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COMBINATIONS OF OPTION SPREADS

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Abstract: Having been constructed as trading strategies, option spreads are also used in margin calculations for offsetting positions in options. All option spreads that appear in trading and margining practice have two, three or four legs. It is well-known that the option spreads with three and four legs are combinations of two option spreads with two legs, and that hedging mechanisms of these combinations consolidate hedging mechanisms of their components. Although more complex combinations with similar properties can be traced in regulatory literature of 2003, they have not yet been studied and used. In this paper we develop a theory for the construction of multi-leg option spreads as combinations of well-known option spreads with two, three and four legs. We show how multi-leg option spreads with extreme properties can maximize arbitrage opportunities in trading options and substantially reduce margin requirements for option portfolios.

1 INTRODUCTION

Option spreads with two, three and four legs such as bull and bear spreads, butterfly, condor, iron butterfly, iron condor and box spreads have been known for more than three decades and have become standard in options trading; cf. (McMillan, 2002; Cohen, 2005; Curley, 2008). Descriptions of more complex spreads appeared as efficient means of margin reductions in 2003. It is important to explain how these spreads were motivated.

1.1 Regulatory Breakthrough

By the end of the nineties, it was commonly recognized that margin regulations impose excessively high minimum margin requirements, especially for option portfolios. This can be partially explained by the fact that option spreads permitted for offsetting by margin regulations by that time had at most four legs. [1] However, it is well-known that the more legs an option spread has the more margin reduction it gives. As shown in (Matsypura and Timkovsky, 2011), just one additional leg can save several thousand dollars on margin. Thus, the reduction of minimum margin requirements can be achieved by constructing new option spreads with a larger number of legs.

[1] A leg of an option spread or offset based on this spread is a position in options with the same exercise price and expiry date.

Option spreads with up to 12 legs appeared as combinations of option spreads with two, three and four legs in August 2003 when the CBOE [2] proposed new margin rules based on these combinations that were called *complex spreads* (CBOE, 2003). After two revisions of this proposal (CBOE, 2004; CBOE, 2005), the SEC [3] approved these rules (SEC, 2005) and added them to NYSE Rule 431 in December 2005. In August 2007, these rules were also recognized in Canada (IDA, 2007).

1.2 Motivation

The regulatory breakthrough of 2005, however, received a limited response of the brokerage industry by the following two reasons: firstly, the definition of the complex spreads was given in a text form that does not allow for complete understanding of their structure, and hence how these spreads can be utilized; secondly, the interest to multi-leg option spreads had been lost because the risk-based margining methodology that had become popular in the U.S. in 2005 offered computationally easier solutions. Consequently, option spreads with more than four legs are still not being used, primarily because they have neither been studied nor properly understood.

Multi-leg option spreads thus call for academic research that shall explain how they can be constructed,

[2] The Chicago Board Options Exchange.

[3] The U.S. Securities and Exchange Commission.

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what advantage they give, and how they can be utilized in options trading and margining practice. To the best of our knowledge, this kind of research has never been attempted. As we show in this paper, 12 legs is not the final step. We discover new multi-leg option spreads that have the same hedging mechanism as that of complex option spreads and propose a full characterization of option spreads with any number of legs. We also formulate integer programs that demonstrate that multi-leg option spreads maximize arbitrage opportunities in options trading and substantially reduce margin requirements in margin accounts with options.

2 MAIN SPREADS

A vector model of option spreads with up to four legs was proposed in (Matsypura and Timkovsky, 2011). In this section we give an extension of this model that deals with option spreads of different width.

2.1 Vector Model of Option Spreads

Let $d \geq 2$ be a positive integer. *Option spreads* of dimension d are integer vectors

$$\mathbf{v} = (\ c_1 \quad c_2 \quad \cdots \quad c_d \quad p_1 \quad p_2 \quad \cdots \quad p_d\ )$$

whose components are associated with positions in options in a margin account as follows.

The component c_j, $1 \leq j \leq d$, is the number of option contracts in the jth call option series, with the exercise price e_j. Similarly, the component p_j is the number of option contracts in the jth put option series, with the same exercise price e_j.

Nonzero components represent legs. A positive, negative or zero component means that it is a long, short leg or a leg is absent, respectively. A *zero spread*, denoted $\mathbf{0}$, is a spread without legs.

Let a be a nonnegative integer. Then $a\mathbf{v}$ is a *multiple* of $\mathbf{v}$ with *factor a*. A spread is said to be *prime* if it is not a multiple of another spread with factor more than one. Thus, $\mathbf{0}$ is a prime spread. If $\mathbf{v}$ is a prime spread, then a is a *multiplicity* of $a\mathbf{v}$. If not stated otherwise, we assume further only prime spreads.

Treating spreads as vectors we can add and subtract them, multiply by an integer scalar, cyclicly shift their components and take their *transpositions*, i.e., create the spreads $\overline{\mathbf{v}}$, where the components c_i and p_i are transposed for all $i = 1, 2, \ldots, d$.

We assume that the exercise prices are all different and placed in increasing order, i.e., $e_1 < e_2 < \cdots < e_d$. The set $\{e_1, e_2, \ldots, e_d\}$ is called an *exercise domain*. If the exercise prices are separated by the same price interval, then the length of the interval, D, is the *exercise differential* of the domain, and the exercise domain is said to be *uniform*. [4]

In what follows, we consider only uniform exercise domains and option spreads on the same exercise domain. Therefore, it will be convenient to normalize all prices and costs by divisor D. Thus, we will further assume that all exercise prices and all option prices have been normalized, and hence all exercise domains have exercise differential 1.

Definition 1. *Let w and k be positive integers such that $w < d$ and $k \leq 2d$, and let $v_1, v_2, \ldots, v_k$ be the sequence of leg indices in a spread* $\mathbf{v}$ *of dimension d such that*

$$e_{v_1} \leq e_{v_2} \leq \cdots \leq e_{v_k}$$

If $e_{v_{j+1}} - e_{v_j} = 0$ or w for each $j = 1, 2, \ldots, k-1$, then $\mathbf{v}$ *is a uniform spread of width w.*

We consider only uniform spreads because only they are being used in practice. Besides, as we consider only normalized prices, the width of spreads will always be integer in the set $\{1, 2, \ldots, d-1\}$. Simplest uniform spreads are basic spreads. They can be defined as follows:

Definition 2. *A basic spread is uniform and has two legs, 1 and -1, such that both legs are on the same side, call or put. A basic spread is a basic call/put spread if all its legs are on the call/put side. A basic spread is a basic bull spread if its first leg is long; otherwise it is a basic bear spread.*

The first 12, 8, 4 spreads in Tables 1, 2, 3, present all basic spreads of width 1, 2, 3 and dimension 4, respectively. The abbreviations “dr” and “cr” mark *debit spreads* and *credit spreads*. [5]

Definition 3. *All basic spreads are two-leg main spreads. Let* $\mathbf{u}$ *and* $\mathbf{v}$*, where $\mathbf{u} \neq -\mathbf{v}$, be a basic bull spread and a basic bear spread, respectively, of the same width w, and let $\mathbf{u}+\mathbf{v}$ be a uniform spread of width w. Then $\mathbf{u}+\mathbf{v}$ is a three- or four-leg main spread of width w.*

Although our attention will be focused on the case of dimension four, all further results are valid for any dimension higher than four. The set of all main spreads of width 1, 2, 3 and dimension 4 is presented in Tables 1, 2, 3, respectively. Note that butterfly and

[4] Exercise prices of listed options of the same expiration date generate a uniform exercise domain. For example, according to http://finance.google.com, as of 02-AUG-2011, 5:50PM, exercise prices of options on the IBM stock listed in NYSE and expiring on 20-AUG-2011 generated the uniform exercise domain $\{85, 90, \ldots, 270\}$ of dimension 38.

