01204211 Discrete Mathematics
Lecture 14: Binomial Coefficients (2)

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The binomial coefficients[1]

In this lecture, we shall study the function $\binom{n}{k}$ itself. First, let's see the actual value of the binomial coefficients $\binom{n}{k}$ for various values of n.

[1] This lecture mostly follows Chapter 3 of [LPV].

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- Why? Can we prove that?

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that is,

- if $k < (n-1)/2$, $\binom{n}{k} < \binom{n}{k+1}$; and
- if $k > (n-1)/2$, $\binom{n}{k} > \binom{n}{k+1}$.

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We can also get a lower bound by noting that the maximum must be at least the average, i.e.,

$$\binom{n}{n/2} \geq \frac{2^n}{n+1}$$

Combining both bounds, we get that

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Let's plug in $n = 200$, and calculate the number of digits to see how close these bounds.

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Can we get a better approximation?

Concentration

- We know that the maximum of $\binom{n}{k}$ is obtained when $k = n/2$. From the graph, you can see that, as you move further from the middle, the value of the function drops rapidly.
- Since we consider even n, we let $2m = n$. One way to quantify how fast the values drop is to think about the ratio

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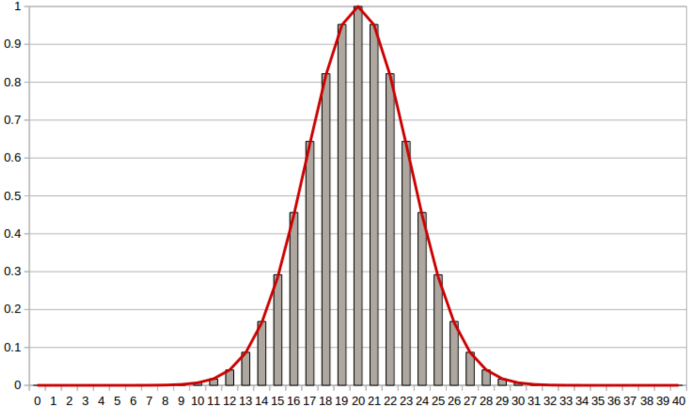
- We will use our basic tools to obtain weaker bounds.

How close is the approximation?

The estimation $e^{-t^2/m}$ is extremely close as shown in the figure below, where the gray bars are the actual value of $\binom{2m}{m-t}/\binom{2m}{m}$ and the red line is $e^{-t^2/m}$.



The actual values

Let's try to calculate the ratio

$$\binom{2m}{m-t} \Big/ \binom{2m}{m} = \frac{(2m)!}{(m-t)!(2m-m+t)!} \times \frac{m!m!}{(2m)!}$$
$$= \frac{m(m-1)(m-2)\cdots(m-t+1)}{(m+t)(m+t-1)\cdots(m+1)}.$$

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We can use the same logarithm trick. We have that the log of the ratio is

$$\ln\left(\frac{m}{m+t}\right) + \ln\left(\frac{m-1}{m+t-1}\right) + \cdots + \ln\left(\frac{m-t+1}{m+1}.\right)$$