

01204211 Discrete Mathematics

Lecture 13: Binomial Coefficients

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The binomial coefficients

There is a reason why the term $\binom{n}{k}$ is called the binomial coefficients. In this lecture, we will discuss

- ▶ the Pascal's triangle,
- ▶ the binomial theorem, and
- ▶ advanced counting with binomial coefficients.

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The table

We shall use the fact that $\binom{n}{0} = 1$ and $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ to fill in the following table.

n	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1						

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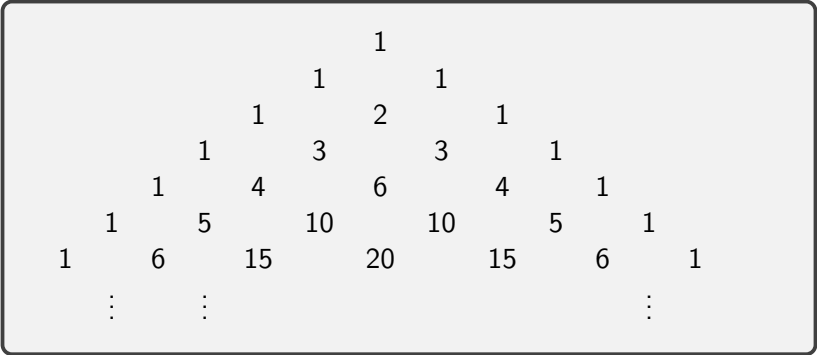
You can note that the table is left-right symmetric. This is true because of the fact that $\binom{n}{k} = \binom{n}{n-k}$.

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A diagram of Pascal's Triangle with 6 rows of numbers. The numbers are arranged in a triangular shape, with each row shifted one position to the right relative to the row above it. The numbers are: Row 1: 1; Row 2: 1, 1; Row 3: 1, 2, 1; Row 4: 1, 3, 3, 1; Row 5: 1, 4, 6, 4, 1; Row 6: 1, 5, 10, 10, 5, 1. Below the first two numbers of the first two rows, there are vertical ellipses (three dots) indicating the pattern continues.

				1						
			1		1					
		1		2		1				
	1		3		3		1			
1		4		6		4		1		
	1		5		10		10		5	
		1		6		15		15		1
			⋮							
			⋮							
									⋮	

The table and the binomial coefficients have many other interesting properties.

Polynomial expansions

Let's start by looking at polynomial of the form $(x + y)^n$. Let's start with small values of n :

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- ▶ $(x + y)^3 = x^3 + 3 \cdot x^2y + 3 \cdot xy^2 + y^3$
- ▶ $(x + y)^4 = x^4 + 4 \cdot x^3y + 6 \cdot x^2y^2 + 4 \cdot xy^3 + y^4.$

Let's focus on the coefficient of each term. You may notice that terms x^n and y^n always have 1 as their coefficients. *Why is that?*

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Let's focus on the coefficient of each term. You may notice that terms x^n and y^n always have 1 as their coefficients. *Why is that?* Let's look further at the coefficients of terms $x^{n-1}y$. Do you see any pattern in their coefficients? *Can you explain why?*