[5] The term *debit*/*credit* indicates that the spread is a result of a *net debit*/*credit* transaction, respectively.

Table 1: Main spreads of width 1 and dimension 4.

spread	spread name	calls				puts				legs	net
a	1st bull call	1	-1							2	dr
b	2nd bull call		1	-1						2	dr
c	3rd bull call			1	-1					2	dr
e	1st bull put					1	-1			2	cr
f	2nd bull put						1	-1		2	cr
g	3rd bull put							1	-1	2	cr
-a	1st bear call	-1	1							2	cr
-b	2nd bear call		-1	1						2	cr
-c	3rd bear call			-1	1					2	cr
-e	1st bear put					-1	1			2	dr
-f	2nd bear put						-1	1		2	dr
-g	3rd bear put							-1	1	2	dr
$\mathbf{a}-\mathbf{b}$	1st long call butterfly	1	-2	1						3	dr
$\mathbf{b}-\mathbf{a}$	1st short call butterfly	-1	2	-1						3	cr
$\mathbf{b}-\mathbf{c}$	2nd long call butterfly		1	-2	1					3	dr
$\mathbf{c}-\mathbf{b}$	2nd short call butterfly		-1	2	-1					3	cr
$\mathbf{e}-\mathbf{f}$	1st long put butterfly					1	-2	1		3	cr
$\mathbf{f}-\mathbf{e}$	1st short put butterfly					-1	2	-1		3	dr
$\mathbf{f}-\mathbf{g}$	2nd long put butterfly						1	-2	1	3	cr
$\mathbf{g}-\mathbf{f}$	2nd short put butterfly						-1	2	-1	3	dr
$\mathbf{a}-\mathbf{c}$	long call condor	1	-1	-1	1					4	dr
$\mathbf{c}-\mathbf{a}$	short call condor	-1	1	1	-1					4	cr
$\mathbf{e}-\mathbf{g}$	long put condor					1	-1	-1	1	4	dr
$\mathbf{g}-\mathbf{e}$	short put condor					-1	1	1	-1	4	cr
$\mathbf{a}-\mathbf{e}$	1st long box	1	-1			-1	1			4	dr
$\mathbf{e}-\mathbf{a}$	1st short box	-1	1			1	-1			4	cr
$\mathbf{b}-\mathbf{f}$	2nd long box		1	-1			-1	1		4	dr
$\mathbf{f}-\mathbf{b}$	2nd short box		-1	1			1	-1		4	cr
$\mathbf{c}-\mathbf{g}$	3rd long box			1	-1			-1	1	4	dr
$\mathbf{g}-\mathbf{c}$	3rd short box			-1	1			1	-1	4	cr
$\mathbf{a}-\mathbf{f}$	1st long call iron butterfly	1	-1				-1	1		4	dr
$\mathbf{f}-\mathbf{a}$	1st short call iron butterfly	-1	1				1	-1		4	cr
$\mathbf{b}-\mathbf{g}$	2nd long call iron butterfly		1	-1				-1	1	4	dr
$\mathbf{g}-\mathbf{b}$	2nd short call iron butterfly		-1	1				1	-1	4	cr
$\mathbf{e}-\mathbf{b}$	1st long put iron butterfly		-1	1		1	-1			4	cr
$\mathbf{b}-\mathbf{e}$	1st short put iron butterfly		1	-1		-1	1			4	dr
$\mathbf{f}-\mathbf{c}$	2nd long put iron butterfly			-1	1		1	-1		4	cr
$\mathbf{c}-\mathbf{f}$	2nd short put iron butterfly			1	-1		-1	1		4	dr
$\mathbf{e}-\mathbf{c}$	long put iron condor			-1	1	1	-1			4	cr
$\mathbf{c}-\mathbf{e}$	short put iron condor			1	-1	-1	1			4	dr
$\mathbf{a}-\mathbf{g}$	long call iron condor	1	-1					-1	1	4	dr
$\mathbf{g}-\mathbf{a}$	short call iron condor	-1	1					1	-1	4	cr

condor spreads, iron butterfly and iron condor spreads of width 2 or 3 and dimension 4 do not exist.

Theorem 1. *The number of main spreads of width w and dimension d is* $n(w,d) =$

$$6(d-w) + 8\max\{0, d-2w\} + 8\max\{0, d-3w\}.$$

Proof A direct count shows that for fixed w and d the numbers of bull, bear or box spreads, butterfly or iron butterfly spreads, and condor or iron condor spreads are $2(d-w)$, $4\max\{0, d-2w\}$, and $4\max\{0, d-3w\}$, respectively. □

2.2 Portfolios and Linear Combinations of Main Spreads

Let $\boldsymbol{A}$ denote the $2d \times n$ matrix, where

$$n = \sum_{w=1}^{d-1} n(w,d)$$

whose columns are all main spreads of dimension d. If $\boldsymbol{A}(w,d)$ is the matrix of main spreads of width w and dimension d, then

Table 2: Main spreads of width 2 and dimension 4.

spread	spread name	calls				puts				legs	net
a	1st bull call	1		-1						2	dr
b	2nd bull call		1		-1					2	dr
e	1st bull put					1		-1		2	cr
f	2nd bull put						1		-1	2	cr
-a	1st bear call	-1		1						2	cr
-b	2nd bear call		-1		1					2	cr
-e	1st bear put					-1		1		2	dr
-f	2nd bear put						-1		1	2	dr
$\mathbf{a}-\mathbf{e}$	1st long box	1		-1		-1		1		4	dr
$\mathbf{e}-\mathbf{a}$	1st short box	-1		1		1		1		4	cr
$\mathbf{b}-\mathbf{f}$	2nd long box		1		-1		-1		1	4	dr
$\mathbf{f}-\mathbf{b}$	2nd short box		-1		1		1		-1	4	cr

Table 3: Main spreads of width 3 and dimension 4.

spread	spread name	calls				puts				legs	net
a	1st bull call	1			-1					2	dr
e	1st bull put					1			-1	2	cr
-a	1st bear call	-1			1					2	cr
-e	1st bear put					-1			1	2	dr
$\mathbf{a}-\mathbf{e}$	1st long box	1			-1	-1			1	4	dr
$\mathbf{e}-\mathbf{a}$	1st short box	-1			1	1			-1	4	cr

$$\boldsymbol{A} = [\, \boldsymbol{A}(1,d)\; \boldsymbol{A}(2,d)\; \cdots\; \boldsymbol{A}(d-1,d)\,]$$

In what follows, an integer column vector of size n with nonnegative components will be associated with the portfolio of main spreads taken in quantities equal to the components of this vector. Such vectors constitute a *portfolio space*.

