

01204211 Discrete Mathematics

Lecture 8: Mathematical Induction 3

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Review: Mathematical Induction

Suppose that you want to prove that property $P(n)$ is true for every natural number n .

Suppose that we can prove the following two facts:

Base case: $P(1)$

Inductive step: For any $k \geq 1$, $P(k) \Rightarrow P(k + 1)$

The **Principle of Mathematical Induction** states that $P(n)$ is true for every natural number n .

The assumption $P(k)$ in the inductive step is usually referred to as **the Induction Hypothesis**.

The Induction Hypothesis

Theorem 1

For any integer $n \geq 1$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$.

The Induction Hypothesis

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Proof.

The statement $P(n)$ that we want to prove is

“ $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$ ”.

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The induction hypothesis is: $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2$.

We want to show $P(k+1)$, i.e.,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2.$$

Then...



Strengthening the Induction Hypothesis (1)

- Is the assumption

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2.$$

“strong” enough to prove

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 \quad ?$$

Why?

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Why?

- To prove $P(k+1)$, we need a “gap” between the LHS and 2, so that we can add $1/(k+1)$ without blowing up the RHS.

Strengthening the Induction Hypothesis (2)

- ▶ Let's see a few values of the sum:
 - ▶ $1/1 = 1$.

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- $1/1 + 1/4 + 1/9 + 1/16 \approx 1.4236.$

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Yes, there is a gap. But how large?

- ▶ We need the gap to be large enough to insert $1/(k+1)^2$.
- ▶ After a “mysterious” moment, we observe that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}.$$

Strengthening the Induction Hypothesis (3)

Theorem 2

For any integer $n \geq 1$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$.

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For any integer $n \geq 1$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$.

Proof.

(... the beginning is left out ...)

Inductive step: For $k \geq 1$, assume that $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} < 2 - \frac{1}{k}$.

Adding $1/(k+1)^2$ on both sides, we get

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2} \right).$$

Since $1/k - 1/(k+1) = 1/(k(k+1))$, we have that

$$1/(k+1) = 1/k - 1/(k(k+1)) < 1/k - 1/(k+1)^2.$$

Therefore, we conclude that

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2} \right) < 2 - \frac{1}{k+1},$$

as required. □

A Lesson learned

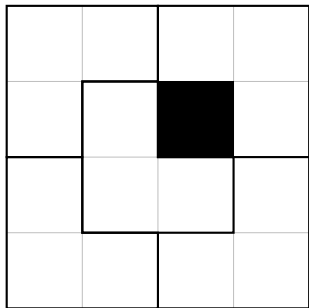
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A Lesson learned

- ▶ Is a stronger statement easier to prove?
- ▶ In this case, the statement is indeed stronger, but the induction hypothesis gets stronger as well. Sometimes, this works out nicely.

L-shaped tiles

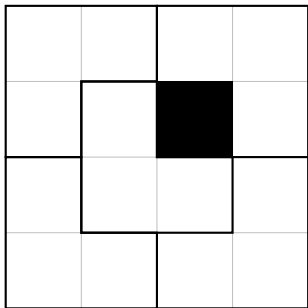
A 4x4 area with a hole in the middle can be tiled with L-shaped tiles.



This is true for 2x2 area, 8x8 area, even 16x16 area.

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This motivates us to try to prove that it is possible to use L-shaped tiles to tile a $2^n \times 2^n$ area.