \forall \upsilon \in H^1_*(0,L)$$

and such that:

$$\begin{cases} w(x,0) = w_0(x) & x \in (0,L), \\ w_t(x,0) = \dot{w_0}(x) & x \in (0,L) \end{cases} \tag{5}$$

2.1.2 Galerkin formulation

We are solving the propagation problem through the Spectral Element Method (SEM). To do so, we should start with the *Garlekin formulation*, which takes the *Weak formulation* and projects it into a finite-dimension space, that is, the trial space $V = H^1_*(0,L)$ is defined into a subspace $V_h \subset V$ and it is defined in the following way:

$$V_h = \mathsf{X}^r_{h,*}(0,L) = \{\upsilon \in \mathsf{C}^0(0,L) : \upsilon|_{\Omega_m} \in \mathbb{P}^r, r > 1, \forall m = 0, ..., M-1, \quad \upsilon(0) = 0\} \tag{6}$$

Its dimension is N_h. In this particular case, V_h consist of a set of continuous functions in $(0,L)$ which are high-order polynomials ($\mathbb{P}^r$, with $r > 1$) in each interval Ω_m, which are the intervals of the uniform mesh Ω_h of $(0,L) \subset \mathbb{R}$ with dimensions M (spatial discretization). The intervals Ω_m of the uniform mesh Ω_h are:

$$\begin{cases} \Omega_m = [x_m, x_{m+1}] = [x_m, x_m + h], \quad \forall m = 0, ..., M-1 \\ \Omega_h = \bigcup_{m=0}^{M-1} \Omega_m \end{cases} \tag{7}$$

Basis functions $\psi(x)$ are the basis functions, polynomials of the order $\mathbb{P}^r, r > 1$ in the interval $(-1,1)$ obtained after combining the Legendre polynomials $L_r(x)$ (complete and orthogonal) and the Gauss-Legendre-Lobatto nodes (GLL), which are a particular case of non-equispaced nodes, that can be expressed in terms of $L_r(x)$:

$$x_i = \in (-1,1) : \frac{\partial L_r(x)}{\partial x}\Big|_{x=x_i} = 0, \quad \forall i = 1, ..., r-1 \tag{8}$$

The basis functions $\psi_i(x)$ are then:

$$\psi_i(x) = -\frac{1}{r(r+1)} \frac{1-x^2}{x-x_i} \frac{L_r'(x)}{L_r(x_i)}, \quad x, x_i \in (-1,1) \quad and \quad \forall i = 0, ..., r \tag{9}$$

And they take the following values:

$$\begin{cases} \psi_i(x_j) = 0 & for \quad x_i \neq x_j \\ \psi_i(x_j) = 1 & for \quad x_i = x_j \end{cases}$$

This means that at the GLL nodes only one function of the basis is non-zero. In the following figure we can appreciated the mentioned basis functions for a polynomial $\mathbb{P}^2$ for $r = 2$:

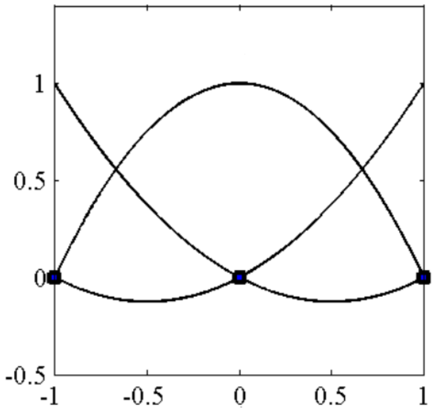


Figure 1: $\mathbb{P}^2(-1,1)$

Gauss-Lobatto quadrature formula Given the complexity of $\psi_i(x)$ there is the need of a formula that helps us to o perform the integration operation on the $(-1,1)$ domain, that, is, the Gauss-Lobatto quadrature formula.

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} \alpha_i f(x_i)$$

In the previous expression, α_i refers to the GL-weights and are calculated as:

$$\alpha_i = \int_{-1}^{1} \psi_i(x)dx = \frac{2}{i(i+1)} \frac{1}{L^2(x_i)}, \quad \forall i = 1, ..., n$$

For a generic interval we should redefine 9 in such a way that we can find the nodes x_i for a generic interval (a,b), according to the following linear transformation:

$$x = \Phi(\xi) = \frac{b-a}{2}\xi + \frac{a+b}{2} \quad \rightarrow \quad \hat{\alpha}_i = \Phi(\alpha_i) = \frac{b-a}{2}\alpha \tag{10}$$

Then, any element $u_h(x,t)$ of the trial subspace $\mathsf{X}^r_{h,*}(0,L)$ can be expressed as a superposition:

$$u_h(x,t) = \sum_{j=0}^{Nh} u_j(t)\phi_j(x) = \sum_{m=0}^{M-1} u_m(t)\psi^*_m(x) + \sum_{m=0}^{M-1}\sum_{i=1}^{r-1} u_i^{(m)}(t)\psi_i^{(m)}(x) \tag{11}$$

The terms in 11 refer to:

- $\psi^*_m(x)$ are the functions obtained that, after the transformation described in 10, are $\psi_i \neq 0$ at each mesh node x_m.
- $\psi_i^{(m)}(x)$ are the remaining basis functions $\psi_i(x)$ of the subinterval.

Final Galerkin formulation Recalling that to carry out the *Galerkin formulation* we should limit the weak formulation 5 to a trial space $V_h = \mathsf{X}^r_{h,\star}$
Find $w_h \in \mathsf{X}^r_{h,\star}$ *such that* $\forall t \in (0,T]$

$$\int_0^L w_{h,tt} v_h dx + c^2 \int_0^L w_{h,x} vh, x dx = \int_0^L f v_h dx - \int_0^L R_{h,tt} v_h dx - c^2 \int_0^L R_{h,x} v_{h,x} dx + g_2(t) v_h(L) \quad \forall v_h \in \mathsf{X}^r_{h,*}$$

And such that

$$\begin{cases} w_h(x,0) = w_0(x) & x \in (0,L), \\ w_{h,t}(x,0) = \dot{w}_0(x) & x \in (0,L) \end{cases} \tag{12}$$

Where w_h is the trial solution, v_h is any test function. The above formulation is valid $\forall v_h \in V_h$. Moreover, the linear and bilinear forms for the *Galerkin formulation* can be done by introducing:

- m:

$$V(0,L) \times V(0,L) \rightarrow \mathbb{R} \quad s.t \quad m(u,v) = \int_0^L uv dx$$

- a:

$$V(0,L) \times V(0,L) \to \mathbb{R} \quad s.t \quad a(u,\upsilon) = \int_0^L c^2 u_x \upsilon_x dx$$

- F:

$$V(0,L) \to \mathbb{R} \quad s.t \quad F(\upsilon) = \int_0^L f\upsilon dx$$

The *Galerkin formulation* can be expressed as:
Find $w_h \in V_h$ *such that* $\forall t \in (0,T]$ *and*

$$m(w_{h,tt}, \upsilon_h) + a(w_{h,x}, \upsilon_{h,x}) = F(\upsilon_h) - m(R_{h,tt}, \upsilon_h) - a(Rh, x, \upsilon_{h,x} + g_2(t)\upsilon_h(L)) \quad \upsilon_h \ in V_h$$

and such that:

$$\begin{cases} w_h(x,0) = w_0(x) & x \in (0,L), \\ w_{h,t}(x,0) = \dot{w}_0(x) & x \in (0,L) \end{cases} \tag{13}$$

Recalling eq.11 and that the just introduced *Galerkin formulation* is valid $\forall \upsilon_h \in V_h$. Making $\upsilon_h = \varphi_i$. We can rewrite w_h, R and the respective derivatives as it follows:

$$m(w_{w,tt}, \upsilon_h) = m\left(\sum_{i=0}^{N} \ddot{w}_j(t)\varphi_j(x), \varphi_i(x)\right)$$

$m(\cdot,\cdot)$ is a bilinear form

$$m(w_{w,tt}, \upsilon_h) = \sum_{j=0}^{N} \ddot{w}_{j,t} \cdot m(\varphi_j(x), \varphi_i(x)) \qquad ; \qquad m(\varphi_j(x), \varphi_i(x)) =: m_{i,j}$$

$$= \sum_{j=0}^{N} m_{i,j} \cdot \ddot{w}_j(t)$$

$$a(w_h, \upsilon_h) = \sum_{j=0}^{N} a_{i,j} \cdot w_j(t)$$

$$F(\upsilon_h) = F(\varphi_i) =: F_i(t)$$

Now for the lifting function R

$$m(R_{h,tt}, \upsilon_h) = \sum_{j=0}^{N} m_{i,j} \cdot \ddot{R}_j(t)$$

$$a(R_h, \upsilon_h) = \sum_{j=0}^{N} a_{i,j} \cdot R_j(t)$$

2.1.3 Algebraic formulation:

Introducing $\mathbb{R}^{N_h+1}$ vectors $\underline{w}(t)$, $\underline{R}(t)$ and $\underline{F}(t)$ as:

$$\underline{w}(t) = \{w_0(t), ..., w_{N_h}(t)\}^T$$

$$\underline{R}(t) = \{R_0(t), ..., R_{N_h}(t)\}^T$$

$$\underline{F}(t) = \{F_0(t), ..., F_{N_h}(t)\}^T$$

And $(N_h+1) \times (N_h+1)$ matrices:

$$M = [m_{i,j}]_{i,j=0}^{N_h} = \left[\int_0^L \varphi_i \varphi_j dx\right]_{i,j=0}^{N_h}$$

$$A = [a_{i,j}]_{i,j=0}^{N_h} = \left[\int_0^L c^2 \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial x} dx\right]_{i,j=0}^{N_h}$$

