A Concise Explanation of the Sphere Packing Problem in Dimension 8

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1 Introduction: History and the Breakthrough in 2003

The sphere packing problem is a meaningful open problem dating back several hundred years. It is related to the subtle symmetry of Euclidean space, appealing to many experts in different subjects. We can explain the problem via the following formal definition.

Definition 1.1. Consider the Euclidean vector space $\mathbb{R}^d$ endowed with Euclidean norm $\|\cdot\|$ and Lebesgue measure Vol(). And we use the notation $B_d(x,r)$ to denote a open ball with center x and radius r in $\mathbb{R}^d$, where $x \in \mathbb{R}^d$ and $r > 0$. Let $X \subset \mathbb{R}^d$ be a discrete subset such that for any two different points x and y, $\|x-y\| \geq 2$. Then we call the union

$$\mathcal{P} = \bigcup_{x \in X} B_d(x,1)$$

a sphere packing. Moreover, if X is a lattice, we will call it a lattice packing. Our goal is to calculate the supremum of every possible density

$$\Delta_{\mathcal{P}} := \limsup_{r\to\infty} \frac{\mathrm{Vol}(\mathcal{P}\cap B_d(0,r))}{\mathrm{Vol}(B_d(0,r))},$$

where we denote the suprenum by Δ_d and call it the sphere packing constant.

The calculation of sphere packing constants are not as easy as we may imagine. Except the trivial case of dimension 1, both value and consturction are unknown for many dimension. For example, although the familiar hexagonal lattice packing is well known, experts still spent over 50 years proving that it gives the best packing in dimension 2. Afterwards, computational mathematicians tried their best to work out the problem in dimension 3. The breakthrough didn't emerge until 2003. Cohn and Elkies, in [1], proved the following theorem and then estimated nicer upper bounds for all dimensions.

Theorem 1.2. We say a smooth function $f : \mathbb{R}^d \to \mathbb{C}$ is admissible, if there exists $\epsilon > 0$ such that $|f|$ and $|\hat{f}|$ are bounded above by a constant times of $(1+|x|)^{-d-\epsilon}$. Here $\hat{f}(x) = \int_{\mathbb{R}^d} f(y)\exp(-2\pi i xy)dy$ is the Fourier transform. Then if we can find a nontrivial admissible f which satisfies

$$f(x) \le 0 \text{ for } |x| \ge 1 \tag{1}$$

and

$$\hat{f}(x) \ge 0 \text{ for all } x \in \mathbb{R}^d, \tag{2}$$

the sphere packing constant will be bounded above by

$$\frac{f(0)}{\hat{f}(0)} \cdot \mathrm{Vol}\left(B_d\left(0, 1/2\right)\right).$$

From then on, many computational mathematicians devoted themselves to finding an optimal function by virtue of computer programing. And these outcomes show that the constants corresponding to dimension 8 and 24 seem to be the densities of E_8-lattice and Leech-lattice packing. More precisely, $\Delta_8 = \frac{\pi^4}{384}$ and $\Delta_{24} = \frac{\pi^{12}}{12!}$. To our surprise, Viazovska constructed the necessary functions via modular forms.

In the next section, we will illustrate the construction for dimension 8 and prove several lemmas on specical modular forms. For the sake of completeness, let us prove the theorem 1.2.

Proof. We need the generalized Poisson formula

$$\sum_{x\in\Lambda} f(x+v) = \frac{1}{|\Lambda|}\sum_{t\in\Lambda^*} \exp(2\pi i t\cdot v)\hat{f}(t),$$

where $|\Lambda|$ is the volmue of fundamental parallelotope and Λ^* is the dual lattice defined by

$$\{y \mid x\cdot y \in \mathbb{Z},\ \forall\, x \in \Lambda\}.$$

First, for a given sphere packing $\mathcal{P}$, the density is the same if we consider a compact set R containing the origin whose closure of its interior is itself instead of the ball. It means that $\Delta_{\mathcal{P}} = \limsup_{r\to\infty} \frac{\mathrm{Vol}(rR\cap\mathcal{P})}{\mathrm{Vol}(rR)}$. Fix R as a fundamental parallelotope of an arbitrary lattice Λ. As a result, we can take a almost optimal packing $\mathcal{P}$ and a sufficiently large r such that the density in rR is near Δ_d. Then we define a new packing $\mathcal{P}'$ by taking all the spheres of $\mathcal{P}$ that lie entirely within rR, and also including all translations of them by $r\Lambda$. Its total density, according to the periodicity, is still near Δ_d.

