01204211 Discrete Mathematics
Lecture 2: Quantifiers and proofs

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This lecture covers:

- More on quantifiers
- How to prove a proposition
- Basic proof techniques

Review: Quantifiers

- A *predicate* is a statement with variables, which can be either true or false, after all its variables are specified.
- If we quantify a predicate completely, the quantified expression now has a truth value, and it is called a quantified proposition.
- Two ways to quantify:
  - **Using universal quantifiers ($\forall$).** This quantifier states that the quantified proposition is true when the predicate is true for every value of the variable in the specified set.
  - **Using existential quantifiers ($\exists$).** This quantifier states that the quantified proposition is true when the predicate is true for at least one value of the variable in the specified set.
- Quantifiers can be nested. E.g.,
  - $\forall x \forall y P(x, y) \equiv \forall x (\forall y (P(x, y)))$
  - $\forall x \exists y P(x, y) \equiv \exists x (\forall y (P(x, y)))$

Quick check 1

Negations (1)

Let consider a set of positive integers $\mathbb{Z}^+$ as our universe. Let predicate $P(x) \equiv$ "x is a prime number."
Consider this proposition

$$(\forall x \in \mathbb{Z}^+)P(x).$$

How can we show that this is false?

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This way of disproving a statement is equivalent to showing that

$$(\exists x)(\neg P(x)).$$

Negations of quantified propositions

Let consider a set of positive integers $\mathbb{Z}^+$ as our universe. Let predicate $Q(x) \equiv$ "if $x > 2$, then $x^2 \leq 2x$."
Consider this proposition

$$(\exists x \in \mathbb{Z}^+)Q(x).$$

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Negations of quantified propositions

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$$(\exists x \in \mathbb{Z}^+)Q(x).$$

How can we show that this is false?
When showing that an existential quantified proposition is false, we need to show that $Q(x)$ is false for every possible values of x. In this case, since $x^2 = x \cdot x > 2 \cdot x$ for every $x > 2$, we have that $(\exists x)Q(x)$ is false.

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Negations (3)

Thus, the following equivalences:

- $\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$
- $\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$

Quick check 2

How to prove a mathematical statement

Given propositions P and Q, these are a very useful logical equivalences (referred to as the De Morgan's Laws).

- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
- $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

(Note that $\neg$ takes precedence over $\vee$ or $\wedge$.)

How can we prove that the first statement is true?

Proof by exhaustion

For any proposition P and Q, $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$.

Proof.

We will prove by exhaustion. There are 4 cases as in the truth table below.

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg Q \wedge \neg P$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

Note that for all possible truth values of P and Q, $\neg(P \vee Q)$ equals $\neg P \wedge \neg Q$. Thus, the statement is true. $\square$