An Elementary Introduction to Kuznecov's Article on Modular Forms Written in 1981

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May, 2022

1 Introduction

History has witnessed the fast development of modular forms, which is common in a number of mathematical branches. There is no denying that due to the idea of modular forms, analytical number theory embraces its brand new era. This article is intended to record the results and some proofs roughly, especially those related to analytical number theory. Our main reference is the classic work that belongs to Kuznecov. His estimates, via the use of modular forms and former conclusions, are more precise than his contemporaries'.

In essence, those basic formulae come from the Fourier expansion of some fundamental functions. However, by virtue of the deformation of integrals, we obtain a series of nontrivial results. The reason for our deformation comes from some masters' estimates and rich properties of Bessel function. We will focus on the deformation but skip some inequalities so that we can make our article seem easy.

This article is based on [1] and [2]. They do help us a lot.

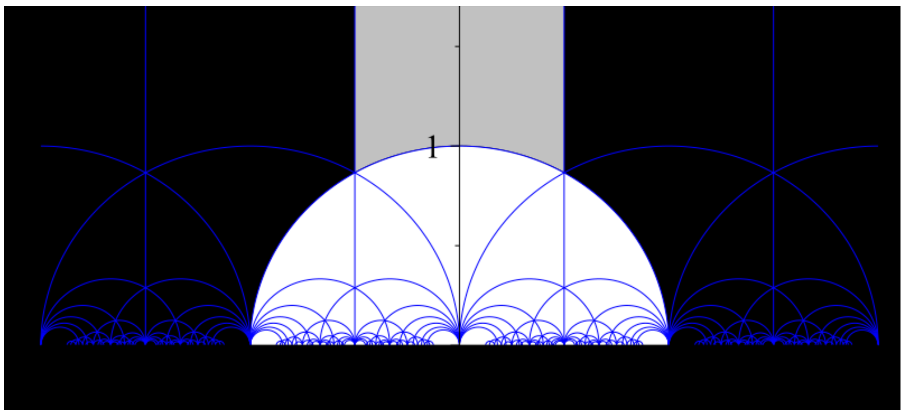
2 Notation

Before we begin our journey, some definitions are important.

Let G be the modular group $PSL(2, \mathbb{Z})$. We equip the upper plane $\mathbb{H}$ with a G-action, that is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Besides, Laplace operator $\mathcal{L} = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ is also frequent. We will consider its eigenfunctions of discrete spectrum, named after cusp forms of weight 0. Here we recognize that they are nontrivial real-analytical automorphic functions, which satisfy the equality $f(gz) = f(z), \forall g \in G$ and finiteness condition $\int_D |f(z)|^2 dz < \infty$, where D is the fundamental field of G and $dz = \frac{dxdy}{y^2}$, the G-invariant measure of $\mathbb{H}$.



The gray domain is the fundamental field.

On the basis of some knowledge on compact operators, we know that the Laplace operator has Lebesgue spectrum of multiplicity one which fills out the semiaxis $\frac{1}{4} \leqslant \lambda < \infty$, and it has a discrete spectrum of finite multiplicity located on the semiaxis $\lambda \geqslant 0$ and having no points of accumulation in every finite interval.

The simplest subgroup of G may be translations $< z \mapsto z + n >$. We call it G_∞.

Hecke defined operators acting on automorphic functions, $T(n), n \in \mathbb{N}_+$. Let us prove some basic propositions.

3 Hecke operators

3.1 Definition and properties

A matrix M is called order n if $\det M = n$. We consider the equivalence relation of M_1 and M_2 if $M_1 = gM_2, g \in G$. It is not difficult to verify that all representatives are of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, ad = n, d > 0, b = 0, 1, ..., d - 1.$$

Hence, the following definition is valid.

Definition 3.1. For every automorphic function f and $n > 0$, we define

$$T(n)(f) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ d>0}} \sum_{b \bmod d} f\left(\frac{az+b}{d}\right). \tag{3.1}$$

Or equivalently,

$$T(n)(f) = \frac{1}{\sqrt{n}} \sum_{M_i} f(M_i z), \tag{3.2}$$

where M_i runs over all representatives.

It is trivial that the images of Hecke operators are also automorphic. And the following theorem implies that they are commutative.