An integer column vector of size $2d$ will be associated with a spread, as we described in Section 2.1. Such vectors constitute a *spread space*. Further, all vectors in the portfolio/spread space will be denoted by italic/direct bold letters.

Thus, the matrix $\boldsymbol{A}$, as a left multiplier, transforms portfolios of main spreads into linear combinations of main spreads. As we show in Section 5, a portfolio of main spreads can have multiple representations in the form of linear combination of main spreads.

According to this assumptions, a main spread can be presented in the following two forms:

- as a column vector of size n whose ith component is 1 and the other components are 0s, that is denoted by $\boldsymbol{e}_i$ (a presentation in the *portfolio space*); the index i will be dropped if the main spread is not specific; or
- as a column of $\boldsymbol{A}$, i.e., as a column vector of size $2d$, that is denoted by $\mathbf{b}_i$, if the main spread is the ith column of $\boldsymbol{A}$, or by $\mathbf{b}$ if the main spread is not specific (a presentation in the *spread space*).

These forms are obviously related by the equation

$$\boldsymbol{A}\boldsymbol{e}_i = \mathbf{b}_i$$

2.3 Market Risk of Main Spreads

It is well known that debit spreads are free of market risk, i.e., they have no loss associated with underlying instrument price changes, cf. (Cohen, 2005) or (McMillan, 2002) for a detailed discussion. Credit spreads, in contrast, are not free of market risk.

The maximum loss on a prime credit spread associated with underlying instrument price changes is its width in all cases except for a *short call iron butterfly* and a *short call iron condor* for which the maximum loss is two widths. Therefore, the market risk of a main spread $\mathbf{b}$ is the integer

$$m(\mathbf{b}) = \begin{cases} 0 & \text{if } \mathbf{b} \text{ is a debit spread,} \\ 2w & \text{if } \mathbf{b} \text{ is} \\ & \text{a short call iron butterfly} \\ & \text{or short call iron condor} \\ & \text{spread,} \\ w & \text{otherwise.} \end{cases} \quad (1)$$

In this paper, we consider the market risk of a main spread to be its maintenance margin requirement.

Moreover, we consider only maintenance margin requirements. Details related to a justification of the market risk as a measure of maintenance margin requirements and discussions on the relationship between maintenance and initial margin requirements for main spreads can be found in (Matsypura and Timkovsky, 2011). In what follows, the term “margin” will stand for a maintenance margin requirement.

3 COMPLEX SPREADS

The regulatory amendment of December 14, 2005, initiated by the CBOE, was motivated by the observation that some combinations of main spreads have the same risk profile as single main spreads. This can be explained by the fact that the summation of main spreads in such a combination turns out to be also a main spread that is a resulting spread. These combinations were named *complex spreads*. The margin of a complex spread is exactly the margin of its resulting spread (CBOE, 2003).

3.1 Table of Complex Spreads

Ten of the complex spreads are presented in Table 4. The other ten are their transpositions, where the names of the components have the words "call" and "put" interchanged. Negations of these 20 produce another 20, where the names of the components have the words "long" and "short" interchanged. Thus, Table 4 defines a total of 40 complex spreads. As complex spreads 1 and 2, 4 and 5, 7 and 8 are isomorphic, there exist only seven types of the complex spreads. [6]

Note that there are no complex spreads of width more than 1 and dimension 4 because otherwise they would involve as components butterfly spreads of width more than 1, which do not exist.

3.2 Advantage of Complex Spreads

The margins of complex spreads are exactly the margins of their resulting spreads, cf. (CBOE, 2003); and a complex spread is an *offset* if its margin is less than the total margin of its components. Hence, not all complex spreads are offsets.

For example, the complex spread 6 in Table 4 has three components: the 1st long call butterfly spreads $\mathbf{b}-\mathbf{c}$, the 2nd long call butterfly spread $\mathbf{a}-\mathbf{b}$ and the 3rd bull call spread $\mathbf{c}$. All the three are debit spreads. By formula (1), the market risks for these spreads are zeros. The resulting spread is the 1st bull call spread $\mathbf{a}$, which is also a debit spread. Therefore, the margin of the complex spread 6 is also zero. Thus, it is not an offset, and there is no advantage of using it for margin reductions. It is not hard to verify that all complex spreads in Table 4 are not offsets. However, their negations are offsets.

For example, since the bear call spread $-\mathbf{a}$ is a credit spread, the margin of the negation of the complex spread 6 is w, while the total margin of $\mathbf{b}-\mathbf{a}$, $\mathbf{c}-\mathbf{b}$ and $-\mathbf{c}$, which are all credit spreads, is $3w$. Thus, the negation of the complex spread 6 is an offset with advantage $2w$.

In general, if a complex spread with the resulting debit or credit spread is an offset, then it reduces the total margin of its components by kw or $(k-1)w$, respectively, where k is the number of credit components. Thus, the negations of the complex spreads 1 through 5, 6 through 9 and 10 reduce the margin by w, $2w$ and $3w$, respectively.

4 BEYOND COMPLEX SPREADS

Complex spreads are constructed as summations of bull/bear spreads, long/short butterfly spreads and long/short box spreads; cf. Table 4.

Developing the idea of complex spreads, we give in this section definitions of other multi-leg spreads as more general combinations of main spreads that we call *centipedes* and *millipedes*. [7]

4.1 Centipedes

Definition 4. *A centipede is a set of main spreads such that their linear combination with nonegative integer coefficients is a nonzero multiple of a main spread, which is the resulting spread of the centipede.*

A nonzero multiple of a main spread, obviously, generates a *trivial centipede* by itself. The margin rule for complex spreads we formulated in the preamble of Section 3 naturally applies to centipedes because they have the same risk profiles as their resulting spreads.

Let a be a positive integer, $\boldsymbol{0}$ be a zero vector of size n, and let $\mathbf{b}$ be a main spread. Then centipedes

[6] The regulatory definitions given in SEC Release 34-52738, the CBOE Regulatory Circular and NYSE Rule 431 imply only these seven types. The CBOE gave some of complex spreads the same names as those of their resulting main spreads. To avoid confusions, we do not use these names. We should also emphasize that this paper presents our mathematical interpretation of CBOE's informal definitions of complex spreads in a text form. Our goal was to follow the idea given in the definitions as close as possible and, at the same time, avoid inconsistencies that we found in them. Any omission that someone may find in our mathematical interpretation of CBOE's complex spreads will be our responsibility.