So it suffices to consider the packing given by the translates of a lattice Λ by vectors $v_1, \cdots, v_N$, whose differences are not in Λ and greater than 1. Then its total density is

$$\delta = \frac{N\,\mathrm{Vol}(B_d(0,1/2))}{|\Lambda|}. \tag{3}$$

By the Poisson formula,

$$\begin{aligned}\sum_{1\le k,j\le N}\sum_{x\in\Lambda} f(x+v_j-v_k) &= \frac{1}{|\Lambda|}\sum_{t\in\Lambda^*}\hat{f}(t)\sum_{1\le k,j\le N} e^{2\pi i v_j\cdot t}\overline{e^{2\pi i v_k\cdot t}} \\ &= \frac{1}{|\Lambda|}\sum_{t\in\Lambda^*}\hat{f}(t)\left|\sum_{1\le j\le N} e^{2\pi i v_j\cdot t}\right|^2 \\ &\ge \frac{N^2\hat{f}(0)}{|\Lambda|},\end{aligned}$$

since $\hat{f}$ is nonnegative everywhere. Besides, as the assumption above, $|x+v_j-v_k|$ is greater than 1, unless $x=0$ and $j=k$. So the restriction on f implies that the left-hand side is bounded above by $Nf(0)$. Hence,

$$Nf(0) \ge \frac{N^2\hat{f}(0)}{|\Lambda|},$$

which gives the estimate in the statement:

$$\delta \le \frac{f(0)}{\hat{f}(0)}\cdot \mathrm{Vol}\left(B_d\left(0,1/2\right)\right)$$

■

2 E_8-Lattice packing

In dimension 8, there is a common lattice packing called E_8 lattice packing. Recall the lattice

$$\Lambda_8 = \left\{ (x_i)\in\mathbb{Z}^8 \cup \left(\mathbb{Z}+\frac{1}{2}\right)^8 \,\middle|\, \sum_{i=1}^{8} x_i \equiv 0 \pmod 2 \right\}.$$

Then E_8-lattice sphere packing is the packing of unit balls with centers at $\frac{1}{\sqrt{2}}\Lambda_8$. The main theorem in [3] is the construction of optimal function which satisfies theorem 1.2 and gives the upper bound equal to the density of E_8-packing. It may be more convenient if we scale the function by $\sqrt{2}$.

Theorem 2.1. There exists a radial Schwartz function $g : \mathbb{R}^8 \to \mathbb{R}$ such that

$$g(x) \leq 0 \ \forall \ |x| \geq \sqrt{2}, \tag{4}$$

$$\hat{g}(x) \geq 0 \ \forall \ x \in \mathbb{R}^8, \tag{5}$$

$$g(0) = \hat{g}(0) = 1. \tag{6}$$

Besides, the values of both $g(x)$ and $\hat{g}(x)$ don't vanish unless $|x|^2 \in 2\mathbb{Z}$.

Then we can get the constant $\Delta_8 = \frac{\pi^4}{384}$ if taking $f(x) = g(\sqrt{2}x)$ in theorem 1.2. In order to prove this theorem, we also need some extra functions.

3 Preliminaries of modular forms

Definition 3.1. The following three modular forms are theta functions, which are so-called "Thetanullwerte":

$$\theta_{00}(z) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$$
$$\theta_{01}(z) := \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i n^2 z}$$
$$\theta_{10}(z) := \sum_{n\in\mathbb{Z}} e^{\pi i \left(n+\frac{1}{2}\right)^2 z}$$

Proposition 3.2. The above functions satisfy the following equalities:

$$z^{-2}\theta_{00}^4\left(\frac{-1}{z}\right) = -\theta_{00}^4(z).$$
$$z^{-2}\theta_{01}^4\left(\frac{-1}{z}\right) = -\theta_{10}^4(z).$$
$$z^{-2}\theta_{10}^4\left(\frac{-1}{z}\right) = -\theta_{01}^4(z). \tag{7}$$
$$\theta_{00}^4(z+1) = \theta_{01}^4(z).$$
$$\theta_{01}^4(z+1) = \theta_{00}^4(z).$$
$$\theta_{10}^4(z+1) = -\theta_{10}^4(z).$$

$$\theta_{01}^4 + \theta_{10}^4 = \theta_{00}^4. \tag{8}$$

As a result, θ_{01}^4, θ_{10}^4, and θ_{00}^4 belong to $M_2(\Gamma(2))$.

Proof. Just take the example of θ_{00}. The outcome of $\theta_{00}^4\left(\frac{-1}{z}\right)$ is directly from Poisson summation formula. As for $\theta_{00}^4(z+1)$, we know that $e^{\pi i n^2} = (-1)^n$. The last identity is actually a combinatorial problem, so we leave it to the reader. ∎

The following propositions are deduced by the main theorem in [2]. For lack of time, we have to skip its proof.

Proposition 3.3. The elliptic j-invariant

$$j = \frac{1728E_4^3}{E_4^3 - E_6^2}$$

has the Fourier expansion with coefficients

$$c_j(n) = \frac{2\pi}{\sqrt{n}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_1\left(\frac{4\pi\sqrt{n}}{k}\right) \quad n \in \mathbb{Z}_{>0}, \tag{9}$$

where

$$A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} e^{\frac{-2\pi i}{k}(nh+h')},\ hh' \equiv -1 \pmod{k},$$

and $I_\alpha(x)$ is the modified Bessel function of the first kind.

