01204211 Discrete Mathematics
Lecture 19: Modular arithmetic

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Review

In previous lectures, we studied various properties of integers and primes, and discussed primality testing algorithms. In this lecture, we will dive deeper into **modular arithmetic**, where we work with integers that "wrap around" when reaching a particular value, called the "modulus".

An example

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$$
\begin{aligned}
a \bmod 12 &= 6 \bmod 12 \\
(a+7) \bmod 12 &= b \bmod 12 \\
(b-3) \bmod 12 &= c \bmod 12 \\
(c-10) \bmod 12 &= d \bmod 12
\end{aligned}
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This is a familiar system of linear equations, but with a little twist: a "modulus" at the end.

Congruence

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Formally, if

$$x \bmod m = y \bmod m,$$

we can write

$$x \equiv y \pmod{m}.$$

The system with the congruence notation

Let's rewrite our previous set of equations using this notation:

$$\begin{aligned} a &\equiv 6 \pmod{12} \\ a+7 &\equiv b \pmod{12} \\ b-3 &\equiv c \pmod{12} \\ c-10 &\equiv d \pmod{12} \end{aligned}$$

Now everything looks fairly much like normal equations. But do they behave the same?