

01204211 Discrete Mathematics

Lecture 1: Introduction

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IMHO, mathematics is a mean to communicate *precise* ideas.

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- ▶ I hope it is also true with this course.

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- ▶ To learn how to make mathematical arguments.
- ▶ To learn various fundamental mathematical concepts that are very useful in computer science.

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    return a  
else:  
    return b
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The author claims that this program takes two variables a and b and returns the larger one.

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Now look at this program.

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for i = 0, 1, ..., n-1:
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    if n <= 1:
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Consider the following code.

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Algorithm CheckPrime2(n): // Input: an integer n
    if n <= 1:
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    let s = square root of n
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Ok, it should be faster. **But is it correct?**

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- ▶ Are we done?

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- ▶ How can we do that?

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- ▶ Before we continue, I'd like to add a bit of formalism to our thinking process.

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- ▶ Note that this statement can either be “true” or “false.” If we can demonstrate, using logical/mathematical arguments that this statement is true, we can say that we **prove** the statement.

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- ▶ Are we doomed?

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- ▶ Are we doom? Not really. The statement above is not precisely the statement we want to prove.

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Revised statement

For all positive composite integer n , and for every divisor a of n such that $\sqrt{n} < a < n$,


$$2 \leq b \leq \sqrt{n},$$

where $b = n/a$.

- ▶ Note that this revised statement is now “quantified,” that is, every variable in the statement has specific scope. Now the statement is either true or false.

Propositions¹

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- ▶ Examples:
 - ▶ Algorithm CheckPrime2 is correct.
 - ▶ $10^2 = 90$.
 - ▶ $\sqrt{2}$ is irrational.
- ▶ Examples of statements which are not propositions (why?):
 - ▶ $x > 10$.
 - ▶ $1 + 2 + \cdots + 10$.
 - ▶ This algorithm is fast.
 - ▶ Run, run quickly.

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- ▶ An expression $P \wedge Q$ is an example of *propositional forms*. The logical value of a propositional form “usually” depends on the truth value of its variables.

Connectives: “and”, “or”, “not”

Given propositions P and Q , we can use connectives to form more complex propositions:

- ▶ **Conjunction:** $P \wedge Q$ (“ P and Q ”),
(True when both P and Q are true)
- ▶ **Disjunction:** $P \vee Q$ (“ P or Q ”),
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If P stands for “today is Tuesday” and Q stands for “dogs are animals”, then

- ▶ $P \wedge Q$ stands for “today is Tuesday and dogs are animals”,
- ▶ $P \vee Q$ stands for “today is Tuesday or dogs are animals”, and
- ▶ $\neg P$ stands for “today is not Tuesday”.

Truth tables

To represents values of propositional forms, we usually use truth tables.

And/Or/Not

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$
T	T	T	T	F
T	F	F	T	
F	T	F	T	T
F	F	F	F	

Quick check 1

For each of these statements, define propositional variables representing each proposition inside the statement and write the proposition form of the statement.

- ▶ All prime numbers are larger than 0 and all natural numbers is at least one.
- ▶ You are smart or you won't be taking this class.

Next lecture...

- ▶ We will discuss other ways to join two propositions, i.e., implications (\Rightarrow) and equivalences (\Leftrightarrow).
- ▶ We will look at two forms of quantifiers: universal quantifiers and existential quantifiers.