

01204211 Discrete Mathematics
Lecture 18: Primality testing (3)

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The first algorithm

This lecture presents the primality testing algorithm based on the Fermat test.

To perform the test if q is a prime, we pick an integer a from the range $\{2, 3, \dots, q - 1\}$ and check that the condition specified by the Fermat's Little Theorem is satisfied.

PROCEDURE FermatTest(q, a)

1. **if** $GCD(q, a) \neq 1$ **then return** "COMPOSITE" **endif**
2. Find $y = a^{q-1} \bmod q$
3. **if** $y \neq 1$ **then**
4. **return** "COMPOSITE"
5. **else**
6. **return** "PRIME"
7. **endif**

Guarantees

Let's consider all input/output possibilities. Let $y = \text{FermatTest}(q, a)$.

	$y = \text{PRIME}$	$y = \text{COMPOSITE}$
q is prime	Correct	Incorrect
q is composite	Incorrect	Correct

However, from the Fermat's Little Theorem, we know that if q is prime, the test always return "PRIME"; thus the test is always correct.

Guarantees

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However, from the Fermat's Little Theorem, we know that if q is prime, the test always return "PRIME"; thus the test is always correct.

We then need to consider the case when q is composite. In this case, the Fermat's Little Theorem does not provide any guarantee. However, we hope that if q is not a prime, we may be able to find a such that $\text{FermatTest}(q, a)$ reveals the truth, i.e., it returns COMPOSITE.

Witness

- ▶ Let's try to be precise. For a composite q , if a is an integer such that $\text{FermatTest}(q,a)$ returns COMPOSITE, we say that a is a **witness** for q .
- ▶ It would be great if, for any composite q , we can quickly find its witness. But unfortunately, there exists a composite q with very few witnesses. I.e., every witness a for q shares its factor (i.e., it is such that $\gcd(a, q) \neq 1$). They are called **Carmichael numbers**.

A **Carmichael** number is a composite number q such that

$$b^{q-1} \bmod q = 1,$$

for all integers $1 < b < n$ which are relatively prime to q .

- ▶ The first Carmichael number is 561. The next ones are 1105, 1729, and 2465.

Witnesses for non-Carmichael numbers (1)

Let's focus on the bright side. Suppose that q is not Carmichael, can we say anything about its witnesses? Can we say that there are plenty of them?

Witnesses for non-Carmichael numbers (2)

Let a be q 's witness such that $\gcd(a, q) = 1$, i.e., we also have that

$$a^{q-1} \bmod q \neq 1.$$

Let's consider a non-witness b , i.e., b is an integer such that $\gcd(b, q) = 1$ and

$$b^{q-1} \bmod q = 1.$$

Now, consider ab . We have that

$$\begin{aligned}(ab)^{q-1} \bmod q &= (a^{q-1}b^{q-1}) \bmod q \\&= (a^{q-1} \bmod q)(b^{q-1} \bmod q) \\&= a^{q-1} \bmod q \\&\neq 1\end{aligned}$$

Can we use this to say that there are a lot of witnesses?

A lot of witnesses

Since we can show that given a witness a and a non-witness b , ab is also witness, we might be able to construct a lot of witnesses from non-witnesses.

Let $A \subseteq \{2, 3, \dots, q-1\}$. Let B be the set of non-witnesses which are relatively prime to q , i.e.,

$$B = \{b \in A : (\gcd(b, q) = 1) \wedge (b^{q-1} \bmod q = 1)\}.$$

Let $C = \{ab \bmod q : b \in A\}$. We know that every $c \in C$ is a witness.

If we know that $|C| = |B|$, can you show that the probability of choosing a witness from the set A is large (i.e, at least $1/2$).

Now, is it obvious that $|C| = |B|$? What is missing?

The missing argument

To show that $|C| = |B|$, we need to argue that when we multiply every element of B , we do not get duplicate elements. I.e., we need to prove that for $x \in B$ and $y \in B$ such that $x \neq y$,

$$x \bmod q \neq y \bmod q.$$

Quick check: prove this statement.

Conclusions

From the previous discussion, we know that for non-Carmichael numbers, the Fermat test succeeds with probability at least $1/2$.

Further developments:

- ▶ In 1976, Miller and Rabin show that one can deal with Carmichael numbers, providing the first randomized algorithm for testing primes.
- ▶ In 2002, Agrawal, Kayal, and Saxena devise an $O(m^{12})$ -time deterministic algorithm for primality testing.