The *Galerkin formulation* can be expressed as an algebraic vectorial one or as a finite element formulation as it follows:
Find $\underline{w}(t) \in \mathbb{R}^{N_h+1}$ *such that* $\forall t \in (0,T]$

$$M\underline{\ddot{w}}(t) + A\underline{w}(t) = \underline{F}(t) - M\underline{\ddot{R}}(t) - A\underline{R}(t) + \underline{G}_2(t)$$

and such that

$$\begin{cases} \underline{w}(0) = \underline{w}_0 = \{w_0(0), ..., w_N(0)\}^T \\ \underline{\dot{w}}(0) = \underline{\dot{w}}_0 = \{w_1(0), ..., w_N(0)\}^T \end{cases} \tag{14}$$

Where $\underline{w}_0$ and $\underline{\dot{w}}_0$ are vectors containing the projections of the initial conditions $w_0(x)$ and $\dot{w}_0(t)$ into the trial subspace $\mathsf{X}^r_{h,*}$. $\underline{G}_2(t)$ is defined, thanks to 2.1.2 , as:

$$\underline{G}_2(t) = \{g_2(t)\varphi_0(L), g_2(t)\varphi_1(L), ..., g_2(t)\varphi_{N_h}(L)\}^T \equiv \{0, 0, ..., g_2(t)\}^T$$

2.1.4 Matrices entries

To define precisely the entries of the matrices M and A and of the right hand side F we should apply the previously described quadrature formulas, from which it is feasible to say that matrices $M = [m_{ij}]$ and $A = [a_{ij}]$ are diagonal and multi-diagonal respectively:

$$m_{ij} = \int_0^L \varphi_i(x)\varphi_j(x) = \sum_{k=0}^{M-1} \int_{\Omega_m} \varphi_i(x)\varphi_j(x)dx$$

$$a_{i,j} = \int_0^L c^2 \frac{\partial \varphi_i(x)}{\partial x}\frac{\partial \varphi_j(x)}{\partial x}dx = \sum_{m=0}^{M-1} \int_{\Omega_m} \frac{\partial \varphi_i(x)}{\partial x}\frac{\partial \varphi_j(x)}{\partial x}dx$$

from this definition we can then define the mass and stiffness matrix as:

$$[M] = \begin{bmatrix} m^{1,1} & \cdots & m^{1,M} \\ \vdots & & \vdots \\ m^{M,1} & \cdots & m^{M,M} \end{bmatrix} \qquad [A] = \begin{bmatrix} a^{1,1} & \cdots & a^{1,M} \\ \vdots & & \vdots \\ a^{M,1} & \cdots & a^{M,M} \end{bmatrix}$$

now we compute each entry for the mass matrix:

$$\begin{aligned} M^{m,m'} &= \left[m^{m,m'}_{k,l}\right]^r_{k,l=1} = \left[\int_0^L \phi^m_k(x)\varphi^{m'}_l(x)dx\right]^r_{k,l=1} = \left[\delta_{m,m'}\int_{\Omega_m} \phi^m_k(x)\phi^m_l(x)dx\right]^r_{k,l=1} \\ &= \left[\delta_{m,m'}\int_{-1}^1 \varphi^m_k\left(\Phi^{-1}(\xi)\right)\varphi^m_l\left(\Phi^{-1}(\xi)\right)\frac{h}{2}d\xi\right]^r_{k,l=1} = \left[\delta_{m,m'}\int_{-1}^1 \psi_k(\xi)\psi_l(\xi)\frac{h}{2}d\xi\right]^r_{k,l=1} \\ &\approx \left[\delta_{m,m'}\sum_{i=1}^r \alpha_i\psi_k(\xi_i)\psi_l(\xi_i)\frac{h}{2}\right]^r_{k,l=1} = \left[\delta_{m,m'}\sum_{i=1}^r \delta_{k,i}\delta_{l,i}\frac{h}{2}\alpha_i\right]^r_{k,l=1} = \left[\delta_{m,m'}\delta_{k,l}\alpha_k\frac{h}{2}\right]^r_{k,l=1} \end{aligned}$$

and for the stiffness matrix:

$$\begin{aligned} A^{m,m'} &= \left[A^{m,m'}_{k,l}\right]^r_{k,l=1} = \left[\int_0^L c^2\frac{\partial\varphi^m_k(x)}{\partial x}\frac{\partial\phi^{m'}_l(x)}{\partial x}dx\right]^r_{k,l=1} = \left[\delta_{m,m'}\int_{\Omega_m} c^2\frac{\partial\varphi^m_k(x)}{\partial x}\frac{\partial\varphi^m_l(x)}{\partial x}dx\right]^r_{k,l=1} \\ &= \left[\delta_{m,m'}\int_{-1}^1 c^2\left(\frac{\partial\varphi^m_k\left(\Phi^{-1}(\xi)\right)}{\partial\xi}\frac{d\xi}{dx}\right)\left(\frac{\partial\varphi^m_l\left(\Phi^{-1}(\xi)\right)}{\partial\xi}\frac{d\xi}{dx}\right)\frac{h}{2}d\xi\right]^r_{k,l=1} \\ &= \left[\delta_{m,m'}\int_{-1}^1 c^2\frac{\partial\psi_k(\xi)}{\partial\xi}\frac{\partial\psi_l(\xi)}{\partial\xi}\frac{2}{h}d\xi\right]^r_{k,l=1} \approx \left[\delta_{m,m'}\sum_{i=1}^r c^2\alpha_i\frac{\partial\psi_k\left(\xi_i\right)}{\partial\xi}\frac{\partial\psi_l\left(\xi_i\right)}{\partial\xi}\frac{2}{h}\right]^r_{k,l=1} \\ &= \left[\delta_{m,m'}\sum_{i=1, i\neq k, i\neq l}^r c^2\alpha_i\frac{\partial\psi_k\left(\xi_i\right)}{\partial\xi}\frac{\partial\psi_l\left(\xi_i\right)}{\partial\xi}\frac{2}{h}\right]^r_{k,l=1} \end{aligned}$$

To define the vector $\underline{F}(t)$ of external forces, we should develop the following integral defined $\forall t \in (0,T]$:

$$F_j(t) = \mathcal{F}\left(\phi_j\right) = \int_0^L f(x,t)\phi_j(x)dx = \sum_{m=0}^{M-1}\int_{\Omega_m} f(x,t)\phi^{(m)}_j(x)dx = \sum_{m=0}^{M-1}\int_{-1}^1 f(\xi,t)\psi_{k(j)}(\xi)\frac{h}{2}d\xi =$$

$$\approx \sum_{m=0}^{M-1} \sum_{i=1}^{r} \alpha_i f\left(\xi_i, t\right) \psi_{k(j)}\left(\xi_i\right) \frac{h}{2} = \sum_{m=0}^{M-1} \alpha_{k(j)} f\left(\xi_{k(j)}, t\right) \frac{h}{2} = M \frac{h}{2} \alpha_{k(j)} f\left(\xi_{k(j)}, t\right)$$

Naming the right-hand side of the equation as $\underline{F}_g$ in such a way that:

$$\underline{F}_g(t) = \underline{F}(t) - [M]\underline{\ddot{R}}(t) - [A]\underline{R} + \underline{G}_2(t)$$

This can be done once the vector $\underline{F}(t)$ is computed and the lifting function vector $\underline{R}(t)$ is chosen, meaning that the right-hand side of the equation can be treated as a single $\mathbb{R}^{N_h+1}$ vector. So the final finite element formulation is this one:
Find $\underline{w} \in \mathbb{R}_h^{N+1}$ *such that* $\forall t \in (0, T]$

$$[M]\underline{\ddot{w}}(t) + [A]\underline{w}(t) = \underline{F}_g(t)$$

and such that

$$\begin{cases} \underline{w}(0) = \underline{w}_0 = \{w_0(0), ..., w_{N_h}(0)\} \\ \underline{\dot{w}}(0) = \underline{\dot{w}}_0 = \{\dot{w}_0(0), ..., \dot{w}_{N_h}(0)\} \end{cases} \tag{15}$$

The computation of the actual solution is done by $\underline{u}(t) = \underline{w}(t) + \underline{R}(t)$

$$\underline{u} = \{g(t), w_1(t), ..., w_{N_h}(t)\}^T \quad \forall t \in (0, T]$$

We can clearly see how the mass matrix $[M]$ is diagonal from the $\delta_{k,l}$ which is derived from the fact that the integral of the product of two different basis functions is zero. In the case of the stiffness matrix $[A]$ we have a different case where we are taking into consideration the derivative of the basis function. This means that in general we don't have the same behaviour as before and so the resulting matrix will not be diagonal but a band matrix where the width of the band is $r + 1$

2.2 Matlab a spectral element solver for the problem

2.2.1 Integration scheme

Now what we need is to perform a time discretization of the system of ordinary linear differential equations with time dependency presented in 15. To do so, we choose a temporal step $\triangle t$ for the time interval $[0, T)$ that for an integer $M_t > 0$:

$$t_k = k\triangle t \qquad k = 0, ..., M_t \to t_{M_t}\triangle t \equiv T$$

By doing so, the trial solution $\underline{w}(t)$ at time $t_k \in [0, T)$ is $\underline{w}_k = \underline{w}(t_k)$ and using the terms $\underline{w}_k$ and $\underline{w}_{k+1}$ it is possible to approximate $\ddot{\underline{w}}(t)$. The following integration scheme allows us to compute the finite element formulation 15

$$\begin{cases} \left(M + \frac{\triangle t^2}{2}A\right)\underline{w}_1 = \frac{\triangle t^2}{2}\underline{F}_1 + M\underline{w}_0 + \triangle t M\dot{\underline{w}}_0, & k = 0 \\ \left(M + \triangle t^2 A\right)\underline{w}_{k+1} = \triangle t^2\underline{F}_{k+1} + 2M\underline{w}_k - M\underline{w}_{k-1}, & k \geq 1 \end{cases} \tag{16}$$

This scheme is different for $k = 0$ to make possible to apply correctly the initial conditions.