[7] Centipedes, as all other creatures, have even number of legs (one pair of legs per body segment), and this number can reach 200 and more. Centipedes usually do not bite humans but a few species, when provoked, can bite inflicting painful wounds. Millipedes are creatures with number of legs multiple of four (two pairs of legs per body segment). Some species have over 400 legs. Millipedes are not predators as centipedes. – *Wikipedia* (terrestrial animals). It can be observed that multi-leg spreads introduced in this section have similar properties.

Table 4: CBOE's complex spreads, their components and resulting spreads.

complex spread: component sum = resulting spread	calls				puts				net
1. $\mathbf{b}+(\mathbf{a}-\mathbf{b})=\mathbf{a}$:	5 legs								
2nd bull call		1	-1						dr
+ 1st long call butterfly	1	-2	1						dr
= 1st bull call	1	-1							dr
2. $\mathbf{c}+(\mathbf{b}-\mathbf{c})=\mathbf{b}$:	5 legs								
3nd bull call			1	-1					dr
+ 2st long call butterfly		1	-2	1					dr
= 2nd bull call		1	-1						dr
3. $(\mathbf{b}-\mathbf{c})+(\mathbf{a}-\mathbf{b})=\mathbf{a}-\mathbf{c}$:	6 legs								
2nd long call butterfly		1	-2	1					dr
+ 1st long call butterfly	1	-2	1						dr
= long call condor	1	-1	-1	1					dr
4. $(\mathbf{a}-\mathbf{b})+(\mathbf{e}-\mathbf{a})=\mathbf{e}-\mathbf{b}$:	7 legs								
1st long call butterfly	1	-2	1						dr
+ 1st short box	-1	1			1	-1			cr
= 1st long put iron butterfly		-1	1		1	-1			cr
5. $(\mathbf{b}-\mathbf{c})+(\mathbf{f}-\mathbf{b})=\mathbf{f}-\mathbf{c}$:	7 legs								
2nd long call butterfly		1	-2	1					dr
+ 2nd short box		-1	1			1	-1		cr
= 2nd long put iron butterfly			-1	1		1	-1		cr
6. $\mathbf{c}+(\mathbf{b}-\mathbf{c})+(\mathbf{a}-\mathbf{b})=\mathbf{a}$:	8 legs								
3rd bull call			1	-1					dr
+ 2nd long call butterfly		1	-2	1					dr
+ 1st long call butterfly	1	-2	1						dr
= 1st bull call	1	-1							dr
7. $\mathbf{b}+(\mathbf{a}-\mathbf{b})+(\mathbf{e}-\mathbf{a})=\mathbf{e}$:	9 legs								
2nd bull call		1	-1						dr
+ 1st long call butterfly	1	-2	1						dr
+ 1st short box	-1	1			1	-1			cr
= 1st bull put					1	-1			cr
8. $\mathbf{c}+(\mathbf{b}-\mathbf{c})+(\mathbf{f}-\mathbf{b})=\mathbf{f}$:	9 legs								
3rd bull call			1	-1					dr
+ 2nd long call butterfly		1	-2	1					dr
+ 2nd short box		-1	1			1	-1		cr
= 2nd bull put						1	-1		cr
9. $(\mathbf{b}-\mathbf{c})+(\mathbf{a}-\mathbf{b})+(\mathbf{e}-\mathbf{a})=\mathbf{e}-\mathbf{c}$:	10 legs								
2nd long call butterfly		1	-2	1					dr
+ 1st long call butterfly	1	-2	1						dr
+ 1st short box	-1	1			1	-1			cr
= long put iron condor			-1	1	1	-1			cr
10. $\mathbf{c}+(\mathbf{b}-\mathbf{c})+(\mathbf{a}-\mathbf{b})+(\mathbf{e}-\mathbf{a})=\mathbf{e}$:	12 legs								
3rd bull call			1	-1					dr
+ 2nd long call butterfly		1	-2	1					dr
+ 1st long call butterfly	1	-2	1						dr
+ 1st short box	-1	1			1	-1			cr
= 1st bull put					1	-1			cr

with resulting spread $a\mathbf{b}$ can be identified with integer solutions to the system $\boldsymbol{Ax}=a\mathbf{b}$, $\boldsymbol{x}\geq\boldsymbol{0}$, where components of $\boldsymbol{x}$ represent *multiplicities* of main spreads in the centipede $\boldsymbol{x}$.

If m is the margin of $\mathbf{b}$, then the margin of $\boldsymbol{x}$ is am. Note that multiplicities of main spreads involved in complex spreads are 1 or 0. Centipedes with a resulting spread $a\mathbf{b}$ can be considered as synthetic counterparts of $a\mathbf{b}$ that are possible to build from main spreads. We will relate centipedes to the same *type* $\mathbf{b}$ if their resulting spreads are multiples of the same main spread $\mathbf{b}$.

4.2 Millipedes

Now we consider combinations of main spreads that are market risk-free. They are based on the concept of a *horizontal spread*, i.e., a long option combined with a short option on the same underlying security of the same type and exercise price. It is well-known that such a spread is invariant to underlying security market price changes and therefore market risk-free.

Lemma 1. *A horizontal spread is market risk-free.*

Proof Let us consider a *horizontal call spread* where the long position in a call option lC and the short position in a call option sC have the same exercise price e. Each option contracts, say, 100 underlying units.

If sC is exercised, then the spread holder is obliged to sell 100 underlying units to the holder of sC at the price e. In this case, the spread holder can exercise lC, i.e., buy 100 underlying units at the same price, and deliver them to the holder of sC with no loss. If sC is not exercised and lC is out-of-the-money, then lC can be kept unexercised. A *horizontal put spread* has the same hedging mechanism except that exercising put options triggers the sell of underlying units. □

Definition 5. *A millipede is a set of main spreads such that their linear combination with nonegative integer coefficients is a zero spread.*

Thus, $\boldsymbol{0}$ is a trivial millipede. This definition implies that millipedes can be found as integer solutions to the system $\boldsymbol{Ax} = \mathbf{0}$, $\boldsymbol{x} \geq \boldsymbol{0}$, where $\boldsymbol{x}$ is the same variable vector as in Section 4.1. A *submillipede* is a subset of a millipede which is also a millepede. A submillipede $\boldsymbol{y}$ of a millipede $\boldsymbol{x}$ is *proper* if $\boldsymbol{0} \neq \boldsymbol{y} \neq \boldsymbol{x}$.

Theorem 2. *A millipede is market risk-free.*

Proof Using induction on the number of components of a millipede and Lemma 1, it is easy to verify that the set of legs of a millipede can be partitioned into pairs such that each pair is a horizontal call or put spread. Therefore a millipede is a market risk-free option combination. □

Thus, the margin of a millipede is zero. There exists a simple relationship between centipedes and millipedes: if $\boldsymbol{x}$ is a centipede with the resulting spread $a\mathbf{b}_i$, where $\mathbf{b}_i$ is the ith column of $\boldsymbol{A}$, and $\boldsymbol{Ae}_i = \mathbf{b}_i$, then $\boldsymbol{x} - a\boldsymbol{e}_i$ is a millipede.

