

# 01204211 Discrete Mathematics

## Lecture 8: Mathematical Induction 3

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September 21, 2015

# Review: Mathematical Induction

Suppose that you want to prove that property  $P(n)$  is true for every natural number  $n$ .

Suppose that we can prove the following two facts:

**Base case:**  $P(1)$

**Inductive step:** For any  $k \geq 1$ ,  $P(k) \Rightarrow P(k + 1)$

The **Principle of Mathematical Induction** states that  $P(n)$  is true for every natural number  $n$ .

The assumption  $P(k)$  in the inductive step is usually referred to as **the Induction Hypothesis**.

# The Induction Hypothesis

## Theorem 1

*For any integer  $n \geq 1$ ,  $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$ .*

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The induction hypothesis is:  $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2$ .

We want to show  $P(k+1)$ , i.e.,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2.$$

Then...



# Strengthening the Induction Hypothesis (1)

- Is the assumption

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2.$$

“strong” enough to prove

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 \quad ?$$

Why?



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Why?

- To prove  $P(k+1)$ , we need a “gap” between the LHS and 2, so that we can add  $1/(k+1)$  without blowing up the RHS.

## Strengthening the Induction Hypothesis (2)

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- $1/1 + 1/4 + 1/9 + 1/16 \approx 1.4236.$

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Yes, there is a gap. But how large?

- ▶ We need the gap to be large enough to insert  $1/(k+1)^2$ .
- ▶ After a “mysterious” moment, we observe that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$



# Strengthening the Induction Hypothesis (3)

## Theorem 2

*For any integer  $n \geq 1$ ,  $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ .*

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**Inductive step:** For  $k \geq 1$ , assume that  $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$ .

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Adding  $1/(k+1)^2$  on both sides, we get

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \left( \frac{1}{k} - \frac{1}{(k+1)^2} \right).$$

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Since  $1/k - 1/(k+1) = 1/(k(k+1))$ , we have that

$$1/(k+1) = 1/k - 1/(k(k+1)) < 1/k - 1/(k+1)^2.$$

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Since  $1/k - 1/(k+1) = 1/(k(k+1))$ , we have that

$$1/(k+1) = 1/k - 1/(k(k+1)) < 1/k - 1/(k+1)^2.$$

Therefore, we conclude that

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \left( \frac{1}{k} - \frac{1}{(k+1)^2} \right) \leq 2 - \frac{1}{k+1},$$

as required. □

# A Lesson learned

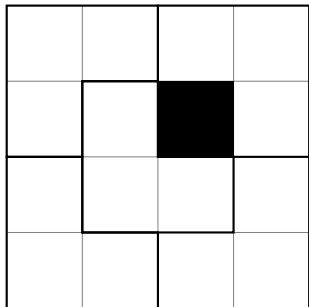
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## A Lesson learned

- ▶ Is a stronger statement easier to prove?
- ▶ In this case, the statement is indeed stronger, but the induction hypothesis gets stronger as well. Sometimes, this works out nicely.

## L-shaped tiles (1)<sup>1</sup>

A 4x4 area with a hole in the middle can be tiled with L-shaped tiles.



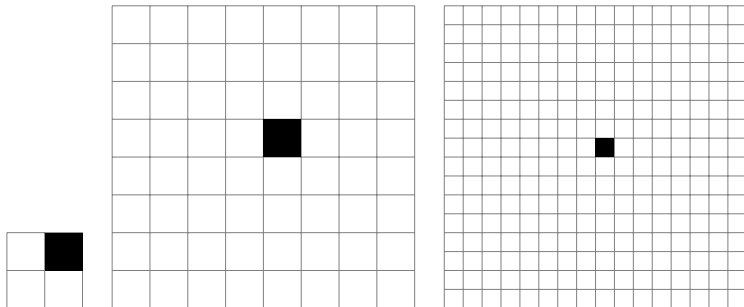
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<sup>1</sup>This section is from Berkeley CS70 lecture notes.



## L-shaped tiles (2)

This is true for  $2 \times 2$  area,  $8 \times 8$  area, even  $16 \times 16$  area.



This motivates us to try to prove that it is possible to use L-shaped tiles to tile a  $2^n \times 2^n$  area.