2.2.2 Implementation test

To verify the accuracy of the solution introduced in 15 we are performing a convergence test by means of the norm $||u - u_{EX}||_{L^2(0,1)}$. The mock problem and the corresponding solution u_{EX} we adopted can be seen here below:

$$\begin{cases} u_{EX} = sin(2\pi x) \cdot sin(2\pi t) \\ L = (0, 1] \\ T = (0, 1] \\ c = 1 \end{cases} \tag{17}$$

We need to set proper values for f, u_0, v_0, $g_1(t)$ and $g_2(t)$. We do so by starting from the expression of the exact solution $u_{EX}(x, t)$ and then derive the other results, starting from the time derivative:

$$\frac{\partial u_{EX}}{\partial t} = 2\pi sin(2\pi x)cos(2\pi t) = u_{EX,t}$$

Then its second derivative is:

$$u_{EX,tt}(x, t) = -4\pi^2 sin(2\pi x)sin(2\pi t)$$

Now for the spatial derivatives:

$$\frac{\partial u_{EX}}{\partial x} = 2\pi cos(2\pi x)cos(2\pi t) = u_{EX,x}$$

$$u_{EX,xx}(x, t) = -4\pi^2 sin(2\pi x)sin(2\pi t)$$

Summarized, we have:

$$\begin{cases} u_{EX}(x, 0) = 0 = u_0(x) \\ u_{EX,t}(x, 0) = 2\pi sin(2\pi x) = v_0(x) \\ u_{EX}(0, t) = 0 = g_1(t) \\ c^2 u_{EX,x}(1, t) = sin(2\pi t) = g_2(t) \\ f(x, t) = 4\pi^2(c^2 - 1)sin(2\pi x)sin(2\pi t) = 0 \end{cases} \tag{18}$$

We performed several convergence test that helped us to understand the influence of the different parameters in the implementation. Specifically, the evaluation was made by means of the norm $||u - u_{EX}||_{L^2(0,1)}$ and $||u - u_{EX}||_{H^1_\star(0,1)}$ with respect to the mesh dimension (h). We took two values for the time step $\triangle t = 5\times^{-4}$ and $\triangle t = 1 \times 10^{-4}$ and we tested each of them with polynomials degrees $\mathbb{P}^2$ and $\mathbb{P}^5$. The convergence plots can be inspected in the following figures:

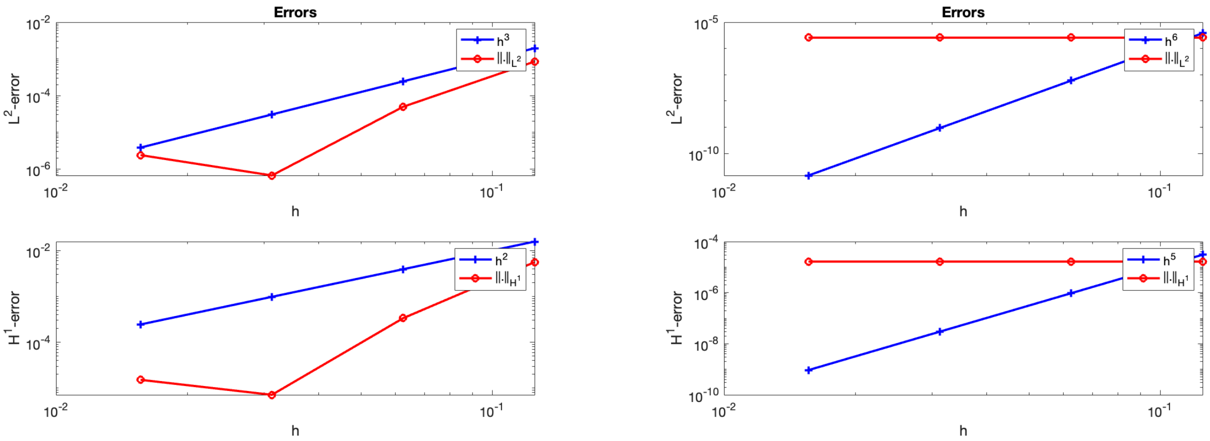


Figure 2: Test for: $\triangle t = 5 \times 10^{-4}$ and polynomials $\mathbb{P}^2$ and $\mathbb{P}^5$ respectively

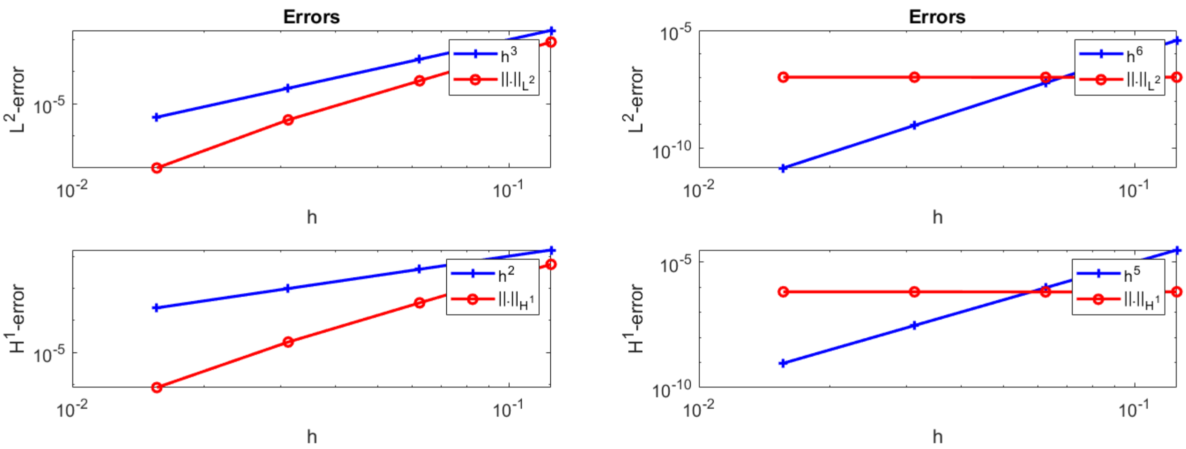


Figure 3: Test for: $\triangle t = 1 \times 10^{-4}$ and polynomial $\mathbb{P}^2$ and $\mathbb{P}^5$

comments For $\triangle t = 5 \times 10^{-4}$ and polynomial of the order $\mathbb{P}^2$ it can be observed that for higher mesh values h the model converge as it follows the reference curve, however, for both calculated errors we can see how the model diverges for lower mesh values. When testing for the same $\triangle t$ but for $\mathbb{P}^5$ fails to converge for the mesh dimensions under observation.

In contrast, when the parameters were set to $\triangle t = 1 \times 10^{-4}$ and $\mathbb{P}^2$ the model converge in a good extent to h^2 and h^3 respectively and for the mesh values under proof. For the final validation test, we set as basis function a polynomial of the order $\mathbb{P}^5$ and we fixed the $\triangle t$, even so, similar considerations on the behaviour of the errors as for the case $\triangle t = 5 \times 10^{-4}$ and $\mathbb{P}^2$ can be made since the convergences ratios are far from h^5 and h^6.

