NUMERICAL MODELING AND SIMULATION FOR ACOUSTICS
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HOMEWORK 3

Consider the following one dimensional Burger's equation:

$$u_t(x,t) + uu_x(x,t) = \mu u_{xx}(x,t) \quad (x,t) \in (0,L) \times (0,T], \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x) \quad x \in (0,L), \tag{2}$$

and, the Dirichlet boundary conditions

$$u(0,t) = u(L,t) = 0 \quad t \in (0,T], \tag{3}$$

or the mixed boundary conditions

$$u_x(0,t) = 0, \quad u(L,t) = f(t) \quad t \in (0,T]. \tag{4}$$

Consider a cartesian mesh for the domain $[0,L]\times[0,T]$ by dividing the interval $[0,T]$ into N steps $0 = t_0 \le t_1 \le ... \le t_N = T$ with constant time step $\Delta t = T/N$ and $t_n = n\Delta t$ for $n = 1,2,...,N$ and the space interval $[0,L]$ into M nodes $0 \le x_0 \le x_1 \le ... \le x_M = L$ with constant spacing step $h = ih$ for $i = 1,2,...,M$.

1. Show that if you consider in (1) a second order backward differentiation formula (BDF-2) with respect to the time you obtain:

$$u^{n+1} = \frac{4}{3}u^n - \frac{1}{3}u^{n-1} + \frac{2}{3}\Delta t(\mu u_{xx}^{n+1} - u^{n+1}u_x^{n+1}), \quad n > 1. \tag{5}$$

   For $n = 1$ show instead that the Backward Euler formula gives

$$u^1 = u^0 + \Delta t(\mu u_{xx}^0 - u^0 u_x^0). \tag{6}$$

2. Prove that, for the generic step n, by applying the following linear extrapolation in time

$$u^{n+1}u_x^{n+1} \approx (2u^n - u^{n-1})u_x^{n+1}$$

   and a central finite difference discretization in space fo the terms u_{xx}^{n+1} and u_x^{n+1} one can obtain the following scheme

$$\alpha_i u_i^{n+1} + \beta_i u_{i+1}^{n+1} + \gamma_i u_{i-1}^{n+1} = f_i, \tag{7}$$

   whrere

$$\begin{cases} \alpha_i = 1 + \frac{4}{3}\frac{\mu}{h^2}\Delta t, \\ \beta_i = -\frac{2}{3}\Delta t\left(\frac{\mu}{h^2} - \frac{(2u_i^n - u_i^{n-1})}{2h}\right), \\ \gamma_i = -\frac{2}{3}\Delta t\left(\frac{\mu}{h^2} + \frac{(2u_i^n - u_i^{n-1})}{2h}\right), \\ f_i = \frac{4}{3}u_i^n - \frac{1}{3}u_i^{n-1}. \end{cases}$$

3. Solve Burger's equation (1)–(3) with initial condition $u(x,0) = \sin(\pi x)$, for $x \in [0,1]$ and the boundary conditions $u(0,t) = u(1,t) = 0$ for $t \in [0,T]$ whose exact solution is given by the Cole Hopf transformation as

$$u(x,t) = 2\mu\pi \frac{\sum_{n=1}^{\infty} c_n n e^{-n^2\pi^2\mu t} \sin(n\pi x)}{c_0 + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\mu t} \cos(n\pi x)}, \tag{8}$$

with Fourier coefficients

$$\begin{aligned} c_0 &= \int_0^1 \mathtt{exp}\{-\frac{1}{2\mu\pi}(1-\cos(\pi x))\}\, dx, \\ c_n &= 2\int_0^1 \mathtt{exp}\{-\frac{1}{2\mu\pi}(1-\cos(\pi x))\} \cos(n\pi x)\, dx. \end{aligned}$$

For $\mu = 10, 1, 0.1$ plot the finite difference solution obatined with (6) and (7) and the exact solution (8) at final times $T = 0.1, 0.5, 2.3$. Compute the L^2 and L^∞ norm of the error at $T = 2.5$ for different values of the space h and time step Δt. Give an estimate of the order of convergence of the method with respect to the discretization parameters h and Δt. Report the results and comment on them.

4. Repeat point 3. considering now the following initial

$$u(x,0) = \frac{1}{4}\cos(\pi x), \quad x \in [0,1],$$

and boundary conditions

$$u_x(0,t) = 0, \quad u(1,t) = -\frac{1}{4}e^{-\mu t}, \quad t \in [0,T].$$

As a reference solution consider the finite difference approxiamtion obtained with the finest space-time grid (e.g. $h = 0.0005 = \Delta t = 0.0005$).