Definition 6. *A millipede is minimal if it does not contain proper submillipedes. A centipede is minimal if it does not contain nontrivial millipedes. A centipede obtained from another centipede by deleting a nontrivial millipede is a subcentipede.*

We should notice that the absence of linear combinations with negative coefficients in the definitions of centipedes and millipedes does not affect the generality of these multi-leg option spreads because negations of main spreads are also main spreads. Thus, a main spread $\mathbf{b}$ with negative coefficient $-c$ can be replaced by $-\mathbf{b}$ with positive coefficient c.

As mentioned in Section 2.2, the matrix $\boldsymbol{A}$ defines a *linear transformation* of vectors in the portfolio space to vectors in the spread space.

It is important to observe that the set of millipedes is the *kernel*, the set of centipedes with the same resulting spread is an *equivalence class*, the set of linear combinations of main spreads is the *image*, and the set of coefficient vectors of these combinations is the *coimage* of this transformation.

5 USING CENTIPEDES AND MILLIPEDES

As we show in this section, centipedes, as synthetic counterparts of main spreads, can increase their profit if the set of options with the same expiration date is mispriced; while millipedes represent “white holes” of option portfolios, i.e. a group of positions whose margin is zero because, as shown in Section 4.2, millipedes are market risk-free option combinations.

5.1 Maximizing Option Arbitrage Opportunities

Let $\boldsymbol{p}$ be the column vector of the prices of main spreads including the transaction costs. Assume that a main spread $\mathbf{b}$ is chosen for trading with quantity a. Then a solution to

$$\min\{\boldsymbol{p}^\top \boldsymbol{x} : \boldsymbol{Ax} = a\mathbf{b}\} \tag{2}$$

answers the question whether there exist a synthetic counterpart $\boldsymbol{x}$ of $a\mathbf{b}$ that is less expensive than $a\mathbf{b}$.

If the answer is positive, then it is probably better to trade the synthetic counterpart. Note that if $a\mathbf{b}$ is a credit spread, then its synthetic counterpart can give an advantage only if the minimum (2) is negative.

If $a\mathbf{b}$ is a multiple of a box spread, then solving the above integer program, as we show in this section, can maximize an option arbitrage opportunity. We should note here that known arbitrage strategies involving only options are based on box spreads which are market risk-free; cf. (Cohen, 2005).

A long box spread is a debit spread because its long (buy) side is more expensive than its short (sell) side. The difference between the prices of these sides is the long box spread price. An arbitrage opportunity appears when the long box spread price is lower than

$we^{-r(\tau-t)}$, where w is the box spread width. i.e., the amount that can be invested in a risk-free asset paying interest r, where τ is the expiration date of the options involved in the long box spread, and t is the present date; cf. (Ronn and Ronn, 1989; Bharadwaj and Wiggins, 2001; Benzion et al., 2005). Therefore, catching the arbitrage opportunity implies finding a long box spread in the options market with a *minimum* price.

A short box spread is a symmetrical image of a long box spread in relation to adjectives "long" and "short". Therefore, a short box spread is a credit spread, i.e. of a negative price, and hence gives a risk-free profit right on entering into it. Thus, catching an arbitrage opportunity by a short box spread implies finding a short box spread in the options market with also a *minimum* but *negative* price.

The box spread arbitrage has been well studied. A recent study and literature review can be found in (Benzion et al., 2005). It is well-known that a box spread is a synthetic position in a short position in a stock and a long position in the same stock. A box spread can also be viewed as a synthetic position in other option spreads as follows:

Definition 7. *A synthetic box spread is a centipede whose resulting spread is a multiple of a box spread.*

Let $p_{\mathbf{b}}$ be the price of a box spread $\mathbf{b}$. We assume that $p_{\mathbf{b}} > 0$ and then $\mathbf{b}$ is a long box (debit) spread, or $p_{\mathbf{b}} < 0$ and then $\mathbf{b}$ is a short box (credit) spread. We exclude the case $p_{\mathbf{b}} = 0$ which means an obvious error in pricing of options.

Let $\boldsymbol{x}^*$ be the synthetic box spread with the resulting spread $a^*\mathbf{b}$ that is found by solving the integer program (2) with variables $\boldsymbol{x}$ and $1 \le a \le a_{\max}$, where $a_{\max}$ is a chosen multiplicity upper bound.

As a multiple of a box spread is a trivial synthetic box spread, $\boldsymbol{p}^\top \boldsymbol{x}^* \le a^* p_{\mathbf{b}}$. We assume that $a^* p_{\mathbf{b}} > 0$ implies $\boldsymbol{p}^\top \boldsymbol{x}^* > 0$; otherwise we have again an obvious error in pricing of options. Thus, if $a^*\mathbf{b}$ gives an arbitrage opportunity, then its synthetic counterpart $\mathbf{x}^*$ gives a better arbitrage opportunity only if $\boldsymbol{p}^\top \boldsymbol{x}^* < a^* p_{\mathbf{b}}$.

While a box spread can find an arbitrage opportunity by capturing only four mispriced options, a synthetic box spread is a much more powerful tool because it captures mispriced options in the whole option chain.

5.2 Decompositions of Option Portfolios

Let us recall that, according to the definition given in

Section 2.2, a portfolio of main spreads is an integer column vector of size n whose components q_i represent quantities of main spreads. Thus, if m_i is the margin of main spread i, then $m_i q_i$ is the margin of the ith component of the portfolio, where $1 \le i \le n$.

We assume that $q_i > 0$ implies that the portfolio has a position in main spread i with quantity q_i and that $q_i = 0$ implies that the portfolio has no position in the main spread i.

Let us introduce the following constant vectors:

$$\begin{array}{rcl} \boldsymbol{m}^\top & = & (\; m_1 \; m_2 \; \cdots \; m_n \;) \\ \boldsymbol{q}^\top & = & (\; q_1 \; q_2 \; \cdots \; q_n \;) \\ \boldsymbol{1}^\top & = & (\; 1 \; 1 \; \cdots \; 1 \;) \end{array}$$

and the following nonnegative integer variable vectors

$$\begin{array}{rcl} \boldsymbol{a}^\top & = & (\; a_1 \; a_2 \; \cdots \; a_n \;) \\ \boldsymbol{x_i}^\top & = & (\; x_{i1} \; x_{i2} \; \cdots \; x_{in} \;) \\ \boldsymbol{y_j}^\top & = & (\; y_{j1} \; y_{j2} \; \cdots \; y_{jr} \;) \\ \boldsymbol{z}^\top & = & (\; z_1 \; z_2 \; \cdots \; z_r \;) \end{array}$$

where components of $\boldsymbol{a}$ are associated with multiplicities of main spreads that are not necessarily in the portfolio, components of $\boldsymbol{x}_i$ and $\boldsymbol{y}_j$ are associated with multiplicities of main spreads in centipedes and millipedes, respectively, that are in the portfolio, and components of $\boldsymbol{z}$ are 0-1 variables for counting minimal millipedes in the portfolio. We assume that $r = \lfloor n/2 \rfloor$ because a nontrivial millipede involves at least two main spreads, hence $1 \le j \le r$.