Theorem 3.2. *n, $m \in \mathbb{N}_+$, then*

$$T(n)T(m) = \sum_{d|(m,n)} T\left(\frac{mn}{d}\right). \tag{3.3}$$

Proof. **Step 1:** If $(m, n) = 1$, then,

$$T(n)T(m)f = \frac{1}{\sqrt{mn}} \sum_{ad=n} \sum_{b \bmod d} \sum_{a'd'=m} \sum_{b' \bmod d'} f\left(\frac{aa'z + a'b + b'd}{dd'}\right). \tag{3.4}$$

Because m and n are coprime, aa' and dd' run over every divisor of mn and $a'b + b'd$ runs over the residue system of dd'. As a result, $T(n)T(m) = T(mn)$.

Step 2: If $m = p$ and $n = p^r$ and p prime, then,

$$T(p)f = \frac{1}{\sqrt{p}} \left(f(pz) + \sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right)\right). \tag{3.5}$$

Then,

$$p^{\frac{r+1}{2}} T(p^r)T(p)f = \sum_{k=0}^{r} \sum_{t=0}^{p^k-1} \left(f\left(\frac{p^{r-k+1}z + tp}{p^k}\right) + \sum_{b=0}^{p-1} f\left(\frac{p^{r-k} + t + bp^k}{p^{k+1}}\right)\right). \tag{3.6}$$

Similarly, basic number theory tells us that,

$$T(p^r)T(p) = T(p^{r+1})T(p^{r-1}). \tag{3.7}$$

Step 3: If $m = p^s$ and $n = p^r$ and p prime, then we can assume $s \leqslant r$.

If $s < r$, $T(p)T(p^r)T(p^s) = T(p^r)\left(T(p^{s+1}) + T(p^{s-1})\right)$. Hence, we can do induction on $T(p^r)T(p^{s+1})$. If $s = r$, the same as the above case.

All the three steps have told us all. ■

Lemma 3.3. If we let Chebyshev polynomial be

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \tag{3.8}$$

then

$$T(p^r) = U_r\left(\frac{1}{2}T(p)\right) = \sum_{0\leqslant k\leqslant r/2} \frac{(-1)^k(r-k)!}{k!(r-2k)!}\left(T(p)\right)^{r-2k}. \tag{3.9}$$

Proof. It is trivial. ■

Corollary 3.4. p prime and $2\cos\theta$ is an eigenvalue of $T(p)$, where $\theta \in \mathbb{C}$. Then $\frac{\sin(r+1)\theta}{\sin\theta}$ is an eigenvalue of $T(p^r)$.

3.2 Inner product of automorphic funtions

Lemma 3.5. $T(n)$ is Hermitian with respect to this inner product of automorphic functions,

$$(f_1, f_2) = \int_D f_1(z)\overline{f_2(z)}dz, \tag{3.10}$$

where D is the fundamental field and dz is the G-invariant measure.

Proof. It suffices to considering the case when p prime. Then,

$$(T(p)f)(z) = \frac{1}{\sqrt{p}}\sum_{j=0}^{p} f(\alpha g_j z) = \frac{1}{\sqrt{p}}\sum_{j=0}^{p} f(\widetilde{\alpha}\widetilde{g_j}z), \tag{3.11}$$

where,

$$\begin{aligned}
\alpha &= \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \\
g_i &= \begin{cases} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} & j = 0, ..., p-1 \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & j = p \end{cases} \\
\widetilde{\alpha} &= \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \\
\widetilde{g_j} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_j
\end{aligned}$$

As a result, if we change variables by $z' = \alpha g_j z$ and conform the integral domain, we can get,

$$(T(p)f_1, f_2) = \frac{1}{\sqrt{p}} \sum_{j=0}^{p} \int_D f_1(\alpha g_j z)\overline{f_2(z)}dz \tag{3.12}$$

$$= \frac{1}{\sqrt{p}} \int_B f_1(z)\overline{f_2(\alpha^{-1}z)}dz, \tag{3.13}$$

where $B = \bigcup_{j=0}^{p} \alpha g_j D$.

The same as this, we cam get,

$$(f_1, T(p)f_2) = \frac{1}{\sqrt{p}} \int_{\widetilde{B}} f_1(z)\overline{f_2(\widetilde{\alpha}z)}dz, \tag{3.14}$$

where $\widetilde{B} = \bigcup_{j=0}^{p} \widetilde{g}_j D$.

