01204211 Discrete Mathematics
Lecture 13: Binomial Coefficients

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The binomial coefficients

There is a reason why the term $\binom{n}{k}$ is called the binomial coefficients. In this lecture, we will discuss

- the Pascal's triangle,
- the binomial theorem, and
- advanced counting with binomial coefficients.

The equation

Last time we proved that, for $n, k > 0$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

While we can prove this equation algebraically using definitions of binomial coefficients, proving the fact by describing the process of choosing k-subsets reveals interesting insights. This equation also hints us how to compute the value of $\binom{n}{k}$ using values of $\binom{n}{\cdot}$'s. So, let's try to do it.

The table

We shall use the fact that $\binom{n}{0} = 1$ and $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ to fill in the following table.

n	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

You can note that the table is left-right symmetric. This is true because of the fact that $\binom{n}{k} = \binom{n}{n-k}$.

The Triangle

If we move the numbers in the table slightly to the right, the table becomes the Pascal's triangle.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & \vdots & & \vdots & & & & & & & & \vdots &
\end{array}
$$

The table and the binomial coefficients have many other interesting properties.

Polynomial expansions

Let's start by looking at polynomial of the form $(x+y)^n$. Let's start with small values of n:

- $(x+y)^1 = x+y$
- $(x+y)^2 = x^2 + 2 \cdot xy + y^2$
- $(x+y)^3 = x^3 + 3 \cdot x^2y + 3 \cdot xy^2 + y^3$
- $(x+y)^4 = x^4 + 4 \cdot x^3y + 6 \cdot x^2y^2 + 4 \cdot xy^3 + y^4$.

Let's focus on the coefficient of each term. You may notice that terms x^n and y^n always have 1 as their coefficients. *Why is that?* Let's look further at the coefficients of terms $x^{n-1}y$. Do you see any pattern in their coefficients? *Can you explain why?*