01204211 Discrete Mathematics
Lecture 13: Binomial Coefficients

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The binomial coefficients[1]

There is a reason why the term $\binom{n}{k}$ is called the binomial coefficients. In this lecture, we will discuss

- the Pascal's triangle,
- the binomial theorem, and
- advanced counting with binomial coefficients.

[1] This lecture mostly follows Chapter 3 of [LPV].

The equation

Last time we proved that, for $n, k > 0$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

While we can prove this equation algebraically using definitions of binomial coefficients, proving the fact by describing the process of choosing k-subsets reveals interesting insights. This equation also hints us how to compute the value of $\binom{n}{k}$ using values of $\binom{n}{\cdot}$'s. So, let's try to do it.

The table

We shall use the fact that $\binom{n}{0} = 1$ and $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ to fill in the following table.

n	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

You can note that the table is left-right symmetric. This is true because of the fact that $\binom{n}{k} = \binom{n}{n-k}$.

The Triangle

If we move the numbers in the table slightly to the right, the table becomes the Pascal's triangle.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & \vdots & & \vdots & & & & & & & & \vdots &
\end{array}
$$

The table and the binomial coefficients have many other interesting properties.

Polynomial expansions

Let's start by looking at polynomial of the form $(x+y)^n$. Let's start with small values of n:

- $(x+y)^1 = x + y$
- $(x+y)^2 = x^2 + 2 \cdot xy + y^2$
- $(x+y)^3 = x^3 + 3 \cdot x^2y + 3 \cdot xy^2 + y^3$
- $(x+y)^4 = x^4 + 4 \cdot x^3y + 6 \cdot x^2y^2 + 4 \cdot xy^3 + y^4$.

Let's focus on the coefficient of each term. You may notice that terms x^n and y^n always have 1 as their coefficients. *Why is that?* Let's look further at the coefficients of terms $x^{n-1}y$. Do you see any pattern in their coefficients? *Can you explain why?*

Another way to look at it

Let's take a look at $(x+y)^4$ again. It is

$$(x+y)(x+y)(x+y)(x+y).$$

- How do we get x^4 in the expansion? For every factory, you have to pick x.
- How do we get x^3y in the expansion? Out of the 4 factors, you have to pick y in one of the factor (or you have to pick x in 3 of the factors). Thus there are $\binom{4}{3} = \binom{4}{1}$ ways to do so.

The binomial theorem

Theorem: If you expand $(x+y)^n$, the coefficient of the term $x^k y^{n-k}$ is $\binom{n}{k}$.

That is,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} =$$

$$\binom{n}{n} x^n + \binom{n}{n-1} x^{n-1} y^1 + \binom{n}{n-2} x^{n-2} y^2 + \cdots + \binom{n}{1} x y^{n-1} + \binom{n}{0} y^n.$$

Additional applications of the binomial theorem

The binomial theorem can be used to prove various identities regarding the binomial coefficients. For example, if we let $x = 1$ and $y = 1$, we get that

$$(1+1)^n = 2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

Quick check. Can you prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = 0.$$

Note that this statements says that the number of odd subsets equals the number of even subsets.

More on counting

We shall see more techniques for counting when we consider the following problems.

- How many anagrams does the word "KASETSARTUNIVERSITY" have? (They do not have to be real English words.)
- How can you give out n different presents to k students when student i has to get n_i pieces of presents?
- How many ways can you distribute n baht coins to k children?

Easy anagrams

- An anagram of a particular word is a word that uses the same set of alphabets. For example, the anagrams of ADD are ADD, DAD, and DDA.
- How many anagrams does "$ABCD$" have?
  - $4!$, because every permutation of A B C or D is a different anagram.

Harder anagrams

- How many anagrams does "$ABCC$" have? Is it $4!$?
  - This time we have to be careful because the answer of $4!$ is too large as it over counts many anagrams, i.e., it "distinguishes" the two C's.
  - Let's try to be concrete. How many times does "$CABC$" get counted in $4!$?
  - If we treat two C's differently as C_1 and C_2, we can see that $CABC$ is counted twice as C_1ABC_2 and C_2ABC_1. This is true for any anagram of $ABCC$.
  - Since each anagram is counted in $4!$ twice, the number of anagrams is $4!/2 = 4 \cdot 3 = 12$.

General anagrams

Let's try to use the same approach to count the anagram of $HELLOWORLD$. (It has 3 L's, 2 O's, H, E, W, R, and D.)

The number of permutation of alphabets in $HELLOWORLD$, treating each character differently is $10!$. However, each anagram is counted for $3!2!$ times because of the 3 copies of L and the 2 copies of O. Therefore, the number of anagrams is

$$\frac{10!}{3!2!}.$$

Distributing presents

I have 9 different presents. I want to give them to 3 students: A, B, and C. I want to give each student 3 presents. In how many ways can I do it?

