NOTES ON NORMS IN MOTIVIC HOMOTOPY THEORY

ZHENPENG LI

Abstract. For the convenience of young men in motivic homotopy theory, I write these notes, introducing Bachmann-Hoyois's construction on norm functors, a generalization of many norms in various contexts. Afterwards, I give the enhancement of norm functors to a coherent system and then define the notion of normed motivic spectra as a more comprehensive and reasonable replacement of motivic $\mathbb{E}_\infty$-ring spectra. Finally, my notes culminate with several compatibilities among norm functors, étale fundamental groupoid theory, Voevodsky's slice filtration, and so on, leading to some nontrivial instances of normed motivic spectra, like $\mathrm{H}\mathbb{Z}$, KGL, and MGL, and enlarging our horizons.

Contents

1. Introduction

The notes are intended to introduce Bachmann and Hoyois's work [BH20] on norm functors in motivic homotopy theory. In unstable motivic homotopy theory, it was first developed by Voevodsky in lectures [Del09]. His main target was to generalize the construction of symmetric power functors to the category of motives, which stimulates us to do so in stable motivic homotopy theory. The

Date: 2024 August.

most important functor they defined is the norm functor, for $f : T \to S$ a finite and étale morphism between schemes,

$$f_\otimes : \mathcal{SH}(T) \to \mathcal{SH}(S),$$

where $\mathcal{SH}(-)$ means Voevodsky's ∞-category of motivic spectra. Similar to Voevodsky's theory, when the morphism f is the fold map $S \sqcup S \to S$, $f_\otimes$ is equivalent to the smash product functor.

As is known, stable motivic homotopy theory satisfies the six-functor formalism. In other words, given a certain morphism between schemes $f : T \to S$, one has several different functors between $\mathcal{SH}$ like $f_\sharp$, f^*,f_*, $f_!$, and $f^!$, compatible with other functors from other morphisms in the sense that some principles, for instance, base change and projection formula, hold. The norm construction is thus a new basic functoriality of stable motivic homotopy theory beyond the common six-functor formalism. Hence, there should be some brand-new and meaningful compatible laws interacting with traditional f_*, f^*, and so on. To understand it better, we have the following slogan from the calculation of $f_\otimes$ and f_* for fold maps.

Slogan: Fix a morphism f. In the parametrized world, the norm functor of f behaves like tensor products (when f is finite étale), while $*$-pushforward behaves like direct sums (when f is proper).

Based on this slogan, I think it's not hard to construe what the compatible laws in Subsection 3.3, like distributivity, mean to us.

Simultaneously, it is useful to express these properties in a coherent manner, i.e., by constructing a complicated functor from a complicated category to Cat_∞. This motivation makes us study the (2,1)-category of spans, leading to this functor below

$$\mathcal{SH}^\otimes : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{fét}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty), \quad (X \xleftarrow{f} Y \xrightarrow{p} Z) \mapsto p_\otimes f^*.$$

Although it absolutely cannot encode natural equivalences like distributivity laws, the huge functor $\mathcal{SH}^\otimes$ inspires us to consider normed motivic spectra, an enhancement of $\mathbb{E}_\infty$ motivic spectra. In traditional $\mathbb{E}_\infty$-ring theory, one needs the operad $\mathrm{Fin}_* = \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$ to define $\mathbb{E}_\infty$-rings as certain sections of the straightening of $\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \to \mathrm{Cat}_\infty$, whose multiplication is reflected via active edges, i.e., the right edges. Now in the category of motivic spectra, since we already believe that norm functors induced from finite étale morphisms are a counterpart of monoidal structure between categories, it is reasonable to pay more attention to sections of $\mathcal{SH}^\otimes$ that are cocartesian along left edges and study generalized multiplication given by finite étale morphisms. This immediately suggests a notion of normed spectrum.

More precisely speaking, a normed spectrum over S is a motivic spectrum $E \in \mathcal{SH}(S)$ equipped with maps

$$\mu_f : f_\otimes E_X \to E_Y,$$

subject to some coherence conditions which in particular makes E an $\mathbb{E}_\infty$-ring, where $f : X \to Y$ is a finite étale morphism and E_X (resp. E_Y) means the pullback of E along the structure map. To be more sensible, one can consider these μ_f's as multiplicative transfers among $[\mathbf{1}_X, E_X]$ informally.

All the constructions in their paper is nontrivial. For example, the readers might wonder the difference between the norm functor with $*$-pushforward functor. However, the case of fold maps implies that norm functors don't preserve limits, while $*$-pushforward functors preserves. In fact, for general enough finite étale morphisms, norm functor will only be compatible with sift colimits, which makes it subtle to deal with.

Another genius observation in their paper is that many other contexts admit similar norm functors, like stable equivariant homotopy theory of groupoids and noncommutative stable motivic theory. It is thus related to several comparison natural transformations, leading to some fundamental examples of normed spectra like KGL. These examples, to some extent, reflect the significance and generality of normed spectra.