Proposition 3.4. The following functions have similar Fourier coefficients.

$$\varphi_{-2} := \frac{-1728E_4E_6}{E_4^3 - E_6^2}, \tag{10}$$

$$\varphi_{-4} := \frac{1728E_4^2}{E_4^3 - E_6^2}. \tag{11}$$

$$c_{\varphi_\kappa}(n) = 2\pi n^{\frac{\kappa-1}{2}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{1-\kappa}\left(\frac{4\pi\sqrt{n}}{k}\right) \quad n \in \mathbb{Z}_{>0} \tag{12}$$

4 The Constructions of a and b

Now we are ready for our tedious construction of g. The author decomposed it into two parts a and b. We will check the details one by one. The Schwartz functions a, b are constructed to satisfy the conditions $\mathcal{F}(a) = a$, $\mathcal{F}(b) = -b$ and their double zeroes at all Λ_8-vectors of length greater then $\sqrt{2}$. And g will be constructed as a linear combination of a and b. The reason for this laborious construction is that when we use modular forms we can make our function hold nice symmetric properties.

4.1 The Construction of a

4.1.1 Properties of ϕ_κ

Definition 4.1.

$$\phi_{-4} := \varphi_{-4},$$
$$\phi_{-2} := \varphi_{-4}E_2 + \varphi_{-2},$$
$$\phi_0 := \varphi_{-4}E_2^2 + 2\varphi_{-2}E_2 + j - 1728.$$

Proposition 4.2.

$$\phi_0\left(\frac{-1}{z}\right) = \phi_0(z) - \frac{12i}{\pi}\frac{1}{z}\phi_{-2}(z) - \frac{36}{\pi^2}\frac{1}{z^2}\phi_{-4}(z). \tag{13}$$

What's more, we have

$$\begin{aligned} \phi_{-2} &= -3D\left(\varphi_{-4}\right) + 3\varphi_{-2}, \\ \phi_0 &= 12D^2\left(\varphi_{-4}\right) - 36D\left(\varphi_{-2}\right) + 24j - 17856, \end{aligned} \tag{14}$$

where $D(f)(z) = \frac{1}{2\pi i}\frac{d}{dx}f(z)$.

Proof. Since E_2 isn't stable under the action of hyperbolic inversion but satisfies a different identity, we can obtain the identity (13). The second part follows directly from corresponding Fourier expansions. ■

Combined with the Fourier coefficients of φ_κ and the asymptotic properties of various functions, we can get this corrolary.

Corrolary 4.3.

$$|c_{\phi_0(n)}| \le 2e^{4\pi\sqrt{n}}.$$

$$|\phi_0(z)| \le Ce^{-2\pi \mathrm{Im}\ z},\ \forall\ \mathrm{Im}\ z > \frac{1}{2}.$$

$$\phi_0\left(\frac{-1}{it}\right) = O\left(e^{-2\pi/t}\right) \qquad \text{as } t \to 0.$$

$$\phi_0\left(\frac{-1}{it}\right) = O\left(t^{-2}e^{2\pi t}\right),\ \text{as } t \to \infty.$$

4.1.2 Properties of a

According to the estimates in subsection 4.1.1, the following function is well-defined in $\mathbb{R}^8$.

$$\begin{aligned} a(x) :=& \int_{-1}^{i} \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2 e^{\pi i\|x\|^2 z}dz + \int_{1}^{i} \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2 e^{\pi i\|x\|^2 z}dz \\ &- 2\int_{0}^{i} \phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i\|x\|^2 z}dz + 2\int_{i}^{\infty} \phi_0(z) e^{\pi i\|x\|^2 z}dz. \end{aligned} \tag{15}$$

Proposition 4.4. a is a Schwartz function and statisfies $\hat{a}(x) = a(x)$.

Proof. For the first summand, we have

$$\begin{aligned} &\left|\int_{-1}^{i} \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2 e^{\pi i r^2 z}dz\right| = \left|\int_{i\infty}^{-1/(i+1)} \phi_0(z) z^{-4} e^{\pi i r^2(-1/z-1)}dz\right| \\ &\le C_1\int_{1/2}^{\infty} e^{-2\pi t}e^{-\pi r^2/t}dt \le C_1\int_{0}^{\infty} e^{-2\pi t}e^{-\pi r^2/t}dt = C_2 r K_1(2\sqrt{2}\pi r), \end{aligned}$$

where C_1 and C_2 are positive constants and K_α is the modified Bessel function of the second kind. Similar estimates holds for the second and third summand. For the last summand, the following inequality is true.

$$\left|\int_i^{i\infty} \phi_0(z)e^{\pi i r^2 z}dz\right| \leq C\int_1^{\infty} e^{-2\pi t}e^{-\pi r^2 t}dt = C_3\frac{e^{\pi\left(r^2+2\right)}}{r^2+2}.$$

So,

$$|a(r)| \leq 4C_2 r K_1(2\sqrt{2}\pi r) + 2C_3\frac{e^{-\pi\left(r^2+2\right)}}{r^2+2},$$

which decays faster enough. Moreover, the partial derivatives of a are analogous to a itself because their differences are at most some exponential factors. Consequently, it is a Schwartz function.

