

01204211 Discrete Mathematics

Lecture 3: Inference rules


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How to prove a mathematical statement?

This lecture covers two fundamental concepts in mathematical proofs:

- ▶ Proofs by exhaustion
- ▶ Inference rules¹

¹The materials on inference rules are from [Rosen]. 

De Morgan's Laws

Given propositions P and Q , these are a very useful logical equivalences (referred to as the De Morgan's Laws).

- ▶ $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

- ▶ $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

(Note that \neg takes precedence over \vee or \wedge .)

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In this case, since there are not too many cases to consider, we can enumerate all the possibilities to show that the proposition is true.

Proof by exhaustion

For any proposition P and Q , $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$.

Proof.

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P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg Q \wedge \neg P$
T	T			
T	F			
F	T			
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Note that for all possible truth values of P and Q , $\neg(P \vee Q)$ equals $\neg P \wedge \neg Q$. Thus, the statement is true. □

Quick check 1

Prove the following statement by exhaustion.

For any proposition P and Q , $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$.

Quick check 2

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I have 2 pairs of socks in 2 colors: black and white. If I pick any 3 socks, I will have at least a pair of socks of the same color.

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This is clearly a brute force method. Sometimes, even in small cases, proofs by exhaustion can be very tedious and error-prone.

Logical deduction (1)

Consider the following statements:

- ▶ It rains.
- ▶ If it rains, then the road will get wet.
- ▶ If the road is wet, it will be dangerous to drive very fast.

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If we believe in these statements (i.e., if we believe that they are all true), is it OK to conclude that:

- ▶ It is dangerous to drive very fast.

Quick check 3

Define propositional variables representing each proposition inside these statements and write proposition forms of them.

- ▶ It rains.
- ▶ If it rains, then the road will get wet.
- ▶ If the road is wet, it will be dangerous to drive very fast.
- ▶ It is dangerous to drive very fast.

Logical deduction (2)

Using that proposition variables, our problem translate to the following.

Let's try to prove by exhaustion

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There are 3 variables. These are all possible cases.

R	W	D
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
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We believe that R , $R \Rightarrow W$, and $W \Rightarrow D$ are true, and we want to conclude that D must be true.

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Proofs by exhaustion can be exhausted...

Valid arguments (1)

Very often, the statement we want to prove is in the form:

Given:

- ▶ Hypothesis 1,
- ▶ Hypothesis 2,
- ▶ ...
- ▶ Hypothesis n

Then:

- ▶ Conclusion

Valid arguments (1)

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Given:

- ▶ Hypothesis 1,
- ▶ Hypothesis 2,
- ▶ ...
- ▶ Hypothesis n

Then:

- ▶ Conclusion

We say that the statement is **valid** if when all hypotheses are true, the conclusion must be true as well. In that case, we say that the conclusion **logically follows** from the hypotheses.

Valid arguments (2)

More precisely, to show that conclusion Q logically follows from hypotheses P_1, P_2, \dots, P_n , we need to show that

$$(P_1 \wedge P_2 \wedge \dots \wedge P_n) \Rightarrow Q,$$

is always true, i.e., is a tautology.

An example

Consider the following argument:

- ▶ Hypotheses: P and $P \Rightarrow Q$
- ▶ Conclusion: Q

Is this a valid argument?

An example

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Is this a valid argument?

It is. See the following truth table.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$R/W/D$ again

Since we know that the previous argument is valid, maybe we can use that “small” step in our previous example.

Recall our hypotheses:

- ▶ R
- ▶ $R \Rightarrow W$
- ▶ $W \Rightarrow D$

$R/W/D$ again

Since we know that the previous argument is valid, maybe we can use that “small” step in our previous example.

Recall our hypotheses:

- ▶ R
- ▶ $R \Rightarrow W$
- ▶ $W \Rightarrow D$

Using the same reasoning, we can say that from R and $R \Rightarrow W$, W logically follows.

$R/W/D$ again

Since we know that the previous argument is valid, maybe we can use that “small” step in our previous example.

Recall our hypotheses:

- ▶ R
- ▶ $R \Rightarrow W$
- ▶ $W \Rightarrow D$

Using the same reasoning, we can say that from R and $R \Rightarrow W$, W logically follows.

Then, since we know that W is now true, and $W \Rightarrow D$, we can conclude that D must follow.

A rule of inference

The previous “small” valid step that we can use in our argument is extremely useful when making arguments. It is called *Modus ponens*, and is one of many useful rules of inference.