Guide for readers. We will begin with a glance of tranditional construction and property of Weil restriction in algebraic geometry in Section 2. Then based on it, in Section 3, we will subsequently define norms functor between motivic homotopy theory, both unstably and stably, and then analysize its properties like distributivity laws. Next, Section 4 introduces to readers the other most important objects in their paper, normed spectra, and then shows the existence of several canonical constructions on them. In Section 5, we consider the comparison of stable motivic homotopy of one scheme and the equivariant stable homotopy theory of its étale groupoid, which enables us to recover Rost norm from our norm functor. Last but not the least, we verify the compatibility among norm functors, slice filtration, noncommutative motivic theory, and motivic Thom spectra, and show that many examples in motivic theory can be lifted to normed spectra.

We will freely use the language of ∞-categores. So this text assumes in particular that the reader is familiar with the basics of ∞-category theory, escpecially limits and colimits, Kan extensions, cartesian and cocartesian fiberations, presentable ∞-categories, sheaves, and $\mathbb{E}_\infty$-rings. In particular, we will denote by $\mathcal{P}(-)$ the presheaves $\mathrm{Fun}(-,\mathcal{S})$ and by $\mathcal{P}_\Sigma(-)$ the nonabelian derived category consisting of presheaves sending coproducts to products. What's more, throughout our notes, we always talk about motivic homotopy theory. So I hope the readers have an overall grasp of the construction process of $\mathcal{SH}(-)$, its six-functor formalism, and motivic Poincare duality (i.e., ambidexterity theorem and purity theorem).

All schemes considered are qcqs. We note however that essentially all results generalize to arbitrary schemes: often the same proofs work, and sometimes one needs a routine Zariski descent argument to reduce to the qcqs case. It seemed far too tedious to systematically include this generality, as additional (albeit trivial) arguments would be required in many places.

2. Preliminaries on Algebraic Geometry

2.1. **Weil restriction functor.** One of the most important tools used to construct our norms in motivic homotopy is Weil restriction. It is easy to define it from the perspective of functors of points, whose representability, finiteness, and smoothness seem quite tedious though. For the sake of completeness, I will prove each relevant property in detail.

Definition 2.1. Given a morphism of schemes $p : T \to S$, there is a pushforward functor $p_* : \mathcal{P}(\mathrm{Sch}_T) \to \mathcal{P}(\mathrm{Sch}_S)$ defined by

$$p_*X(Y) = X(Y \times_S T),$$

where Y is an S-scheme. When X is a scheme over T and the image p_*X is representable in $\mathcal{P}(\mathrm{Sch}_S)$, we call that scheme the Weil restriction of X along p, denoted by R_pX.

Example 2.2. In the case of $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$, the Weil restriction of $\mathbb{A}^n_{\mathbb{C}}$ exists and is exactly $\mathbb{A}^{2n}_{\mathbb{R}}$. However, not all of the schemes admit such a restriction. For example, still in the case, we consider the restriction of line with two origins. For more details, please see mathoverflow.

Fortunately, we have this theorem guaranteeing the existence of Weil restriction for most of the nice T-schemes.

Theorem 2.3. *Let $p : T \to S$ be a finite and locally free morphism between schemes and X be a T-scheme. Assume that for every point $s \in S$ and every finite set of points in the fiber X_s, there is an affine open subscheme U of X containing these points (e.g., quasi-projective T-schemes). Then the Weil restriction of X is a well-defined scheme.*

In order to prove this theorem, first let's study the behavior of open or closed immersions under p_*.

Lemma 2.4. *Let $p : T \to S$ be a morphism between schemes, and $u : F \to G$ be a morphism between set-valued presheaves on T.*

(1) *If u is an open immersion and p is proper, then p_*u is also an open immersion.*
(2) *If u is a closed immersion and p is proper, flat, and of finite presentation, then p_*u is also a closed immersion.*

Proof. Only take (1) as an example. For an S-scheme W and a section $W \to p_*G$, set W' as the pullback of W to T. Unwinding the adjunction property of p_*, I claim that we have a natural map $W' \to G$ and $W \to p_*G$ can be factored as the composition $W \to p_*W' \to p_*G$. Hence, we could build a commutative diagram

$$\begin{tikzcd}
F_{W'} \arrow[r] & W' \arrow[d] \\
(p_*F)_W \arrow[r] \arrow[d] & W \arrow[d] \\
(p_*F)_{p_*W'} \arrow[r] \arrow[d] & p_*W' \arrow[d] \\
p_*F \arrow[r] & p_*G.
\end{tikzcd}$$