- Let's think about the process of distributing the presents. We can first let A choose 3 presents, then B chooses the next 3 presents, and C chooses the last 3 presents. If we distinguish the order which each child chooses the presents, then there are $9!$ ways. However, in this case, we consider the distribution of presents, i.e., we consider the set of presents each child gets.
- To see how many times each distribution is counted in the $9!$ ways, we can let children form a line and let each child permute his or her presents. Each child has $3!$ choices. Thus, one distribution appears $3!3!3!$ times.
- Thus, the number of ways we can distribute presents is

$$\frac{9!}{3!3!3!}$$

Another way to look at the present distribution

- Let's look closely at a particular present distribution in the previous question. Let $\{1, 2, \ldots, 9\}$ be the set of presents.
- Consider the case where A gets $\{1, 3, 8\}$, B gets $\{2, 4, 6\}$, and C gets $\{5, 7, 9\}$.
- Another way to look at this distribution is to fix the order of the presents and see who gets each of the presents. Thus, the previous distribution is represented in the following table:

Presents	1	2	3	4	5	6	7	8	9
Children	A	B	A	B	C	B	C	A	C

- This is essentially an anagram problem. You can think of one particular way of present distribution as anagram of AAABBBCCC. Thus, we reach the same solution of

$$\frac{9!}{3!3!3!}.$$

Distributing identical presents

Now suppose that I have 9 identical presents. I want to give them to 3 students: A, B, and C. I want to give each student 3 presents. In how many ways can I do it?

- Note that when we state that the presents are identical, we mean that we do not distinguish them, i.e., the first present and the second present are indistinguishable.

Distributing coins (1)

I have 9 indentical coins. I want to give them to 3 students: A, B, and C. In how many ways can I do it so that each student gets at least one coin?

- Let's first try to organize the distribution of coins. We place all 9 coins in a line. We let the first student picks some coin, then the second student, then the last one.
- Since each coin is identical, we can let the first student picks the coin from the beginning of the line. Then the second one pick the next set of coins, and so on.
- One possible distribution is

$$\underbrace{oo}_{1}\underbrace{oooo}_{2}\underbrace{ooo}_{3}$$

- In how many ways can we do that?

Distributing coins (2)

The example below provides us with a hint on how to count.

$$\underbrace{oo}_{1}\underbrace{oooo}_{2}\underbrace{ooo}_{3}$$

Since all coins are identical, what matters are where the first student and the second student stop picking the coins. I.e, the previous example can be depicted as

$$oo|oooo|ooo$$

Thus, in how many ways can we do that?
Since there are 8 places we can mark starting points, and there are 2 starting points we have to place, then there are $\binom{8}{2}$ ways to do so.
This is a fairly surprising use of binomial coefficients.

Distributing coins (3)

Let's consider a general problem where we have n identical coins to give out to k students so that each student gets at least one coin. In how many ways can we do that?

Since there are $n-1$ places between n coins and we need to place $k-1$ starting points, there are $\binom{n-1}{k-1}$ ways to do so.

> There are $\binom{n-1}{k-1}$ ways to distribute n identical coins to k children so that each child get at least one coin.

Distributing coins (4)

I have 9 indentical coins. I want to give them to 3 students: A, B, and C. In how many ways can I do it, given that some student may not get any coins?

Identities in the Triangle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Odd and even subsets

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Let's try to prove this identity with the Pascal's triangle

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0.$$

A more formal proof

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0.$$

The next experiment

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Let's try to compute the sum of squares of numbers in each row.

$$
\begin{aligned}
1^2 &= 1 \\
1^2 + 1^2 &= 2 \\
1^2 + 2^2 + 1^2 &= 6 \\
1^2 + 3^2 + 3^2 + 1^2 &= 20 \\
1^2 + 4^2 + 6^2 + 4^2 + 1^2 &= 70
\end{aligned}
$$

Theorem:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Another identity

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

This suggests

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$

Theorem:

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$

Let's see the actual value of the binomial coefficients $\binom{n}{\cdot}$.

What do you see?

- The function $\binom{n}{\cdot}$ is symmetric around $n/2$.
- Why? This is true because we know that $\binom{n}{k} = \binom{n}{n-k}$.
- The maximum is at the middle, i.e., when n is even the maximum is at $\binom{n}{n/2}$ and when n is odd, the maximum is at $\binom{n}{\lfloor n/2 \rfloor}$ and $\binom{n}{\lceil n/2 \rceil}$.
- Why? Can we prove that?

Largest in the middle

To understand the behavior of $\binom{n}{k}$ as k changes, let's look at two consecutive values:

$$\binom{n}{k} \ \heartsuit \ \binom{n}{k+1}$$

Let's write them out:

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \heartsuit \frac{n(n-1)(n-2)\cdots(n-k)}{(k+1)k!}.$$

Removing common terms, we can see that we are comparing these two terms:

$$1 \heartsuit \frac{n-k}{k+1} \Leftrightarrow k \heartsuit \frac{n-1}{2},$$

that is,

- if $k < (n-1)/2$, $\binom{n}{k} < \binom{n}{k+1}$; and
- if $k > (n-1)/2$, $\binom{n}{k} > \binom{n}{k+1}$.

How large is the middle $\binom{n}{n/2}$

Here, to simplify the calculation, we shall only consider the case when n is even. Let's try to estimate the value of $\binom{n}{n/2}$ by finding its upper and lower bounds.
A simple upper bound can be obtain using the fact that $\binom{n}{n/2}$ counts subsets of certain size:

$$\binom{n}{n/2} < 2^n.$$

We can also get a lower bound by noting that the maximum must be at least the average, i.e.,

$$\binom{n}{n/2} \geq \frac{2^n}{n+1}$$

Combining both bounds, we get that

$$\frac{2^n}{n+1} \leq \binom{n}{n/2} < 2^n.$$

Let's plug in $n = 200$, and calculate the number of digits to see how close these bounds.

$$27.80 \approx 200 \cdot \log 2 - \log 201 \leq \log \binom{n}{n/2} < 200 \cdot \log 2 \approx 30.10$$

Can we get a better approximation?