We also consider the nonnegative integer variable vector

$$\boldsymbol{y}^\top \; = \; (\; y_1 \; y_2 \; \cdots \; y_r \;)$$

instead of the vectors $\mathbf{y}_j$ in portfolio decompositions with a single millipede.

Now we show how the *no-offset* margin $\boldsymbol{m}^\top \boldsymbol{q}$ of the portfolio $\boldsymbol{q}$ can be reduced to obtain an *offset margin* of this portfolio using centipedes and millipedes as offsets. As we show in this section, the reduction follows from a decomposition of the portfolio into centipedes and millipedes.

Let $\mathbf{b}$ be a main spread. We say that the portfolio $\boldsymbol{q}$ contains a centipede or millipede $\boldsymbol{x}$ with the resulting spread $a\mathbf{b}$, where $a > 0$ or $a = 0$, respectively, if $\boldsymbol{A}\boldsymbol{x} = a\mathbf{b}$ and $\boldsymbol{x} \le \boldsymbol{q}$. Centipedes and/or millipedes $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_s$ generate a decomposition of the portfolio $\boldsymbol{q}$ if $\boldsymbol{x}_1 + \boldsymbol{x}_2 + \ldots + \boldsymbol{x}_s = \boldsymbol{q}$.

If μ_i denotes the margin of the ith component of this decomposition, where $\mu_i = 0$ if $\boldsymbol{x}_i$ is a millipede, then $\mu_1 + \mu_2 + \ldots + \mu_s$ is the *decomposition margin*.

Lemma 2. *Any decomposition of an option portfolio into centipedes and millipedes can be transformed into a decomposition with at most n nonzero components, at most one centipede of each type, and the same decomposition margin.*

Proof We can sum up centipedes of each type, arbitrarily add to these sums all nontrivial millipedes, if any, and thus obtain a decomposition with at most n nonzero components. The margin of the sum $\boldsymbol{x}_i + \boldsymbol{x}_j$ is $\mu_i + \mu_j$, therefore the new decomposition has the same margin. □

To obtain a maximum reduction of the margin $\boldsymbol{m}^\top \boldsymbol{q}$ using centipedes and millipedes as offsets we can decompose the portfolio $\boldsymbol{q}$ as follows.

Let $\mathbf{b}_i$ be the ith main spread, i.e., the ith column of $\boldsymbol{A}$, and let $\{\boldsymbol{a}, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n,\}$ be the variable set. Then the set of solutions to the system

$$\begin{gathered} \boldsymbol{A}\boldsymbol{x}_i = a_i \mathbf{b}_i,\ 1 \le i \le n \\ \sum_{i=1}^{n} \boldsymbol{x}_i = \boldsymbol{q} \end{gathered} \tag{3}$$

defines a set of decompositions of the portfolio $\boldsymbol{q}$ that contains, by Lemma 2, a decomposition with a minimum margin. Note that setting $\boldsymbol{a} = \boldsymbol{q}$ and $\boldsymbol{x}_i = q_i \boldsymbol{e}_i$ for all i gives a trivial solution to this system.

It is important to observe that a solution to the system (3) defines a portfolio $\boldsymbol{a}$ from the coimage of the transformation by the matrix $\boldsymbol{A}$. The no-offset margin of $\boldsymbol{a}$, i.e., $\boldsymbol{m}^\top \boldsymbol{a}$, is the offset margin of $\boldsymbol{q}$. Since $\boldsymbol{a}$ carries the same risk profile as $\boldsymbol{q}$ we call it a *representative* of $\boldsymbol{q}$ for the decomposition $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$.

Note that $\boldsymbol{a}$ is not always a subportfolio of $\boldsymbol{q}$ because the inequality $\boldsymbol{a} \le \boldsymbol{q}$ may not hold. If it holds, then we call $\boldsymbol{a}$ a *proper representative* of $\boldsymbol{q}$.

Lemma 3. *If $\boldsymbol{a}$ is a representative of $\boldsymbol{q}$, then $\boldsymbol{q} - \boldsymbol{a}$ is a millipede.*

Proof Multiplying the lower equation in (3) by $\boldsymbol{A}$ from the left and using the upper equation we obtain

$$\boldsymbol{A}\boldsymbol{a} = \sum_{i=1}^{n} a_i \mathbf{b}_i = \boldsymbol{A}\boldsymbol{q}$$

which implies $\boldsymbol{A}(\boldsymbol{q} - \boldsymbol{a}) = \mathbf{0}$. □

Obviously, if $\boldsymbol{a}$ is a proper representative of $\boldsymbol{q}$, then the millipede $\boldsymbol{q} - \boldsymbol{a}$ is a subportfolio of $\boldsymbol{q}$.

Definition 8. *A solution $\boldsymbol{a}^*, \boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \ldots, \boldsymbol{x}_n^*$ to the integer program of minimizing the objective*

$$\boldsymbol{m}^\top \boldsymbol{a} \tag{4}$$

under the constraints (3) *defines a main decomposition $\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \ldots, \boldsymbol{x}_n^*$ of the portfolio $\boldsymbol{q}$ and a main representative $\boldsymbol{a}^*$ of $\boldsymbol{q}$ for this decomposition.*

The following theorem establishes that the margin of a main decomposition cannot be reduced using centipedes and millipedes as offsets.

Theorem 3. *A main representative of a main representative of $\boldsymbol{q}$ is also a main representative of $\boldsymbol{q}$.*

Proof Let $\boldsymbol{a}'$ be a main representative of $\boldsymbol{a}^*$ for the decomposition $\boldsymbol{x}_1', \boldsymbol{x}_2', \ldots, \boldsymbol{x}_n'$. Obviously, $\boldsymbol{m}^\top \boldsymbol{a}' \le \boldsymbol{m}^\top \boldsymbol{a}^*$. Then it is not hard to verify that $\boldsymbol{a}'$ is also a representative of $\boldsymbol{q}$ for the decomposition $\boldsymbol{x}_1'', \boldsymbol{x}_2'', \ldots, \boldsymbol{x}_n''$, where

$$\boldsymbol{x}_i'' = \boldsymbol{x}_i^* + \boldsymbol{x}_i' - a_i^* \boldsymbol{e}_i$$

for all i. Therefore, $\boldsymbol{m}^\top \boldsymbol{a}' = \boldsymbol{m}^\top \boldsymbol{a}^*$, and hence $\boldsymbol{a}'$ is a main representative of $\boldsymbol{q}$. □

A main decomposition contains millipedes not only among its own millipedes but also inside its centipedes which, therefore, may not be minimal. However, solving the following modified version of the problem (3)(4) we can extract all millipedes from the centipedes that are not minimal and collect all millipedes in a single subportfolio we call a *white hole*.

