

# 01204211 Discrete Mathematics

## Lecture 8: Mathematical Induction 3

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# Review: Mathematical Induction

Suppose that you want to prove that property  $P(n)$  is true for every natural number  $n$ .

Suppose that we can prove the following two facts:

**Base case:**  $P(1)$

**Inductive step:** For any  $k \geq 1$ ,  $P(k) \Rightarrow P(k + 1)$

The **Principle of Mathematical Induction** states that  $P(n)$  is true for every natural number  $n$ .

The assumption  $P(k)$  in the inductive step is usually referred to as **the Induction Hypothesis**.

# The Induction Hypothesis

## Theorem 1

*For any integer  $n \geq 1$ ,  $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$ .*

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## Proof.

The statement  $P(n)$  that we want to prove is

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The induction hypothesis is:  $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2$ .

We want to show  $P(k+1)$ , i.e.,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2.$$

Then...



# Strengthening the Induction Hypothesis (1)

- Is the assumption

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“strong” enough to prove

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 \quad ?$$

Why?



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Why?

- To prove  $P(k+1)$ , we need a “gap” between the LHS and 2, so that we can add  $1/(k+1)$  without blowing up the RHS.

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- $1/1 + 1/4 + 1/9 + 1/16 \approx 1.4236.$

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Yes, there is a gap. But how large?

- ▶ We need the gap to be large enough to insert  $1/(k+1)^2$ .
- ▶ After a “mysterious” moment, we observe that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}.$$



# Strengthening the Induction Hypothesis (3)

## Theorem 2

*For any integer  $n \geq 1$ ,  $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ .*

# Strengthening the Induction Hypothesis (3)

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## Proof.

(... the beginning is left out ...)

**Inductive step:** For  $k \geq 1$ , assume that  $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2 - \frac{1}{k}$ .  
Adding  $1/(k+1)^2$  on both sides, we get

$$\begin{aligned} \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &< 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\ &= 2 - \left( \frac{(k+1)^2 - k}{k(k+1)^2} \right). \end{aligned}$$

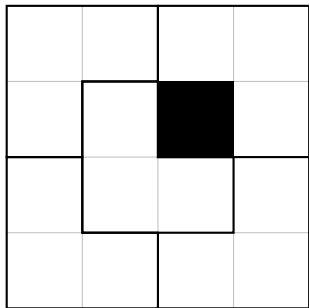
Since one can verify that  $\frac{(k+1)^2 - k}{k(k+1)^2} > \frac{1}{k+1}$ , we conclude that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1},$$

as required. □

## L-shaped tiles

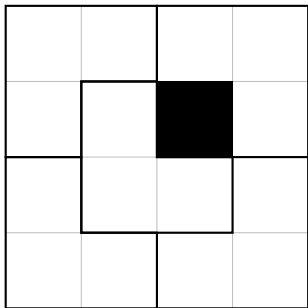
A 4x4 area with a hole in the middle can be tiled with L-shaped tiles.



This is true for 2x2 area, 8x8 area, even 16x16 area.

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This is true for 2x2 area, 8x8 area, even 16x16 area.

This motivates us to try to prove that it is possible to use L-shaped tiles to tile a  $2^n \times 2^n$  area.