# 01204211 Discrete Mathematics
# Lecture 14: Binomial Coefficients (2)

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# The binomial coefficients[1]

[1] This lecture mostly follows Chapter 3 of [LPV].

Let's see the actual value of the binomial coefficients $\binom{n}{\cdot}$.

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- Why? Can we prove that?

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that is,

- if $k < (n-1)/2$, $\binom{n}{k} < \binom{n}{k+1}$; and
- if $k > (n-1)/2$, $\binom{n}{k} > \binom{n}{k+1}$.

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We can also get a lower bound by noting that the maximum must be at least the average, i.e.,

$$\binom{n}{n/2} \geq \frac{2^n}{n+1}$$

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Let's plug in $n = 200$, and calculate the number of digits to see how close these bounds.

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Can we get a better approximation?