01204211 Discrete Mathematics
Lecture 15: Fibonacci sequence

Jittat Fakcharoenphol

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The Fibonacci sequence[1]

Source:

https://en.wikipedia.org/wiki/

File:Fibonacci.jpg

In 1202, Leonardo Bonacci (known as Fibonacci) asked the following question.

> "[A]ssuming that: a newly born pair of rabbits, one male, one female, are put in a field; rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on."
>
> "The puzzle that Fibonacci posed was: how many pairs will there be in one year?"

From https://en.wikipedia.org/wiki/Fibonacci_number

[1] This lecture mostly follows Chapter 4 of [LPV].

Let's try to solve Fibonacci's question.

Let ♠ denote a newly born rabit pair, and ♡ denote a mature rabit pair.

Month	Rabits	
1	♠	1
2	♡	1
3	♡ ♠	2
4	♡ ♡ ♠	3
5	♡ ♡ ♡ ♠ ♠	5
6	♡ ♡ ♡ ♡ ♡ ♠ ♠ ♠	8
7	♡ ♡ ♡ ♡ ♡ ♡ ♡ ♡ ♠ ♠ ♠ ♠ ♠	13

How many rabit pairs do we have at the beginning of the 8th month?

- Surely all 13 rabit pairs we have in the 7th month remain there and are all mature. So, the question is how many newly born rabbit pairs that we have.
- The number of newly born rabbit pairs equals the number of mature rabbit pairs we have. This is also equal to the number of rabit pairs that we have in the 6th month: 8.

Thus, we will have 13+8 rabit pairs at the beginning of the 8th month.
If we write down the sequence, we get the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Again, what's the next number in this sequence? How can you compute it?
21+13 $=$ 34 is the answer. You take the last two numbers and add them up to get the next number. Why?

To be precise, let F_n be the n-th number in the Fibonacci sequence. (That is, $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$ and so on.) We can define the $(n+1)$-th number as

$$F_{n+1} = F_n + F_{n-1},$$

for $n = 2, 3, \ldots$. Is this enough to completely specify the sequence? No, because we do not know how to start. To get the Fibonacci sequence, we need to specify two starting values: $F_1 = 1$ and $F_2 = 1$ as well.

Now, you can see that the equation and these special values uniquely determine the sequence. It is also convenient to define $F_0 = 0$ so that the equation works for $n = 1$.

A recurrence

The equation

$$F_{n+1} = F_n + F_{n-1}$$

and the initial values $F_0 = 0$ and $F_1 = 1$ specify all values of the Fibonacci sequence. With these two initial values, you can use the equation to find the value of any number in the sequence.

This definition is called a **recurrence**. Instead of defining the value of each number in the sequence explicitly, we do so by using the values of other numbers in the sequence.

Tilings with 1x1 and 2x1 tiles

You have a walk way of length n units. The width of the walk way is 1 unit. You have unlimited supplies of 1x1 tiles and 2x1 tiles. Every tile of the same size is indistinguishable. In how many ways can you tile the walk way?
Let's consider small cases.

- When $n = 1$, there are 1 way.
- When $n = 2$, there are 2 ways.
- When $n = 3$, there are 3 ways.
- When $n = 4$, there are 5 ways.

Let's define J_n to be the number of ways you can tile a walk way of length n. From the example above, we know that $J_1 = 1$ and $J_2 = 2$.
Can you find a formula for general J_n?

Figuring out the recurrence for J_n

To figure out the general formula for J_n, we can think about the first choice we can make when tiling a walk way of length n. There are two choices:

- (1) We can start placing a 1x1 tile at the beginning, or
- (2) We can start placing a 2x1 tile at the beginning.

In each of the cases, let's think about how many ways we can tile the rest of the walk way, provided that the first step is made.

Note that if we start by placing a 1x1 tile, we are left with a walk way of length $n - 1$. From the definition of J_n, we know that there are J_{n-1} ways to tile the rest of the walk way of length $n - 1$. Using similar reasoning, we know that if we start with a 2x1 tile, there are J_{n-2} ways to tile the rest of the walk way.

The recurrence for J_n

From the discussion, we have that

$$J_n = J_{n-1} + J_{n-2},$$

where $J_1 = 1$ and $J_2 = 2$.