But note that $\alpha^{-1}z = \widetilde{\alpha z} = pz$. So what we only need to do is to compare B and $\widetilde{B}$. We claim that they are both the fundamental field of G_∞ and the proof is reserved for practice. ■

3.3 Relation to eigenfunctions of Laplace operator

We denote ψ as the eigenfunction of the discrete spectrum of Laplace operator equipped with the eigenvalue $\lambda > \frac{1}{4}$. And let $\kappa = \sqrt{\lambda - \frac{1}{4}}$. The Fourier expansion of ψ is clear. And the regular property makes the following formula appropriate,

$$\psi(z) = \sum_{-\infty}^{+\infty} c_n(y)e^{2\pi inx}. \tag{3.15}$$

So, apply this to characteristic equation, we can get,

$$-y^2c_n'' + 4\pi^2n^2y^2c_n = \lambda c_n, \tag{3.16}$$

which is the classical Bessel equation. Then,

$$c_n(y) = \rho(n)\sqrt{y}K_{i\kappa}(2\pi|n|y) + \widetilde{\rho}(n)\sqrt{y}I_{i\kappa}(2\pi|n|y), \tag{3.17}$$

with $\rho(n)$ and $\widetilde{\rho}(n)$ to be decided.

But the second component is always unbounded. Finiteness condition request it to be zero.

Simultaneously, when $n = 0$, the concrete calculation suggests that $\rho(0) = 0$. So,

$$\psi(z) = \sum_{n\neq 0} \rho(n)\sqrt{y}K_{i\kappa}(2\pi|n|y)e^{2\pi inx}. \tag{3.18}$$

Beside, $K_\nu(y)$, considered as a function of ν, is even and entire on the complex plane. When ν purely imaginary and y positive, $K_\nu(y)$ is a real number. As a result, if $\psi(z)$ takes on real value, an extra condition is inevitable,

$$\rho(n) = \overline{\rho(-n)}. \tag{3.19}$$

This part provide a case of Hecke operators acting on special functions.

Lemma 3.6. The same as above and let $n \geqslant 1$, then,

$$(T(n)\psi)(z) = \sum_{m\neq 0} t_n(m)\sqrt{y}K_{i\kappa}(2\pi|n|y)e^{2\pi inx}, \tag{3.20}$$

where,

$$t_n(m) = \sum_{\substack{d|(m,n)\\ d>0}} \rho\left(\frac{mn}{d^2}\right). \tag{3.21}$$

Proof.

$$T(n)\psi = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n\\ a>0}} \sum_{b \bmod d} \sum_{m\neq 0} \rho(m)\sqrt{\frac{ay}{d}}K_{i\kappa}\left(\frac{2\pi|m|ay}{d}\right)\exp\left(2\pi im\frac{ax+b}{d}\right), \tag{3.22}$$

and,

$$\sum_{b \bmod d} \exp\left(2\pi im\frac{b}{d}\right) = \begin{cases} d, & \text{if } d \mid m, \\ 0, & \text{else.} \end{cases}, \tag{3.23}$$

tell us all. ∎

Then, using the following proposition, we can choose a special basis in an attempt for simplification.

Proposition 3.7. Hecke operators commute with the Laplace operator.

If V_λ is the λ-characteristic space of $\mathcal{L}$, V_λ is also the invariant space of $T(n)$. We have told readers $\dim V_\lambda < \infty$ and $T(n)$ Hermitian and commutative. Hence, we pose induction on $T(n)$ and make them diagonal under some basis. Note that the dimension is finite, our induction will stop finally. That means, we can choose a basis such that every $T(n)$ act as a stretch on them. That is, if ψ_j is the eigenfunction of λ_j and,

$$T(n)\psi_j = \mu_j(n)\psi_j, \tag{3.24}$$

we can get,

$$\psi_0(n) = \text{const.} \tag{3.25}$$

If we use our Fourier coefficients, we can get,

$$\sum_{\substack{d|(m,n)\\ d>0}} \rho_j\left(\frac{mn}{d^2}\right) = \mu_j(n)\rho_j(m). \tag{3.26}$$

If we take $m = 1$, we get,

$$\mu_j(n) = \frac{\rho_j(n)}{\rho_j(1)}. \tag{3.27}$$

As a result, we get the matrix form of the formula (3.3),

$$\rho_j(n)\rho_j(m) = \rho_j(1) \sum_{d|(m,n)} \rho_j\left(\frac{mn}{d^2}\right). \tag{3.28}$$

Example 3.8.

$$\left(T(n)E\right)(z, s) = \tau_s(n)E(z, s), \tag{3.29}$$

where $E(z, s)$ is Eisenstein series that will be discussed later. Its definition is that,

$$E(z, s) = y^s + \frac{1}{2} \sum_{\substack{(c,d)=1\\ c\neq 0}} \frac{y^s}{|cz + d|^{2s}}. \tag{3.30}$$

