01204211 Discrete Mathematics
Lecture 2: Quantifiers

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This lecture covers:

- Implications
- Quantifiers

Review (1)

- A *proposition* is a statement which is either **true** or **false**.
- We can use variables to stand for propositions, e.g., $P =$ "today is Tuesday".
- We can use connectives to combine variables to get propositional forms.
  - **Conjunction:** $P \wedge Q$ ("P and Q"),
  - **Disjunction:** $P \vee Q$ ("P or Q"), and
  - **Negation:** $\neg P$ ("not P")

Review (2)

To represents values of propositional forms, we usually use truth tables.

And/Or/Not

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$
T	T	T	T	F
T	F	F	T	
F	T	F	T	T
F	F	F	F	

Quick check 1

As we said before, the truth value of propositional forms may not depend on the values of its variables. As you can see in this exercise.
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Note that $P \wedge \neg P$ is always false and $P \vee \neg P$ is always true.

A propositional form which is always true regardless of the truth values of its variables is called a *tautology*. On the other hand, a propositional form which is always false regardless of the truth values of its variables is called a *contradiction*.

Implications

Given P and Q, an implication

$$P \Rightarrow Q$$

stands for "if P, then Q". This is a very important propositional form.

It states that "when P is true, Q must be true". Let's try to fill in its truth table:

Implications

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What?

- Yes, when P is false, $P \Rightarrow Q$ is **always true** no matter what truth value of Q is.
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- Yes, when P is false, $P \Rightarrow Q$ is **always true** no matter what truth value of Q is.
- We say that in this case, the statement $P \Rightarrow Q$ is *vacuously true.*
- You might feel a bit uncomfortable about this, because in most natural languages, when we say that if P, then Q we sometimes mean something more than that in the logical expression "$P \Rightarrow Q$."

One explanation

- But let's look closely at what it means when we say that:

  > if P is true, Q must be true.

- Note that this statement does not say anything about the case when P is false, i.e., it only considers the case when P is true.

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- Therefore, having that $P \Rightarrow Q$ is true is OK with the case that (1) Q is false when P is false, and (2) Q is true when P is false.
- This is an example when mathematical language is "stricter" than natural language.

Noticing if-then

We can write "if P, then Q" for $P \Rightarrow Q$, but there are other ways to say this. E.g., we can write (1) Q if P, (2) P only if Q, or (3) when P, then Q.

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Quick check 2

For each of these statements, define propositional variables representing each proposition inside the statement and write the proposition form of the statement.

- If you do not have enough sleep, you will feel dizzy during class.
- If you eat a lot and you do not have enough exercise, you will get fat.
- You can get A from this course, only if you work fairly hard.

Only-if

Let P be "you get A from this course."
Let Q be "you work fairly hard."
Let R be "You can get A from this course, only if you work fairly hard."
Let's think about the truth values of R.

Only if you work fairly hard.

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Thus, R should be logically equivalent to $P \Rightarrow Q$. (We write $R \equiv P \Rightarrow Q$ in this case.)

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the statement "P if and only if Q." It is logically equivalent to

$$(P \Leftarrow Q) \wedge (P \Rightarrow Q),$$

i.e., $P \Leftrightarrow Q \equiv (P \Leftarrow Q) \wedge (P \Rightarrow Q)$.
Let's fill in its truth table.

P	Q	$P \Rightarrow Q$	$P \Leftarrow Q$	$P \Leftrightarrow Q$
T	T			
T	F			
F	T			
F	F			

An implication and its friends

When you have two propositions

- $P =$ "I own a cell phone", and
- $Q =$ "I bring a cell phone to class".

We have

- an implication $P \Rightarrow Q \equiv$
  "If I own a cell phone, I'll bring it to class",
- its **converse** $Q \Rightarrow P \equiv$
  "If I bring a cell phone to class, I own it", and
- its **contrapositive** $\neg Q \Rightarrow \neg P \equiv$
  "If I do not bring a cell phone to class, I do not own one".

Quick check 3

Let's consider the following truth table:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$
T	T			
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Quick check 3

Let's consider the following truth table:

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Do you notice any equivalence?

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Do you notice any equivalence?

Right, $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$.