2.3 Finite Difference Problem Discretization

2.3.1 Hyperbolic Problem

To reformulate the problem in a hyperbolic system fashion, let us state the two dimensional vector U that contains two first-order equations:

$$U = \left\{ \begin{array}{c} u_t \\ u_x \end{array} \right\} \tag{19}$$

From Schwarz's theorem it is feasible to state that $u_{xt} = u_{tx}$, meaning that for the continuous wave function $u(x,t)$ its second partial derivatives are also continuous $\forall (x,t) \in \Omega$. Then we can do the following matrix formulation:

$$\left\{ \begin{array}{l} u_{tt} - c^2 u_{xx} = f \\ u_{xt} - u_{tx} = 0 \end{array} \right. \tag{20}$$

$$\left\{ \begin{array}{c} u_t \\ u_x \end{array} \right\}_t - \begin{bmatrix} 0 & c^2 \\ 1 & 0 \end{bmatrix} \left\{ \begin{array}{c} u_t \\ u_x \end{array} \right\}_x = \left\{ \begin{array}{c} f \\ 0 \end{array} \right\}$$

Which can be easily expressed as:

$$U_t - AU_x = F \tag{21}$$

The initial and boundary conditions initial propagation problem 1 can be written in terms of U:

- Initial conditions

$$\begin{cases} u_t(x,0) = v_0(x) & x \in (0,L) \\ u_x = \partial_x u_0(x) & x \in (0,L) \end{cases}$$

Becomes:

$$U(x,0) = U_0(x) = \begin{Bmatrix} u_t \\ u_x \end{Bmatrix}, x \in (0,L) \tag{22}$$

- Boundary conditions

$$\begin{cases} u_t(0,t) = \partial_t g_1(x) & x \in (0,L) \\ u_x = \partial_x u_0(x) & x \in (0,L) \end{cases}$$

Becomes

$$\begin{cases} U_1(0,t) = \partial_t g_1(t) & t \in (0,T) \\ U_2(L,t) = g_2(t)/c^2 & t \in (0,T) \end{cases} \tag{23}$$

The propagation problem can be reformulated in terms of U as well:

$$\begin{cases} U_t(x,t) - AU_x(x,t) = F(x,t) & (x,t) \in \Omega \\ U_1(0,t) = \partial_t g_1(t) & t \in (0,T] \\ U_2(L,t) = g_2(t)/c^2 & t \in (0,T) \\ U(x,0) = U_0(x) & x \in (0,L) \end{cases} \tag{24}$$

The system of equations of the problem is hyperbolic, meaning that A must be a diagonalizable matrix in the way showed here below:

$$T^{-1}AT = D \rightarrow A = TDT^{-1}$$

Replacing the definition of A into the problem's system of equation 21 and multiplying it by T^{-1}:

$$T^{-1}U_t - DT^{-1}U_x = T^{-1}F(x,t)$$

The system obtained is composed by decoupled equations, it looks as it follows:

$$V_t - DV_x = T^{-1}F(x,t) \tag{25}$$

Where $V = T^{-1}U$. Before redefining the initial and boundary conditions in terms of the hyperbolic system V we need to define the elements D and T. D is a matrix that has on its main diagonal the eigenvalues $eig(A) = \{c, -c\}$, so it is in the following form:

$$D = \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix}$$

T is then matrix of eigenvectors:

$$T = \frac{1}{\sqrt{1+c^2}} \begin{bmatrix} c & -c \\ 1 & 1 \end{bmatrix} \qquad T^{-1} = \frac{\sqrt{1+c^2}}{2c} \begin{bmatrix} 1 & c \\ -1 & c \end{bmatrix}$$

Consequently:

$$T^{-1}F = \frac{\sqrt{1+c^2}}{2c} \begin{Bmatrix} f \\ -f \end{Bmatrix}$$

Let us start with the boundary conditions, they are obtained as it follows:

$$\begin{cases} V_1(0,t) - V_2(0,t) = \frac{\sqrt{1+c^2}}{c} U_1(0,t) \equiv \frac{\sqrt{1+c^2}}{c} \partial_t g_1(t), & t \in (0,T) \\ V_1(L,t) + V_2(L,t) = \sqrt{1+c^2} U_2(L,t) \equiv \frac{\sqrt{1+c^2}}{c^2} g_2(t), & t \in (0,T) \end{cases} \tag{26}$$

For the initial conditions we use the matrix T^{-1}

$$V_0(x) = T^{-1}U_0(x) = \begin{Bmatrix} \frac{\sqrt{1+c^2}}{2c}(v_0(x) + c\partial_x u_0(x)) \\ \frac{\sqrt{1+c^2}}{2c}(-v_0(x) + c\partial_x u_0(x)) \end{Bmatrix}, x \in (0,L) \tag{27}$$

Recalling the conditions given in 24 , we can finally formulate them in terms of V:

$$\begin{cases} V_t(x,t) - DV_x(x,t) = T^{-1}F(x,t) & (x,t) \in \Omega \\ V_1(0,t) - V_2(0,t) = \frac{\sqrt{1+c^2}}{c}\partial_t g_1(t) & t \in (0,T] \\ V_1(L,t) + V_2(L,t) = \frac{\sqrt{1+c^2}}{c^2} g_2(t) & t \in (0,T) \\ V(x,0) = V_0(x) & x \in (0,L) \end{cases} \tag{28}$$

Where the solution $u(x,t)$ can retrieved from

$$TV(x,t) = \left\{ \begin{array}{c} u_t(x,t) \\ u_x(x,t) \end{array} \right\}$$

2.3.2 Discretization by finite difference scheme

The objective of the finite difference formulation is to take the problem previously introduced (see 28) and with the use of rules transforming the time and space derivatives into finite differences. To do so, we should create a space-time mesh, in which the space interval $[0,L]$ is divided into N_x equidistant nodes and each spatial step is long $h = h_i$ for $i = 1,2,...,N_x$. On regards time, the interval $[0,T]$ consists in N_t steps of constant size of $\triangle t = T/N_t$ and $t^n = n\triangle t$ for $n = 1,2,...,N_t$. We are going to estimate the solutions for each of the grid points (x_i,t^n), so we can do the following approximation:

$$v^n_{k,i} \approx v_k(x_i,t^n)$$

We implement the *Lax-Wendroff scheme* in order to solve the *transport problem*, in which the term $H^n_{i\pm\frac{1}{2}}$ is the so called *numerical flux*.

$$v^{n+1}_i = v^n_i - \lambda(H^n_{i+\frac{1}{2}} - H^n_{i-\frac{1}{2}}) \tag{29}$$

λ is defined as $\lambda = \triangle t/h$. For the solutions V_1 and V_2 the Lax-Wendroff finite difference formulation on the mesh nodes (x_i,t^n) in Ω:

$$\begin{cases} V^{n+1}_{1,i} = V^n_{1,i} + \frac{\lambda}{2}c(V^n_{1,i+1} - V^n_{1,i-1}) + \frac{\lambda^2}{2}c^2(V^n_{1,i+1} - 2V^n_{1,i} + V^n_{1,i-1}) \\ V^{n+1}_{2,i} = V^n_{2,i} + \frac{\lambda}{2}c(V^n_{2,i+1} - V^n_{2,i-1}) + \frac{\lambda^2}{2}c^2(V^n_{2,i+1} - 2V^n_{2,i} + V^n_{2,i-1}) \end{cases} \tag{30}$$

If we recall the notation given at the beginning of this section, $V^i_{k,i}$ states that the solutions V_k are evaluated on the mesh nodes (x_i,t^n), namely, $V^n_{k,i} = V_k(x_i,t^n)$. On regards the *Numerical flux*, it can be expressed adopting the Lax-Wendroff scheme and terms of V_1 and V_2:

$$\begin{cases} H^n_{i+\frac{1}{2}} = -\frac{1}{2}[c(V^n_{1,i+1} + V^n_{1,i}) + \lambda c^2(V^n_{1,i+1} + V^n_{1,i}]) \\ H^n_{i+\frac{1}{2}} = \frac{1}{2}[c(V^n_{2,i+1} + V^n_{2,i}) + \lambda c^2(V^n_{2,i+1} + V^n_{2,i}]) \end{cases} \tag{31}$$

2.3.3 Boundary and initial conditions

We should also discretized the boundary and initial conditions presented in 21 to include them correctly into the finite difference formulation we just introduced in 30 . Starting from 27 , at the mesh time $t^{n=0}$ and for all nodes x_i the initial conditions might be expressed in such a way that for the discrete time $t^{n+1=1}$ they are correctly implemented under the Lax-Wendroff scheme:

$$V_0(x_i) = \begin{cases} V^0_{1,i} = \frac{\sqrt{1+c^2}}{2c}(v_0(x_i) + c\partial_x u_0(x_i)) \\ V^0_{2,i} = \frac{\sqrt{1+c^2}}{2c}(-v_0(x_i) + c\partial_x u_0(x_i)) \end{cases}$$

Similarly on what accounts for the boundary conditions 26 in both nodes, $x_0 = 0$ and $x_{N_x} = L$ and for all $n > 0$, by using the constant extrapolation method that allow us to consider the ghost nodes x_{-1} and $x_{N_{x+1}}$ for each time step, we obtain:

$$\begin{cases} V^{n+1}_{1,0} = \frac{\sqrt{1+c^2}}{c}\partial_t g_1(t^{n+1}) + V^{n+1}_{2,1} \\ V^{n+1}_{2,N_x} = \frac{\sqrt{1+c^2}}{c} g_2(t^{n+1}) - V^{n+1}_{1,N_{x-1}} \end{cases}$$

Formulating the boundary conditions in this fashion allows to the values $V^{n+1}_{2,0}$ and V^{n+1}_{1,N_x} to be known $\forall n > 0$