Note that this is exactly the same recurrence as the Fibonacci sequence, but with different initial values. In fact, we have that

$$J_n = F_{n+1}.$$

Identities on Fibonacci numbers

There are a lot of identities related to Fibonacci numbers. Let's see the first few values in the sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$$

Now, let's add the first few numbers:

$$\begin{aligned}
0 + 1 &= 1 \\
0 + 1 + 1 &= 2 \\
0 + 1 + 1 + 2 &= 4 \\
0 + 1 + 1 + 2 + 3 &= 7 \\
0 + 1 + 1 + 2 + 3 + 5 &= 12 \\
0 + 1 + 1 + 2 + 3 + 5 + 8 &= 20 \\
0 + 1 + 1 + 2 + 3 + 5 + 8 + 13 &= 33
\end{aligned}$$

From this we can formulate the following conjecture:

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1.$$

Theorem: For $n \geq 0$, we have that

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1.$$

Proof: We shall prove by induction on n. The base case has already been demonstrated when we consider small values of n.

Inductive Step: Let's assume that the statement is true for $n = k$, for $k \geq 0$, i.e., assume that

$$F_0 + F_1 + \cdots + F_k = F_{k+2} - 1.$$

We shall prove that the statement is true when $n = k + 1$. This is not hard to show. We write

$$\begin{aligned} (F_0 + F_1 + \cdots + F_k) + F_{k+1} &= (F_{k+2} - 1) + F_{k+1} \\ &= F_{k+3} - 1, \end{aligned}$$

as required. Note that the first step follows from the induction hypothesis. ■

Another harder identity

The following identity is harder to prove:

$$F_n^2 + F_{n-1}^2 = F_{2n-1}.$$

Let's try a few values as a sanity check.

$$F_1^2 + F_2^2 = 1^2 + 1^2 = 2 = F_3$$

$$F_2^2 + F_3^2 = 1^2 + 2^2 = 5 = F_5$$

$$F_3^2 + F_4^2 = 2^2 + 3^2 = 13 = F_7$$

To see how hard it is to prove the identity, let's try to prove it by induction. (Let's jump to the inductive step.)

We use strong induction. Assume that the statement is true for $n = k, k-1, k-2, \ldots, 0$. We prove the statement for $n = k+1$.

Let's work on the left hand side.

$$\begin{aligned}
F_{k+1}^2 + F_k^2 &= (F_k + F_{k-1})^2 + (F_{k-1} + F_{k-2})^2 \\
&= F_k^2 + 2F_kF_{k-1} + F_{k-1}^2 + F_{k-1}^2 + 2F_{k-1}F_{k-2} + F_{k-2}^2 \\
&= (F_k^2 + F_{k-1}^2) + 2F_kF_{k-1} + (F_{k-1}^2 + F_{k-2}^2) + 2F_{k-1}F_{k-2} \\
&= F_{2k-1} + F_{2k-3} + 2F_kF_{k-1} + 2F_{k-1}F_{k-2},
\end{aligned}$$

where the last step follows from the induction hypothesis.

Note that we end up with the terms like: $F_kF_{k-1} + F_{k-1}F_{k-2}$. We can keep expanding the terms, but we will end up with the same cross terms like this.

So, let's take a look at a few values of this expression. Maybe we can guess its values.

Let's plug in a few values:

$$F_3F_2 + F_2F_1 = 2 \cdot 1 + 1 \cdot 1 = 3 = F_4$$

$$F_4F_3 + F_3F_2 = 3 \cdot 2 + 2 \cdot 1 = 8 = F_6$$

$$F_5F_4 + F_4F_3 = 5 \cdot 3 + 3 \cdot 2 = 21 = F_8$$

$$F_6F_5 + F_5F_4 = 8 \cdot 5 + 5 \cdot 3 = 55 = F_{10}$$

From this, we can make another conjecture:

Conjecture 2:

$$F_{n+1}F_n + F_nF_{n-1} = F_{2n}.$$

Let's assume that Conjecture 2 is true and see if we can prove the identity that we want.
Recall that we have

$$\begin{aligned} F_{k+1}^2 + F_k^2 &= F_{2k-1} + F_{2k-3} + 2F_kF_{k-1} + 2F_{k-1}F_{k-2} \\ &= F_{2k-1} + F_{2k-3} + 2(F_kF_{k-1} + F_{k-1}F_{k-2}) \\ &= F_{2k-1} + F_{2k-3} + 2F_{2k-2} \qquad \text{(from Conj 2)} \\ &= (F_{2k-1} + F_{2k-2}) + (F_{2k-2} + F_{2k-3}) \\ &= F_{2k} + F_{2k-1} \\ &= F_{2k+1}, \end{aligned}$$

as required. We use Conjecture 2 to show the second step.

