

01204211 Discrete Mathematics

Lecture 5: Proof techniques 2


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Proof techniques¹

In this lecture, we will focus on two other proof techniques.

- ▶ Proofs by contradiction
- ▶ Proofs by cases

¹This lecture mostly follows Berkeley CS70 lecture notes. 

Proofs by contradiction

We want to prove that proposition P is true. To do so, we first assume that P is false, and show that this logically leads to a contradiction. This means that it is impossible for P to be false; hence, P has to be true. This is called a proof by contradiction or *reductio ad absurdum*.

Direct proofs

Theorem:

P

Proof.

We use prove by contradiction.

Assume $\neg P$.

... (then show that R and $\neg R$ follows from $\neg P$)

This is a contradiction. Therefore, P must be true.



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In this case, a and b share no common factors.

Let's square both terms. We get $2 = a^2/b^2$, or

$$a^2 = 2b^2.$$

(cont. in next slide)



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Again by definition, there exists integer k such that $a = 2k$. We then obtain

$$2b^2 = (2k)^2 = 4k^2,$$

i.e., $b^2 = 2k^2$.

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This contradicts the fact that we choose the pair a and b that share no common factor.

Therefore, $\sqrt{2}$ must be irrational.



Proofs by cases

- ▶ The last proof technique that we shall discuss is closely related to proofs by exhaustion we tried before.
- ▶ Sometimes when we want to prove a statement, there are many possible cases. Also, we might not know which cases are true.
- ▶ We might still be able to prove the statement if we can show that the statement is true in every case.

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Theorem 2

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Proof.

Let's split the process of picking 4 socks into 2 steps. First, pick 3 socks, then pick the last sock.

After we pick the first 3 socks. There are 2 possible cases: either I have a pair of socks with the same color, or I do not have such a pair. We shall consider each case separately.

(cont. in the next slide)



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- ▶ **Case 1:** *I have a pair of socks with the same color.*

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Proof. (cont.)

- ▶ **Case 1:** *I have a pair of socks with the same color.*
In this case, the theorem is true.
- ▶ **Case 2:** *I do not have a pair of socks with the same color.*
In this case, since I have 3 colors and 3 socks, I must have one sock for each color. Now, after we pick the last sock, whatever color the last one is, we have a color-matching sock in our first 3 socks. Therefore, the theorem is also true in this case.

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Proof. (cont.)

- ▶ **Case 1:** *I have a pair of socks with the same color.*
In this case, the theorem is true.
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In this case, since I have 3 colors and 3 socks, I must have one sock for each color. Now, after we pick the last sock, whatever color the last one is, we have a color-matching sock in our first 3 socks. Therefore, the theorem is also true in this case.

Since these two cases cover all possibilities, we conclude that the theorem is true. □

Proofs by cases in propositional logic

In propositional logic, the following describe a proof by cases.

$$P \vee Q \vee R$$

$$P \Rightarrow S$$

$$Q \Rightarrow S$$

$$R \Rightarrow S$$

$$S$$

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Sometimes, when we have 2 cases, we also see:

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Sometimes, when we have 2 cases, we also see:

$$\begin{array}{l} P \vee \neg P \\ P \Rightarrow S \\ \neg P \Rightarrow S \\ \hline S \end{array}$$

Note that we can leave $P \vee \neg P$ out, because it is always true.