# Voluntary Exercises 2

Gabriel Pons Fontoira

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## 1 Voluntary exercises from chapter 2 and 3

### 1.1 Let $\{\mathbf{X_1}, ..., \mathbf{X_n}\}$ a s.r.s. of the r.v. X with mean and variance finite. Find a value of $\alpha$ for which the estimator T is an unbiased estimator of the variance.

$$T = \alpha \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2$$

We need to prove that the expectation of our estimator T minus the parameter to estimate it's equal to zero:

$$Bias_T(\sigma^2) = E(T) - \sigma^2 = 0$$

We also know that:

$$Var(X_i) = \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2 \quad and \quad E[X_i] = \mu$$

The calculation for $E[T]$ proceeds as it follows:

$$E[T] = E[\alpha \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2] = \alpha \sum_{i=1}^{n-1} E[(X_{i+1} - X_i)^2] = \alpha \sum_{i=1}^{n-1} E[(X_{i+1})^2 + (X_i)^2 - (2X_{i+1}X_i)] =$$

$$= \alpha \sum_{i=1}^{n-1} E[(X_{i+1})^2] + E[(X_i)^2] - 2E[(X_{i+1})]E[(X_i)]$$

From the definition of variance in terms of expectation we obtain:

$$Var(X) = E[X^2] - E[X]^2 \Leftrightarrow Var(X) + E[X]^2 = E[X^2]$$

For every $E[X^2]$ we will use the previous expression.

$$\alpha \sum_{i=1}^{n-1} Var(X_{i+1}) + E[Xi+1]^2 + Var(X_i) + E[X_i]^2 - 2[E_{X_i+1}]E[X_i]$$

As we are working with a s.r.s, $X_{i+1}$ has the same expectation and variance as $X_i$.

$$\alpha \sum_{i=1}^{n-1} \sigma^2 + \mu^2 + \sigma^2 + \mu^2 - 2\mu^2 = \alpha(n-1)2\sigma^2 = E[T]$$

$$Bias_T(\sigma^2) = \alpha(n-1)2\sigma^2 - \sigma^2 = 0$$

So $\alpha$ will be:

$$\alpha = \mathbf{\frac{1}{2(n-1)}}$$

### 1.2 Let $\mathbf{X_1 \sim Bin(n, p)}$ where $\mathbf{n \in \{2, 3\}}$ and $\mathbf{p \in \{\frac{1}{2}, \frac{1}{3}\}}$. If $\mathbf{x_1 = 1}$, find the maximum likelihood estimation of the parameters (n,p)

The probability mass function is:

$$P\binom{n}{x}p^x(1-p)^{n-x}$$

Then:

$$\mathcal{L}(X_1 = 1; n, p) = \binom{n}{1}p(1-p)^{n-1}$$

To find the maximum likelihood we must try the 4 parameter possible combinations which are:

$$(2, \frac{1}{2}), (2, \frac{1}{3}), (3, \frac{1}{2}), (3, \frac{1}{3})$$

$$\mathcal{L}(X_1 = 1; 2, \frac{1}{2}) = \binom{2}{1} 0.25 = 0.5$$

$$\mathcal{L}(X_1 = 1; 2, \frac{1}{3}) = \binom{2}{1} \frac{1}{3} (\frac{2}{3}) = \frac{4}{9}$$

$$\mathcal{L}(X_1 = 1; 3, \frac{1}{2}) = \binom{3}{1} 0.5^3 = \frac{3}{8}$$

$$\mathcal{L}(X_1 = 1; 3, \frac{1}{3}) = \binom{3}{1} \frac{1}{3} (\frac{2}{3})^2 = \frac{4}{9}$$

The values for **n** and **p** that maximize the likelihood function are $(\mathbf{2}, \mathbf{\frac{1}{2}})$