# Proving the fact?

## Theorem 3

*For integer  $n \geq 1$ , an area of size  $2^n \times 2^n$  with one hole in the middle can be tiled with L-shaped tiles.*

**Proof:** We prove by induction on  $n$ .

**Base case:** For  $n = 1$ ,  $2^1 \times 2^1$  area with a hole in the middle can be tiled.

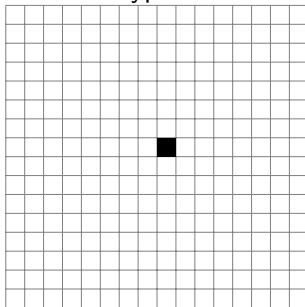
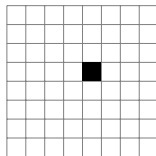
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Proof: (cont.)

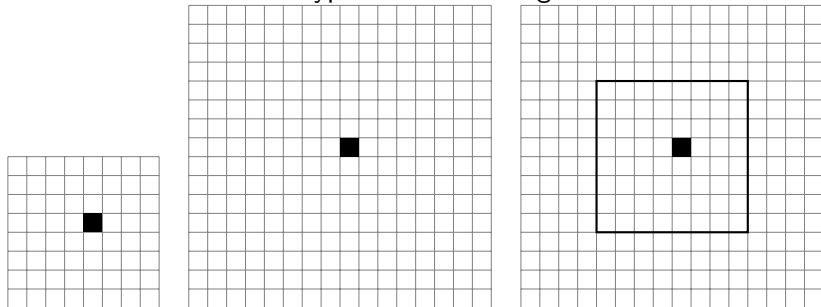
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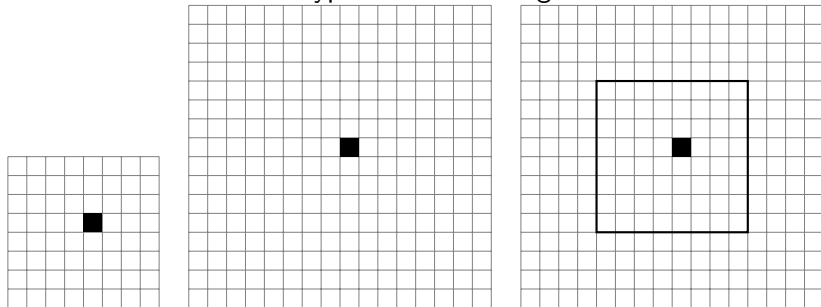


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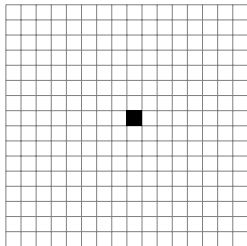
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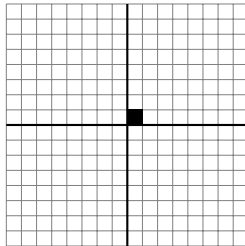
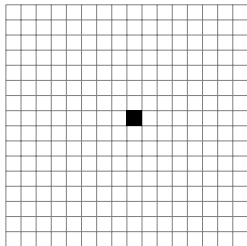


With the current form of the Induction Hypothesis, this is probably the way to use it. But it seems hard to go further with this approach....

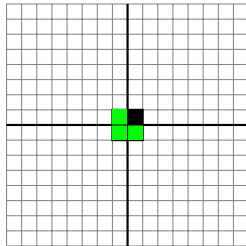
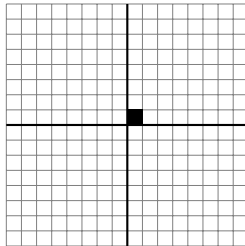
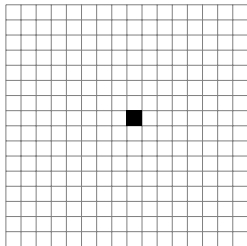
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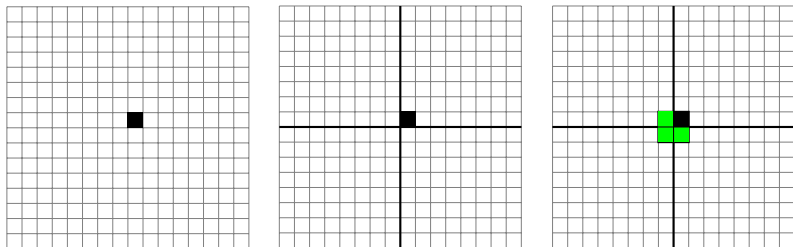


