

# 01204211 Discrete Mathematics

## Lecture 7: Mathematical Induction 2

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# Review: Mathematical Induction

Suppose that you want to prove that property  $P(n)$  is true for every natural number  $n$ .

Suppose that we can prove the following two facts:

**Base case:**  $P(1)$

**Inductive step:** For any  $k \geq 1$ ,  $P(k) \Rightarrow P(k + 1)$

The **Principle of Mathematical Induction** states that  $P(n)$  is true for every natural number  $n$ .

The assumption  $P(k)$  in the inductive step is usually referred to as **the Induction Hypothesis**.

# Example 1

**Theorem:** For every natural number  $n$ ,

$$\sum_{i=1}^n i^2 = \frac{n}{6}(n+1)(2n+1)$$

**Proof:** We prove by induction. The property that we want to prove  $P(n)$  is " $\sum_{i=1}^n i^2 = \frac{n}{6}(n+1)(2n+1)$ ."

**Base case:** We can plug in  $n = 1$  to check that  $P(1)$  is true:  
 $1^2 = \frac{1}{6}(1+1)(2 \cdot 1 + 1)$ .

**Inductive step:** We assume that  $P(k)$  is true for  $k \geq 1$  and show that  $P(k+1)$  is true.

We first assume the Induction Hypothesis  $P(k)$ :

$$\sum_{i=1}^k i^2 = \frac{k}{6}(k+1)(2k+1)$$

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## Example 1 (cont.)

Let's show  $P(k+1)$ . We write  $\sum_{i=1}^{k+1} i^2 = \left(\sum_{i=1}^k i^2\right) + (k+1)^2$ .

Using the Induction Hypothesis, we know that this is equal to

$$\begin{aligned}(k/6)(k+1)(2k+1) + (k+1)^2 &= \frac{(k+1)}{6}(k(2k+1) + 6(k+1)) \\ &\quad \text{(In this step, we factor out } (k+1)/6\text{)} \\ &= \frac{(k+1)}{6}(2k^2 + 7k + 6) \\ &= \frac{(k+1)}{6}((k+1) + 1)(2(k+1) + 1).\end{aligned}$$

This implies  $P(k+1)$  as required.

From the Principle of Mathematical Induction, this implies that  $P(n)$  is true for every natural number  $n$ . ■

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## Theorem 1

*For any set of cows, all cows have the same color.*

**Proof.**

We prove by induction on the size  $n$  of the set of cows.

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**Base case:** For  $n = 1$ , clearly for any set of a single cow, every cow in the set has the same color.

**Inductive step:** Suppose that for every set of size  $k$  of cows, all cows in the set have the same color.

We will show that every set of size  $k + 1$  of cows, all cows in this set have the same color.

## Not an example (2)

**Inductive step (cont.):** Consider set  $A$  of  $k + 1$  cows.



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**Inductive step (cont.):** Consider set  $A$  of  $k + 1$  cows.

Because we have established that the base case and the inductive step is true, we can conclude that for any set of cows, all cows have the same color. ■

## Not an example (3)

Clearly the following theorem cannot be true.

### Theorem 2

*For any set of cows, all cows have the same color.*

What is wrong with its proof based on mathematical induction?

## Unused facts

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- ▶ Then why don't we use them as well?

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Then  $P(n)$  is true for every natural number  $n$ .

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**Inductive step:** Assume that for  $k \geq 5$ , we can obtain exactly  $\ell$  baht, for  $4 \leq \ell \leq k$ , using only 2-baht and 3-baht coins. We will show how to obtain a set of  $k + 1$  baht.

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Since  $k \geq 5$ , we have that  $k - 1 \geq 4$ . Therefore from the Induction Hypothesis, we can use only 2-baht coins and 3-baht coins to form a set of coins of total value  $k - 1$  baht.

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From the Principle of Strong Mathematical Induction, we conclude that the theorem is true. ■

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- ▶ Hint: Let  $Q(n) = P(1) \wedge P(2) \wedge \cdots \wedge P(n)$ .