And,

$$\tau_s(n) = |n|^{s-1/2} \sum_{\substack{d|n\\ d>0}} d^{1-2s}. \tag{3.31}$$

4 Inner product between real-analytic Poincare series

When $z \in \mathbb{H}$ and $s \in \mathbb{C}$, Poincare defined the series $P_n(z,k) = n^{k-1} \sum\limits_{g \in G_\infty \backslash G} \frac{e^{2\pi i n g z}}{(cz+d)^k}$, where $gz = \frac{az+b}{cz+d}$. Similarly, Selberg defined $U_n(z,s) = \sum\limits_{g \in G_\infty \backslash G} (\mathrm{Im} gz)^s e^{2\pi i n g z}$, which is called real-analytic Poincare series.

This section is the highlight of the entire paper. Due to calculating this inner product from two basis, principal properties of Poincare series are described in two senses, both numerically and analytically. Corollary 5.1 and theorem 5.3 can be understood easily in this way. As for Poincare series, it can be regarded as the expansion of group representation theory, where sums with respect to group elements exist everywhere. Moreover, in the theory of Riemann surfaces, the Riemann θ functions share the sane philosophy.

Remark. When $\mathrm{Re}\, s > 1$, the series absolutely converge. And $U_0(z,s) = E(z,s)$. Besides, we can verify that U_n is automorphic, and if we let $\sigma = \mathrm{Re}\, s$,

$$\mathcal{L}U_n(z,s) = s(1-s)U_n(z,s) + 4\pi n s U_n(z,s+1), \tag{4.1}$$

$$|U_n(z,s)| \leqslant y^\sigma e^{-2\pi n y} + E(z,\sigma) - y^\sigma. \tag{4.2}$$

Theorem 4.1. $s_1,\ s_2 \in \mathbb{C}$, and $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_2 > \frac{3}{4}$, $\mathrm{Re}\,(s_1+s_2) < \frac{5}{2}$. Then,

$$\begin{aligned}(U_n(\cdot,s_1), U_m(\cdot,\overline{s_2})) &= \delta_{mn} \frac{\Gamma(s_1+s_2-1)}{(4\pi n)^{s_1+s_2-1}} \\ &+ \left(\sqrt{\frac{n}{m}}\right)^{s_2-s_1} \frac{2^{3-s_1-s_2}}{\sin\pi(s_1-s_2)} \sum_{c=1}^{\infty} \frac{S(n,m;c)}{c^{s_1+s_2}} \Phi(s_1,s_2;\frac{4\pi\sqrt{mn}}{c}),\end{aligned} \tag{4.3}$$

where $S(n,m;c) = \sum\limits_{\substack{1 \leqslant d \leqslant |c| \\ (c,d)=1 \\ dd' \equiv 1 \bmod c}} \exp\left(2\pi i \left(\frac{nd}{c} + \frac{md'}{c}\right)\right)$ is the Klooster's sum and,

$$\Phi(s_1,s_2;x) = \pi \int_1^\infty (u-1/u)^{s_1+s_2-2} \left(-\sin(\pi s_1) J_{s_1-s_2}(xu) + \sin(\pi s_2) J_{s_2-s_1}(xu)\right) \frac{du}{u}. \tag{4.4}$$

The idea of this proof is in center of Fourier expansion. So, we can first calculate the Fourier coefficients of U_n.

4.1 Fourier expansion of U_n

Lemma 4.2. $\operatorname{Re} s > 1$, $n \in \mathbb{N}$, $z \in \mathbb{H}$, then,

$$U_n(z,s) = \sum_{m=-\infty}^{+\infty} e^{2\pi i m x} B_n(m;y,s), \tag{4.5}$$

where,

$$\begin{aligned} B_n(m;y,s) &= \delta_{mn} y^s e^{-2\pi n y} \\ &\quad + \frac{1}{2}\sum_{c\neq 0} \frac{S(n,m;c)}{|c|^{2s}} y^{1-s} \int_{-\infty}^{+\infty} \exp\left(-2\pi i m y\xi - \frac{2\pi n}{c^2 y(1-i\xi)}\right) \frac{d\xi}{(1+\xi^2)^s}. \end{aligned} \tag{4.6}$$

Moreover, when $\operatorname{Re} s > \frac{3}{4}$, the formula is well-defined and holomorphic.

