Voluntary Exercises - Module 1

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18/09/2024

1 Consider a s.r.s. of size n = 48 of the r.v. X with uniform distribution in (0, θ).

1.1 Find an approximate distribution for $\bar{X}$.

Let $\{X_1, ..., X_{48}\}$ be a s.r.s of the r.v. $X_i \sim U(0, \theta)$.

where:

$$E[X_i] = \frac{a+b}{2} = \frac{\theta}{2}$$

$$Var[X_i] = \frac{(b-a)^2}{12} = \frac{\theta^2}{12}$$

$\bar{X}$ of the s.r.s follows a normal distribution with parameters μ and $\sigma^2 \Rightarrow \bar{X} \sim N(\mu, \sigma^2)$.

$$\mu = E[\bar{X}] = E[\frac{1}{n}\sum_{i=1}^{n} X_i] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\sum_{i=1}^{n}\frac{\theta}{2} = \frac{n\theta}{2n} = \frac{\theta}{2}$$

$$\sigma^2 = Var(\bar{X}) = Var(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) + \frac{1}{n^2}2\sum_{i<j}^{n} Cov(X_i, X_j)$$

Note: As we are working with independent r.v. the covariance of any pair i,j will be zero.

It proceeds as it follows:

$$\frac{1}{n^2}\sum_{i=1}^{n}\frac{\theta}{12} = \frac{n\theta^2}{12n^2} = \frac{\theta^2}{12n} = \frac{\theta^2}{576}$$

So the the mean distribution is:

$$\bar{X} \sim N(\frac{\theta}{2}, \frac{\theta^2}{576})$$

1.2 If $\theta = 2$, what is the probability that the sample mean is between 5/6 and 7/6?

$$\bar{X} \sim N(1, 4/576)$$

In terms of s.t. deviation:

$$\bar{X} \sim N(1, 1/12)$$

To calculate the probability we must typify our values and the mean distribution.

$$P(5/6 \leq N(1,1/12) \leq 7/6) \Rightarrow P((\frac{5}{6}-1)12 \leq N(0,1) \leq (\frac{7}{6}-1)12) = P(-2 \leq Z \leq 2) = 0.9545$$

2 (Sampling distribution of the maximum). Let $\{X1, ..., X_n\}$ be a s.r.s. of a r.v. X with cumulative distribution function F. Calculate the sampling distribution of the statistic $T(X_1, ..., X_n) = X_{(n)}$.

The sample maximum is given by:

$$T(X_1, ..., X_n) = X_{(n)} = \max(X_1, ..., X_n)$$

The cumulative distribution function of $X_{(n)}$ is denoted by $F_{X_{(n)}}(x)$.

This means that the sample maximum $X_{(n)}$ is less than or equal to x if and only if all the sample values are less than or equal to x.

Hence, we have:

$$F_{X_{(n)}}(x) = P(X_{(n)}(x)) \Leftrightarrow F_{X_{(n)}}(x) = P(X_1 \leq x) \cdot P(X_2 \leq x) \cdots P(X_n \leq x)$$

As every X_i has the same F(x), we have:

$$F_{X_{(n)}}(x) = [F(x)]^n$$

To obtain the probability density function of $X_{(n)}$, denoted $f_{X_{(n)}}(x)$, we have to derivate $FX_{(n)}(x)$.

$$f_{X_{(n)}}(x) = \frac{d}{dx}[F(X)]^n = n[F(X)]^{n-1}f(x)$$

Where $f(x) = \frac{d}{dx}F(X)$ is the probability density function of the original r.v.