Let $\{\boldsymbol{a}, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n, \boldsymbol{y}\}$ be the variable set. Then the set of solutions to the system

$$\begin{gathered} \boldsymbol{A}\boldsymbol{x}_i = a_i \mathbf{b}_i,\ 1 \le i \le n,\ \ \boldsymbol{A}\boldsymbol{y} = \mathbf{0} \\ \sum_{i=1}^{n} \boldsymbol{x}_i + \boldsymbol{y} = \boldsymbol{q} \end{gathered} \tag{5}$$

defines a set of decompositions of $\boldsymbol{q}$ that contains all main decompositions.

Definition 9. *A solution $\boldsymbol{a}^\circ, \boldsymbol{x}_1^\circ, \boldsymbol{x}_2^\circ, \ldots, \boldsymbol{x}_n^\circ, \boldsymbol{y}^\circ$ to the integer program of minimizing the objective*

$$\boldsymbol{m}^\top \boldsymbol{a} - \frac{\boldsymbol{1}^\top \boldsymbol{y}}{\boldsymbol{1}^\top \boldsymbol{q}} \tag{6}$$

under the constraints (5) *defines a white-hole decomposition of the portfolio $\boldsymbol{q}$ into centipedes, which are nonzero vectors among $\boldsymbol{x}_1^\circ, \boldsymbol{x}_2^\circ, \ldots, \boldsymbol{x}_n^\circ$, and a millipede $\boldsymbol{y}^\circ$, which is a white hole in $\boldsymbol{q}$.*

As the following lemma and theorem state, a white-hole decomposition is just an extension of a main decomposition by one component collecting all millipedes; hence, a white-hole decomposition has only one millipede and only minimal centipedes.

Lemma 4. *The vectors $\boldsymbol{x}_1^\circ, \boldsymbol{x}_2^\circ, \ldots, \boldsymbol{x}_n^\circ$ are either minimal centipedes or zero vectors.*

Proof Let a centipede $\boldsymbol{x}_i^\circ$ be not minimal, and let $\boldsymbol{v}$ be a nontrivial millipede in $\boldsymbol{x}_i^\circ$. Then moving $\boldsymbol{v}$ from $\boldsymbol{x}_i^\circ$ to $\boldsymbol{y}^\circ$ adds value

$$-\boldsymbol{m}^\top \boldsymbol{v} - \frac{\boldsymbol{1}^\top \boldsymbol{v}}{\boldsymbol{1}^\top \boldsymbol{q}}$$

to the objective (6). As $\boldsymbol{1}^\top \boldsymbol{v} > 0$, the objective would decrease, which is a contradiction.

Let $\boldsymbol{x}_i^\circ > \boldsymbol{0}$ be not a minimal centipede. Then it is a nontrivial millipede that could be added to $\boldsymbol{y}^\circ$ and decrease the objective again. □

Theorem 4. $\boldsymbol{m}^\top\boldsymbol{a}^\circ = \boldsymbol{m}^\top\boldsymbol{a}^*$, *i.e.,* $\boldsymbol{x}_1^\circ,\boldsymbol{x}_2^\circ,\ldots,\boldsymbol{x}_n^\circ$ *is a main decomposition, and hence* $\boldsymbol{a}^\circ$ *is a main representative of* $\boldsymbol{q}$*, where* $\boldsymbol{q}-\boldsymbol{a}^\circ\pm\boldsymbol{y}^\circ$ *is a millipede.*

Proof If $\boldsymbol{y}^\circ = \boldsymbol{0}$, then the equation $\boldsymbol{m}^\top\boldsymbol{a}^\circ = \boldsymbol{m}^\top\boldsymbol{a}^*$ is evident. Assume that $\boldsymbol{y}^\circ > \boldsymbol{0}$.

Let $\boldsymbol{y}^*$ be the sum of all millipedes inside and among $\boldsymbol{x}_1^*,\boldsymbol{x}_2^*,\ldots,\boldsymbol{x}_n^*$. Note that $\boldsymbol{y}^\circ > \boldsymbol{0}$ implies $\boldsymbol{y}^* > \boldsymbol{0}$ and that $\boldsymbol{x}_1^\circ,\boldsymbol{x}_2^\circ,\ldots,\boldsymbol{x}_n^\circ+\boldsymbol{y}^\circ$ is a decomposition with margin $\boldsymbol{m}^\top\boldsymbol{a}^\circ \geq \boldsymbol{m}^\top\boldsymbol{a}^*$; otherwise $\boldsymbol{x}_1^*,\boldsymbol{x}_2^*,\ldots,\boldsymbol{x}_n^*$ is not a main decomposition.

If $\boldsymbol{m}^\top\boldsymbol{a}^\circ > \boldsymbol{m}^\top\boldsymbol{a}^*$, then

$$\boldsymbol{m}^\top\boldsymbol{a}^* \leq \boldsymbol{m}^\top\boldsymbol{a}^\circ - 1 \leq \boldsymbol{m}^\top\boldsymbol{a}^\circ - \frac{\boldsymbol{1}^\top\boldsymbol{y}^\circ}{\boldsymbol{1}^\top\boldsymbol{q}}$$

because $\boldsymbol{y}^\circ \leq \boldsymbol{q}$. As $\boldsymbol{m}^\top\boldsymbol{a}^* - \dfrac{\boldsymbol{1}^\top\boldsymbol{y}^*}{\boldsymbol{1}^\top\boldsymbol{q}} < \boldsymbol{m}^\top\boldsymbol{a}^*$, we have

$$\boldsymbol{m}^\top\boldsymbol{a}^* - \frac{\boldsymbol{1}^\top\boldsymbol{y}^*}{\boldsymbol{1}^\top\boldsymbol{q}} < \boldsymbol{m}^\top\boldsymbol{a}^\circ - \frac{\boldsymbol{1}^\top\boldsymbol{y}^\circ}{\boldsymbol{1}^\top\boldsymbol{q}}$$

Thus, $\boldsymbol{x}_1^\circ,\boldsymbol{x}_2^\circ,\ldots,\boldsymbol{x}_n^\circ,\boldsymbol{y}^\circ$ is not a white-hole decomposition, which is a contradiction.

By Lemma 3, $\boldsymbol{q}-\boldsymbol{a}^\circ$ is a millipede, therefore $\boldsymbol{q}-\boldsymbol{a}^\circ\pm\boldsymbol{y}^\circ$ is a millipede because it is a sum of two millipedes, $\boldsymbol{q}-\boldsymbol{a}^\circ$ and $\pm\boldsymbol{y}^\circ$. □

A white-hole decomposition partitions a given portfolio into two subportfolios that can be independently closed without affecting the risk profiles of each other. A white hole can be closed because the residual portfolio can be more attractive for returns or, vice versa, because the white hole is risk-free.

Providing the finest partition of the residual portfolio into minimal centipedes (that can also be closed independently), a white-hole decomposition leaves the structure of the white hole unclear. Solving the following integer program, however, decomposes a white hole into minimal millipedes.

