# 01204211 Discrete Mathematics Lecture 8: Mathematical Induction 3

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### Review: Mathematical Induction

Suppose that you want to prove that property P(n) is true for every natural number n.

Suppose that we can prove the following two facts:

Base case: P(1)

**Inductive step:** For any  $k \ge 1$ ,  $P(k) \Rightarrow P(k+1)$ 

The **Principle of Mathematical Induction** states that P(n) is true for every natural number n.

The assumption P(k) in the inductive step is usually referred to as the Induction Hypothesis.

# The Induction Hypothesis

#### Theorem 1

For any integer  $n \ge 1$ ,  $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} < 2$ .

#### Proof.

The statement P(n) that we want to prove is " $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}<2$ ".

Case case: For n = 1, the statement is true because 1 < 2.

**Inductive step:** For  $k \geq 1$ , let's assume P(k) and we prove that P(k+1) is true.

The induction hypothesis is:  $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} < 2$ .

We want to show P(k+1), i.e.,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2.$$

Then...

# Strengtening the Induction Hypothesis (1)

Is the assumption

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} < 2.$$

"strong" enough to prove

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2$$
?

Why?

▶ To prove P(k+1), we need a "gap" between the LHS and 2, so that we can add 1/(k+1) without blowing up the RHS.

# Strengtening the Induction Hypothesis (2)

- Let's see a few values of the sum:
  - ▶ 1/1 = 1.
  - 1/1 + 1/4 = 1.25.
  - $1/1 + 1/4 + 1/9 \approx 1.361.$
  - $1/1 + 1/4 + 1/9 + 1/16 \approx 1.4236.$
  - $1/1 + 1/4 + 1/9 + 1/16 + 1/25 \approx 1.4636.$

Yes, there is a gap. But how large?

- ▶ We need the gap to be large enough to insert  $1/(k+1)^2$ .
- After a "mysterious" moment, we observe that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} \le 2 - \frac{1}{n}.$$

# Strengtening the Induction Hypothesis (3)

#### Theorem 2

For any integer 
$$n \ge 1$$
,  $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} \le 2 - \frac{1}{n}$ .

#### Proof.

(... the beginning is left out ...)

Inductive step: For  $k \geq 1$ , assume that  $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$ .

Adding  $1/(k+1)^2$  on both sides, we get

$$\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \le 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right).$$

Since 1/k - 1/(k+1) = 1/(k(k+1)), we have that

$$1/(k+1) = 1/k - 1/(k(k+1)) < 1/k - 1/(k+1)^{2}.$$

Therefore, we conclude that

$$\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \le 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right) \le 2 - \frac{1}{k+1},$$

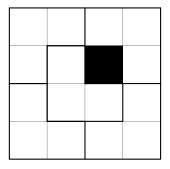
as required.

### A Lesson learned

- Is a stronger statement easier to prove?
- ▶ In this case, the statement is indeed stronger, but the induction hypothesis gets stronger as well. Sometimes, this works out nicely.

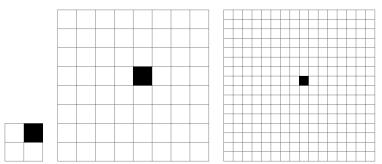
# L-shaped tiles (1)

A 4x4 area with a hole in the middle can be tiled with L-shaped tiles.



## L-shaped tiles (2)

This is true for 2x2 area, 8x8 area, even 16x16 area.



This motivates us to try to prove that it is possible to use L-shaped tiles to tile a  $2^n \times 2^n$  area.

## Proving the fact?

#### Theorem 3

For integer  $n \ge 1$ , an area of size  $2^n \times 2^n$  with one hole in the middle can be tiled with L-shaped tiles.

Proof: We prove by induction on n.

Base case: For n=1,  $2^1\times 2^1$  area with a hole in the middle can be tiled.

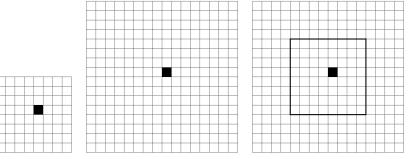
**Inductive step:** Assume that for  $k \ge 1$ , an  $2^k \times 2^k$  area with a hole in the middle can be tiled. We shall prove the statement for n=k+1, i.e., that an  $2^{k+1} \times 2^{k+1}$  area with one hole in the middle can be tiled.

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## Proving the fact?

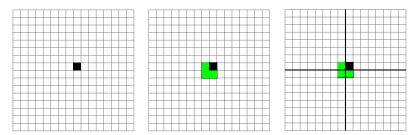
### Proof: (cont.)

Let's see the Induction Hypothesis and the goal:



With the current form of the Induction Hypothesis, this is probably the way to use it. But it seems hard to go further with this approach....

## Let's try a different approach



The last step seems nice, because it shows how we can solve the problem in the  $2^{k+1} \times 2^{k+1}$  area with 4 problems in the  $2^k \times 2^k$  areas. But do you see an issue with this approach regarding the Induction Hypothesis?

Current Inductive Hypothesis: Assume that for  $k \ge 1$ , an  $2^k \times 2^k$  area with "a hole in the middle" can be tiled.

**A Stronger Inductive Hypothesis:** Assume that for  $k \geq 1$ , an  $2^k \times 2^k$  area with one hole can be tiled.

### A stronger statement

Theorem: For integer  $n \ge 1$ , an area of size  $2^n \times 2^n$  with one hole can be tiled with L-shaped tiles.

Proof: We prove by induction on n.

Base case: For  $n=1,\ 2^1\times 2^1$  area with one hole can be tiled; there are 4 cases shown below.



**Inductive step:** Assume that for  $k \geq 1$ , an  $2^k \times 2^k$  area with one hole can be tiled. We shall prove the statement for n=k+1, i.e., that an  $2^{k+1} \times 2^{k+1}$  area with one hole can be tiled. (Try to finish it in homework.)

### Proof of the Principle of Mathematical Induction

#### Theorem 4

If P(1) and for any integer  $k \ge 1$ ,  $P(k) \Rightarrow P(k+1)$ , then P(n) for all natural number n.

#### Proof.

We prove by contradiction. Assume that P(n) is not true for some natural number n. Let m be the smallest positive integer such that P(m) is false. If m=1, we get a contradiction because we know that P(1) is true; therefore, we know that m>1.

Since m is smallest and m>1, then P(m-1) must be true. However, because for any integer  $k\geq 1$ ,  $P(k)\Rightarrow P(k+1)$ , we can conclude that P(m) must be true. Again, we reach a contradiction.

Therefore, P(n) is true for every positive integer n.

Is this proof correct?

### The Well-Ordering Property

▶ The proof of the Principle of Mathematical Induction depends on the following axiom of natural numbers N:

The Well-Ordering Property: Any nonempty subset  $S \subseteq \mathbb{N}$  contains the smallest element.

Previously, we use the well-ordering property of natural numbers to prove the Principle of Mathematical Induction, but it turns out that we can use the induction to prove the well-ordering property as well. Therefore, we can take one as an axiom, and use it to prove the other.