We know that in this case, we can exchange the integral order when we calculate the Fourier transform. First, recall the Fourier transform of a Gaussian function is

$$\mathcal{F}\left(e^{\pi i\|x\|^2 z}\right)(y) = z^{-4}e^{\pi i\|y\|^2\left(\frac{-1}{z}\right)}. \tag{16}$$

Using this, we can compute

$$\begin{aligned}\widehat{a}(y) =& \int_1^i \phi_0\left(1-\frac{1}{w-1}\right)\left(\frac{-1}{w}+1\right)^2 w^2 e^{\pi i\|y\|^2 w}dw\\ &+\int_{-1}^i \phi_0\left(1-\frac{1}{w+1}\right)\left(\frac{-1}{w}-1\right)^2 w^2 e^{\pi i\|y\|^2 w}dw\\ &-2\int_{i\infty}^i \phi_0(w)e^{\pi i\|y\|^2 w}dw + 2\int_i^0 \phi_0\left(\frac{-1}{w}\right)w^2 e^{\pi i\|y\|^2 w}dw.\end{aligned}$$

Then if we let $w = \frac{-1}{z}$ and take the advantage of *1-periodicity*, we know that

$$\begin{aligned}\widehat{a}(y) =& \int_1^i \phi_0\left(1-\frac{1}{w-1}\right)\left(\frac{-1}{w}+1\right)^2 w^2 e^{\pi i\|y\|^2 w}dw\\ &+\int_{-1}^i \phi_0\left(1-\frac{1}{w+1}\right)\left(\frac{-1}{w}-1\right)^2 w^2 e^{\pi i\|y\|^2 w}dw\\ &-2\int_{i\infty}^i \phi_0(w)e^{\pi i\|y\|^2 w}dw + 2\int_i^0 \phi_0\left(\frac{-1}{w}\right)w^2 e^{\pi i\|y\|^2 w}dw\\ =&\int_1^i \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2 e^{\pi i\|y\|^2 z}dz + \int_{-1}^i \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2 e^{\pi i\|y\|^2 z}dz\\ &+2\int_i^{i\infty}\phi_0(z)e^{\pi i\|y\|^2 z}dz - 2\int_0^i \phi_0\left(\frac{-1}{z}\right)z^2 e^{\pi i\|y\|^2 z}dz\\ =&a(y).\end{aligned}$$

■

Next, we need to check a has double roots at the points of the lattice.

Proposition 4.5. Since a is clearly a radial function, it is unambiguous to use the notation $a(r)$ for $r > 0$. And when $r > \sqrt{2}$,

$$a(r) = -4\sin\left(\pi r^2/2\right)^2 \int_0^{i\infty} \phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i r^2 z} dz. \tag{17}$$

Proof. We denote by $d(r)$ the right-hand side. It is well defined because of the estimates in subsection 4.1.1. Due to the trivial identities

$$e^{\pi i r^2} + e^{-\pi i r^2} - 2 = 2\cos(\pi r^2) - 2,$$
$$-4\sin^2(\pi r^2/2) = 2\cos(\pi r^2) - 2,$$

we get

$$\begin{aligned} d(r) = &\int_{-1}^{i\infty-1} \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2 e^{\pi i r^2 z} dz - 2\int_0^{i\infty} \phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i r^2 z} dz \\ &+ \int_1^{i\infty+1} \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2 e^{\pi r^2} z dz. \end{aligned}$$

From (13), when $r > \sqrt{2}$, we know

$$\phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i r^2 z} \to 0 \quad \text{as } \operatorname{Im}(z) \to \infty.$$

So every integral can be regarded to be integrated to $i\infty$ by paths deformation, i.e.

$$\begin{aligned} d(r) = &\int_{-1}^{i} \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2 e^{\pi i r^2 z} dz + \int_1^{i\infty} \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2 e^{\pi i r^2 z} dz \\ &- 2\int_0^{i} \phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i r^2 z} dz - 2\int_1^{\infty} \phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i r^2} z dz \\ &+ \int_1^{i} \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2 e^{\pi i r^2} z dz + \int_1^{\infty} \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2 e^{\pi i r^2 z} dz. \end{aligned}$$

It closely resembles the definition of a. In order to deal with the extra part, we find the key from the equation (13):

$$\begin{aligned} &\phi_0\left(\frac{-1}{z+1}\right)(z+1)^2 - 2\phi_0\left(\frac{1}{z}\right) z^2 + \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2 \\ &= \phi_0(z+1)(z+1)^2 - 2\phi_0(z)z^2 + \phi_0(z-1)(z-1)^2 \\ &\quad - \frac{12i}{\pi}\left(\phi_{-2}(z+1)(z+1) - 2\phi_{-2}(z)z + \phi_{-2}(z-1)(z-1)\right) \\ &\quad - \frac{36}{\pi^2}\left(\phi - 4(z+1) - 2\phi_{-4}(z) + \phi_{-4}(z-1)\right) \\ &= 2\phi_0(z). \end{aligned}$$

In this way, we proved this identity. ∎

At last, we need some data of a at the special points.