This means that assuming the Conjecture 2, we can show the identity $F_n^2 + F_{n-1}^2 = F_{2n-1}$.

Let's prove Conjecture 2

Conjecture 2: $F_{n+1}F_n + F_nF_{n-1} = F_{2n}$.

Proof: Let's do so by induction. Since we have plugged in many small values, we can only consider the inductive step now. Assume that the statement is true for $n = k, k-1, k-2, \ldots, 0$. We prove the statement for $n = k+1$.

We write

$$\begin{aligned}
F_{k+2}F_{k+1} + F_{k+1}F_k &= (F_{k+1} + F_k)F_{k+1} + (F_k + F_{k-1})F_k \\
&= F_{k+1}^2 + F_kF_{k+1} + F_k^2 + F_{k-1}F_k \\
&= (F_kF_{k+1} + F_{k-1}F_k) + F_{k+1}^2 + F_k^2 \\
&= F_{2k} + F_{k+1}^2 + F_k^2.
\end{aligned}$$

(Note that the 4th step uses the induction hypothesis.) Do you see any familiar terms?

Yes, the terms $F_{k+1}^2 + F_k^2$ is the left hand side of the identity we have just proven. Actually, we cannot use it directly here, because we use Conjecture 2 to prove it and now we are trying to prove the conjecture itself. **Using it results in a circular reasoning.**

We can actually prove the conjecture using that identity, but we first have to break our circular reasoning by proving both statements together. Formally, let's define predicates P and Q:

$$P(n) : F_n^2 + F_{n-1}^2 = F_{2n-1}$$

$$Q(n) : F_{n+1}F_n + F_nF_{n-1} = F_{2n}$$

We will prove that for all integer $n \geq 0$, $P(n) \wedge Q(n)$.
Base Case: We have shown that $P(1)$ and $Q(1)$ are true.
Inductive Step: Assume that the statements are true for $n = k, k-1, \ldots, 1$ for $k \geq 1$. We will prove $P(k+1) \wedge Q(k+1)$.

- $P(k+1)$ can be proved as in the proof of the identity previously.
- To prove $Q(k+1)$, we can use the induction hypotheses and also $P(k+1)$.

Simultaneous induction

Let's prove $Q(k+1)$. We can continue from our "broken" proof. We have that

$$\begin{aligned} F_{k+2}F_{k+1} + F_{k+1}F_k &= F_{2k} + (F_{k+1}^2 + F_k^2) \\ &= F_{2k} + F_{2k+1} = F_{2k+2}, \end{aligned}$$

as required. Note that the second step uses $P(k+1)$. ■

The technique we use to prove P and Q together is called **simultaneous induction**.

An explicit form of the Fibonacci sequence

While the recurrence for F_n completely specifies the sequence, it is hard to find the value of, say, F_{20} quickly. We really have to enumerate the sequence from $F_0, F_1, \ldots,$ to get to F_{20}. Also, with the definition based on the recurrence, other properties of the sequcen is unclear (e.g., how fast the sequence grows). Therefore, it might be useful to find the explicit definition of the Fibonacci sequence.

Ratios

To get started, we might want to look for a common form of the function. We can start by looking at the numbers in the sequence.

n	F_n	ratio F_n/F_{n-1}
1	1	
2	1	1.0000000000
3	2	2.0000000000
4	3	1.5000000000
5	5	1.6666666667
6	8	1.6000000000
7	13	1.6250000000
8	21	1.6153846154
9	34	1.6190476190
10	55	1.6176470588
11	89	1.6181818182
12	144	1.6179775281

n	F_n	ratio F_n/F_{n-1}
13	233	1.6180555556
14	377	1.6180257511
15	610	1.6180371353
16	987	1.6180327869
17	1597	1.6180344478
18	2584	1.6180338134
19	4181	1.6180340557
20	6765	1.6180339632
21	10946	1.6180339985
22	17711	1.6180339850
23	28657	1.6180339902
24	46368	1.6180339882

The 1st guess: a^n

We can see that the ratio between two consecutive Fibonacci numbers is close to 1.61803. We may guess that the explicit form for F_n is an exponential function a^n. (While we know that this is not true, it may give us hints on the correct function.)