Here, $F_{W'}$ and other objects mean the corresponding base change. We only need to show that the middle map $(p_*F)_W \to W$ is open, while the upper arrow $F_{W'} \to W'$ is already an open immersion due to the property of u. Denote the image of $F_{W'} \to W'$ as U' and its complement as V'. Since p is proper, its base change $W' \to W$ is also proper. Hence the image of V', denoted by V, is a closed subscheme in W. Take the complement U, and then I claim that $(p_*F)_W \to W$ is isomorphic to $U \to W$. Spelling out the middle pullback square, we find that

$$(p_*F)_W \cong \mathrm{Hom}_T(\cdot \times_S T, U') \times_{\mathrm{Hom}_T(\cdot \times_S T, W')} \mathrm{Hom}_S(\cdot, W),$$

while the right-hand side is canonically isomorphic to U as we desire. □

Proof of Theorem 2.3. First, assume we could do so in the affine case. Let me show you how to glue them together and get the Weil restriction. The gluing data of X are preserved by the functor p_*. So we construct an S-scheme Y over p_*X which is also an open immersion by the lemma above. The only thing remaining is to show that they are isomorphic. Note that the universal property guarantees that p_*X is also a Zariski sheaf. We can thus check it Zariski locally. For any map $a : W \to p_*X$ where W is an arbitrary S-scheme, we want to show that a will uniquely factor through Y in a neighborhood of each point $z \in W$. Obviously, a gives us a morphism between schemes $W \times_S T \to X$. The finiteness condition of p implies that the fiber of z consists of finite points and so does its image in X. Due to the assumption of X, we could find such an affine open subscheme U containing these points. Then Zariski locally, a will uniquely factor through p_*U, one open subscheme of Y.

Next, as for the affine case, local freeness implies that we can additionally assume that T is induced by a free module with generators $e_1, \cdots, e_n$. Because every affine scheme is a closed subscheme of affine spaces (possibly generated by infinite indeterminates), the second part of the lemma above enables us to reduce the problem to $\mathbb{A}^I_T$. At this time, the Weil restriction is actually $\mathbb{A}^{n\times I}_S$. □

Several properties are necessary in the construction of norm functors. The first one is the direct corollary in the proof of Theorem 2.3.

Corollary 2.5. *For $p : T \to S$ a finite and locally free morphism and a smooth T-scheme X, R_pX is always smooth over S if existing.*

Proof. It's formally smooth by trivial diagram chasing and locally of finite presentation as shown above. □

Proposition 2.6. *For $p : T \to S$ a finite and locally free morphism and a quasi-projective T-scheme X, $R_p X$ exists and is also quasi-projective.*

Proof. $R_p X$ is of finite type because of the explicit construction. So it suffices to find one relatively ample line bundle of $R_p X$. Assume that S is affine. Take the ample line bundle $\mathcal{L}$ over X, pull it back to $R_p X \times_S T$, and then consider the norm bundle along p. Those global sections generating $\mathcal{L}$ and of affine nonvanishing locus will induce analogous sections of the norm bundle. Hence, the norm bundle is ample. The detailed analysis about these loci could be found in [CGP15]. □

Remark 2.7. All the proofs above don't heavily rely on the Zariski site. One can develop an analogous functor in the category of algebraic spaces and prove the corresponding representability theorem. See [Ryd11].

3. Norms in Motivic Homotopy

In this section, we will construct norms functors in several contexts and show the homotopy coherence laws. Our strategy is to decompose it via three steps, since stable motivic homotopy theory is defined from nonabelian derived categories from Nisnevich localization, $\mathbb{A}^1$-localization, and stabilization. At last, after restricting to the simplest pointed schemes, we find that norm functor is derived from $*$-pushforward, but they are unlikely to be the same.

Among all coherence laws, it is distributivity laws that are the most abstract ones. But distributivity plays a role in decomposing complicated objects into tensor product of simpler objects, leading to some nice properties in the sense of Goodwillie calculus.

3.1. **Norms of pointed motivic spaces.** We begin by a useful lemma. In the sequel we will say that a morphism $X \to Y$ in $\mathcal{P}(\mathrm{Sm}_S)$ is relatively representable if for every $U \to Y$ where U belongs to Sm_S, the pullback $U \times_Y X$ is representable.

Moreover, for $p : T \to S$ between schemes, $X \in \mathcal{P}_\Sigma(\mathrm{Sm}_T)$, and $Y_1, \cdots, Y_k \to X$ a series of relatively representable morphisms, define a subpresheaf of $p_*(X)$

$$p_*(X|Y_1, \cdots, Y_k) := (U \mapsto \{s : U \times_S T \to X \mid s^{-1}(Y_i) \to U \text{ is surjective for all } i\}).$$

Lemma 3.1. *Additionally assume that p is universally closed and open, then every coproduct decomposition $X \simeq X' \sqcup X''$ in $\mathcal{P}_\Sigma(\mathrm{Sm}_S)$ induces a decomposition*

$$p_*(X|Y_1, \cdots, Y_k) \simeq p_*(X'|Y_1', \cdots, Y_k') \sqcup p_*(X|X'', Y_1, \cdots, Y_k)$$

in which Y_i' means $Y_i \times_X X'$.

Proof. By definition, we have the following pullback diagram

$$\begin{tikzcd} p_*(X'|Y_1', \cdots, Y_k') \sqcup p_*(X|X'', Y_1, \cdots, Y_k) \arrow[r] \arrow[d] & p_*(X|Y_1, \cdots, Y_k) \arrow[d] \\ p_*(X') \sqcup p_*(X|X'') \arrow[r, "\phi"] & p_*(X). \end{tikzcd}$$

The horizontal morphisms are given by the inclusion of subpresheaves, and the verticle ones are also inclusions. Hence, we may assume $k = 0$ without any hurt.