Proposition 4.6. When $r \geq 0$,

$$a(r) = 4i \sin\left(\pi r^2/2\right)^2 \left(\frac{36}{\pi^3\left(r^2-2\right)} - \frac{8640}{\pi^3 r^4} + \frac{18144}{\pi^3 r^2} + \int_0^\infty \left(t^2 \varphi_0\left(\frac{i}{t}\right) - \frac{36}{\pi^2}e^{2\pi t} + \frac{8640}{\pi}t - \frac{18144}{\pi^2}\right) e^{-r^2 t} dt\right).$$

So the value of a is always purely imaginary, and

$$a(0) = \frac{-8640i}{\pi}, a(\sqrt{2}) = 0, a'(\sqrt{2}) = \frac{72\sqrt{2}i}{\pi}.$$

Proof. By Proposition 4.5, it is obvious if we use the expansion of $\phi_0(i/t)t^2$. ■

4.2 The Construction of b

4.2.1 Properties of ψ_I, ψ_T, ψ_S

To define b, another modular form h is considered.

$$h := 128\frac{\theta_{00}^4 + \theta_{01}^4}{\theta_{10}^8} \tag{18}$$

Two generators of $\Gamma_0(2)$ are $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so $h \in M_{-2}^{!}\left(\Gamma_0(2)\right)$ follows from (7). It's well known that θ_{10} have no zeros in the upper-half plane, therefore f has poles only at cusps.

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma(1)$.

And define

$$\begin{aligned} \psi_I &:= h - h|_{-2}ST, \\ \psi_T &:= \psi_I|_{-2}T \\ \psi_S &:= \psi_I|_{-2}S. \end{aligned} \tag{19}$$

$\psi_I, \psi_T, and \psi_S$ satisfy the relation

$$\psi_T + \psi_S = \psi_I \tag{20}$$

which can be checked as follows. By the definition of ψ_I, ψ_T, ψ_S, we have

$$\psi_T + \psi_S = h|_{-2}T - h|_{-2}ST^2 + h|_{-2}S - h|_{-2}STS.$$

Note that $ST^2S, T, STS(ST)^{-1} \in \Gamma_0(2)$ and $S^2 = -I$, then $\psi_T + \psi_S = h|_{-2}T - h|_{-2}STS = h|_{-2}T - h|_{-2}STS = h - h|_{-2}ST = \psi_I$.

The Fourier expansions of these functions are from the Fourier expansions of theta functions:

$$\begin{aligned}\psi_I(z) =& q^{-1} + 144 - 5120q^{1/2} + 70524q - 626688q^{3/2} + 4265600q^2 + O\left(q^{5/2}\right)\\ \psi_T(z) =& q^{-1} + 144 + 5120q^{1/2} + 70524q + 626688q^{3/2} + 4265600q^2 + O\left(q^{5/2}\right),\\ \psi_S(z) =& -10240q^{1/2} - 1253376q^{3/2} - 48328704q^{5/2} - 1059078144q^{7/2} + O\left(q^{9/2}\right).\end{aligned} \tag{21}$$

4.2.2 Properties of b

The function b is defined as for every $x \in \mathbb{R}^8$

$$b(x) := \int_{-1}^{i} \psi_T(z)e^{\pi i\|x\|^2 z}dz + \int_{1}^{i} \psi_T(z)e^{\pi i\|x\|^2 z}dz - 2\int_{0}^{i} \psi_I(z)e^{\pi i\|x\|^2 z}dz - 2\int_{i}^{i\infty} \psi_S(z)e^{\pi i\|x\|^2 z}dz.$$

Now we check b satisfies the desired properties. The process is just the same as in the case of the function a, so we work out the differences between a and b to fit the case of b into the arguments used in the last section. And this benefits the understanding of the construction of a and b.

Proposition 4.7. The function b is a Schwarts function and satisfies $\widehat{b}(x) = -b(x)$.

Proof. We estimate the first summand as follows.

$$\begin{aligned}\int_{-1}^{i} \psi_T(z)e^{\pi i\|x\|^2 z}dz &= \int_{i\infty}^{-1/(i+1)} \psi_I\left(\frac{-1}{z}\right) e^{\pi i r^2(-1/z-1)} z^{-2}dz\\ &= \int_{i\infty}^{-1/(i+1)} \psi_S(z)z^{-4}e^{\pi i r^2(-1/z-1)}dz\end{aligned}, \tag{22}$$

and from (21), $|\psi_S(z)| \le Ce^{-\pi \operatorname{Im} z}$ for $\operatorname{Im} z > \frac{1}{2}$ where C is a positive constant. So as in the proof of Proposition 4.4, the first summand is controlled by $\left|\int_{-1}^{i} \psi_T(z)e^{\pi i r^2 z}dz\right| \le C_1 r K_1(2\pi r)$. And similarly, the other three summands can be estimated in this way and we get $|b(r)| \le C_2 r K_1(2\pi r) + C_3 \frac{e^{-\pi\left(r^2+1\right)}}{r^2+1}$, where C_1, C_2, and C_3 are some positive constants. And all the higher derivatives of b have similar estimates.