2.3.4 Final hyperbolic problem statement

Getting together all that was developed in this section, the finite element formulation of the wave equation in hyperbolic form is: *Compute* $V_{1,i}^n$ *and* $V_{2,i}^n$ $\forall i = 0, ..., N_x$ *and* $\forall n = 0, ..., N_t$ *such that:*

$$\begin{cases} V_{1,i}^0 = \frac{\sqrt{1+c^2}}{2c}[v_0(x_i) + c\partial_x u_0(x_i)] & including \quad i=-1 \quad and \quad i = N_x+1 \\ V_{2,i}^0 = \frac{\sqrt{1+c^2}}{2c}[-v_0(x_i) + c\partial_x u_0(x_i)] & including \quad i=-1 \quad and \quad i = N_x+1 \\ V_{1,i}^{n+1} = V_{1,i}^n + \frac{\lambda}{2}c(V_{1,i+1}^n - V_{1,i-1}^n) + \frac{\lambda^2}{2}c^2(V_{1,i+1}^n - 2V_{1,i}^n + V_{1,i-1}^n) & for \quad i = 1, ..., N_x - 1 \\ V_{2,i}^{n+1} = V_{2,i}^n + \frac{\lambda}{2}c(V_{2,i+1}^n - V_{2,i-1}^n) + \frac{\lambda^2}{2}c^2(V_{2,i+1}^n - 2V_{2,i}^n + V_{2,i-1}^n) & for \quad i = 1, ..., N_x - 1 \\ V_{1,0}^{n+1} = \frac{\sqrt{1+c^2}}{c}\partial_t g_1(t^{n+1}) + V_{2,1}^{n+1} & \\ V_{2,N_x}^{n+1} = \frac{\sqrt{1+c^2}}{c} g_2(t^{n+1}) - V_{1,N_{x-1}}^{n+1} & \end{cases}$$

The actual solution is retrieve from:

$$\begin{Bmatrix} u_t(x_i, t^n) \\ u_x(x_i, t^n) \end{Bmatrix} = T \begin{Bmatrix} V_{1,i}^n \\ V_{2,i}^n \end{Bmatrix} \tag{32}$$

2.4 Validation of the implementation

To verify the accuracy of the numerical scheme introduced in 32 we are performing a convergence test by means of the norm $||u - u_{EX}||_{L^2(0,1)}$. The mock problem and the corresponding solution u_{EX} we adopted can be seen here below:

$$\begin{cases} u_{EX} = sin(2\pi x) \cdot sin(2\pi t) \\ L = (0, 1] \\ T = (0, 1] \\ c = 1 \end{cases} \tag{33}$$

We need to set proper values for f, u_0, v_0, $g_1(t)$ and $g_2(t)$. We do so by starting from the expression of the exact solution $u_{EX}(x, t)$ and then derive the other results:

$$\frac{\partial u_{EX}}{\partial t} = 2\pi cos(2\pi x) cos(2\pi t) = u_{EX,t}$$

$$\frac{\partial u_{EX}}{\partial x} = 2\pi cos(2\pi x) cos(2\pi t) = u_{EX,x}$$

$$\begin{cases} u_{EX}(x, 0) = 0 = u_0(x) \\ u_{EX,t}(x, 0) = 2\pi cos(2\pi x) = v_0(x) \\ u_{EX}(0, t) = sin(2\pi t) = g_1(t) \\ g_2(t) = 0 \\ f(x, t) = 0 \end{cases} \tag{34}$$

2.4.1 Results

The said norms behaviour were evaluated at time $t = T$ (final observation time) and the tests were also performed for different values of h and $\triangle t$

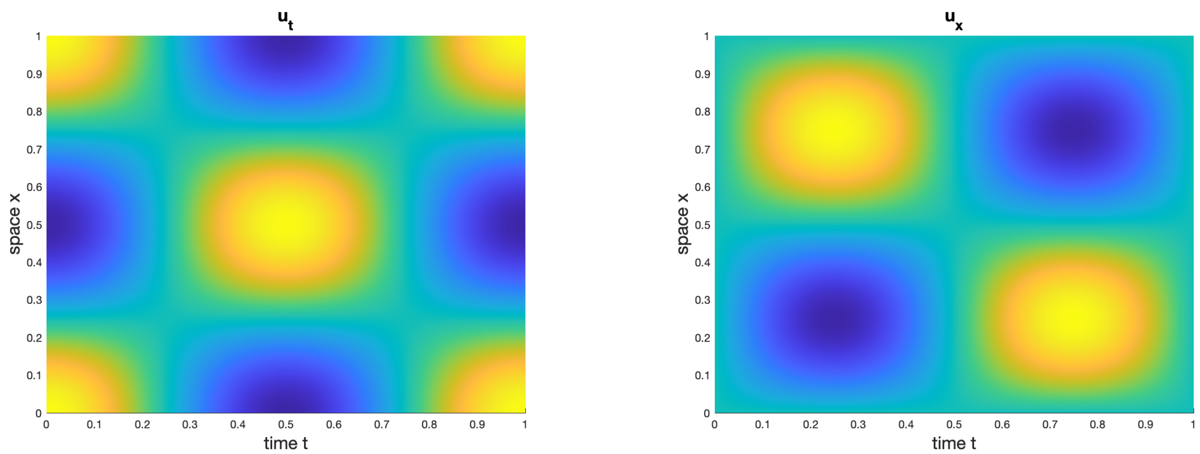


Figure 4: Numerical solutions obtained from the implementation of the formulation in 32

2.5 Spectral element method comparison against finite difference scheme

Considering the following data: $L = 1$, $c = 2$, $T = 1$, $f = u_0 = v_0 = 0$ while

$$g_1(t) = \begin{cases} \frac{1}{2}(1 + cos(\pi(t - 0.2)/0.1)) & |t - 0.1| \leq 0.05, \\ 0 & otherwise \end{cases} \tag{35}$$

The expression above corresponds to a half cosine pulse with unitary amplitude and width 0.05 applied to the left-hand side of the domain.

2.5.1 Spectral Element Method

The solution u is obtained by means of the formulation given in 15 in the end of section 1. We are presenting two particular cases for the SEM calculation: For the first one u is calculated with a polynomial degree $\mathbb{P}^2$ and for the second one we set the degree for the polynomial to be $\mathbb{P}^3$. Moreover, with $N_x = 256$ and $N_t = 1000$ the values for h and $\triangle t$ are the following:

$$h = \frac{L}{N_x} = \frac{1}{256} \qquad \triangle t = \frac{T}{N_t} = \frac{1}{1000}$$

The results u_x and u_t obtained with the polynomial basis chosen can be seen in the figures below:

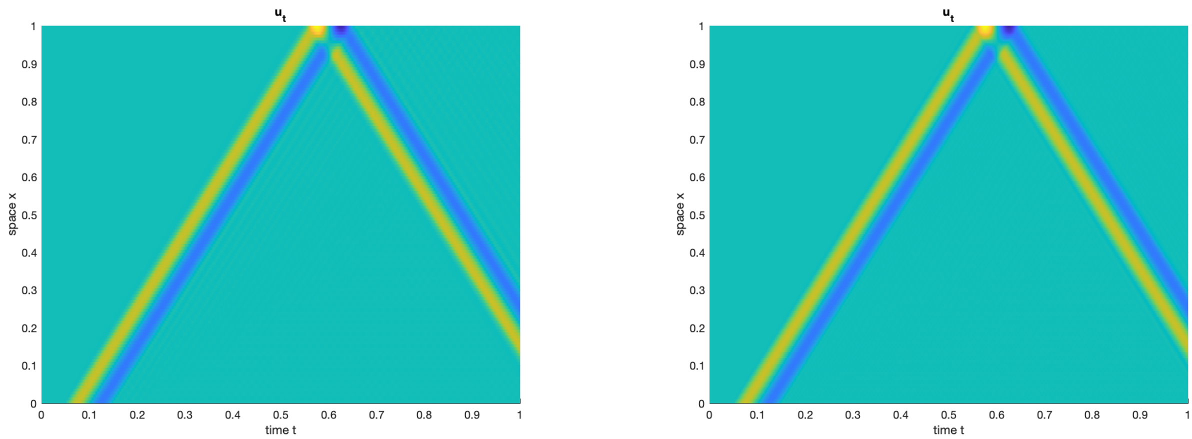


Figure 5: Temporal development of u_x for polynomials $\mathbb{P}^2$ and $\mathbb{P}^3$

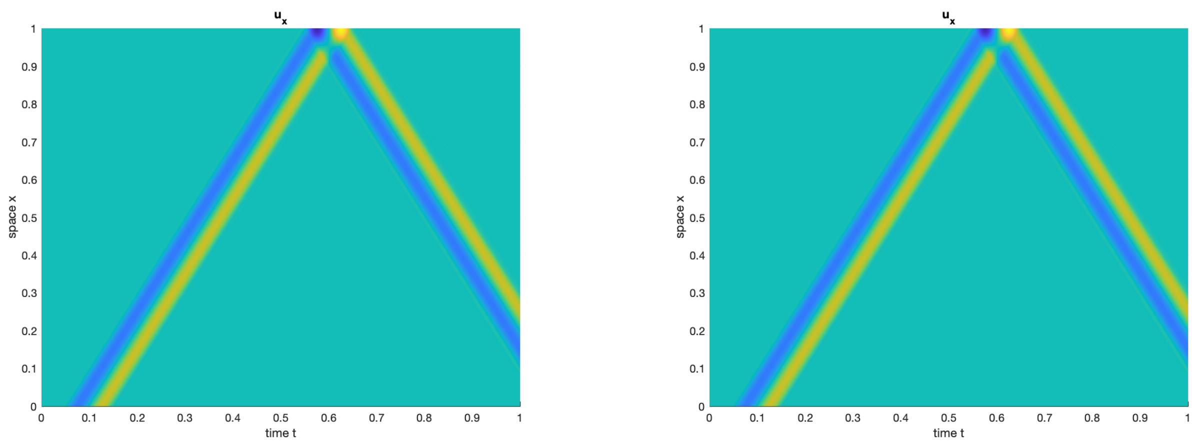


Figure 6: Spatial development of u_x for polynomials $\mathbb{P}^2$ and $\mathbb{P}^3$