On one hand, $p_*(X') \times_{p_*(X)} p_*(X|X'')$ is exactly the initial presheaf by obvious reason. It follows from the universality of colimits that ϕ is a monomorphism. On the other hand, given $U \in \mathrm{Sm}_S$ and $s : U \times_S T \to X$, consider $U' := \{x \in U \mid p_U^{-1}(x) \subset s^{-1}(X')\}$ and its complement U''. So $U'' = p_U(s^{-1}(X''))$. It is a closed and open subscheme of U since p_U is a base change of p. Thus we have a coproduct decomposition $U \simeq U' \sqcup U''$ in which the restriction of s to U' belongs to $p_*(X')(U')$

and the restriction to U'' is actually in $p_*(X|X'')(U'')$. These two sections could be extended to U by zero and combined together, implying that ϕ is an objectwise effective epimorphism. $\square$

As a result of this lemma, we can now construct the norm functor and analyze its properties. From then on, for an object X in $\mathcal{P}(\mathrm{Sm}_S)$, we shall denote $X \sqcup S$ by X_+ which is also a pointed object. Besides, we define Sm_{S+} as the full subcategory of $\mathrm{Sm}_{S\bullet}$ spanned by X_+ where X belongs to Sm_S. Here, it is obvious that $\mathcal{P}(\mathrm{Sm}_{S+}) \cong \mathcal{P}(\mathrm{Sm}_S)_\bullet$ and other similar isomorphisms hold.

Theorem 3.2. *There is a unique symmetric monoidal functor, for every $p : T \to S$ universally closed and open morphism,*

$$p_\otimes : \mathcal{P}_\Sigma(\mathrm{Sm}_T)_\bullet \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_\bullet,$$

satisfying that:

(1) *$p_\otimes$ preserves sifted colimit;*
(2) *after being restricted to $\mathcal{P}_\Sigma(\mathrm{Sm}_T)$ via $(-)_+$, we have a symmetric monoidal natural equivalence $p_\otimes(X_+) \cong p_*(X)_+$;*
(3) *in terms of a morphism $f : Y_+ \to X_+$, $p_\otimes(f)$ is equivalent to the composition*

$$p_*(Y)_+ \to p_*(f^{-1}(X))_+ \xrightarrow{f} p_*(X)_+$$

in which the first map collapses $p_*(Y|Y\backslash f^{-1}(X))$ by lemma 3.1;
(4) *When p is integral, $p_\otimes$ will preserve Nisnevich and even motivic equivalences and then induce a functor*

$$p_\otimes : \mathcal{H}_\bullet(T) \to \mathcal{H}_\bullet(S);$$

(5) *For those morphisms p that are universal homeomorphisms, $p_\otimes \cong p_*$.*

Proof. In order to define such a functor, we only need to determine what it is in the subcategory Sm_{T+} by virtue of the universal property of nonabelian derived categories. In this case, we choose the functor just like what the second and third points display. Hence, for general presheaf, (3) is also true because all the operations can be decomposed into sifted colimits of representable objects.

Similarly, to verify (4), it suffices, when p is integral, to illustrate the descent property for p_*. That's what [BH20, Proposition 2.11] implies. Last but not least, for (5), we have to prove that $p_\otimes(f) \simeq p_*(f)$ for $f : Y_+ \to X_+$ as in (3). Now since p is a universal homeomorphism, p_* preserves finite coproduct. So the composition in (3) is exactly $p_*(f)$. $\square$

Example 3.3. When $T = S^{\sqcup n}$, $p_\otimes$ is the n-fold smash product because one can use (3) in theorem 3.2 to verify it for representables.

Now we can deal with some nontrivial calculations about $p_\otimes$. First of all, we will prove:

Proposition 3.4. *Let $p : T \to S$ be a finite étale morphism. For $X \in \mathrm{Sm}_T$ such that R_pX exists and a closed subscheme $Z \subset X$,*

$$p_\otimes\left(\frac{X}{X\backslash Z}\right) \simeq \frac{R_pX}{R_pX\backslash R_pZ}$$

as a pointed Nisnevich sheaf.

Remark 3.5. The condition that f is étale is quite crucial here. For instance, if we take $p : S[\epsilon]/\epsilon^2 \to S$, $R_p(X[\epsilon]/\epsilon^2)$ for $X \in \mathrm{Sm}_S$ is isomorphic to the relative tangent bundle of X. Then under mild conditions, Proposition 3.4 can be translated into that the relative tangent bundle $T_{X\backslash Z/S}$ is isomorphic to the complement $T_{X/S}\backslash T_{Z/S}$. In most cases, this is absolutely wrong.

Actually, we can do better in the calculation of norms of quotients.

Lemma 3.6. *Let $p : T \to S$ be an integral universally open morphism. Then every presheaf $X \in \mathcal{P}_\Sigma(\mathrm{Sm}_T)$ and one of its open subsheaves Y, we have*

$$p_\otimes(X/Y) \simeq p_*(X)/p_*(X|Y)$$

in the category of pointed Nienevich sheaves.

