01204211 Discrete Mathematics
Lecture 15: Fibonacci sequence

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The Fibonacci sequence[1]

Source:

https://en.wikipedia.org/wiki/

File:Fibonacci.jpg

In 1202, Leonardo Bonacci (known as Fibonacci) asked the following question.

> "[A]ssuming that: a newly born pair of rabbits, one male, one female, are put in a field; rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on."
>
> "The puzzle that Fibonacci posed was: how many pairs will there be in one year?"

From https://en.wikipedia.org/wiki/Fibonacci_number

[1]This lecture mostly follows Chapter 4 of [LPV].

Let's try to solve Fibonacci's question.

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Let ♠ denote a newly born rabit pair, and ♡ denote a mature rabit pair.

Month	Rabits	
1	♠	1

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Let $\spadesuit$ denote a newly born rabit pair, and $\heartsuit$ denote a mature rabit pair.

Month	Rabits	
1	$\spadesuit$	1
2	$\heartsuit$	1

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Month	Rabits	
1	♠	1
2	♡	1
3	♡	

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Month	Rabits	
1	♠	1
2	♡	1
3	♡ ♠	2

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Month	Rabits	
1	♠	1
2	♡	1
3	♡ ♠	2
4	♡ ♡	

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1	♠	1
2	♡	1
3	♡ ♠	2
4	♡ ♡ ♠	3

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Month	Rabits	
1	♠	1
2	♡	1
3	♡ ♠	2
4	♡ ♡ ♠	3
5	♡ ♡ ♡	

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Month	Rabits	
1	♠	1
2	♡	1
3	♡ ♠	2
4	♡ ♡ ♠	3
5	♡ ♡ ♡ ♠ ♠	5

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Month	Rabits	
1	♠	1
2	♡	1
3	♡ ♠	2
4	♡ ♡ ♠	3
5	♡ ♡ ♡ ♠ ♠	5
6	♡ ♡ ♡ ♡ ♡	

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Month	Rabits	
1	♠	1
2	♡	1
3	♡ ♠	2
4	♡ ♡ ♠	3
5	♡ ♡ ♡ ♠ ♠	5
6	♡ ♡ ♡ ♡ ♡ ♠ ♠ ♠	8

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1	♠	1
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3	♡ ♠	2
4	♡ ♡ ♠	3
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7	♡ ♡ ♡ ♡ ♡ ♡ ♡ ♡ ♠ ♠ ♠ ♠ ♠	13

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1	♠	1
2	♡	1
3	♡ ♠	2
4	♡ ♡ ♠	3
5	♡ ♡ ♡ ♠ ♠	5
6	♡ ♡ ♡ ♡ ♡ ♠ ♠ ♠	8
7	♡ ♡ ♡ ♡ ♡ ♡ ♡ ♡ ♠ ♠ ♠ ♠ ♠	13

How many rabit pairs do we have at the beginning of the 8th month?

Let's try to solve Fibonacci's question.

Let $\spadesuit$ denote a newly born rabit pair, and $\heartsuit$ denote a mature rabit pair.

Month	Rabits	
1	$\spadesuit$	1
2	$\heartsuit$	1
3	$\heartsuit\ \spadesuit$	2
4	$\heartsuit\ \heartsuit\ \spadesuit$	3
5	$\heartsuit\ \heartsuit\ \heartsuit\ \spadesuit\ \spadesuit$	5
6	$\heartsuit\ \heartsuit\ \heartsuit\ \heartsuit\ \heartsuit\ \spadesuit\ \spadesuit\ \spadesuit$	8
7	$\heartsuit\ \heartsuit\ \heartsuit\ \heartsuit\ \heartsuit\ \heartsuit\ \heartsuit\ \heartsuit\ \spadesuit\ \spadesuit\ \spadesuit\ \spadesuit\ \spadesuit$	13

How many rabit pairs do we have at the beginning of the 8th month?

- Surely all 13 rabit pairs we have in the 7th month remain there and are all mature. So, the question is how many newly born rabbit pairs that we have.

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3	♡ ♠	2
4	♡ ♡ ♠	3
5	♡ ♡ ♡ ♠ ♠	5
6	♡ ♡ ♡ ♡ ♡ ♠ ♠ ♠	8
7	♡ ♡ ♡ ♡ ♡ ♡ ♡ ♡ ♠ ♠ ♠ ♠ ♠	13

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- Surely all 13 rabit pairs we have in the 7th month remain there and are all mature. So, the question is how many newly born rabbit pairs that we have.
- The number of newly born rabbit pairs equals the number of mature rabbit pairs we have.

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5	♡ ♡ ♡ ♠ ♠	5
6	♡ ♡ ♡ ♡ ♡ ♠ ♠ ♠	8
7	♡ ♡ ♡ ♡ ♡ ♡ ♡ ♡ ♠ ♠ ♠ ♠ ♠	13

How many rabit pairs do we have at the beginning of the 8th month?

- Surely all 13 rabit pairs we have in the 7th month remain there and are all mature. So, the question is how many newly born rabbit pairs that we have.
- The number of newly born rabbit pairs equals the number of mature rabbit pairs we have. This is also equal to the number of rabit pairs that we have in the 6th month: 8.

Thus, we will have 13+8 rabit pairs at the beginning of the 8th month.

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If we write down the sequence, we get the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

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Again, what's the next number in this sequence? How can you compute it?

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21+13 $=$ 34 is the answer.

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Again, what's the next number in this sequence? How can you compute it?
$21+13 = 34$ is the answer. You take the last two numbers and add them up to get the next number. Why?

To be precise, let F_n be the n-th number in the Fibonacci sequence. (That is, $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$ and so on.) We can define the $(n+1)$-th number as

$$F_{n+1} = F_n + F_{n-1},$$

for $n = 2, 3, \ldots$.

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$$F_{n+1} = F_n + F_{n-1},$$

for $n = 2, 3, \ldots$. Is this enough to completely specify the sequence? No, because we do not know how to start. To get the Fibonacci sequence, we need to specify two starting values: $F_1 = 1$ and $F_2 = 1$ as well.
Now, you can see that the equation and these special values uniquely determine the sequence. It is also convenient to define $F_0 = 0$ so that the equation works for $n = 1$.

A recurrence

The equation

$$F_{n+1} = F_n + F_{n-1}$$

and the initial values $F_0 = 0$ and $F_1 = 1$ specify all values of the Fibonacci sequence. With these two initial values, you can use the equation to find the value of any number in the sequence.

This definition is called a **recurrence**. Instead of defining the value of each number in the sequence explicitly, we do so by using the values of other numbers in the sequence.