01204211 Discrete Mathematics
Lecture 13: Binomial Coefficients

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The binomial coefficients[1]

There is a reason why the term $\binom{n}{k}$ is called the binomial coefficients. In this lecture, we will discuss

- the Pascal's triangle,
- the binomial theorem, and
- advanced counting with binomial coefficients.

[1] This lecture mostly follows Chapter 3 of [LPV].

The equation

Last time we proved that, for $n, k > 0$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

While we can prove this equation algebraically using definitions of binomial coefficients, proving the fact by describing the process of choosing k-subsets reveals interesting insights. This equation also hints us how to compute the value of $\binom{n}{k}$ using values of $\binom{n}{\cdot}$'s. So, let's try to do it.

The table

We shall use the fact that $\binom{n}{0} = 1$ and $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ to fill in the following table.

n	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

You can note that the table is left-right symmetric. This is true because of the fact that $\binom{n}{k} = \binom{n}{n-k}$.

The Triangle

If we move the numbers in the table slightly to the right, the table becomes the Pascal's triangle.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & \vdots & & \vdots & & & & & & & & \vdots &
\end{array}
$$

The table and the binomial coefficients have many other interesting properties.

Polynomial expansions

Let's start by looking at polynomial of the form $(x+y)^n$. Let's start with small values of n:

- $(x+y)^1 = x+y$
- $(x+y)^2 = x^2 + 2 \cdot xy + y^2$
- $(x+y)^3 = x^3 + 3 \cdot x^2y + 3 \cdot xy^2 + y^3$
- $(x+y)^4 = x^4 + 4 \cdot x^3y + 6 \cdot x^2y^2 + 4 \cdot xy^3 + y^4$.

Let's focus on the coefficient of each term. You may notice that terms x^n and y^n always have 1 as their coefficients. *Why is that?* Let's look further at the coefficients of terms $x^{n-1}y$. Do you see any pattern in their coefficients? *Can you explain why?*

Another way to look at it

Let's take a look at $(x+y)^4$ again. It is

$$(x+y)(x+y)(x+y)(x+y).$$

- How do we get x^4 in the expansion? For every factory, you have to pick x.
- How do we get xy^3 in the expansion? Out of the 4 factors, you have to pick y in one of the factor (or you have to pick x in 3 of the factors). Thus there are $\binom{4}{3} = \binom{4}{1}$ ways to do so.

The binomial theorem

Theorem: If you expand $(x+y)^n$, the coefficient of the term $x^k y^{n-k}$ is $\binom{n}{k}$.

That is,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} =$$

$$\binom{n}{n} x^n + \binom{n}{n-1} x^{n-1} y^1 + \binom{n}{n-2} x^{n-2} y^2 + \cdots + \binom{n}{1} x y^{n-1} + \binom{n}{0} y^n.$$

Additional applications of the binomial theorem

The binomial theorem can be used to prove various identities regarding the binomial coefficients. For example, if we let $x = 1$ and $y = 1$, we get that

$$(1+1)^n = 2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

Quick check. Can you prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = 0.$$

Note that this statements says that the number of odd subsets equals the number of even subsets.

More on counting

We shall see more techniques for counting when we consider the following problems.

- How many anagrams does the word "KASETSARTUNIVERSITY" have? (They do not have to be real English words.)
- How can you give out n different presents to k students when student i has to get n_i pieces of presents?
- How many ways can you distribute n baht coins to k children?

Easy anagrams

- An anagram of a particular word is a word that uses the same set of alphabets. For example, the anagrams of ADD are ADD, DAD, and DDA.
- How many anagrams does "ABC" have?
  - $3!$, because every permutation of A B or C is a different anagram.
- How many anagrams does "$ABCC$" have?
  - This time we have to be careful because the answer of $4!$ is too large as it over counts many anagrams.
  - How many times does "$CABC$" get counted in $4!$?
  - If we treat two C's differently as C_1 and C_2, we can see that $CABC$ is counted twice as C_1ABC_2 and C_2ABC_1. This is true for any anagram of $ABCC$.
  - Since each anagram is counted in $4!$ twice, the number of anagrams is $4!/2 = 4 \cdot 3 = 12$.

General anagrams

Let's try to use the same approach to count the anagram of $HELLOWORLD$. (It has 3 L's, 2 O's, H, E, W, R, and D.)

The number of permutation of alphabets in $HELLOWORLD$, treating each character differently is $10!$. However, each anagram is counted for $3!2!$ times because of the 3 copies of L and the 2 copies of O. Therefore, the number of anagrams is

$$\frac{10!}{3!2!}.$$

Distributing presents

I have 9 different presents. I want to give them to 3 students: A, B, and C. I want to give each student 3 presents. In how many ways can I do it?

This is essentially an anagram problem. You can think of one particular way of present distribution as anagram of AAABBBCCC. Thus, the number is

$$\frac{9!}{3!3!3!}.$$