Proof of Proposition 3.4 assuming Lemma 3.6. By Corollary 2.5, R_pX is smooth over S and so is $R_pX\backslash R_pZ$ as an open subscheme. Note that R_pX is just another notation of p_*X. So by Lemma 3.6, it suffices to show that $p_*(X|X\backslash Z) \simeq R_pX\backslash R_pZ$.

Unwinding definition, the former one is contained in the latter one. Now we will show the opposite direction. Given a morphism $s : U \to R_pX$, it factors through $R_pX\backslash R_pZ$ if and only if its base change $U \times_S T \to X$ doesn't send some fiber over $u \in U$ into Z. According to reducedness, this condition is equivalent to that each fiber interacts with $X\backslash Z$, i.e., that s factors through $p_*(X|X\backslash Z)$. □

Proof of Lemma 3.6. We first define such a subpresheaf of $p_*(X)$ for $X \in \mathcal{P}(\mathrm{Sm}_T)$ and a subpresheaf $Y \subset X$:

$$p_*(X||Y) := (U \mapsto \{s : U \times_S T \to X \mid s \text{ sends a clopen subset that covers } U \text{ to } Y\}).$$

Now we will show that:

(1) $p_\otimes(X/Y) \simeq p_*(X)/p_*(X||Y)$ in $\mathcal{P}_\Sigma(\mathrm{Sm}_S)_\bullet$. (We just need the universal closed property of p instead of the integrality.)
(2) $p_*(X||Y) \hookrightarrow p_*(X|Y)$ is a Nisnevich equivalence.

For (1), Bar constructions realizes $p_\otimes(X/Y)$ as the sifted colimit

$$\cdots \rightrightarrows p_*(X \sqcup Y)_+ \rightrightarrows p_*(X)_+$$

in which higher terms can be, by Lemma 3.1, identified with

$$p_*(X) \sqcup p_*(X \sqcup Y|Y) \sqcup p_*(X \sqcup Y \sqcup Y|Y) \sqcup \cdots$$

and is sent to $p_*(X)$ by sending these $p_*(X \sqcup Y^{\sqcup i}|Y)$ to $p_*(X||Y)$. Hence, after checking compatibility, one can obtain a natural map $p_\otimes(X/Y) \to p_*(X)/p_*(X||Y)$. Reduce it to the case of representables, then we can assume that $X \in \mathrm{Sm}_T$. At this time, it follows from Theorem 3.2(1)(2) that $p_\otimes(X/Y)$ is also 0-truncated. So the geometric realization diagram above can be cut into the first two terms. By inspection, that is the quotient $p_*(X)/p_*(X||Y)$.

For (2), it suffices to show that the sections over a henselian local scheme U of every smooth S-scheme are equivalent to each other. Simultaneously, by a colimit process, we may assume that p is finite. So $U \times_S T$ splits into several local schemes. Then any open subset of $U \times_S T$ covering U contains a clopen subset covering U. It means that $p_*(X||Y)(U) \to p_*(X|Y)(U)$ is an effective epimorphism and thus an equivalence. □

3.2. **Norms of motivic spectra.** For every vector bundle V over a scheme S, we define S^V in $\mathcal{H}_\bullet(S)$ as $V/(V\backslash 0)$ and write $\Sigma^V = \mathrm{S}^V \wedge (-)$ and $\Omega^V = \mathrm{Hom}(\mathrm{S}^V, -)$. Note that $\mathrm{S}^{\mathbb{A}^1} \simeq \mathbb{P}^1$, an tensor invertible object in $\mathcal{SH}(S)$. Actually, in $\mathcal{SH}(S)$, we have:

Proposition 3.7 ([CD19, Corollary 2.4.19])**.** *The objects S^V for vector bundles become invertible in $\mathcal{SH}(S)$ as well.*

As a result, in an attempt to extend our norm construction to $\mathcal{SH}(S)$, we need to calculate the image of S^V.

Proposition 3.8. *Let $p : T \to S$ be a finite étale morphism and V be a vector bundle over T. Then the Weil restriction R_pV is also a vector bundle, and $p_\otimes(\mathrm{S}^V) \simeq \mathrm{S}^{R_pV}$ in $\mathcal{H}_\bullet(S)$. In particular, according to the universal property of motivic spectra, $p_\otimes$ can be uniquely extended to $\mathcal{SH}(S)$, preserving sifted colimits.*

Proof. Due to the finite étale property, the stalk of p_*V is the direct sum of the stalks at each point in the fiber, which implies that it has the structure of a vector bundle. So using Lemma 3.4, we have a series of equivalences

$$p_\otimes(\mathrm{S}^V) = p_\otimes\left(\frac{V}{V\backslash 0}\right) \simeq \frac{R_pV}{R_pV\backslash R_p0} \simeq \frac{R_pV}{R_pV\backslash 0} = \mathrm{S}^{R_pV}.$$

The extension follows from the invertibility of S^{R_pV}. □

Remark 3.9. One simple corollary of this proposition is that we have the following commutative diagram connecting $K(\mathrm{Vect}(-))$ and $\mathrm{Pic}(\mathcal{SH}(-))$

$$\begin{tikzcd} K(\mathrm{Vect}(T)) \arrow[r, "\mathrm{S}^{(-)}"] \arrow[d, "p_*"'] & \mathrm{Pic}(\mathcal{SH}(T)) \arrow[d, "p_\otimes"] \\ K(\mathrm{Vect}(S)) \arrow[r, "\mathrm{S}^{(-)}"] & \mathrm{Pic}(\mathcal{SH}(S)) \end{tikzcd}$$

where $p : T \to S$ is finite and étale. We will enhance this diagram by substituting $\mathrm{Vect}(-)$ with the more general categories $\mathrm{Perf}(-)$ in the sequel.

