

# 01204211 Discrete Mathematics

## Lecture 10: Counting 2

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In this case, we order by their cardinalities, then use dictionary ordering for subsets with the same numbers of elements.

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  - ▶  $\{b, c\}$  is rep. as:  $011_2 = 3$ ,  $\{\}$  is rep. as:  $000_2 = 0$
- ▶ Also, this representation can be considered backwards, i.e., if we start with an integer 6, we can write down its binary representation:  $110_2$  and turns it into a subset  $\{a, b\}$ .

# A correspondence

Let's see a full list of correspondence between  $\{0, 1, 2, \dots, 7\}$  and subsets of  $\{a, b, c\}$ .

- ▶  $0 \leftrightarrow 000_2 \leftrightarrow \{\}$
- ▶  $1 \leftrightarrow 001_2 \leftrightarrow \{c\}$
- ▶  $2 \leftrightarrow 010_2 \leftrightarrow \{b\}$
- ▶  $3 \leftrightarrow 011_2 \leftrightarrow \{c, b\}$
- ▶  $4 \leftrightarrow 100_2 \leftrightarrow \{a\}$
- ▶  $5 \leftrightarrow 101_2 \leftrightarrow \{a, c\}$
- ▶  $6 \leftrightarrow 110_2 \leftrightarrow \{a, b\}$
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Do you notice anything interesting?

## A general case

Similarly, we can describe a representation for each subset of a set  $A$  with  $n$  elements. As we consider each element  $a$  of  $A$ , we put 1 if  $a \in A$  and put 0 if  $a \notin A$ .



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How many bit strings of length  $n$  are there?

There are  $2^n$  bit strings; hence, the number of subsets is also  $2^n$ . This is another proof of the following theorem:

**Theorem:** The number of subsets of a set with  $n$  elements is  $2^n$ .

# Two proofs

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Really, it does not make a statement stronger, truer, “more” correct. But each proof usually reveals additional facts related to the statement.

- ▶ The first proof considers a procedure for constructing subsets.
- ▶ The second proof introduces a nice technique for counting. I.e., instead of counting subsets directly, we show that we have a “special” correspondence between subsets and binary numbers, and then just count the numbers.

# A bijection

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- ▶ For each number, there is only **one** subset that corresponds to it.
- ▶ For each subset, there is only **one** number that it corresponds to.

With these two properties, we can conclude that both sets have the same cardinality.

This type of correspondence is called a **one-to-one correspondence** or **bijection**.



# Sequences of choices

Previously, when we want to count the number of bit strings of length  $n$ , we use this argument:

Suppose that to select an object, you have to make  $k$  decisions. The first decision has  $n_1$  choices, the second decision has  $n_2$  choices, and so on. More precisely, for  $1 \leq i \leq k$ , the  $i$ -th decision has  $n_i$  choices. Then the number of ways you can select an object is  $n_1 \cdot n_2 \cdots n_{k-1} \cdot n_k$ .

## Example 1

A car license number consists of two English letters and one number from 1 to 9999. How many possible license numbers are there?

## Example 2

10 students stand in a line. You want to give them ice cream. There are 4 flavours, but you don't want to give the same flavour to any consecutive students. In how many ways can you give out the ice cream to these students?

# Permutations

# Counting permutations: an example

We want to count the number of permutations. Let's try with a small example: permutations of set  $\{a, b, c\}$ .

# Counting permutations

# Number of permutations

We have proved this theorem.

**Theorem:** The number of permutations of a set with  $n$  elements is  $n!$ .