01204211 Discrete Mathematics
Lecture 6: Mathematical Induction 1

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Mathematical Induction

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or "We can pay any integer amount $x \geq 4$ baht with 2-baht coins and 5-baht coins."

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- The range of the index may be sets. For example, let $A = \{1, 2, 4, 15\}$, we have that $\displaystyle\sum_{i \in A} i^2 = 1^2 + 2^2 + 4^2 + 15^2$.
- What is $\sum_{i=5}^{2} i$? Note that in this case, the range is empty. This sum is called an **empty sum**. By convention, we define it to be zero.

Informal arguments (1)

- Let's try to check that $\sum_{i=1}^{n} i = n(n+1)/2$, for any integer $n \geq 1$, by experimentation.
- Try $n = 1$:

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- With this trying-all approach, we can't actually prove this statement.

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- Try $n = 2$: LHS: $1 + 2 = 3$, RHS: $2(2+1)/2 = 3$.
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&= (2+2)(2+1)/2 = (3+1)(3)/2,
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which is equal to $3(3+1)/2$.

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- If we compare these two lines, we can see that

$$\begin{aligned} 1+2+3 &= (1+2)+3 \\ &= 2(2+1)/2+3 \qquad (*) \\ &= 2(2+1)/2+(2+1) \\ &= 2(2+1)/2+2\cdot(2+1)/2 \\ &= (2+2)(2+1)/2 = (3+1)(3)/2, \end{aligned}$$

  which is equal to $3(3+1)/2$.
- Line (*) is important here. That is because we use the fact that the statement is true when $n = 2$ there.

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- Can we show that, with this assumption, the statement is true for $n = k + 1$? I.e., can we show that

$$\sum_{i=1}^{k+1} i = (k+1)((k+1)+1)/2?$$

Informal arguments (4)

Let's try...
Assumption: $\sum_{i=1}^{k} i = k(k+1)/2$.
Goal: $\sum_{i=1}^{k+1} i = (k+1)((k+1)+1)/2$.

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&= k(k+1)/2 + (k+1) \\
&= k(k+1)/2 + 2 \cdot (k+1)/2 \\
&= (k+2)(k+1)/2 \\
&= (k+1)((k+1)+1)/2,
\end{aligned}
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as required.

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We have all the ingredients required to prove this statement:

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Informally, these chain of reasoning will eventually reach any natural number n. Therefore, we can conclude that $P(n)$ for any natural number n.

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Informally, these chain of reasoning will eventually reach any natural number n. Therefore, we can conclude that $P(n)$ for any natural number n.
We have just shown the statement with mathematical induction.

Mathematical Induction

Suppose that you want to prove that property $P(n)$ is true for every natural number n.

Suppose that we can prove the following two facts:
Base case: $P(1)$
Inductive step: For any $k \geq 1$, $P(k) \Rightarrow P(k+1)$

The **Principle of Mathematical Induction** states that $P(n)$ is true for every natural number n.

The assumption $P(k)$ in the inductive step is usually referred to as **the Induction Hypothesis**.

Let's re-write the proof again

Theorem 1

For every natural number n, $\sum_{i=1}^{n} i = n(n+1)/2$

Proof: We prove by induction. The property that we want to prove $P(n)$ is "$\sum_{i=1}^{n} i = n(n+1)/2$."

Base case: We can plug in $n = 1$ to check that $P(1)$ is true: $1 = 1(1+1)/2$.

Inductive step: We assume that $P(k)$ is true for $k \geq 1$ and show that $P(k+1)$ is true.

Let's state the Induction Hypothesis $P(k)$: $\sum_{i=1}^{k} i = k(k+1)/2$.

Let's show $P(k+1)$. We write $\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$. Using the Induction Hypothesis, we know that this is equal to

$$\begin{aligned} k(k+1)/2 + (k+1) &= k(k+1)/2 + 2 \cdot (k+1) \\ &= (k+2)(k+1)/2, \end{aligned}$$

which implies $P(k+1)$ as required.

From the Principle of Mathematical Induction, this implies that $P(n)$ is true for every natural number n.