2.5.2 Finite Difference Method

To compare the results obtained with the SEM method against with the finite difference scheme, we used the formulation proposed in 32 along with Lax-Wendroff scheme. We fixed the values N_x and N_t to test both methods under the same conditions (there is no counterpart in the FDM for the polynomial degree in SEM)

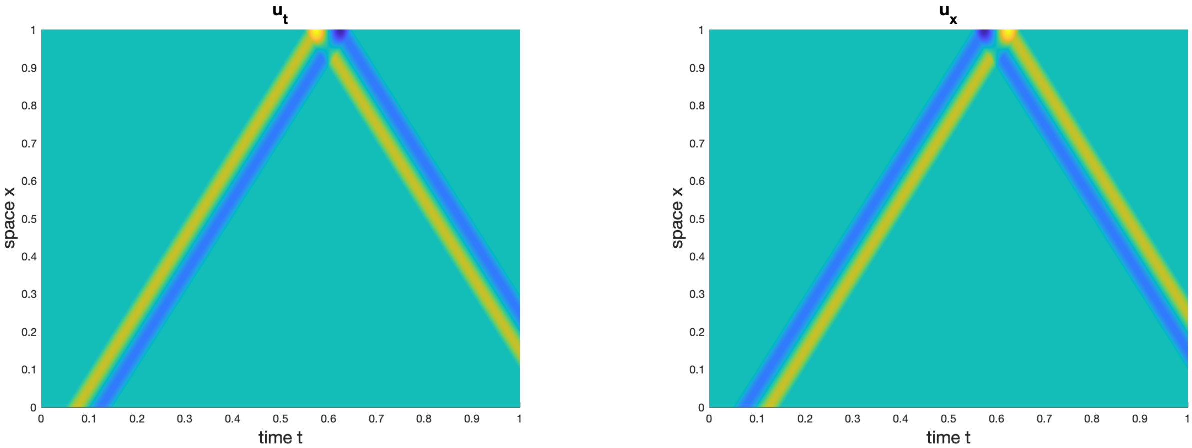


Figure 7: Spatial and temporal evolution of u_x and u_t

3 Homework 3

3.1 Introduction

The proposed problem takes into consideration a one dimensional Burger's equation

$$u_t(x,t) + uu_x(x,t) = \mu u_{xx}(x,t) \quad (x,t) \in (0,L) \times (0,T] \tag{1}$$

with initial condition

$$u(x,0) = u_0(x) \tag{2}$$

and Dirichlet condition

$$u(0,t) = u(L,t) = 0 \tag{3}$$

or with mixed boundary conditions

$$u_x(0,t) = 0 \quad u(L,t) = f(t) \tag{4}$$

and a Cartesian mesh for the domain $(x,t) \in (0,L) \times (0,T]$ that is obtained by dividing the interval $(0,L)$ using M equispaced nodes with distance h from one another so that $x_i = ih$ for $i = 0, 1, ..., M$. The same is done with the time interval $(0,T]$ by dividing it into N time steps of length $\Delta t = T/N$ so that $t_n = n\Delta t$ for $n = 0, 1, ..., N$

Burger's equation is a particular case of a non linear hyperbolic problem which can be written in its conservative form as

$$u_t + F(u)_x = 0 \tag{5}$$

where $F(u)$ is the flux. Otherwise it is also possible to write it in the non conservative form as

$$u_t + uu_x = 0 \tag{6}$$

in this report we'll focus on the conservative form.
We can easily compute the flux for this case by noticing that

$$F(u)_x = uu_x - \mu u_{xx}$$

which, by a simple integration with respect to the x coordinate, gives us

$$F(u) = \frac{u^2}{2} - \mu u_x \tag{7}$$

This result where $F(u)$ is non linear and convex confirms that we are in fact dealing with the case of a Burger's equation

3.1.1 Backward Differentiation Formula

The Burger's equation showed in Eq.1 is a case of a viscous Burger's Equation where μ represents the viscosity. For higher values of μ we get that the solution will be more regular while for lower values we get that the viscous equation will be more and more similar to an unviscid Burger's equation. To solve this kind of equation it is useful to introduce a Backward Differentiation Formula (BDF-2) in order to approximate the time derivative

$$\frac{\partial u(x,t^{n+1})}{\partial t} \approx \frac{\alpha_0}{\Delta t} u(x,t^{n+1}) - \sum_{k=1}^{p} \frac{\alpha_k}{\Delta t} u(x,t^{n+1-k}) \quad \forall x \in (0,L) \tag{8}$$

with

$$\alpha_0 = \frac{3}{2}, \quad \alpha_1 = 2, \quad \alpha_2 = \frac{1}{2}, \quad p = 2$$

In particular we want to show that by applying Eq.1 in Eq.8 what we obtain is

$$u^{n+1} = \frac{4}{3}u^n - \frac{1}{3}u^{n-1} + \frac{2}{3}\Delta t(\mu u_{xx}^{n+1} - u^{n+1}u_x^{n+1}) \quad n > 1 \tag{9}$$

and that for the Backward Euler Formula for $n = 0$ it is true that

$$u^1 = u^0 + \Delta t(\mu u_{xx}^0 - u^0 u_x^0) \tag{10}$$

For this we start from considering Eq.8 with our data.

$$u_t^{n+1} \approx \frac{3}{2\Delta t}u^{n+1} - \left(\frac{2}{\Delta t}u^n - \frac{1}{2\Delta t}u^{n-1}\right)$$

now that we have a numerical approximation of the time derivative we can substitute it into Eq.1 and obtain

$$\frac{3}{2\Delta t}u^{n+1} - \frac{2}{\Delta t}u^n + \frac{1}{2\Delta t}u^{n-1} + u^{n+1}u_x^{n+1} = \mu u_{xx}^{n+1}$$

$$u^{n+1} = \frac{4}{3}u^n - \frac{1}{3}u^{n-1} + \frac{2}{3}\Delta t\left(\mu u_{xx}^{n+1} - u^{n+1}u_x^{n+1}\right)$$

which is exactly what we expected as from Eq.9.
Now we instead want to show the equality shown in Eq.10. Starting from the Backward Euler Formula

$$\frac{\partial u^{n+1}}{\partial t} \approx \frac{u^{n+1} - u^n}{\Delta t} \tag{11}$$

if we the substitute the previous equation into Eq.1 what we obtain is

$$\frac{u^{n+1} - u^n}{\Delta t} + u^n u_x^n = \mu u_{xx}^n$$

now imposing $n = 0$ and by moving some of the terms around

$$\frac{u^1 - u^0}{\Delta t} + u^0 u_x^0 = \mu u_{xx}^0$$

$$u^1 = u^0 + \Delta t(\mu u_{xx}^0 - u^0 u_x^0)$$

which is exactly the solution we were expecting from the Backward Euler Formula.

3.2 Integration Scheme

The aim is to prove that, by applying this linear extrapolation in time

$$u^{n+1}u_x^{n+1} \approx (2u^n - u^{n-1})u_x^{n+1} \tag{12}$$

and a central finite difference discretization scheme in space for the terms u_x and u_{xx} it is possible to obtain the scheme

$$\alpha_i u_i^{n+1} + \beta_i u_{i+1}^{n+1} + \gamma_i u_{i-1}^{n+1} = f_i \tag{13}$$

with

$$\begin{cases} \alpha_i = 1 + \frac{4\mu}{3h^2}\Delta t \\ \beta_i = -\frac{2}{3}\Delta t\left(\frac{\mu}{h^2} - \frac{2u_i^n - u_i^{n-1}}{2h}\right) \\ \gamma_i = -\frac{2}{3}\Delta t\left(\frac{\mu}{h^2} + \frac{2u_i^n - u_i^{n-1}}{2h}\right) \\ f_i = \frac{4}{3}u_i^n - \frac{1}{3}u_i^{n-1} \end{cases} \tag{14}$$