Next, prove b is an eigenfunction of the Fourier transform. Again using the formula (16) and exchanging the order of integral, we get

$$\begin{aligned}\mathcal{F}(b)(x) =& \int_{-1}^{i} \psi_T(z)z^{-4}e^{\pi i\|x\|^2\left(\frac{-1}{z}\right)}dz + \int_{1}^{i} \psi_T(z)z^{-4}e^{\pi i\|x\|^2\left(\frac{-1}{z}\right)}dz\\ &- 2\int_{0}^{i} \psi_I(z)z^{-4}e^{\pi i\|x\|^2\left(\frac{-1}{z}\right)}dz - 2\int_{i}^{i\infty} \psi_S(z)z^{-4}e^{\pi i\|x\|^2\left(\frac{-1}{z}\right)}dz\\ =& \int_{1}^{i} \psi_T\left(\frac{-1}{w}\right) w^2 e^{\pi i\|x\|^2 w}dw + \int_{-1}^{i} \psi_T\left(\frac{-1}{w}\right) w^2 e^{\pi i\|x\|^2 w}dw\\ &- 2\int_{i\infty}^{i} \psi_I\left(\frac{-1}{w}\right) w^2 e^{\pi i\|x\|^2 w}dw - 2\int_{i}^{0} \psi_S\left(\frac{-1}{w}\right) w^2 e^{\pi i\|x\|^2 w}dw.\end{aligned} \tag{23}$$

The second equality is the change of variables $w = \frac{-1}{z}$.

And definition (19) indicates $\psi_T|_{-2}S = -\psi_T$, $\psi_I|_{-2}S = \psi_S$, $\psi_S|_{-2}S = \psi_I$. Combining this with (23), we finally obtain $\mathcal{F}(b)(x) = -b(x)$. ■

Next, we check b has double roots at Λ_8 points. As usual, we give another expression of b.

Proposition 4.8. For $r > \sqrt{2}$, $b(r) = -4\sin\left(\pi r^2/2\right)^2 \int_0^{i\infty} \psi_I(z)e^{\pi i r^2 z}dz$.

Proof. The right-hand side of the above equation to prove is donated by $c(r)$. By definition (19), the analysis of h, we have the estimates $\psi_I(it) = O\left(t^2 e^{-\pi/t}\right)$ as $t \to 0, \psi_I(it) = O\left(e^{2\pi t}\right)$ as $t \to \infty$. Therefore, c is well-defined. Next we show $c(r) = b(r)$.

$$c(r) = \int_{-1}^{i\infty-1} \psi_I(z+1)e^{\pi i r^2 z}dz - 2\int_0^{i\infty} \psi_I(z)e^{\pi i r^2 z}dz + \int_1^{i\infty+1} \psi_I(z-1)e^{\pi i r^2 z}dz$$

Here the equality $-4\sin\left(\pi r^2/2\right)^2 = e^{\pi i r^2} + e^{-\pi i r^2} - 2$ is used.

From the Fourier expansions (21), $\psi_I(z) = e^{-2\pi i z} + O(1)$ as $\mathrm{Im}(z) \to \infty$. Hence, the path of integration can be deformed:

$$\int_{-1}^{i\infty-1} \psi_I(z+1)e^{\pi i r^2 z}dz = \int_{-1}^{i\infty} \psi_I(z+1)e^{\pi i r^2 z}dz = \int_{-1}^{i} \psi_T(z)e^{\pi i r^2 z}dz + \int_i^{i\infty} \psi_T(z)e^{\pi i r^2 z}dz,$$
$$\int_{1}^{i\infty+1} \psi_I(z-1)e^{\pi i r^2 z}dz = \int_{-1}^{i\infty} \psi_I(z+1)e^{\pi i r^2 z}dz = \int_{-1}^{i} \psi_T(z)e^{\pi i r^2 z}dz + \int_i^{i\infty} \psi_T(z)e^{\pi i r^2 z}dz.$$

Hence,

$$c(r) = \int_{-1}^{i} \psi_T(z)e^{\pi i r^2 z}dz + \int_1^{i} \psi_T(z)e^{\pi i r^2 z}dz - 2\int_0^{i} \psi_I(z)e^{\pi i r^2 z}dz + 2\int_i^{i\infty} \left(\psi_T(z) - \psi_I(z)\right)e^{\pi i r^2 z}dz.$$

From (20), we reach $c(r) = b(r)$. ■

Since $\theta_{ij}(z)$ has real value when z is purely imaginary, the above Proposition shows b is totally imaginary. From this expression, we can arrive at another expression of $b(x)$ that helps to compute the special value of b by expanding a few terms.