Example 3.10. The restrictions of the morphism p are crucial. Take S the affine spectrum of a discrete valuation ring whose residue field isn't of characteristic 2. Let $p : T \to S$ be a finite *locally free* morphism with generic fiber étale of degree $n > 1$ and an isomorphic special fiber. We claim that $E := \Sigma^\infty p_\otimes(\mathbb{P}^1)$ isn't invertible any more. Let $i : \{x\} \to S$ denote the closed point of S and $j : \{\eta\} \to S$ denote the generic point. By base change theorem we have $j^*E \simeq \mathrm{S}^{\mathbb{A}^n}$ and $i^*E \simeq \mathrm{S}^{\mathbb{A}^1}$. Hence, consider the zero-extension fiber sequence

$$j_!j^*E \to E \to i_*i^*E \xrightarrow{\alpha} j_!j^*E[1].$$

We only need to show that $\alpha \simeq 0$ since if E were invertible, such a nullhomotopy would imply that $[E, E] \simeq [\mathbf{1}_S, \mathbf{1}_S] \simeq \mathrm{GW}(S)$ (Theorem 5.9) must have idempotents, contrary to the results of [Gil19, Theorem 2.4] and [KK82, Proposition II.2.22]. Use the equivalences above, and then it suffices to show that $[\mathrm{S}^{\mathbb{A}^1}, i^!j_!\,\mathrm{S}^{\mathbb{A}^n}[1]] \simeq 0$, which is a direct corollary of Morel's $\mathbb{A}^1$-connectivity theorem.

3.3. **Homotopy coherence.** Now we want to construct a complete functor to encode all of the huge data of norms and pullbacks, instead of a single norm functor. More precisely speaking, it would be much better if there's a functor

$$\mathcal{SH}^\otimes : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{f\acute{e}t}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

sending S to $\mathcal{SH}(S)$ and $U \xleftarrow{f} T \xrightarrow{p} S$ where p is finite and étale to $p_\otimes f_*$. Here, the notation Span denotes a (2,1)-category whose 2-morphisms are isomorphisms of spans. Once such a functor is constructed, we can formally obtain different kinds of coherence equivalences including composition laws as proved in [BH20, Section 5]. In addition, some properties of norm functors like polynomial and excisive properties can be resolved via a systematical dévissage procedure. I encourage readers unfamiliar with it to refer to Peter Scholze's [Sch22]. We will also prove some relevant propositions in certain generality.

It seems convenient to decompose the functor into the steps

$$\mathrm{SmQP}_{S+} \rightarrowtail \mathcal{P}_\Sigma(\mathrm{SmQP}_S)_\bullet \rightarrowtail \mathcal{H}_\bullet(S) \rightarrowtail \mathcal{SH}(S),$$

where the letters SmQP mean smooth and quasi-projective schemes. Note that every smooth S-scheme is Zariski locally covered by quasi-projective smooth ones and in this case, p_* is realized by Weil restriction. So the change we make here is reasonable and effective.

First and foremost, the functor from $\mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{fét})$ to $\mathrm{CAlg}(\mathrm{Cat}_1)$ sending S to SmQP_{S+} is easy to demonstrate by hand.

Next, since nonabelian derived categories determine a functor from symmetric monoidal ∞-categories to sifted-cocomplete symmetric monoidal ∞-categories, the inclusion of $\mathrm{Cat}_1 \hookrightarrow \mathrm{Cat}_\infty$ induces a functor

$$\mathcal{P}_\Sigma(\mathrm{SmQP})^\otimes_\bullet : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{fét}) \to \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{sift}}_\infty)$$

sending S to $\mathcal{P}_\Sigma(\mathrm{SmQP}_S)_\bullet$ with Day convolution.

