01204211 Discrete Mathematics
Lecture 1: Introduction

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What is this course about?

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Ah... that's a philosopical question.
IMHO, mathematics is a mean to communicate *precise* ideas.

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- I hope it is also true with this course.

The goals of this course

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- To learn how to make mathematical arguments.
- To learn various fundamental mathematical concepts that are very useful in computer science.

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    return a
else:
    return b
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    if c > b:
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Finally, look at this program.

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// Input: array A with n elements: A[0],...,A[n-1]
m = 0
for i = 0, 1, ..., n-1:
  if A[i] > m:
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    if n <= 1:
        return False
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Can we do better?

Another example: testing primes (2)

Consider the following code.

```
Algorithm CheckPrime2(n):  // Input: an integer n
    if n <= 1:
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    let s = square root of n
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Ok, it should be faster. **But is it correct?**

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- Are we done?

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- How can we do that?

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- Before we continue, I'd like to add a bit of formalism to our thinking process.

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> **"Algorithm `CheckPrime2` is correct."**

- Note that this statement can either be "true" or "false." If we can demonstrate, using logical/mathematical arguments that this statement is true, we can say that we **prove** the statement.

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- Are we doom? Not really. The statement above is not precisely the statement we want to prove.

The (sub) goal (second try)

- **Current (sub) goal:** Consider a positive composite n and its positive divisor a, where $a > \sqrt{n}$. Let $b = n/a$. We want to show that $2 \leq b \leq \sqrt{n}$.
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Revised statement

For all positive composite integer n, and for some of its divisor a such that $a > \sqrt{n}$,

$$2 \leq b \leq \sqrt{n},$$

where $b = n/a$.

- Note that this revised statement is now "quantified," that is, every variable in the statement has specific scope. Now the statement is either true or false.

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  - Algorithm `CheckPrime2` is correct.
  - $10^2 = 100$.
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- Examples:
  - Algorithm `CheckPrime2` is correct.
  - $10^2 = 100$.
  - $\sqrt{2}$ is irrational.
- Examples of statements which are not propositions (why?):
  - $x > 10$.
  - $1 + 2 + \cdots + 10$.
  - This algorithm is fast.
  - Run, run fast.

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- For example, we can join P and Q with "and" (denoted by "$\wedge$") and get
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which stands for "it rains and the road is wet".
- An expression $P \wedge Q$ is an example of *propositional forms*. The logical value of a propositional form "usually" depends on the truth value of its variables.

Connectives: “and”, “or”, “not”

Given propositions P and Q, we can use connectives to form more complex propositions:

- **Conjunction:** $P \wedge Q$ (“P and Q”),
  (True when both P and Q are true)
- **Disjunction:** $P \vee Q$ (“P or Q”),
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- **Negation:** $\neg P$ (“not P”)
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If P stands for "today is Tuesday" and Q stands for "dogs are animals", then

- $P \wedge Q$ stands for "today is Tuesday and dogs are animals",
- $P \vee Q$ stands for "today is Tuesday or dogs are animals", and
- $\neg P$ stands for "today is not Tuesday".

Truth tables

To represents values of propositional forms, we usually use truth tables.

And/Or/Not

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$
T	T	T	T	F
T	F	F	T	
F	T	F	T	T
F	F	F	F	

Quick check 1

For each of these statements, define propositional variables representing each proposition inside the statement and write the proposition form of the statement.

- All prime numbers are larger than 0 and all natural numbers is at least one.
- You are smart or you won't be taking this class.

Quick check 2

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Note that $P \wedge \neg P$ is always false and $P \vee \neg P$ is always true.
A propositional form which is always true regardless of the truth values of its variables is called a *tautology*. On the other hand, a propositional form which is always false regardless of the truth values of its variables is called a *contradiction*.

Implications

Given P and Q, an implication

$$P \Rightarrow Q$$

stands for "if P, then Q". This is a very important propositional form.

It states that "when P is true, Q must be true". Let's try to fill in its truth table:

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What?

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- We say that in this case, the statement $P \Rightarrow Q$ is *vacuously true.*

What?

- Yes, when P is false, $P \Rightarrow Q$ is **always true** no matter what truth value of Q is.
- We say that in this case, the statement $P \Rightarrow Q$ is *vacuously true.*
- You might feel a bit uncomfortable about this, because in most natural languages, when we say that if P, then Q we sometimes mean something more than that in the logical expression "$P \Rightarrow Q$."

One explanation

- But let's look closely at what it means when we say that:

  > if P is true, Q must be true.

- Note that this statement does not say anything about the case when P is false, i.e., it only considers the case when P is true.

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- This is an example when mathematical language is "stricter" than natural language.

Noticing if-then

We can write "if P, then Q" for $P \Rightarrow Q$, but there are other ways to say this. E.g., we can write (1) Q if P, (2) P only if Q, or (3) when P, then Q.

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Quick check 3

For each of these statements, define propositional variables representing each proposition inside the statement and write the proposition form of the statement.

- If you do not have enough sleep, you will feel dizzy during class.
- If you eat a lot and you do not have enough exercise, you will get fat.
- You can get A from this course, only if you work fairly hard.

Only-if

Let P be "you get A from this course."
Let Q be "you work fairly hard."
Let R be "You can get A from this course, only if you work fairly hard."
Let's think about the truth values of R.

Only if you work fairly hard.

P	Q	R
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F	F	

Thus, R should be logically equivalent to $P \Rightarrow Q$. (We write $R \equiv P \Rightarrow Q$ in this case.)

If and only if: ($\Leftrightarrow$)

Given P and Q, we denote by

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the statement "P if and only if Q." It is logically equivalent to

$$(P \Leftarrow Q) \wedge (P \Rightarrow Q),$$

i.e., $P \Leftrightarrow Q \equiv (P \Leftarrow Q) \wedge (P \Rightarrow Q)$.
Let's fill its truth table.

P	Q	$P \Rightarrow Q$	$P \Leftarrow Q$	$P \Leftrightarrow Q$
T	T			
T	F			
F	T			
F	F			