01204211 Discrete Mathematics
Lecture 12: The pigeonhole principle and the birthday problem

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September 18, 2015

The sock problem

I have n pairs of socks. Each pair is different from the other pair. How many socks do I have to pick out to be sure that I have at least one matching pair.

The Pigeonhole Principle

The answer of the previous question seems obvious. But it appears to be very useful in numerous cases. It is called **the pigeonhole principle**.

The pigeonhole principle

If we put $n+1$ objects into n boxes, at least one box gets more than one objects.

Example

Assume that nobody is taller than 250 cm. In a group of 251 people, there are at least two people whose heights differ by at most 1cm.

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- But that's the worst case scenario, as it is more common to find people with the same birthday. (In the next class, we will try to see if there is a pair of students in the class with the same birthday.)
- So, let's think about the probability that there are two students with the same birthday in a room with 40 students.

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- Thus, the probability is $\frac{366}{366^2} = 0.0027$, very unlikely.

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  - The first person has 366 choices. The second one has $366 - 1 = 365$ choices. The third one has $366 - 2 = 364$ choices. Thus, the number of ways they do not share any birthdays is $366 \cdot 365 \cdot 364$.
  - Notice that this is the number of ordered subsets.
- Thus, the probability that they do not share birthdays is $\frac{366 \cdot 365 \cdot 364}{366^3} = 0.9918$. Thus the probability that two of them share a birthday is $1 - 0.9918 = 0.0082$.

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- Umm... how small is it?
- Again you can use a computer to compute the exact value of this quantity. For example, you may want to use Wolfram Alpha.
- Anyway, we will try to estimate it using basic mathematical tools.

General case: n days k people

- Let's continue on the general case. When we have k people and a year contains n days, the probability that no two people share the same birthday is

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$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{n^k}.$$
- If this number is very close to 0, then it is very unlikely that no two people share the same birthday, i.e., it is very likely that there exists two people with the same birthday.

A few tweaks

- Dealing with small numbers is sometimes troublesome. (The reason will be more apparent later when we start introducing the tools.) So let's consider the reciprocal instead:

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- The top term looks easy to deal with; the bottom one does not. Let's break up the product:

$$\left(\frac{n}{n}\right) \cdot \left(\frac{n}{n-1}\right) \cdot \left(\frac{n}{n-2}\right) \cdots \left(\frac{n}{n-k+1}\right).$$

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- If you look closely at this product, you can see that each term is at least one. In the beginning, the terms are very close to one and they get larger at the end.

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$$\ln\left(\left(\frac{n}{n}\right)\cdot\left(\frac{n}{n-1}\right)\cdot\left(\frac{n}{n-2}\right)\cdots\left(\frac{n}{n-k+1}\right)\right)$$

$$=\ln\left(\frac{n}{n}\right)+\ln\left(\frac{n}{n-1}\right)+\ln\left(\frac{n}{n-2}\right)+\cdots+\ln\left(\frac{n}{n-k+1}\right).$$

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- The terms do not look that much better. But there's a nice fact about the natural logarithms.

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So let's do that at Wolfram Alpha.

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We know that

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If we use the fact that $\ln \frac{1}{x} = -\ln x$, we can obtain the lower bound.

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Let's conclude by stating the lemma:

Lemma 1

$$\frac{x-1}{x} \leq \ln x \leq x - 1.$$

The lower bound

Let's look at each term in the sum: $\ln\left(\frac{n}{n-j}\right)$. Using the lower bound in Lemma 1, we get that

$$\ln\left(\frac{n}{n-j}\right) \geq \frac{\frac{n}{n-j} - 1}{\frac{n}{n-j}} = \frac{\frac{n-n+j}{n-j}}{\frac{n}{n-j}} = \frac{j}{n}.$$

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Thus,

$$\ln\left(\left(\frac{n}{n}\right)\cdot\left(\frac{n}{n-1}\right)\cdot\left(\frac{n}{n-2}\right)\cdots\left(\frac{n}{n-k+1}\right)\right)$$

$$\begin{aligned}
&= \ln\left(\frac{n}{n}\right)+\ln\left(\frac{n}{n-1}\right)+\ln\left(\frac{n}{n-2}\right)+\cdots+\ln\left(\frac{n}{n-k+1}\right)\\
&\geq \frac{0}{n}+\frac{1}{n}+\frac{2}{n}+\cdots+\frac{k-1}{n}\\
&= \frac{1}{n}\left(1+2+\cdots+(k-1)\right) = \frac{k(k-1)}{2n}.
\end{aligned}$$

The upper bound

Again, let's look at each term in the sum: $\ln\left(\frac{n}{n-j}\right)$. Using the upper bound in Lemma 1, we get that

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$$\begin{aligned}
&\leq \frac{0}{n-0} + \frac{1}{n-1} + \frac{2}{n-2} + \cdots + \frac{k-1}{n-k+1} \\
&\leq \frac{0}{n-k+1} + \frac{1}{n-k+1} + \frac{2}{n-k+1} + \cdots + \frac{k-1}{n-k+1} \\
&= \frac{1}{n-k+1}\left(1 + 2 + \cdots + (k-1)\right) = \frac{k(k-1)}{2(n-k+1)}.
\end{aligned}$$

Both

Using the derived upper and lower bounds, we get

$$e^{\frac{k(k-1)}{2n}} \leq \frac{n^k}{n(n-1)(n-2)\cdots(n-k+1)} \leq e^{\frac{k(k-1)}{2(n-k+1)}}$$

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So the probability that we get no two people with the same birthday is between $1/8.42 \approx 0.118$ and $1/10.86 \approx 0.092$. So we have high chance of finding two students with the same birthday.

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