Let's try to figure out the exact value for a. The value a must satisfy the recurrence $F(n+1) = F(n) + F(n-1)$, i.e.,

$$a^{n+1} = a^n + a^{n-1}.$$

We can try to solve for a. Dividing the equation by a^{n-1}, we get

$$a^2 = a + 1,$$

i.e., $a^2 - a - 1 = 0$.

Solutions (1)

We can use a standard formula to get the values of a, i.e., a can be $\frac{1+\sqrt{1^2+4\cdot 1\cdot 1}}{2\cdot 1}$, $\frac{1-\sqrt{1^2+4\cdot 1\cdot 1}}{2\cdot 1}$, or

$$\tfrac{1+\sqrt{5}}{2}, \tfrac{1-\sqrt{5}}{2}.$$

These look nice because $\frac{1+\sqrt{5}}{2} \approx 1.61803$.

These solutions give us two candidates for F_n:

$$g(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n,$$

and

$$h(n) = \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

But we can see that while both $g(n)$ and $h(n)$ satisfy $g(n+1) = g(n) + g(n-1)$ and $h(n+1) = h(n) + h(n-1)$, they are not the correct function for F_n. (We can just plug in various values of n to check.)

Solutions (2)

To get the actual function, we observe that if both $g(n)$ and $h(n)$ are solutions to our recurrence, then for any α and β,

$$\ell(n) = \alpha \cdot g(n) + \beta \cdot h(n)$$

is also a solution to the recurrence, because

$$\begin{aligned}
\ell(n+1) &= \alpha g(n+1) + \beta h(n+1) \\
&= \alpha(g(n) + g(n-1)) + \beta(h(n) + h(n-1)) \\
&= \alpha g(n) + \beta h(n) + \alpha g(n-1) + \beta h(n-1) \\
&= \ell(n) + \ell(n-1).
\end{aligned}$$

This opens another possibility for us, i.e., $g(n)$ and $h(n)$ may be useful, but we need to find α and β. How?

The 2nd guess

Let $\ell(n) = \alpha g(n) + \beta h(n)$. We may use two initial values for F_n to set up the system of equations:

$$0 = \ell(0) = \alpha g(0) + \beta h(0),$$

and

$$1 = \ell(1) = \alpha g(1) + \beta h(1).$$

Plugging in $g(0), h(0), g(1)$, and $h(1)$, we get

$$\alpha \left(\tfrac{1+\sqrt{5}}{2}\right)^0 + \beta \left(\tfrac{1-\sqrt{5}}{2}\right)^0 = \alpha + \beta = 0,$$

and

$$\alpha \left(\tfrac{1+\sqrt{5}}{2}\right)^1 + \beta \left(\tfrac{1-\sqrt{5}}{2}\right)^1 = \alpha \left(\tfrac{1+\sqrt{5}}{2}\right) + \beta \left(\tfrac{1-\sqrt{5}}{2}\right) = 1.$$

The first equation gives $\beta = -\alpha$. Put that in the second equation to get

$$\alpha \left(\tfrac{1+\sqrt{5}}{2}\right) - \alpha \left(\tfrac{1-\sqrt{5}}{2}\right) = \tfrac{2\alpha\sqrt{5}}{2} = \alpha\sqrt{5} = 1,$$

implying that $\alpha = 1/\sqrt{5}$ and $\beta = -1/\sqrt{5}$.

The final solution

Using the obtained α and β, our solution to F_n becomes

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Now, we can check that this is indeed the correct solution to F_n.

Note that $\frac{1+\sqrt{5}}{2} \approx 1.61803$ is the golden ratio. Also observe that $|\frac{1-\sqrt{5}}{2}| \approx |-0.61803| < 1$; therefore, the term $\left(\frac{1-\sqrt{5}}{2}\right)^n$ goes to zero as n goes to infinity. This explains why we only observe only the ratio $\frac{1+\sqrt{5}}{2}$ in F_n as n gets large.