Then we should coherently invert Nisnevich equivalences and $\mathbb{A}^1$-equivalences. This auxiliary ∞-category is necessary:

Informally, $\mathcal{M}\mathrm{Cat}_\infty$ denotes the ∞-category of ∞-categories with a collection of equivalence classes of morphisms which can defined as a categorical pullback

$$\begin{tikzcd}
\mathcal{M}\mathrm{Cat}_\infty \arrow[rrr] \arrow[d] & & & \mathrm{St}(\mathrm{Poset} \hookrightarrow \mathrm{Cat}_\infty) \arrow[d] \\
\mathrm{Cat}_\infty \arrow[r, "(-)^{\Delta^1}"] & \mathrm{Cat}_\infty \arrow[r, "\mathrm{Ob}"] & \mathrm{Set} \arrow[r, "2^{(-)}"] & \mathrm{Poset}.
\end{tikzcd}$$

Both f^* and $p_\otimes$ for p finite and étale preserve motivic equivalences, so our functor from $\mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{fét})$ can be lifted to $\mathrm{CAlg}(\mathcal{M}\mathrm{Cat}^{\mathrm{sift}}_\infty)$ naturally. Simultaneously, as a partial left adjoint functor, the localization functor from $\mathrm{CAlg}(\mathcal{M}\mathrm{Cat}^{\mathrm{sift}}_\infty)$ to $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{sift}}_\infty)$ is well-defined if the equivalences satisfy the condition of [Lur17a, Proposition 4.1.7.4]. Since fortunately, motivic equivalences satisfy corresponding conditions and the functoriality holds by the universal property, we extend the functor to $\mathcal{H}^\otimes_\bullet : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{fét}) \to \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{sift}}_\infty)$ sending a scheme S to $\mathcal{H}_\bullet(S)$. Until now all the properties of $p_\otimes$ only rely on the local freeness, rather than unramifiedness.

Last, an analogous process can help us invert each $\mathbb{P}^1$ coherently whose well-defined property follows from [Rob15] and Lemma 3.8. Now the étale property is indispensable.

Remark 3.11. In the first step, one can take advantage of Barwick's unfurling and functorialities of 2-categories of spans instead of direct verification. See [BH20, Subsection 6.1].

So far we have constructed an abstract and huge functor. Let me show you how to extract some seemingly nontrivial and obviously tedious equivalences readily from $\mathcal{SH}^\otimes$ and basic motivic theory. Since we will only deal with the cases of stable motivic homotopy, I will pay attention to finite étale morphisms for norms and $\mathcal{SH}$. If readers want to relieve conditions to get the generality of unstable cases, please see [BH20, Section 5] for formal statements or apply a functor slightly stronger than $\mathcal{H}^\otimes_\bullet$ we constructed to simple diagrams by themselves.

3.3.1. *Composition and base change.*

Proposition 3.12 (Composition)**.** *For $p : T \to S$ and $q : S \to R$ finite étale morphisms, $(pq)_\otimes \simeq p_\otimes q_\otimes$ as functors between stable motivic categories.*

Proof. It follows from the 2-simplex

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & S \arrow[r, "q"] \arrow[d] & R \\
T \arrow[r, "p"] \arrow[d] & S & \\
T & &
\end{tikzcd}$$

in which all unlabeled maps are identities. □

Proposition 3.13 (Base change)**.** *Consider a Cartesian square of schemes*

$$\begin{tikzcd} T' \arrow[r, "g"] \arrow[d, "q"'] & T \arrow[d, "p"] \\ S' \arrow[r, "f"] & S, \end{tikzcd}$$

where p is finite and étale, then we have a symmetric monoidal natural equivalence

$$\mathrm{Ex}^*_\otimes : f^* p_\otimes \to q_\otimes g^*$$

in $\mathcal{SH}$.

Proof. Still, consider the compatibility of composition. □

The transformation $\mathrm{Ex}^*_\otimes : f^* p_\otimes \to q_\otimes g^*$ induces by adjunction another natural transformation

$$\mathrm{Ex}_{\otimes *} : p_\otimes g_* \to f_* q_\otimes.$$

Besides, when f is smooth, Poincaré duality gives a left adjunction of f^* denoted by $f_\sharp$. Then the inverse of $\mathrm{Ex}^*_\otimes$ induces a natural transform

$$\mathrm{Ex}_{\sharp\otimes} : f_\sharp q_\otimes \to p_\otimes g_\sharp.$$

Of course, they are $\mathcal{SH}(T)$-linear because $\mathrm{Ex}^*_\otimes$ is symmetric monoidal.

3.3.2. *Distribuitivity.*

Proposition 3.14. *In the following diagram*

$$\begin{tikzcd} U & R_pU \times_S T \arrow[l, "e"'] \arrow[r, "q"] \arrow[d, "g"] & R_pU \arrow[d, "f"] \\ & T \arrow[from=1-1, "h"'] \arrow[r, "p"] & S, \end{tikzcd}$$

p is finite and étale, h is quasi-projective, e means the counit of adjunction, $f = R_p h$, and other maps are the canonical ones. Then we have the following equivalences in $\mathcal{SH}$:

1. *if h is also smooth,* $\mathrm{Dis}_{\sharp\otimes} : f_\sharp q_\otimes e^* \xrightarrow{\text{base change}} p_\otimes g_\sharp e^* \xrightarrow{\text{counit}} p_\otimes h_\sharp$*;*
2. *if h is also proper,* $\mathrm{Dis}_{\otimes *} : p_\otimes h_* \xrightarrow{\text{unit}} p_\otimes g_* e^* \xrightarrow{\text{base change}} f_* q_\otimes e^*$*.*

Proof. (1) is almost formal. First, we have an analogous equivalence in the category of quasi-projective schemes, i.e., the one obtained by substituting norms with pushforward ([BH20, Lemma 5.6]), whose proof relies on triangular identities. The natural morphisms restricted on representables in $\mathcal{SH}$ are still true, and (1) follows via a colimit process and projection formula for general objects.

