01204211 Discrete Mathematics
Lecture 14: Binomial Coefficients (2)

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The binomial coefficients[1]

[1] This lecture mostly follows Chapter 3 of [LPV].

Let's see the actual value of the binomial coefficients $\binom{n}{\cdot}$.

What do you see?

- The function $\binom{n}{\cdot}$ is symmetric around $n/2$.
- Why? This is true because we know that $\binom{n}{k} = \binom{n}{n-k}$.
- The maximum is at the middle, i.e., when n is even the maximum is at $\binom{n}{n/2}$ and when n is odd, the maximum is at $\binom{n}{\lfloor n/2 \rfloor}$ and $\binom{n}{\lceil n/2 \rceil}$.
- Why? Can we prove that?

Largest in the middle

To understand the behavior of $\binom{n}{k}$ as k changes, let's look at two consecutive values:

$$\binom{n}{k} \ \heartsuit \ \binom{n}{k+1}$$

Let's write them out:

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \heartsuit \frac{n(n-1)(n-2)\cdots(n-k)}{(k+1)k!}.$$

Removing common terms, we can see that we are comparing these two terms:

$$1 \heartsuit \frac{n-k}{k+1} \Leftrightarrow k \heartsuit \frac{n-1}{2},$$

that is,

- if $k < (n-1)/2$, $\binom{n}{k} < \binom{n}{k+1}$; and
- if $k > (n-1)/2$, $\binom{n}{k} > \binom{n}{k+1}$.

How large is the middle $\binom{n}{n/2}$

Here, to simplify the calculation, we shall only consider the case when n is even. Let's try to estimate the value of $\binom{n}{n/2}$ by finding its upper and lower bounds.
A simple upper bound can be obtain using the fact that $\binom{n}{n/2}$ counts subsets of certain size:

$$\binom{n}{n/2} < 2^n.$$

We can also get a lower bound by noting that the maximum must be at least the average, i.e.,

$$\binom{n}{n/2} \geq \frac{2^n}{n+1}$$

Combining both bounds, we get that

$$\frac{2^n}{n+1} \leq \binom{n}{n/2} < 2^n.$$

Let's plug in $n = 200$, and calculate the number of digits to see how close these bounds.

$$27.80 \approx 200 \cdot \log 2 - \log 201 \leq \log \binom{n}{n/2} < 200 \cdot \log 2 \approx 30.10$$

Can we get a better approximation?