# 01204211 Discrete Mathematics Lecture 5: Proof techniques 2

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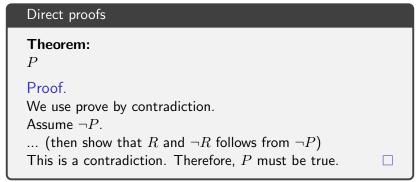
## Proof techniques<sup>1</sup>

In this lecture, we will focus on two other proof techniques.

- Proofs by contradiction
- Proofs by cases

### Proofs by contradiction

We want to prove that proposition P is true. To do so, we first assume that P is false, and show that this logically leads to a contradiction. This means that it is impossible for P to be false; hence, P has to be true. This is called a proof by contradiction or reductio ad absurdum.



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Let's square both terms. We get  $2 = a^2/b^2$ , or

$$a^2 = 2b^2.$$

(cont. in next slide)



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By definition, we know that  $a^2$  is an even number. From a theorem from last time, we know that a must also be an even number.

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$$2b^2 = (2k)^2 = 4k^2,$$

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Therefore,  $\sqrt{2}$  must be irrational.

### Proofs by cases

- ► The last proof technique that we shall discuss is closely related to proofs by exhaustion we tried before.
- Sometimes when we want to prove a statement, there are many possible cases. Also, we might not know which cases are true.
- We might still be able to prove the statement if we can show that the statement is true in every case.

#### Theorem 2

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#### Proof.

Let's split the process of picking 4 socks into 2 steps. First, pick 3 socks, then pick the last sock.

After we pick the first 3 socks. There are 2 possible cases: either I have a pair of socks with the same color, or I do not have such a pair. We shall consider each case separately.

(cont. in the next slide)

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- ▶ Case 2: I do not have a pair of socks with the same color. In this case, since I have 3 colors and 3 socks, I must have one sock for each color. Now, after we pick the last sock, whatever color the last one is, we have a color-matching sock in our first 3 socks. Therefore, the theorem is also true in this case.

### Proof. (cont.)

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Since these two cases cover all possibilities, we conclude that the theorem is true.

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P \lor Q \lor R
P \Rightarrow S
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Sometimes, when we have 2 cases, we also see:

$$P \vee \neg P$$

$$P \Rightarrow S$$

$$\neg P \Rightarrow S$$

$$S$$

Note that we can leave  $P \vee \neg P$  out, because it is always true.