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- In many cases, the statement we are interested in contains variable.
- For example, "x is even," "p is prime," or "s is a student."
- As we previously did with propositions, we can use variables to represent these statements. E.g.,
  - let $E(x) \equiv$ "x is even",
  - let $P(y) \equiv$ "y is prime, and
  - let $S(w) \equiv$ "w is a student.

  We call $E(x)$, $P(y)$, and $S(w)$ *predicates*. (You can think of predicates as statements that may be true of false depending on the values of its variables.)

Quantifiers (1)

- As we note before, these predicates are not propositions. But if we know the value of the variables, then they becomes propositions. For example, if we let $x = 5$, then $E(5)$ is a proposition which is false. Also, $P(7)$ is true.
- Since the truth values of predicates depend on the assignments of its variables, we can put *quantifiers* to specify the scope of these variables and how to interprete the truth values of the predicates over these values.

Quantifiers (2): universal quantifiers

- Let $A = \{2, 4, 6, 8\}$.
- Note that $E(2), E(4), E(6)$, and $E(8)$ are true, i.e., $E(x)$ is true for every $x \in A$.

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- The quantifier $\forall$ is called a universal quantifier. (We usually pronounce "for all x", or "for every x.")

Quantifiers (3): existential quantifiers

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> When the universe A is clear, we can leave it out and just write $\forall x E(x)$ or $\exists y P(y)$.

The main goal

- Let's try to be more specific about our main goal:

  Algorithm `CheckPrime2` is correct.

- Can we re-write this statement so that the input/output of the algorithm are explicit?
- Note that the set of its input n is an integer. Thus, we are interested in every $n \in \mathbb{Z}$, where $\mathbb{Z}$ denote the set of all integers.
- Let's rewrite the goal as:

  $$\forall n \in \mathbb{Z},\ C(n) \Leftrightarrow P(n),$$

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where $C(n) \equiv$ "`CheckPrime2(n)` returns True", and
$P(n) \equiv$ "n is a prime."

Quantified propositions with more than one variables

Let our universe be integers ($\mathbb{Z}$). Which of the following statements is true?

- $\forall x \forall y (x = y)$
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When you have many quantifiers, we can interprete the statement by nesting the quantifiers. E.g,

$$\exists x \forall y P(x, y) \equiv \exists x (\forall y (P(x, y))).$$

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Also note that usually, $\exists x \forall y P(x, y) \not\equiv \forall y \exists x P(x, y)$.

Quick check 4

Quick check 5

- Let's consider the current subgoal. (Note that in this version, variable b is replaced with n/a.)

Another revised statement

For all positive composite integer n, and for every divisor a of n such that $\sqrt{n} < a < n$,

$$2 \leq n/a \leq \sqrt{n}.$$

- Define all required predicates and describe a quantified proposition equivalent to the revised statement above.

Negations (1)

Let consider a set of positive integers $\mathbb{Z}^+$ as our universe. Let predicate $P(x) \equiv$ "x is a prime number."
Consider this proposition

$$(\forall x \in \mathbb{Z}^+)P(x).$$

How can we show that this is false?

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This way of disproving a statement is equivalent to showing that

$$(\exists x)(\neg P(x)).$$

Negations of quantified propositions

Let consider a set of positive integers $\mathbb{Z}^+$ as our universe. Let predicate $Q(x) \equiv$ "if $x > 2$, then $x^2 \leq 2x$."
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How can we show that this is false?
When showing that an existential quantified proposition is false, we need to show that $Q(x)$ is false for every possible values of x. In this case, since $x^2 = x \cdot x > 2 \cdot x$ for every $x > 2$, we have that $(\exists x) Q(x)$ is false.

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Negations (3)

Thus, the following equivalences:

- $\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$
- $\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$

Quick check 6

How to prove a mathematical statement

Given propositions P and Q, these are a very useful logical equivalences (referred to as the De Morgan's Laws).

- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
- $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

(Note that $\neg$ takes precedence over $\vee$ or $\wedge$.)

How can we prove that the first statement is true?

Proof by exhaustion

For any proposition P and Q, $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$.

Proof.

We will prove by exhaustion. There are 4 cases as in the truth table below.

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg Q \wedge \neg P$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

Note that for all possible truth values of P and Q, $\neg(P \vee Q)$ equals $\neg P \wedge \neg Q$. Thus, the statement is true. $\square$