Proof. Dismiss all the strictness, we can get,

$$\begin{aligned} U_n(z,s) &= y^s e^{2\pi i n z} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ c\neq 0 \\ ad\equiv 1 \bmod c}} \frac{y^s}{|cz+d|^{2s}} \exp\left(\frac{2\pi i n a}{c} - \frac{2\pi i n}{c(cz+d)}\right) \\ &= y^s e^{2\pi i n z} + \frac{1}{2}\sum_{c\neq 0} \frac{y^s}{|c|^{2s}} \sum_{\substack{1\leqslant d\leqslant |c| \\ (c,d)=1}} e^{2\pi i n \frac{a}{c}} f_n\left(x + \frac{d}{c}; c, y, s\right), \end{aligned} \tag{4.7}$$

where,

$$f_n(x;c,y,s) = \sum_{-\infty}^{+\infty} |iy+m+x|^{-2s} \exp\left(\frac{-2\pi i n}{c^2(iy+m+x)}\right). \tag{4.8}$$

Note that f_n has period one, so,

$$f_n = \sum_{m=-\infty}^{+\infty} e^{2\pi i m x} b_n(m; c.y, s), \tag{4.9}$$

where,

$$\begin{aligned} b_n(m;c,y,s) &= \int_0^1 e^{-2\pi i m \xi} f_n(\xi;c,y,s) d\xi \\ &\xlongequal{\text{Change order}} y^{1-2s} \int_{-\infty}^{+\infty} \exp\left(-2\pi i m y\xi - \frac{2\pi n}{c^2 y(1-i\xi)}\right) \frac{d\xi}{(1+\xi^2)^s}. \end{aligned} \tag{4.10}$$

We may as well omit $[i, i\infty)$ and $(-i\infty, -i]$ for one-valued branch. And we can change the toy contour to $\operatorname{Im}\xi = \Delta$, $-1 < \Delta < 1$. Then,

$$\operatorname{Re}(-i\xi) = \Delta, \quad \operatorname{Re}\left(\frac{-1}{1-i\xi}\right) < 0, \quad \left|\left(1+\xi^2\right)\right|^{-s} < \left((1-|\Delta|)^2 + (\operatorname{Re}\xi)^2\right)^{-\sigma}. \tag{4.11}$$

Hence, for any s for which $\operatorname{Re} s > 1/2$ and any $\Delta \in (-1, 1)$ we have,

$$|b_n(m; c, y, s)| \leqslant A(y, \sigma) e^{-2\pi|m\Delta|y}, \quad \sigma = \operatorname{Re} s. \tag{4.12}$$

As a result, this series converge absolutely when $y > 0$ and $\sigma > 1/2$ and if the following series converges, we can substitute (4.9) in (4.7).

$$\sum_{c\neq 0} \frac{S(n, m; c)}{|c|^{2\sigma}}. \tag{4.13}$$

It really converges when $\sigma > 3/4$. In fact, Kloosterman sums satisfy Weil's estimate,

$$|S(n, m; c)| \leqslant |c|^{1/2} \min\left\{\sqrt{(n, c)}\, d\left(\frac{c}{(n, c)}\right), \sqrt{(m, c)}\, d\left(\frac{c}{(m, c)}\right)\right\}. \tag{4.14}$$

So, for any fixed n, this series is determined by $\sqrt{n} \sum_{c\neq 0} \frac{d(c)}{|c|^{2\sigma - 1/2}}$, which converges when $\sigma > 3/4$.

Replace f_n by its Fourier expansion, and our target will be reached.

■

4.2 One way of calculating inner product

It is easy to see that, the inner product I above satisfies that,

$$I = \sum_{g \in G_\infty \backslash G} \int_D U_n(z, s_1) (\operatorname{Im} gz)^{s_2} \overline{e^{2\pi i m g z}} dz \tag{4.15}$$

$$\overset{G\text{-invariant}}{=\!=\!=} \sum_{g} \int_{gD} U_n(z, s_1) y^{s_2} \overline{e^{2\pi i m z}} dz \tag{4.16}$$

$$= \int_B U_n(z, s_1) y^{s_2} \overline{e^{2\pi i m z}} dz, \text{ where B is the strip of } [0, 1] \times [0, \infty) \tag{4.17}$$

$$= \delta_{mn} \frac{\Gamma(s_1 + s_2 - 1)}{(4\pi n)^{s_1 + s_2 - 1}} + \frac{1}{2} \int_0^\infty y^{s_2 - 2} e^{-2\pi m y} \sum_{c \neq 0} \frac{S(n, m; c)}{|c|^{2s_1}} y^{s_1} b_n(m; c, y, s_1). \tag{4.18}$$

