

01204211 Discrete Mathematics  
Lecture 14: Binomial Coefficients (2)

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# The binomial coefficients<sup>1</sup>

In this lecture, we shall study the function  $\binom{n}{k}$  itself. First, let's see the actual value of the binomial coefficients  $\binom{n}{k}$  for various values of  $n$ .

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<sup>1</sup>This lecture mostly follows Chapter 3 of [LPV].

## What do you see?

- ▶ The function  $\binom{n}{\cdot}$  is symmetric around  $n/2$ .
- ▶ Why? This is true because we know that  $\binom{n}{k} = \binom{n}{n-k}$ .
- ▶ The maximum is at the middle, i.e., when  $n$  is even the maximum is at  $\binom{n}{n/2}$  and when  $n$  is odd, the maximum is at  $\binom{n}{\lfloor n/2 \rfloor}$  and  $\binom{n}{\lceil n/2 \rceil}$ .
- ▶ Why? Can we prove that?

## Largest in the middle

To understand the behavior of  $\binom{n}{k}$  as  $k$  changes, let's look at two consecutive values:

$$\binom{n}{k} \heartsuit \binom{n}{k+1}$$

Let's write them out:

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \heartsuit \frac{n(n-1)(n-2)\cdots(n-k)}{(k+1)k!}.$$

Removing common terms, we can see that we are comparing these two terms:

$$1 \heartsuit \frac{n-k}{k+1} \Leftrightarrow k \heartsuit \frac{n-1}{2},$$

that is,

- ▶ if  $k < (n-1)/2$ ,  $\binom{n}{k} < \binom{n}{k+1}$ ; and
- ▶ if  $k > (n-1)/2$ ,  $\binom{n}{k} > \binom{n}{k+1}$ .

## How large is the middle $\binom{n}{n/2}$

Here, to simplify the calculation, we shall only consider the case when  $n$  is even. Let's try to estimate the value of  $\binom{n}{n/2}$  by finding its upper and lower bounds.

A simple upper bound can be obtained using the fact that  $\binom{n}{n/2}$  counts subsets of certain size:

$$\binom{n}{n/2} < 2^n.$$

We can also get a lower bound by noting that the maximum must be at least the average, i.e.,

$$\binom{n}{n/2} \geq \frac{2^n}{n+1}$$

Combining both bounds, we get that

$$\frac{2^n}{n+1} \leq \binom{n}{n/2} < 2^n.$$

Let's plug in  $n = 200$ , and calculate the number of digits to see how close these bounds.

$$27.80 \approx 200 \cdot \log 2 - \log 201 \leq \log \binom{200}{100} < 200 \cdot \log 2 \approx 30.10$$

Can we get a better approximation?

# Concentration

- ▶ We know that the maximum of  $\binom{n}{k}$  is obtained when  $k = n/2$ . From the graph, you can see that, as you move further from the middle, the value of the function drops rapidly.
- ▶ Since we consider even  $n$ , we let  $2m = n$ . One way to quantify how fast the values drop is to think about the ratio

$$\binom{2m}{m-t} / \binom{2m}{m}.$$

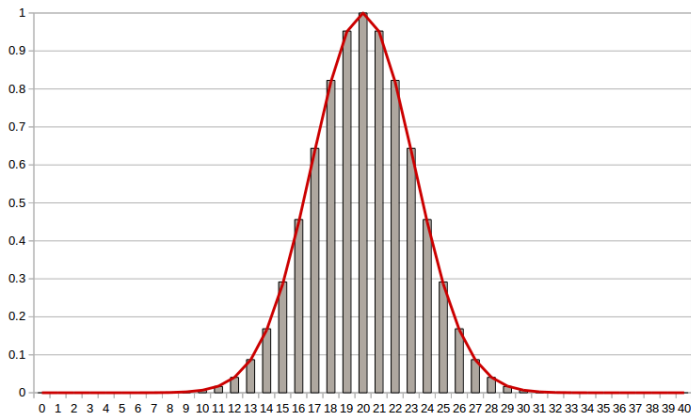
- ▶ In fact, it is known that

$$\binom{2m}{m-t} / \binom{2m}{m} \approx e^{-t^2/m}$$

- ▶ We will use our basic tools to obtain weaker bounds.

