01204211 Discrete Mathematics
Lecture 1: Introduction

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August 7, 2018

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Ah... that's a philosopical question.
IMHO, mathematics is a mean to communicate *precise* ideas.

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- I hope it is also true with this course.

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- To learn how to make mathematical arguments.
- To learn various fundamental mathematical concepts that are very useful in computer science.

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    return a
else:
    return b
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Finally, look at this program.

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// Input: array A with n elements: A[0],...,A[n-1]
m = 0
for i = 0, 1, ..., n-1:
  if A[i] > m:
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Can we do better?

Another example: testing primes (2)

Consider the following code.

```
Algorithm CheckPrime2(n):  // Input: an integer n
    if n <= 1:
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Ok, it should be faster. **But is it correct?**

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- Are we done?

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- How can we do that?

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- Before we continue, I'd like to add a bit of formalism to our thinking process.

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> **"Algorithm `CheckPrime2` is correct."**

- Note that this statement can either be "true" or "false." If we can demonstrate, using logical/mathematical arguments that this statement is true, we can say that we **prove** the statement.

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- Are we doom? Not really. The statement above is not precisely the statement we want to prove.

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- **Current (sub) goal:** Consider a positive composite n and its positive divisor a, where $a > \sqrt{n}$. Let $b = n/a$. We want to show that $2 \leq b \leq \sqrt{n}$.
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Revised statement

For all positive composite integer n, and for every divisor a of n such that $\sqrt{n} < a < n$,

$$2 \leq b \leq \sqrt{n},$$

where $b = n/a$.

- Note that this revised statement is now "quantified," that is, every variable in the statement has specific scope. Now the statement is either true or false.

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- Examples:
  - Algorithm `CheckPrime2` is correct.
  - $10^2 = 90$.
  - $\sqrt{2}$ is irrational.
- Examples of statements which are not propositions (why?):
  - $x > 10$.
  - $1 + 2 + \cdots + 10$.
  - This algorithm is fast.
  - Run, run quickly.

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Combining propositions

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- For example, we can join P and Q with "and" (denoted by "$\wedge$") and get
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which stands for "it rains and the road is wet".
- An expression $P \wedge Q$ is an example of *propositional forms*. The logical value of a propositional form "usually" depends on the truth value of its variables.

Connectives: "and", "or", "not"

Given propositions P and Q, we can use connectives to form more complex propositions:

- **Conjunction:** $P \wedge Q$ ("P and Q"), (True when both P and Q are true)
- **Disjunction:** $P \vee Q$ ("P or Q"), (True when at least one of P and Q is true)
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If P stands for "today is Tuesday" and Q stands for "dogs are animals", then

- $P \wedge Q$ stands for "today is Tuesday and dogs are animals",
- $P \vee Q$ stands for "today is Tuesday or dogs are animals", and
- $\neg P$ stands for "today is not Tuesday".

Truth tables

To represents values of propositional forms, we usually use truth tables.

And/Or/Not

P	Q	$P \land Q$	$P \lor Q$	$\neg P$
T	T	T	T	F
T	F	F	T	
F	T	F	T	T
F	F	F	F	

Quick check 1

For each of these statements, define propositional variables representing each proposition inside the statement and write the proposition form of the statement.

- All prime numbers are larger than 0 and all natural numbers is at least one.
- You are smart or you won't be taking this class.

Next lecture...

- We will discuss other ways to join two propositions, i.e., implications ($\Rightarrow$) and equivalences ($\Leftrightarrow$).
- We will look at two forms of quantifiers: universal quantifiers and existential quantifiers.