The last step uses Fourier expansion of U_n. And if we can change the order, we will get the target form of (4.3). To reach this, we note that,

$$|y^{s_1} b_n(m; c, y, s_1)| \leqslant y^{1-\sigma_1} \int_{-\infty}^{\infty} \frac{d\xi}{(1+\xi^2)^{\sigma_1}}, \quad \sigma_1 = \operatorname{Re} s_1. \tag{4.19}$$

So, when we temporarily assume $\sigma_2 > \sigma_1 > 1$ and $m \geqslant 1$, the integrand in (4.18) is majorized by $y^{\sigma_2-\sigma_1-1} e^{-2\pi m y} \sum_{c \neq 0} \frac{|S(n,m;c)|}{c^{2\sigma_1}}$. As a result, change the order of summation over c and integration over y and use the representation of b_n, we can obtain that the inner integral is equal to,

$$\int_0^{\infty} y^{s_2-s_1-1} \exp\left(-2\pi m(1+i\xi)y - \frac{2\pi n}{c^2(1-i\xi)y}\right) dy. \tag{4.20}$$

Here, the writer skipped a lot of calculation and claimed that using the well-known integral representation for the Hankel function of the first kind of a purely imaginary argument,

$$K_v(z) = \frac{1}{2} \int_0^{\infty} \exp\left(-\frac{z}{2}\left(u + \frac{1}{u}\right)\right) \frac{du}{u^{v+1}} \quad \operatorname{Re} z > 0, \tag{4.21}$$

we can get, the integral in (4.20) is equal to,

$$\frac{2}{|c|^{s_2-s_1}} \left(\frac{n}{(1+\xi^2)\, m}\right)^{\frac{s_2-s_1}{2}} K_{s_1-s_2}\left(\frac{4\pi\sqrt{nm}}{|c|} \sqrt{\frac{1+i\xi}{1-i\xi}}\right). \tag{4.22}$$

Substitute it into the above formula, the second term in (4.18) is,

$$2\left(\sqrt{\frac{n}{m}}\right)^{s_2-s_1} \sum_{c=1}^{\infty} \frac{S(n,m;c)}{c^{s_1+s_2}} \int_{-\infty}^{\infty} K_{s_1-s_2}\left(\frac{4\pi\sqrt{nm}}{c} \sqrt{\frac{1+i\xi}{1-i\xi}}\right) \left(1+\xi^2\right)^{-\frac{s_1+s_2}{2}} d\xi. \tag{4.23}$$

And if we let $v = \sqrt{\frac{1+i\xi}{1-i\xi}}$ which changes along right unit semicircle from $-i$ to i, we can obtain,

$$(4.23) = (-i) 2^{2-s_1-s_2} \int_{-i}^{i} K_{s_1-s_2}(xv) \left(v + \frac{1}{v}\right)^{s_1+s_2-2} \frac{dv}{v}, \quad x = \frac{4\pi\sqrt{nm}}{c}. \tag{4.24}$$

Cut the complex $v-$plane along the negative real semiaxis, and deform the path of the integral above. We can obtain a path from the imaginary axis from $-i\infty$ to $-i$ and from i to $i\infty$, because for each fixed v and fixed $x > 0$, when $|v| \to \infty$ in the right half-plane, $K_v(xv) \ll |v|^{-1/2}$. So the integral along the bigger semicircle converges to 0 when $\sigma_1 + \sigma_2 \leq 5/2$.

And In the integral from i to $i\infty$, we have,

$$K_v(z) = \frac{\pi}{2\sin\pi v}\left\{e^{-i\pi v/2}J_{-v}\left(ze^{-i\pi/2}\right) - e^{i\pi v/2}J_v\left(ze^{-i\pi/2}\right)\right\}, \tag{4.25}$$

and in the integral from $-i\infty$ to $-i$, we have,

$$K_v(z) = \frac{\pi}{2\sin\pi v}\left\{e^{i\pi v/2}J_{-v}\left(ze^{i\pi/2}\right) - e^{-i\pi v/2}J_v\left(ze^{i\pi/2}\right)\right\}. \tag{4.26}$$

Hence, substitute the above expressions and combine same terms, we can get the formula (4.4).

