

01204211 Discrete Mathematics

Lecture 19: Modular arithmetic 1

Jittat Fakcharoenphol

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The jug puzzle

Possibilities

Integer linear combinations

The minimum integer linear combinations

Review

In previous lectures, we studied various properties of integers and primes, and discussed primality testing algorithms. In this lecture, we will dive deeper into **modular arithmetic**, where we work with integers that “wrap around” when reaching a particular value, called the “modulus”.

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$$\begin{aligned}a \bmod 12 &= 6 \bmod 12 \\(a + 7) \bmod 12 &= b \bmod 12 \\(b - 3) \bmod 12 &= c \bmod 12 \\(c - 10) \bmod 12 &= d \bmod 12\end{aligned}$$

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This is a familiar system of linear equations, but with a little twist: a “modulus” at the end.

Congruence

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Formally, if

$$x \bmod m = y \bmod m,$$

we can write

$$x \equiv y \pmod{m}.$$

The system with the congruence notation

Let's rewrite our previous set of equations using this notation:

$$\begin{aligned}a &\equiv 6 \pmod{12} \\ a + 7 &\equiv b \pmod{12} \\ b - 3 &\equiv c \pmod{12} \\ c - 10 &\equiv d \pmod{12}\end{aligned}$$

Now everything looks fairly much like normal equations. But do they behave the same?

Addition, subtraction, and multiplication

Suppose that, for a positive integer q , we know that

$$a \equiv b \pmod{q},$$

and

$$c \equiv d \pmod{q}.$$

It is not hard to show that

$$a + c \equiv b + d \pmod{q},$$

$$a - c \equiv b - d \pmod{q},$$

and

$$ac \equiv bd \pmod{q}.$$

Thus, we can treat a system of congruences in the same way we deal with a system of linear equations, except the division.