## How close is the approximation?

The estimation  $e^{-t^2/m}$  is extremely close as shown in the figure below, where the gray bars are the actual value of  $\binom{2m}{m-t}/\binom{2m}{m}$  and the red line is  $e^{-t^2/m}$ .





## The actual values

Because dealing with numbers less than 1 with logarithms is error-prone, we will work on the reciprocal. Let's try to calculate the ratio

$$\begin{aligned}\binom{2m}{m} / \binom{2m}{m-t} &= \frac{(2m)!}{m!m!} \times \frac{(2m-m+t)!(m-t)!}{(2m)!} \\ &= \frac{(m+t)(m+t-1)\cdots(m+1)}{m(m-1)(m-2)\cdots(m-t+1)}.\end{aligned}$$

We can use the same logarithm trick. We have that the log of the ratio is

$$\ln\left(\frac{m+t}{m}\right) + \ln\left(\frac{m+t-1}{m-1}\right) + \cdots + \ln\left(\frac{m+1}{m-t+1}\right).$$

Then we can apply the bounds we have for  $\ln x$ :

$$\frac{x-1}{x} \leq \ln x \leq x-1$$

## The upper bound on the reciprocal

Each term in the sum is in this form  $\ln((m-i)/(m+t-i))$ .

Applying the upper bound, we get

$$\ln\left(\frac{m+t-i}{m-i}\right) \leq \frac{m+t-i}{m-i} - 1 = \frac{m+t-i-m+i}{m-i} = \frac{t}{m-i}.$$

Let's sum them up to get

$$\begin{aligned} \ln\left(\frac{m+t}{m}\right) + \ln\left(\frac{m+t-1}{m-1}\right) + \cdots + \ln\left(\frac{m+1}{m-t+1}\right) \\ \leq \frac{t}{m} + \frac{t}{m-1} + \cdots + \frac{t}{m-t+1} \\ \leq \frac{t}{m-t+1} + \frac{t}{m-t+1} + \cdots + \frac{t}{m-t+1} \\ = \frac{t^2}{m-t+1}. \end{aligned}$$

This implies that

$$\ln \left( \frac{(m+t)(m+t-1)\cdots(m+1)}{m(m-1)(m-2)\cdots(m-t+1)} \right) \leq \frac{t^2}{m-t+1},$$

i.e.,

$$\begin{aligned} \binom{2m}{m} / \binom{2m}{m-t} &= \left( \frac{(m+t)(m+t-1)\cdots(m+1)}{m(m-1)(m-2)\cdots(m-t+1)} \right) \\ &\leq e^{t^2/(m-t+1)}. \end{aligned}$$

Taking the reciprocal, we get

$$e^{-t^2/(m-t+1)} \leq \binom{2m}{m-t} / \binom{2m}{m}.$$

## Upper bounds

Using the same approach, we can show that

$$\binom{2m}{m-t} / \binom{2m}{m} \leq e^{-t^2/(m+t)}.$$

Thus, we derived the estimates:

$$e^{-t^2/(m-t+1)} \leq \binom{2m}{m-t} / \binom{2m}{m} \leq e^{-t^2/(m+t)},$$

which is fairly close to the estimate of  $e^{-t^2/m}$ .

## How fast?

- ▶ Let's return to the question on how fast do the values of the binomial coefficients decrease as you move further from the middle. Let's use the better estimate  $\binom{2m}{m-t} / \binom{2m}{m} \approx e^{-t^2/m}$ .
- ▶ Given a constant  $C$ , we want to estimate the value of  $t$  such that  $\binom{2m}{m-t}$  is less than  $\binom{2m}{m}/C$ . (E.g., we can set  $C = 2$  to see when the value drops by 50%.) Therefore, we want to find  $t$  such that

$$1/C \geq \binom{2m}{m-t} / \binom{2m}{m} \approx e^{-t^2/m}$$

Taking the logs, we get

$$\ln 1/C = -\ln C \geq \ln \binom{2m}{m-t} / \binom{2m}{m} \approx -t^2/m.$$

This is true when

$$t \geq \sqrt{m \ln C}.$$

## What does this means?

As an example, let  $m = 20$  and  $C = 2$ . We know that when  $t$  is approximately  $\sqrt{20 \cdot \ln 2} = 3.723$  the value of  $\binom{2m}{m-t}$  drops by 50%.