We have that the central finite difference discretization in space can be expressed as

$$u_x \approx \frac{u_{i+1} - u_{i-1}}{2h} \tag{15}$$

$$u_{xx} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{16}$$

Starting from Eq.9 we can substitute all the previous equations inside it so to obtain

$$u_i^{n+1} = \frac{4}{3}u_i^n - \frac{1}{3}u_i^{n-1} + \frac{2}{3}\Delta t\left[\mu\frac{u_{i+1}^{n+1} - 2u_i^{n+1}u_{i-1}^{n+1}}{h^2} - (2u_i^n - u_i^{n-1})\frac{u_{i+1}^n - i_{i-1}^n}{2h}\right]$$

$$u_i^{n+1} = \frac{4}{3}u_i^n - \frac{1}{3}u_i^{n-1} + \frac{2}{3}\Delta t\left[\mu\frac{u_{i+1}^{n+1} - 2u_i^{n+1}u_{i-1}^{n+1}}{h^2} - \frac{2u_i^n u_{i+1}^n - 2u_i^n u_{i-1}^n - u_i^{n-1}u_{i+1}^n + u_i^{n-1}u_{i-1}^n}{2h}\right]$$

now it is possible to collect all terms relative to $u_i^{n+1}, u_{i+1}^{n+1}, u_{i-1}^{n+1}$ to obtain α_i, β_i and γ_i and the remains will constitute the entries of f_i

$$\left[1 + \frac{4\mu}{3h^2}\Delta t\right]u_i^{n+1} + \left[-\frac{2}{3}\Delta t\left(\frac{\mu}{h^2} - \frac{2u_i^n - u_i^{n-1}}{2h}\right)\right]u_{i+1}^{n+1} + \left[-\frac{2}{3}\Delta t\left(\frac{\mu}{h^2} + \frac{2u_i^n - u_i^{n-1}}{2h}\right)\right]u_{i-1}^{n+1} = \frac{4}{3}u_i^n - \frac{1}{3}u_i^{n-1}$$

from which we can clearly recognize that the equation takes the form stated in Eq.13

3.3 Validation of the difference scheme

It is now required to validate the scheme presented in the previous section by solving Eq.1 with the following initial and Dirichlet boundary conditions:

$$\begin{cases} u(x,0) = \sin(\pi x) \quad \forall x \in [0,1] \\ u(0,t) = u(1,t) = 0 \quad \forall t \in [0,T] \end{cases} \tag{17}$$

The exact solution is given by the Cole Hopf transformation as

$$u(x,t) = 2\mu\pi\frac{\sum_{n=1}^{\infty} c_n n e^{-n^2\pi^2\mu t}\sin(n\pi x)}{c_0 + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\mu t}\cos(n\pi x)}$$

with Fourier coefficients

$$\begin{cases} c_0 = \int_0^1 e^{-\frac{1}{2\mu\pi}(1-\cos(\pi x))}dx \\ c_n = 2\int_0^1 e^{-\frac{1}{2\mu\pi}(1-\cos(\pi x))}\cos(n\pi x)dx \end{cases}$$

for different values of μ, N, M.

3.3.1 Order of Convergence

The error between the exact solution and the numerical solution will be evaluated through L^2 and L^∞ errors where

$$L^2error = ||u - u_{EX}||_{L^2}(T) = \left(h \sum_{i=1}^{N_x} |u_i^{N_t} - u_{EX}(x_i,T)|^2 \right)^{1/2}$$
$$L^\infty error = ||u - u_{EX}||_{L^\infty}(T) = \max_{i=1,\dots,N_X} |u_i^{N_t} - u_{EX}(x_i,T)|$$

The way that the error behave with different values of h and Δt is described through a value γ known as Order of Convergence which is such that

$$||u - u_{EX}||_{.} \propto h^{\gamma_h} + \Delta t^{\gamma_{\Delta t}}$$

We now assume a fixed value of Δt and a varying value of our choice for h, for example h_1 and h_2 which result in the corresponding errors E_1 and E_2. Since now the term proportional to Δt is fixed we can say that in this case

$$E_1 \propto h_1^{\gamma_h}$$
$$E_2 \propto h_2^{\gamma_h}$$

now we can compute the ratio between the two errors and than take the logarithm of the ratio to make the dependency on γ more explicit.

$$\log\left(\frac{E_1}{E_2}\right) = \log\left(\frac{h_1^{\gamma_h}}{h_2^{\gamma_h}}\right) = \log\left(\left(\frac{h_1}{h_2}\right)^{\gamma_h}\right) = \gamma_h \log\left(\frac{h_1}{h_2}\right)$$

this can be done analogously in the opposite case where h is fixed and Δt is chosen at will. From this we can obtain an explicit value for γ

$$\gamma_h = \frac{\log(E_1/E_2)}{\log(h_1/h_2)}$$
$$\gamma_{\Delta t} = \frac{\log(E_1/E_2)}{\log(\Delta t_1/\Delta t_2)}$$

3.3.2 Implementation of the numerical scheme

The numerical scheme obtained in Eq.13 now needs to be implemented to solve the numerical problem. This scheme is implicit at time t^n $\forall n > 1$. This means that the variables that are needed to compute the next time step $n+1$ are coupled in an equation and an iterative technique is needed to obtain the solution. For this reason we can formulate Eq.13 in a matrix-vector form so for each time step we have a solution given by:

$$[A]^n \underline{u}^{n+1} = \underline{f}^n \tag{18}$$

so that at each time step if we can safely assume that $[A]^n$ is non-singular (i.e. that exists $([A]^n)^{-1}$ we can compute the solution at the next time step as

$$\underline{u}^{n+1} = ([A]^n)^{-1} \underline{f}^n \tag{19}$$

In order to compute the entries of $[A]^n$ we need to take into account both Eq.13 and the boundary conditions shown in Eq.17. In particular in order to correctly apply the boundary conditions we know that we need to introduce 2 "ghost nodes", one at $i = -1$ and one at $i = N_x + 1$ which implies that $[A]^n$ is a $[N_x + 2 \times N_x + 2]$ matrix and both $\underline{f}^n$ and $\underline{u}^{n+1}$ are vectors of length $N_x + 2$. In order to correctly apply homogeneous Dirichlet conditions at both boundaries we set $A_{1,1} = A_{N_x,N_x} = 1$ and we apply constant extrapolation to the "ghost nodes". Finally the structure of matrix $[A]^n$ is obtained by putting the values of α_i on the diagonal, β_i on the supradiagonal and γ_i on the subdiagonal. The resulting matrix is shown in Eq.20

$$[A]^n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \beta_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \gamma_2 & \alpha_2 & \beta_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_3 & \alpha_3 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \gamma_{N_{x-1}} & \alpha_{N_{x-1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

which is tridiagonal and non-singular and we can compute the inverse of $[A]^n$ to find the solution at the next time step. We also have that $\underline{f}^n$ is in the form

$$\underline{f}^i = \begin{bmatrix} 0 & 0 & f_1 & f_2 & \cdots & f_{N_x-1} & f_{N_x} & 0 & 0 \end{bmatrix}' \tag{21}$$

Now implementing this scheme into Matlab and computing the solution for values of $\mu = 0.1$, 1 and 10 and showing the computed and reference solution at times $T = 0.1$, 0.5 and 2.3 gives us the results shown in Fig.1

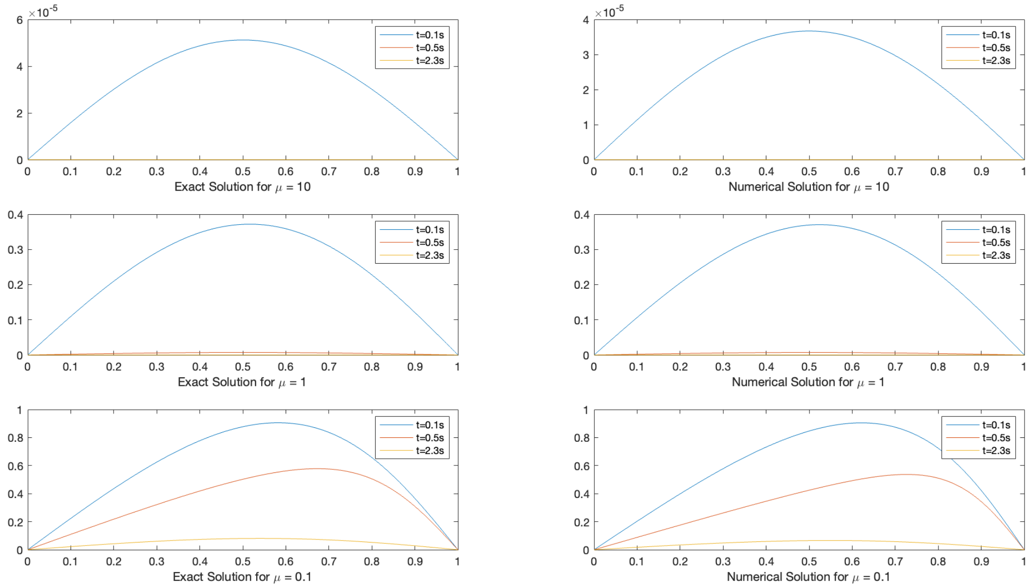
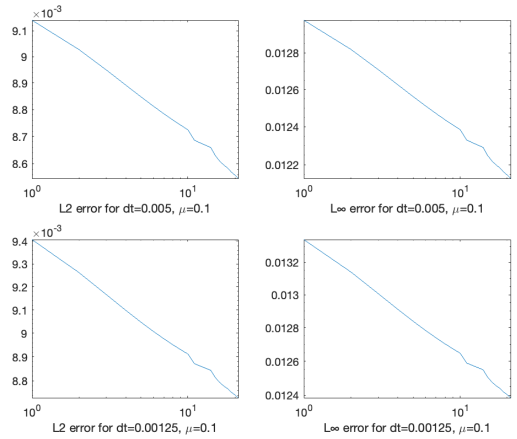
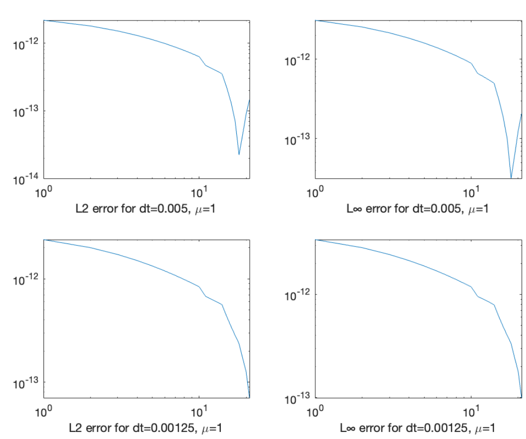