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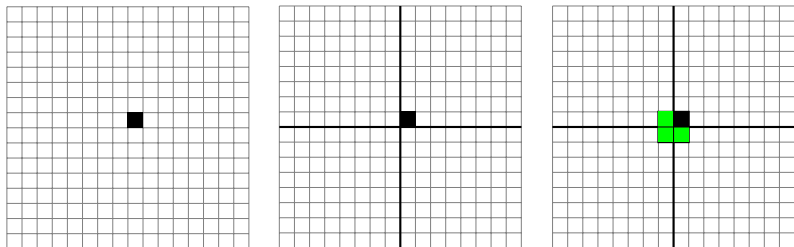


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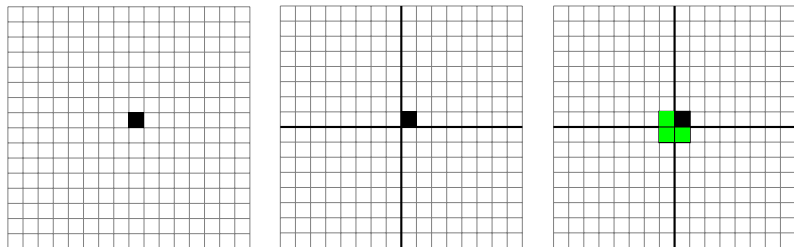
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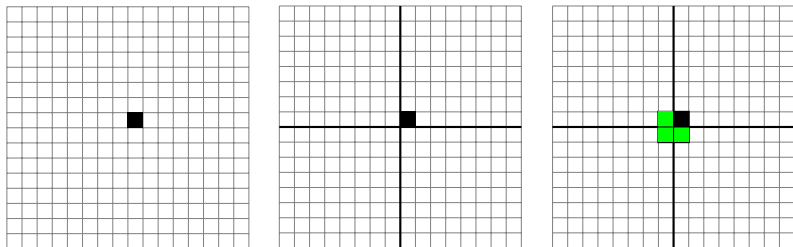
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**Current Inductive Hypothesis:** Assume that for  $k \geq 1$ , an  $2^k \times 2^k$  area with “a hole in the middle” can be tiled.

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**Current Inductive Hypothesis:** Assume that for  $k \geq 1$ , an  $2^k \times 2^k$  area with “a hole in the middle” can be tiled.

**A Stronger Inductive Hypothesis:** Assume that for  $k \geq 1$ , an  $2^k \times 2^k$  area with **one hole** can be tiled.

## A stronger statement

**Theorem:** For integer  $n \geq 1$ , an area of size  $2^n \times 2^n$  with one hole can be tiled with L-shaped tiles.

**Proof:** We prove by induction on  $n$ .

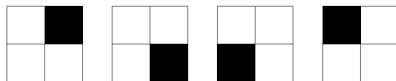
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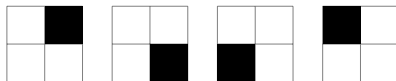
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(Try to finish it in homework.)




# Proof of the Principle of Mathematical Induction<sup>2</sup>

## Theorem 4

*If  $P(1)$  and for any integer  $k \geq 1$ ,  $P(k) \Rightarrow P(k + 1)$ , then  $P(n)$  for all natural number  $n$ .*

Proof.

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Since  $m$  is smallest and  $m > 1$ , then  $P(m - 1)$  must be true.

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Since  $m$  is smallest and  $m > 1$ , then  $P(m-1)$  must be true.

However, because for any integer  $k \geq 1$ ,  $P(k) \Rightarrow P(k+1)$ , we can conclude that  $P(m)$  must be true. Again, we reach a contradiction.

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# Proof of the Principle of Mathematical Induction<sup>2</sup>

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*If  $P(1)$  and for any integer  $k \geq 1$ ,  $P(k) \Rightarrow P(k+1)$ , then  $P(n)$  for all natural number  $n$ .*

## Proof.


We prove by contradiction. Assume that  $P(n)$  is not true for some natural number  $n$ . Let  $m$  be the smallest positive integer such that  $P(m)$  is false. If  $m = 1$ , we get a contradiction because we know that  $P(1)$  is true; therefore, we know that  $m > 1$ .

Since  $m$  is smallest and  $m > 1$ , then  $P(m-1)$  must be true.

However, because for any integer  $k \geq 1$ ,  $P(k) \Rightarrow P(k+1)$ , we can conclude that  $P(m)$  must be true. Again, we reach a contradiction.

Therefore,  $P(n)$  is true for every positive integer  $n$ . □

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Is this proof correct?

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- ▶ Previously, we use the well-ordering property of natural numbers to prove the Principle of Mathematical Induction, but it turns out that we can use the induction to prove the well-ordering property as well. Therefore, we can take one as an axiom, and use it to prove the other.