Let $\boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_r,\boldsymbol{z}$ be the variable set. Then the set of solutions to the system

$$\begin{aligned} A\boldsymbol{y}_j = \boldsymbol{0},\ \ \boldsymbol{1}^\top\boldsymbol{y}_j \geq z_j,\ \ 1 \leq j \leq r \\ \sum_{j=1}^{r}\boldsymbol{y}_j = \boldsymbol{y}^\circ \end{aligned} \tag{7}$$

defines all decompositions of the white hole $\boldsymbol{y}^\circ$ into at most r millipedes.

A solution $\boldsymbol{y}_1^\dagger,\boldsymbol{y}_2^\dagger,\ldots,\boldsymbol{y}_r^\dagger,\boldsymbol{z}^\dagger$ to the integer program of maximizing the score $\boldsymbol{1}^\top\boldsymbol{z}$ under the constraints (7) defines a decomposition of the white hole $\boldsymbol{y}^\circ$ into $\boldsymbol{1}^\top\boldsymbol{z}$ nontrivial millipedes, which are nonzero vectors among $\boldsymbol{y}_1^\dagger,\boldsymbol{y}_1^\dagger,\ldots,\boldsymbol{y}_r^\dagger$.

Definition 10. $\boldsymbol{a}^\circ$, $\boldsymbol{x}_1^\circ,\boldsymbol{x}_2^\circ,\ldots,\boldsymbol{x}_n^\circ$, $\boldsymbol{y}_1^\dagger,\boldsymbol{y}_2^\dagger,\ldots,\boldsymbol{y}_r^\dagger$, $\boldsymbol{z}^\dagger$ *define a main representative* $\boldsymbol{a}^\circ$ *and a prime decomposition of the portfolio* $\boldsymbol{q}$ *into centipedes, which are nonzero vectors among* $\boldsymbol{x}_1^\circ,\boldsymbol{x}_2^\circ,\ldots,\boldsymbol{x}_n^\circ$*, and* $\boldsymbol{1}^\top\boldsymbol{z}$ *nontrivial millipedes, which are nonzero vectors among* $\boldsymbol{y}_1^\dagger,\boldsymbol{y}_2^\dagger,\ldots,\boldsymbol{y}_r^\dagger$*.*

The following theorem establishes that a prime decomposition of an option portfolio is a finest decomposition that completely reveals its structure in terms of centipedes and millipedes.

Theorem 5. *All centipedes and millipedes in a prime decomposition are minimal.*

Proof Lemma 4 implies that all centipedes in a prime decomposition are minimal. Let us show that all nontrivial millipedes in there are also minimal.

Let a nontrivial millipede $\boldsymbol{y}_k^\dagger$ be not minimal. Then $\boldsymbol{y}_k^\dagger$ contains a proper submillipede $\boldsymbol{v}$ and hence $\boldsymbol{u}=\boldsymbol{y}_k^\dagger-\boldsymbol{v}$ is also a proper submillipede.

Besides, there exists a positive integer $l<r$ such that $\boldsymbol{y}_l^\dagger=\boldsymbol{0}$ and hence $z_l^\dagger=0$; otherwise each millipede among $\boldsymbol{y}_1^\dagger,\boldsymbol{y}_2^\dagger,\ldots,\boldsymbol{y}_r^\dagger$ is minimal and contains two components. Then we can construct a new decomposition replacing $\boldsymbol{y}_k^\dagger$, $\boldsymbol{y}_l^\dagger$ and $z_l^\dagger$ with $\boldsymbol{y}_k^\ddagger=\boldsymbol{u}$, $\boldsymbol{y}_l^\ddagger=\boldsymbol{v}$, and $z_l^\ddagger=1$, where $\boldsymbol{y}_k^\ddagger+\boldsymbol{y}_l^\ddagger=\boldsymbol{y}_k^\dagger$. This replacement decreases the number of nontrivial millipedes by one, which is a contradiction. □

5.3 Decompositions with Proper Representatives

It is not hard to verify that all results obtained in Section 5.2 remain valid if the systems (3) and (5) are complimented by the inequality $\boldsymbol{a}\leq\boldsymbol{q}$. Thus, there exist counterparts of main, white-hole and prime decompositions with proper representatives. The option trader can be interested in this kind of decompositions because the conversion of the portfolio $\boldsymbol{q}$ into its proper representative $\boldsymbol{a}$ requires only selling the subportfolio $\boldsymbol{q}-\boldsymbol{a}$. While the conversion into a representative, which is not proper, requires selling quantities q_i-a_i from positions i with $q_i>a_i$, buying quantities a_j-q_j for positions j with $a_j>q_j>0$ and opening new positions k with quantities a_k if $a_k>q_k=0$. Considering only proper main representatives, however, can reduce savings on margin; see Section 6.

6 COMPUTATIONAL EXPERIMENT

Now we are conducting the computational study on finding option arbitrage opportunities by synthetic box spreads and the estimation of margin reductions for portfolios of main spreads by white-hole decompositions using ILOG CPLEX 12.1 for solving integer programs. We present here only preliminary results.

We experimented with the option chains for different stocks provided by www.google.com/finance and did not detect an option arbitrage. It is not surprising because the price quotes in this web site are delayed by 15 minutes, while option arbitrage opportunities usually last for seconds; cf. (Bharadwaj and Wiggins, 2001; Benzion et al., 2005).

Using the same web site we also estimated the average savings on margin by replacing randomly generated portfolios of different sizes by their main representatives. The results are presented on Fig. 1.

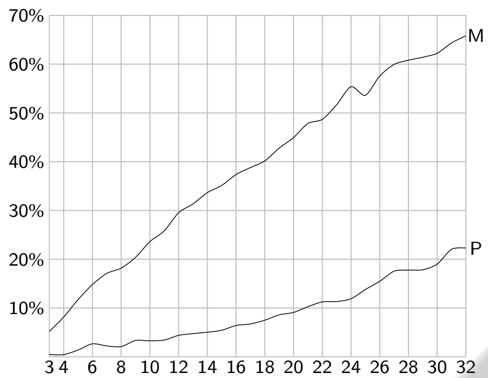

Figure 1: The percentage of average savings on margin by main (M) and proper main (P) representatives of over 500 portfolios of main spreads of each size from 3 to 32. All the main spreads were on the exercise domain of size 10 and exercise differential of $5USD. The portfolios were generated by proportional random sampling with replacement based on the trading volume of options on NASDAQ:AAPL expiring on 22-OCT-2011. The options data were taken from www.google.com/finance on 04-AUG-2011 when the underlying security price was at $377.37USD.

7 CONCLUSIONS

This paper takes only the first step in studying combinations of option spreads and demonstrates how these combinations can be used in trading and margining practice. An important consequence of this study is a sketch of a combinatorial theory of option portfolios that we believe will be useful for developing new techniques for high-frequency trading and new margining methodologies.

Our next step will be devoted to computational experiments for detecting option arbitrage opportunities using live option price quotes and estimating margin reductions for portfolios of main spreads with different underlying stocks.

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