Figure 1: Results for $N_x = N_T = 1000$ and various values of μ

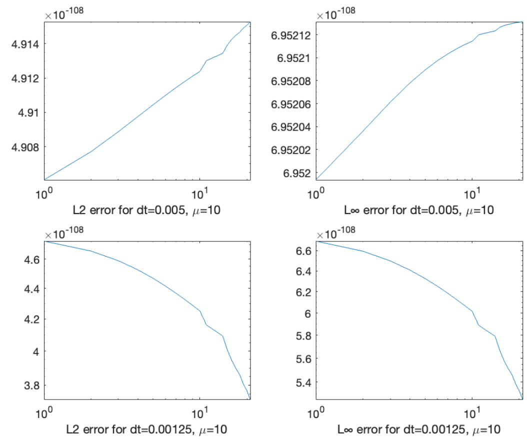
As for the convergence the results are reported in the next Figure.



(a) Results for $\mu = 0.1$



(b) Results for $\mu = 1$



(c) Results for $\mu = 10$

In general we can say that a finer mesh reduces the error with quite good results. The only difference is in the case where $\mu = 10$ where the error is almost constant for different values of h.

3.4 Another implementation of the scheme

Again we use the same integration scheme used previously but now we are presented with a different set of initial and boundary conditions

$$\begin{cases} u(x,0) = \frac{1}{4}\cos(\pi x) \quad \forall x \in [0,1] \\ u_x(0,t) = 0 \quad \forall t \in [0,T] \\ u(1,t) = -\frac{1}{4}e^{-\mu t} \quad \forall t \in [0,T] \end{cases} \tag{22}$$

In this new case we have non-homogeneous Dirichlet conditions applied at the right boundary and homogeneous Neumann conditions applied at the left boundary. We need to discretize the Neumann conditions in order to correctly apply it. Since it involves the space derivative we use again the Backward Difference scheme

$$u_x(0,t) = g_1(t)$$
$$\frac{u_{-1}^n - u_0^n}{2h} = g_1^n$$

which implies

$$u_{-1}^n - u_0^n = -2hg_1^n$$

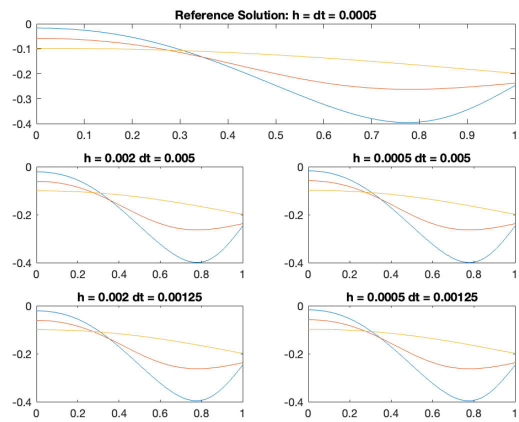
Applying these changes by taking into consideration that $g_1^n = 0$ to the previous formulation shown in Eq.20 we obtain

$$[A]^n = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \gamma_0 & \alpha_0 & \beta_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \beta_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \gamma_2 & \alpha_2 & \beta_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_3 & \alpha_3 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \gamma_{N_x-1} & \alpha_{N_x-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \tag{23}$$

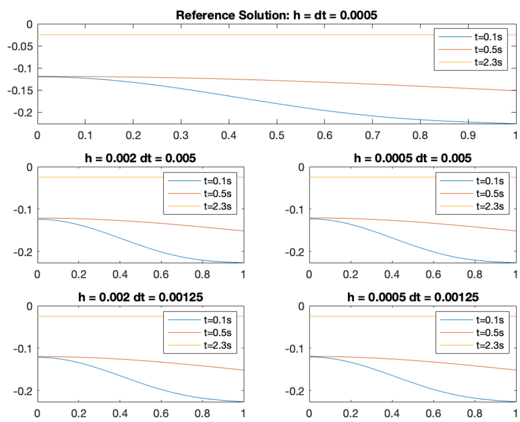
The result by applying this new formulation of the matrix is a solution of the proposed problem that respects the homogeneous Neumann condition at the left side and homogeneous Dirichlet condition at the right side but in this case the Dirichlet condition is not homogeneous and instead it is set as $u(1,t) = -\frac{1}{4}e^{-\mu t}$. To keep track of this we add a constant term equivalent to the boundary condition definition to the whole solution so the final solution has the following form:

$$\begin{cases} \underline{u_i}^{n+1} = ([A]_i^n)^{-1}\underline{f_i}^n + r(i\Delta t) \\ r(t) = -\frac{1}{4}e^{-\mu t} \end{cases} \tag{24}$$

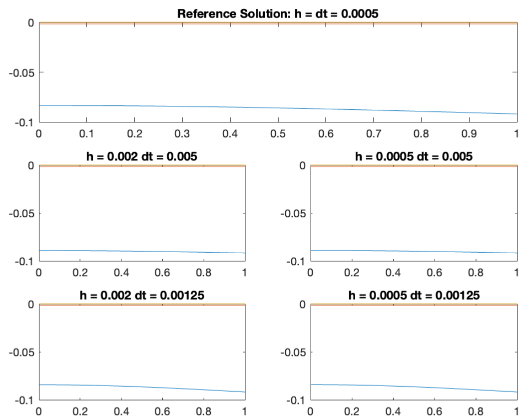
Now taking as reference the solution this scheme with $dt = h = 0.0005$ the results are show in Fig.3 for all combinations of $dt = [0.005, 0.00125]$ and $h = [0.002, 0.0005]$



(a) Results for $\mu = 0.1$



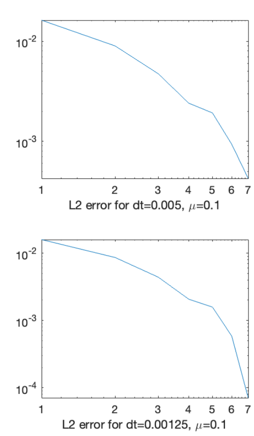
(b) Results for $\mu = 1$



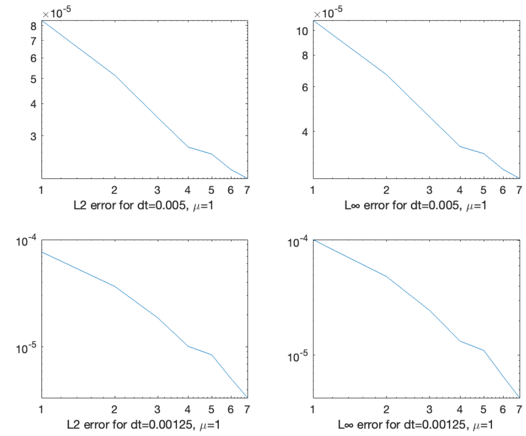
(c) Results for $\mu = 10$

Figure 3: Plot of the solution for reference and test values of h and Δt

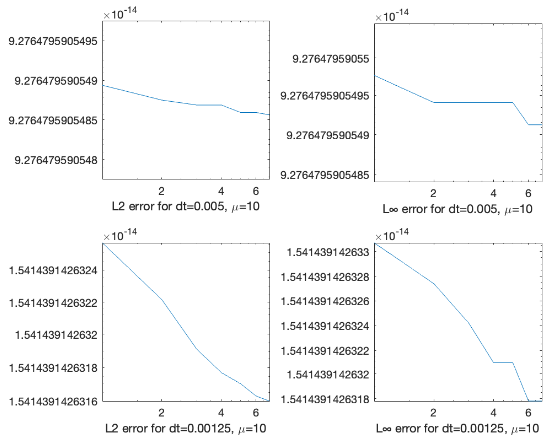
And results of the convergence in L^2 norm and L^∞ norm



(a) Results for $\mu = 0.1$



(b) Results for $\mu = 1$



(c) Results for $\mu = 10$

Figure 4: Plot of the L^2 and L^∞ error as function of h for different values of Δt and μ

As with the previous point we see that in general a finer mesh size improves the result. The choice of Δt seems to impact, at least locally, the slope of the convergence curve which means that for a lower value of Δt we have greater improvements for smaller variations of h.