

01204211 Discrete Mathematics  
Lecture 6: Mathematical Induction 1

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# Mathematical Induction

- ▶ In this lecture, we will focus on how to prove properties on natural numbers.
- ▶ For example, we may want to prove that for any integer  $n \geq 1$ ,

$$\sum_{i=1}^n i = n(n+1)/2,$$

or for any integer  $n \geq 1$ ,

$$\sum_{i=1}^n i^2 = \frac{n}{6}(n+1)(2n+1),$$

or “We can pay any integer amount  $x \geq 4$  baht with 2-baht coins and 5-baht coins.”

## A review of the summation notation (by examples)

- ▶  $\sum_{i=1}^{10} i = 1 + 2 + \cdots + 10.$   
(reads “sum from  $i = 1$  to 10 of  $i$ ” or “sum of  $i$  from  $i = 1$  to 10”)
- ▶  $\sum_{i=7}^9 (i^2 + i) = (7^2 + 7) + (8^2 + 8) + (9^2 + 9).$   
(reads “sum from  $i = 7$  to 9 of  $i^2 + i$ ” or “sum of  $i^2 + i$  from  $i = 7$  to 9”)
- ▶ The range of the index may be sets. For example, let  $A = \{1, 2, 4, 15\}$ , we have that  $\sum_{i \in A} i^2 = 1^2 + 2^2 + 4^2 + 15^2.$
- ▶ What is  $\sum_{i=5}^2 i$ ? Note that in this case, the range is empty. This sum is called an **empty sum**. By convention, we define it to be zero.

# Informal arguments (1)

- ▶ Let's try to check that  $\sum_{i=1}^n i = n(n+1)/2$ , for any integer  $n \geq 1$ , by experimentation.
- ▶ Try  $n = 1$ : LHS<sup>1</sup>: 1, RHS<sup>2</sup>:  $1(1+1)/2 = 1$ , OK
- ▶ Try  $n = 2$ : LHS:  $1 + 2 = 3$ , RHS:  $2(2+1)/2 = 3$ , OK
- ▶ Try  $n = 3$ : LHS:  $1 + 2 + 3 = 6$ , RHS:  $3(3+1)/2 = 6$ , OK
- ▶ Try ...
- ▶ With this trying-all approach, we can't actually prove this statement.

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<sup>1</sup>LHS = left hand side

<sup>2</sup>RHS = right hand side

## Informal arguments (2)

- ▶ Our goal is to show that  $\sum_{i=1}^n i = n(n+1)/2$ , for any integer  $n \geq 1$ .
- ▶ Try  $n = 2$ : LHS:  $1 + 2 = 3$ , RHS:  $2(2+1)/2 = 3$ .
- ▶ Try  $n = 3$ : LHS:  $1 + 2 + 3$ , RHS:  $3(3+1)/2$
- ▶ If we compare these two lines, we can see that

$$\begin{aligned} 1 + 2 + 3 &= (1 + 2) + 3 \\ &= 2(2 + 1)/2 + 3 & (*) \\ &= 2(2 + 1)/2 + (2 + 1) \\ &= 2(2 + 1)/2 + 2 \cdot (2 + 1)/2 \\ &= (2 + 2)(2 + 1)/2 = (3 + 1)(3)/2, \end{aligned}$$

which is equal to  $3(3+1)/2$ .

- ▶ Line (\*) is important here. That is because we use the fact that the statement is true when  $n = 2$  there.

## Informal arguments (3)

- ▶ Goal: show that  $\sum_{i=1}^n i = n(n+1)/2$ , for any integer  $n \geq 1$ .
- ▶ **What we have just done?** We show that the statement is true when  $n = 3$  if it is true when  $n = 2$ .
- ▶ Let's try to make a more general argument.
- ▶ Assume that the statement is true for  $n = k$ . I.e.,

$$\sum_{i=1}^k i = k(k+1)/2.$$

- ▶ Can we show that, with this assumption, the statement is true for  $n = k + 1$ ? I.e., can we show that

$$\sum_{i=1}^{k+1} i = (k+1)((k+1)+1)/2?$$

## Informal arguments (4)

Let's try...

**Assumption:**  $\sum_{i=1}^k i = k(k+1)/2$ .

**Goal:**  $\sum_{i=1}^{k+1} i = (k+1)((k+1)+1)/2$ .

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \left( \sum_{i=1}^k i \right) + (k+1) \\ &= k(k+1)/2 + (k+1) \\ &= k(k+1)/2 + 2 \cdot (k+1)/2 \\ &= (k+2)(k+1)/2 \\ &= (k+1)((k+1)+1)/2,\end{aligned}$$

as required.

## Informal arguments (5)

We have all the ingredients required to prove this statement:

$$\text{For integer } n \geq 1, \sum_{i=1}^n i = n \cdot (n + 1)/2.$$

Let  $P(n) \equiv \text{“}\sum_{i=1}^n i = n \cdot (n + 1)/2\text{”}$ .

The statement we want to prove becomes:

For any natural number  $n$ ,  $P(n)$ .

We have shown:

1.  $P(1)$  (by experimentation)
2.  $P(k) \Rightarrow P(k + 1)$  for any integer  $k \geq 1$ .

What do these two statements imply?



## Informal arguments (6)

We have:

1.  $P(1)$  (by experimentation)
2.  $P(k) \Rightarrow P(k + 1)$  for any integer  $k \geq 1$ .

What do these two statements imply?

$P(1)$  (1st statement itself)

$\Rightarrow P(2)$  (from 2nd statement, let  $k = 1$ )

$\Rightarrow P(3)$  (from 2nd statement, let  $k = 2$ )

$\Rightarrow P(4)$  (from 2nd statement, let  $k = 3$ )

$\Rightarrow P(5) \Rightarrow P(6) \Rightarrow P(7) \dots$

Informally, these chain of reasoning will eventually reach any natural number  $n$ . Therefore, we can conclude that  $P(n)$  for any natural number  $n$ .

We have just shown the statement with mathematical induction.

# Mathematical Induction

Suppose that you want to prove that property  $P(n)$  is true for every natural number  $n$ .

Suppose that we can prove the following two facts:

**Base case:**  $P(1)$

**Inductive step:** For any  $k \geq 1$ ,  $P(k) \Rightarrow P(k + 1)$

The **Principle of Mathematical Induction** states that  $P(n)$  is true for every natural number  $n$ .

The assumption  $P(k)$  in the inductive step is usually referred to as **the Induction Hypothesis**.

# Let's re-write the proof again

## Theorem 1

For every natural number  $n$ ,  $\sum_{i=1}^n i = n(n+1)/2$

**Proof:** We prove by induction. The property that we want to prove  $P(n)$  is " $\sum_{i=1}^n i = n(n+1)/2$ ."

**Base case:** We can plug in  $n = 1$  to check that  $P(1)$  is true:  
 $1 = 1(1+1)/2$ .

**Inductive step:** We assume that  $P(k)$  is true for  $k \geq 1$  and show that  $P(k+1)$  is true.

Let's state the Induction Hypothesis  $P(k)$ :  $\sum_{i=1}^k i = k(k+1)/2$ .

Let's show  $P(k+1)$ . We write  $\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^k i\right) + (k+1)$ . Using the Induction Hypothesis, we know that this is equal to

$$\begin{aligned} k(k+1)/2 + (k+1) &= k(k+1)/2 + 2 \cdot (k+1) \\ &= (k+2)(k+1)/2, \end{aligned}$$

which implies  $P(k+1)$  as required.

From the Principle of Mathematical Induction, this implies that  $P(n)$  is true for every natural number  $n$ .