In order to erase the extra assumption of s, it suffices to verifying the series in (4.3) converges absolutely and use the principle of analytic continuation. Since we have estimates,

$$|J_{\pm v}(x)| \ll x^{-|\operatorname{Re} v|}, \tag{4.27}$$

and,

$$\int_1^\infty u^\mu J_v(u)du, \tag{4.28}$$

is finite for any $\operatorname{Re}\mu < 1/2$, we can obtain, when $x \to 0+$

$$|\Phi(s_1, s_2; x)| \ll \begin{cases} x^{2-\sigma_1-\sigma_2}, & \min(\sigma_1, \sigma_2) > 1, \\ x - |\sigma_1 - \sigma_2| \ln\frac{1}{x}, & \min(\sigma_1, \sigma_2) \leqslant 1. \end{cases} \tag{4.29}$$

Thus, the general term in the series in (4.3) can be dominated by $o\left(|c|^{-2}|S(n,m;c)|\right)$ if $\min(\sigma_1, \sigma_2) > 1$, and by $o\left(|c|^{-2\min(\sigma_1,\sigma_2)}\ln|c||S(n,m;c)|\right)$ if $\min(\sigma_1, \sigma_2) \geqslant 1$.

And then, every proposition in this section has been proved.

4.3 The other way

The subsection 3.3 has stepped forward a lot. Here we can use their information to calculate the inner product the second time.

Lemma 4.3. The eigenfunctions above are complete in this Hilbert space.

Theorem 4.4. Let s_1 and s_2 be complex variables. For any fixed value of one of them, the inner product is a meromorphic function of the second variable in the entire plane, and for all s_1 and s_2 with $\operatorname{Re} s_j > 1$ it equals to,

$$I = \pi(4\pi)^{1-s_1-s_2}\left(\sqrt{\frac{n}{m}}\right)^{s_2-s_1}\left(\sum_{j=1}^{\infty}\rho_j(n)\overline{\rho_j(m)}\Lambda(s_1,s_2;\kappa_j)\right.$$
$$\left.+\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{m}{n}\right)^{ir}\sigma_{2ir}(n)\sigma_{-2ir}(m)\Lambda(s_1,s_2;r)\frac{\cosh\pi r}{|\zeta(1+2ir)|^2}dr\right), \quad (4.30)$$

where $\sigma_s(n) = \sum_{d|n} d^s$ and,

$$\Gamma(s_1,s_2;r) = \frac{\Gamma(s_1-1/2+ir)\Gamma(s_1-1/2-ir)\Gamma(s_2-1/2+ir)\Gamma(s_2-1/2-ir)}{\Gamma(s_1)\Gamma(s_2)}. \quad (4.31)$$

Proof. If we let,

$$\begin{array}{r}\mathcal{E}_i(f) = \int_D f(z)\overline{\psi_j(z)}dz \\ \mathcal{E}(r,f) = \int_D f(z)\overline{E(z,1/2+ir)}dz,\end{array} \quad (4.32)$$

We have Parseval's equality,

$$\left(U_n(\cdot,s_1),U_m(\cdot,\overline{s_2})\right) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\mathcal{E}\left(r,U_n(\cdot,s_1)\right)\overline{\mathcal{E}\left(r,U_m(\cdot,\overline{s_2}\right)}dr$$
$$+\sum_{j=0}^{\infty}\mathcal{E}_j\left(U_n(\cdot,s_1)\right)\overline{\mathcal{E}_j\left(U_m(\cdot,\overline{s_2})\right)}. \quad (4.33)$$

Similarly, using their Fourier coefficients, we will obtain,

$$(U_n(\cdot,s),\psi_j) = \int_0^{\infty} y^{s-2}\int_0^1 e^{2\pi inz}\overline{\psi_j(z)}dxdy \quad (4.34)$$
$$= (2\pi n)^{1/2-s}\overline{\rho_j(n)}\int_0^{\infty} e^{-y}K_{i\kappa_j}(y)y^{s-3/2}dy. \quad (4.35)$$

And we can get a simple form,

$$(U_n(\cdot,s),\psi_j) = 2\pi\sqrt{n}(4\pi n)^{-s}\overline{\rho_j(n)}\frac{\Gamma\left(s-1/2+i\kappa_j\right)\Gamma\left(s-1/2-i\kappa_j\right)}{\Gamma(s)}, \quad (4.36)$$

and,

$$\frac{1}{\pi}\left(U_n(\cdot,s),E(\cdot,1/2+ir)\right)$$
$$= 2^{2-2s}(n\pi)^{1/2-s-ir}\sigma_{2ir}(n)\frac{\Gamma(s-1/2+ir)\Gamma(s-1/2-ir)}{\Gamma(s)\Gamma(1/2-ir)\zeta(1-2ir)} \quad (4.37)$$

Substituting these Fourier coefficients in Parseval's equality, we obtain the main assertion of the lemma. We reserve the others for readers.