Proposition 4.9. For $r \geq 0$,

$$b(r) = 4i\sin\left(\pi r^2/2\right)^2 \left(\frac{144}{\pi r^2} + \frac{1}{\pi\left(r^2-2\right)} + \int_0^{\infty} \left(\psi_I(it) - 144 - e^{2\pi t}\right)e^{-\pi r^2 t}dt\right) \tag{24}$$

And this integral converges absolutely for all $r \in \mathbb{R}_{\geq 0}$.

Proof. This is mainly a result of Proposition 4.8 and Fourier expansion shown in (21). (21) shows $\psi_I(it) = e^{2\pi t} + 144 + O\left(e^{-\pi t}\right)$ as $t \to \infty$, and for $r > \sqrt{2}$, $\int_0^\infty \left(e^{2\pi t} + 144\right)e^{-\pi r^2 t}dt = \frac{1}{\pi(r^2-2)} + \frac{144}{\pi r^2}$. This proves the identity (24) holds for $r > \sqrt{2}$. Because both sides of (24) are analytic in some neighborhood of $[0,\infty)$, the identity (24) holds for all $r \geq 0$. ■

One direct corollary is as follows.

Proposition 4.10. $b(0) = 0 \quad b(\sqrt{2}) = 0 \quad b'(\sqrt{2}) = 2\sqrt{2}\pi i$.

5 The Construction and Properties of g

Theorem 5.1. The function

$$g(x) := \frac{\pi i}{8640} a(x) + \frac{i}{240\pi} b(x) \tag{25}$$

satisfies conditions (4)-(6). And the values $g(x)$ and $\widehat{g}(x)$ do not vanish for all vectors x with $\|x\|^2 \notin 2\mathbb{Z}_{>0}$.

The rest of this section is the proof of this final Theorem. And the solution to these estimates is to consider the Taylor extension to higher summands. In the author's original proof, she uses computer techniques to verify these inequalities. Here we just list the sketch.

Proof. We first prove that Property (4) holds.

By Proposition 4.5 and 4.8, $g(r) = \frac{\pi}{2160} \sin\left(\pi r^2/2\right)^2 \int_0^\infty A(t) e^{-\pi r^2 t} dt$, where $A(t) = -t^2\phi_0(i/t) - \frac{36}{\pi^2}\psi_I(it)$. It will suffice to show $A(x) < 0$ for $x > 0$. We do the estimate with the help of Fourier expansion. From identity (13) and (19), we get the following two expressions of $A(t)$ which are respectively used to estimate $A(t)$ for $t \in [0,1]$ and $t \in [1,\infty]$

$$A(t) = -t^2\phi_0(i/t) + \frac{36}{\pi^2} t^2 \psi_S(i/t), \tag{26}$$

$$A(t) = -t^2\phi_0(it) + \frac{12}{\pi} t\phi_{-2}(it) - \frac{36}{\pi^2}\phi_{-4}(it) - \frac{36}{\pi^2}\psi_I(it). \tag{27}$$

For $n \geq 0$, $A_0^{(n)}$ and $A_\infty^{(n)}$ are defined to be

$$A(t) = A_0^{(n)}(t) + O\left(t^2 e^{-\pi n/t}\right) \qquad \text{as } t \to 0, \tag{28}$$

$$A(t) = A_\infty^{(n)}(t) + O\left(t^2 e^{-\pi n t}\right) \qquad \text{as } t \to \infty. \tag{29}$$

The next step is using Fourier expansions to compute $A_0^{(n)}$ and $A_\infty^{(n)}$. To solve our problem, $n = 6$ is enough. From Fourier expansions listed in (21) and (12),

$$\begin{aligned} A_\infty^{(6)}(t) = &-\frac{72}{\pi^2}e^{2\pi t} - \frac{23328}{\pi^2} + \frac{184320}{\pi^2}e^{-\pi t} - \frac{5194368}{\pi^2}e^{-2\pi t} \\ &+ \frac{22560768}{\pi^2}e^{-3\pi t} - \frac{250583040}{\pi^2}e^{-4\pi t} + \frac{869916672}{\pi^2}e^{-5\pi t} \\ &+ t\left(\frac{8640}{\pi} + \frac{2436480}{\pi}e^{-2\pi t} + \frac{113011200}{\pi}e^{-4\pi t}\right) \\ &- t^2\left(518400e^{-2\pi t} + 31104000e^{-4\pi t}\right). \end{aligned}$$

$$A_0^{(6)}(t) = t^2\left(-\frac{368640}{\pi^2}e^{-\pi/t} - 518400e^{-2\pi/t} - \frac{45121536}{\pi^2}e^{-3\pi/t} - 31104000e^{-4\pi/t} - \frac{1739833344}{\pi^2}e^{-5\pi/t}\right).$$