Modus ponens

$$\frac{P \quad P \Rightarrow Q}{Q}$$

Other rules of inference

Addition

$$\frac{P}{P \vee Q}$$

Simplification

$$\frac{P \wedge Q}{P}$$

Modus tollens

$$\frac{\neg Q \quad P \Rightarrow Q}{\neg P}$$

Hypothetical syllogism

$$\frac{P \Rightarrow Q \quad Q \Rightarrow R}{P \Rightarrow R}$$

Conjunction

$$\frac{P \quad Q}{P \wedge Q}$$

Disjunctive syllogism

$$\frac{P \vee Q \quad \neg P}{Q}$$

Using inference rules

Argue that $P \Rightarrow Q$, $(P \vee R)$, and $\neg R$ logically leads to the conclusion Q .

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Steps

Reasons

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Steps	Reasons
1. $P \vee R$	Hypothesis

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Steps	Reasons
1. $P \vee R$	Hypothesis
2. $\neg R$	Hypothesis

Using inference rules

Argue that $P \Rightarrow Q$, $(P \vee R)$, and $\neg R$ logically leads to the conclusion Q .

Steps	Reasons
1. $P \vee R$	Hypothesis
2. $\neg R$	Hypothesis
3. P	Disjunctive syllogism using Step 1 and 2

Using inference rules

Argue that $P \Rightarrow Q$, $(P \vee R)$, and $\neg R$ logically leads to the conclusion Q .

Steps	Reasons
1. $P \vee R$	Hypothesis
2. $\neg R$	Hypothesis
3. P	Disjunctive syllogism using Step 1 and 2
4. $P \Rightarrow Q$	Hypothesis

Using inference rules

Argue that $P \Rightarrow Q$, $(P \vee R)$, and $\neg R$ logically leads to the conclusion Q .

Steps	Reasons
1. $P \vee R$	Hypothesis
2. $\neg R$	Hypothesis
3. P	Disjunctive syllogism using Step 1 and 2
4. $P \Rightarrow Q$	Hypothesis
5. Q	Modus ponens using Step 3 and 4.

Other useful logical equivalences

We have discussed De Morgan's Laws, which are logical equivalences. The following logical equivalences are also useful when making valid arguments. (Notes: do not get confused with operator \Leftrightarrow and notation $P \equiv Q$.)

Equivalences	Names
$\neg(\neg P) \equiv P$	Double negation law
$(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R)$	Distributive law
$(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$	Distributive law
$P \Rightarrow Q \equiv \neg P \vee Q$	

Another example

Argue that $P \Rightarrow R$ and $Q \Rightarrow R$ logically leads to the conclusion $(P \vee Q) \Rightarrow R$.

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Steps	Reasons
1. $P \Rightarrow R$	Hypothesis

Another example

Argue that $P \Rightarrow R$ and $Q \Rightarrow R$ logically leads to the conclusion $(P \vee Q) \Rightarrow R$.

Steps

1. $P \Rightarrow R$
2. $\neg P \vee R$

Reasons

Hypothesis
Equivalence of Step 1

Another example

Argue that $P \Rightarrow R$ and $Q \Rightarrow R$ logically leads to the conclusion $(P \vee Q) \Rightarrow R$.

Steps	Reasons
1. $P \Rightarrow R$	Hypothesis
2. $\neg P \vee R$	Equivalence of Step 1
3. $Q \Rightarrow R$	Hypothesis

Another example

Argue that $P \Rightarrow R$ and $Q \Rightarrow R$ logically leads to the conclusion $(P \vee Q) \Rightarrow R$.

Steps	Reasons
1. $P \Rightarrow R$	Hypothesis
2. $\neg P \vee R$	Equivalence of Step 1
3. $Q \Rightarrow R$	Hypothesis
4. $\neg Q \vee R$	Equivalence of Step 3

Another example

Argue that $P \Rightarrow R$ and $Q \Rightarrow R$ logically leads to the conclusion $(P \vee Q) \Rightarrow R$.

Steps	Reasons
1. $P \Rightarrow R$	Hypothesis
2. $\neg P \vee R$	Equivalence of Step 1
3. $Q \Rightarrow R$	Hypothesis
4. $\neg Q \vee R$	Equivalence of Step 3
5. $(\neg P \vee R) \wedge (\neg Q \vee R)$	Conjunction of Steps 2 and 4.

Another example

Argue that $P \Rightarrow R$ and $Q \Rightarrow R$ logically leads to the conclusion $(P \vee Q) \Rightarrow R$.

Steps	Reasons
1. $P \Rightarrow R$	Hypothesis
2. $\neg P \vee R$	Equivalence of Step 1
3. $Q \Rightarrow R$	Hypothesis
4. $\neg Q \vee R$	Equivalence of Step 3
5. $(\neg P \vee R) \wedge (\neg Q \vee R)$	Conjunction of Steps 2 and 4.
6. ... (left as homework)	