■

5 Application

In this section, we will only show the outcome from the analysis above without proofs.

Corollary 5.1. m, $n \in \mathbb{N}_+$, $|\mathrm{Im}\, t| \leqslant \frac{1}{4}$, then,

$$\sum_{j=1}^{\infty} \frac{\rho_j(n)\overline{\rho_j(m)}}{\cosh \pi\kappa_j} H(\kappa_j, t) + \frac{1}{\pi}\int_{-\infty}^{+\infty} \left(\frac{m}{n}\right)^{ir} \sigma_{2ir}(n)\sigma_{-2ir}(m) \frac{H(r,t)}{|\zeta(1+2ir)|^2} dr$$
$$= \frac{\delta_{mn}}{\pi^2}\frac{t}{\sinh \pi t} + \frac{2t}{\pi \sinh(2\pi t)} \sum_{c=1}^{\infty} \frac{S(n,m;c)}{c} \Phi(\frac{4\pi\sqrt{mn}}{c}, t), \quad (5.1)$$

where

$$H(r,t) = \frac{\cosh \pi r}{\cosh \pi(r+t)\cosh \pi(r-t)}, \quad (5.2)$$
$$\Phi(x,t) = x\int_{x}^{\infty} \left(J_{2it}(u) + J_{-2it}(u)\right) \frac{du}{u}. \quad (5.3)$$

Proof. Let $s_1 = 1 + it$ and $s_2 = 1 - it$. In this case, compare the two forms of the inner product. Intereted readers can complete the remaining proof. ■

Corollary 5.2. Given $\epsilon > 0$, $X \geqslant 2$ and $n \geqslant 1$,

$$\sum_{\kappa_j \leqslant X} \frac{|\rho_j(n)|^2}{\cosh \pi\kappa_j} = \frac{X^2}{\pi^2} + O\left(X\log(X) + Xn^{\epsilon} + n^{\frac{1}{2}+\epsilon}\right). \quad (5.4)$$

Theorem 5.3. $h(r)$ is an even function of complex variables holomorphic on the strip $\{\mathrm{Im}\, r \leqslant \Delta\}$ with $\Delta > \frac{1}{2}$ and $h(r) = O\left(|r|^{-2-\delta}\right)$ where $\delta > 0$. m, $n \in \mathbb{N}_+$. Then,

$$\sum_{j=1}^{\infty} \frac{\rho_j(n)\overline{\rho_j(m)}}{\cosh \pi\kappa_j} h(\kappa_j) + \frac{1}{\pi}\int_{-\infty}^{+\infty} \left(\frac{m}{n}\right)^{ir} \sigma_{2ir}(n)\sigma_{-2ir}(m) \frac{h(r)}{|\zeta(1+2ir)|^2} dr$$
$$= \frac{\delta_{mn}}{\pi^2}\int_{-\infty}^{+\infty} r\tanh \pi r h(r) dr + \sum_{c=1}^{\infty} \frac{S(n,m;c)}{c} \phi(\frac{4\pi\sqrt{mn}}{c}), \quad (5.5)$$

where,

$$\phi(x) = \frac{2i}{\pi} \int_{-\infty}^{+\infty} J_{2ir}(x) \frac{r}{\cosh \pi r} h(r) dr. \tag{5.6}$$

Remark. The estimate (5.4) can be obtained when we let $m = n$. Now weight function H plays a role of filtration. Moreover, theorem 5.3 can be obtained if we integrate (5.1) over t and change the order of sum and integration.

Theorem 5.4. When n, m fixed and $T \to \infty$,

$$|\sum_{1 \geqslant c \geqslant T} \frac{S(n,m;c)}{c}| \ll T^{1/6} (\ln T)^{1/6}. \tag{5.7}$$

Remark. This estimate is the first nontrivial conclusion all over the world. Ju. V. Linnik conjectured that the average on the left is much smaller than any T^{ϵ}. And Selberg found a counterexample to show that the analog of Linnik's conjecture for an arbitrary discrete subgroup of $SL(2, \mathbb{R})$ is wrong.

6 Summary

We briefly discuss the basic ideas of this topic and some theorems. In this process, we turn to be familiar with modular forms and contemporary analytic number theory. The deeper our grasp of arithmetic group is, the better we can understood the number theory. I think it is what Kuznecov's article suggests. The introduction is still imperfect, and we apologize for all possible errors and fault sentence in advance.

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