The n-th Fourier coefficient can be estimated as follows:

$$\begin{aligned}
|c_{\psi_I}(n)| &\le e^{4\pi\sqrt{n}}, && n \in \tfrac{1}{2}\mathbb{Z}_{>0} \\
|c_{\psi_S}(n)| &\le 2e^{4\pi\sqrt{n}}, && n \in \tfrac{1}{2}\mathbb{Z}_{>0}, \\
|c_{\phi_0}(n)| &\le 2e^{4\pi\sqrt{n}}, && n \in \mathbb{Z}_{>0}, \\
\left|c_{\phi_{-2}}(n)\right| &\le e^{4\pi\sqrt{n}}, && n \in \mathbb{Z}_{>0}, \\
\left|c_{\phi_{-4}}(n)\right| &\le e^{4\pi\sqrt{n}}, && n \in \mathbb{Z}_{>0}.
\end{aligned}$$

Therefore, the remainder terms of $A(t)$ can be estimated as follows:

$$\begin{aligned}
\left|A(t) - A_0^{(m)}(t)\right| &\le \left(t^2 + \frac{36}{\pi^2}\right) \sum_{n=m}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}} e^{-\pi n/t} \\
\left|A(t) - A_\infty^{(m)}(t)\right| &\le \left(t^2 + \frac{12}{\pi}t + \frac{36}{\pi^2}\right) \sum_{n=m}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}} e^{-\pi n t}.
\end{aligned}$$

Using interval arithmetic it can be checked that

$$\begin{aligned}
\left(t^2 + \frac{36}{\pi^2}\right) \sum_{n=6}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}} e^{-\pi n/t} &\le \left|A_0^{(6)}(t)\right| && \text{for } t \in (0,1], \\
\left|R_\infty^{(6)}(t)\right| &\le \left|A_\infty^{(6)}(t)\right| && \text{for } t \in [1,\infty), \\
A_0^{(6)}(t) &< 0 && \text{for } t \in (0,1], \\
A_\infty^{(6)}(t) &< 0 && \text{for } t \in [1,\infty).
\end{aligned}$$

Therefore, $A(t) < 0$ for $t \in (0,\infty)$.

The proof of Property (5) is completely analogous. $\widehat{g}(r) = \frac{\pi}{2160} \sin\left(\pi r^2/2\right)^2 \int_0^\infty B(t) e^{-\pi r^2 t} dt$ where $B(t) = -t^2\phi_0(i/t) + \frac{36}{\pi^2}\psi_I(it)$. The two expressions of $\widehat{g}(r)$ is

$$B(t) = -t^2\phi_0(i/t) - \frac{36}{\pi^2} t^2 \psi_S(i/t), \tag{30}$$

$$B(t) = -t^2\phi_0(it) + \frac{12}{\pi} t\phi_{-2}(it) - \frac{36}{\pi^2}\phi_{-4}(it) + \frac{36}{\pi^2}\psi_I(it). \tag{31}$$

We prove $B(t) > 0$ for $t \in (0,\infty)$ using totally the same method of computing the Fourier expansions of the two expressions of B as we do previously, and the process is omitted.

The property (6) readily follows from Propositions 4.6 and 4.10. ■

6 Summary

Our construction of g can be assumed to take the projections of Schwart function space to these characteristic subspaces of Fourier transformation. And we need modular forms to provide more symmetry to these functions so that we are focused on the core, the behaviors of g at certain parts. Our essay skips some details of analysis and inequalities, but we think it doesn't influence your understanding.

Let us recall the construction process shown in the text. The function g is constructed to satisfy both conditions on g and $\widehat{g}$. To simplify the construction, g is constructed as the function of the radius of the variable vector. It's known any function f can be decomposed to be the sum of f_1 and f_2 such that $f = f_1 + f_2$ and $\mathcal{F}(f_1) = f_1, \mathcal{F}(f_2) = -f_2$ where f_1, f_2 correspond to multiples of a, b in our construction. We can deduce from the desired properties (4)-(6) that g and $\widehat{g}$ should vanish at all the points of Λ_8 and the first derivatives of g and $\widehat{g}$ should vanish at points of Λ_8 of length greater than $\sqrt{2}$. Therefore, a and b have double zeroes at all Λ_8-vectors of length greater than $\sqrt{2}$. The expressions of a and b in Propositions 4.5 and 4.8 satisfy this condition. And the proof of Propositions 4.4, 4.7, 4.5, 4.8 shows the essential use of modular form in this construction. Take the properties of b as an example. The expression in Proposition 4.7 satisfies the vanishing conditions and has a chance to be the eigenfunction of Fourier expansion because of the properties of the Gauss function. If we want a function given by this expression to be an eigenfunction of Fourier expansion, ϕ_I is just needed to be defined as in (19) and satisfy (20), and this is where the modular forms work. The technique of using Fourier expansion to estimate is an important tool through this process, which helps to check the convergence and the core properties (4) and (5) of g.

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