Solutions to Exercises on Graphical Models

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Exercise 1

Suppose that the joint density of a discrete variable X and two continuous variables Y and Z is given (up to proportionality) by

$$f(x,y,z) \propto \frac{z^{\alpha+x}}{x!} \exp\Big[-(\beta+y+1)z\Big].$$

(a) **Conditional Independence of X and Y given Z:**

We rewrite the joint density as

$$f(x,y,z) \propto \underbrace{z^{\alpha} \exp\big[-z(\beta+1)\big]}_{g(z)} \cdot \underbrace{\frac{z^{x}}{x!} \exp\big[-yz\big]}_{h(x,y,z)}.$$

When we condition on $Z = z$, the joint conditional density becomes

$$f(x,y \mid z) \propto \frac{z^{x}}{x!} \exp(-yz).$$

Notice that the factorization separates any dependence on x and y (given z) into a product of a function of x (with z as a parameter) and a function of y (with z as a parameter). Hence, we have

$$f(x,y \mid z) = f(x \mid z)\, f(y \mid z),$$

which shows that X and Y are conditionally independent given Z.

(b) **Marginal Independence of X and Y:**

To check unconditional independence we would need

$$f(x,y) = f(x)f(y).$$

Marginalizing over z, we obtain

$$f(x,y) \propto \frac{1}{x!} \int_0^{\infty} z^{\alpha+x} \exp\big[-z(\beta+y+1)\big]\, dz.$$

The integral yields a Gamma function,

$$\int_0^\infty z^{\alpha+x} e^{-z(\beta+y+1)} dz = \frac{\Gamma(\alpha+x+1)}{(\beta+y+1)^{\alpha+x+1}},$$

so that

$$f(x,y) \propto \frac{\Gamma(\alpha+x+1)}{x!\,(\beta+y+1)^{\alpha+x+1}}.$$

Because the exponent in the denominator depends on x, the marginal $f(x,y)$ does not factor into a product of a function of x only and a function of y only. Thus, X and Y are *not* independent.

(c) **Undirected Graph Representation:**

Since X and Y are conditionally independent given Z, the undirected graph connects Z to both X and Y but does not connect X and Y directly. The graph can be drawn as:

$$X — Z — Y.$$

(d) **Directed Graph Representation:**

A natural directed acyclic graph (DAG) representing the factorization

$$f(x,y,z) = f(z)\,f(x \mid z)\,f(y \mid z)$$

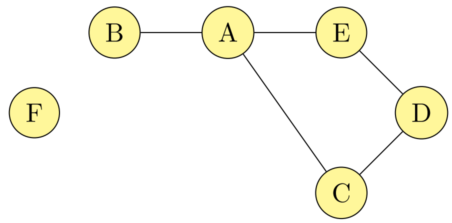
is to let Z be the common parent of both X and Y. In diagram form:

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

This graph indicates that once Z is known, X and Y are independent.

Exercise 2

Consider the following undirected graph:



(a) **Maximal Cliques:**

A *clique* is a subset of nodes where every pair is connected. A *maximal clique* is a clique that cannot be extended by adding any other nodes from the graph.

- The set $\{A, B, C, D, E\}$ forms a clique because all nodes are pairwise connected.
- The set $\{F\}$ forms a trivial clique since F is isolated.

Therefore, the maximal cliques are:

$$\{A, B, C, D, E\} \quad \text{and} \quad \{F\}.$$

(b) **Dependence Structure:**

The dependence structure in this graph can be described as follows:

- Nodes A, B, C, D, E are all directly connected, so they depend on each other.
- Node F is independent of all other nodes because it is isolated.

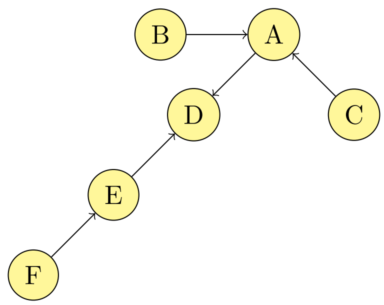
Thus, the dependence structure is:

$$A \longleftrightarrow B \longleftrightarrow C \longleftrightarrow D \longleftrightarrow E,$$

and F is independent of all other nodes.

Exercise 3

Consider the following directed graph:



(a) **Are nodes A and F independent? Why?**

In this graph, there is a directed path from F to A via E and D. Therefore, A and F are *not independent* because there is an indirect path from F to A.

(b) **Are nodes A and F independent given D? Why?**

When we condition on D, the path from F to A is blocked. This is because conditioning on D removes the influence of F on A, as there is no longer any path from F to A once D is observed. Hence, A and F become *independent* given D.