For (2), h can be decomposed into the composition of a smooth projective morphism and a closed immersion Zariski locally by [Sta, Tag 087S]. The problem is Nisnevich local, so it remains to deal with the smooth case and the closed one by the compatibility of composition. Assume that h is a closed immersion so that h^* is essentially surjective. Then we only need to show that $\mathrm{Dis}_{\otimes *} h^*$ is an equivalence. Likewise, it suffices to consider unstable representable cases. For a smooth quasi-projective scheme X over T, using base change theorem and basic fact $h_* h^*(X_+) \simeq X/(X \backslash X_U)$, we should prove that

$$p_\otimes \left(\frac{X}{X \backslash X_U} \right) \simeq \frac{R_p X}{R_p X \backslash R_p(X_U)},$$

which follows from Proposition 3.4. When h is smooth, Poincaré duality and the corresponding compatibility reduce our problem to (1). □

3.3.3. *Compatibility with purity equivalences.*

Proposition 3.15. *Let $p : T \to S$ be a finite étale map, $h : X \to T$ be smooth, and $u : Z \to X$ be a closed immersion such that Z is smooth over T, forming a diagram*

$$\begin{tikzcd}
Z \arrow[dr, "u"'] & Z \times_X R_pX \times_S T \arrow[l, hook', "e'"'] \arrow[dr, hook, "u'"] & R_pZ \times_S T \arrow[l, "d"'] \arrow[r, "r"] \arrow[d, hook, "t"] & R_pZ \arrow[d, hook, "s"] \\
& X \arrow[dr, "h"'] & R_pX \times_S T \arrow[l, "e"'] \arrow[r, "q"] \arrow[d, "g"] & R_pX \arrow[d, "f"] \\
& & T \arrow[r, "p"] & S.
\end{tikzcd}$$

Then the purity equivalence in motivic stable categories $\Pi_s : f_\sharp s_* \simeq (fs)_\sharp \Sigma^{\mathrm{N}_s}$ and $\Pi_u : h_\sharp u_* \simeq (hu)_\sharp \Sigma^{\mathrm{N}_u}$, where each N means normal bundle, along with distributivity maps, satisfy the commutative diagram

$$\begin{tikzcd}
f_\sharp s_* r_\otimes d^* e'^* \arrow[d, "\Pi_s"'] & f_\sharp q_\otimes u'_* e'^* \arrow[l, "\mathrm{Dis}_{\otimes *}"'] & f_\sharp q_\otimes e^* u_* \arrow[l, "\text{base change}"'] \arrow[r, "\mathrm{Dis}_{\sharp\otimes}"] & p_\otimes h_\sharp u_* \arrow[d, "\Pi_u"] \\
(fs)_\sharp \Sigma^{\mathrm{N}_s} r_\otimes d^* e'^* \arrow[r, "\sim"] & (fs)_\sharp \Sigma^{r_* d^* e'^*(\mathrm{N}_u)} r_\otimes d^* e'^* \arrow[r, "\sim"] & (fs)_\sharp r_\otimes d^* e'^* \Sigma^{\mathrm{N}_u} \arrow[r, "\mathrm{Dis}_{\sharp\otimes}"] & p_\otimes (hu)_\sharp \Sigma^{\mathrm{N}_u}.
\end{tikzcd}$$

Proof. It suffices to check the case of representables. Note that by base change technique, we can even only deal with the unit. Then in unstable world, the diagram to be proved, by Proposition 3.4, is translated into the equivalence of these two zig-zag maps

$$\begin{tikzcd}
\frac{R_pX}{R_pX \setminus R_pZ} \arrow[r] \arrow[d, "\sim"'] & \frac{R_p \operatorname{Def}(X,Z)}{R_p \operatorname{Def}(X,Z) \setminus R_p(Z \times \mathbb{A}^1)} & \frac{\mathrm{N}_s}{\mathrm{N}_s \setminus R_pZ} \arrow[l] \arrow[d, "\sim"] \\
\frac{R_pX}{R_pX \setminus R_pZ} \arrow[r] & \frac{\operatorname{Def}(R_pX, R_pZ)}{\operatorname{Def}(R_pX, R_pZ) \setminus (R_pZ \times \mathbb{A}^1)} & \frac{\mathrm{N}_s}{\mathrm{N}_s \setminus R_pZ} \arrow[l]
\end{tikzcd}$$

where Def is Verdier's deformation spaces. The proposition thus follows from the natural commutativity map of Def and Weil restriction, which we leave to the readers. □

3.3.4. *Compatibility with ambidexterity equivalences.* Because in motivic stable homotopy theory, ambidexterity and purity theorems, when relevant morphism is both smooth and proper, are almost the same equivalence up to a twist, the compatibility of purity will deduce that of ambidexterity after (tedious) diagram chasing.

Proposition 3.16. *In the diagram*

$$\begin{tikzcd}
U \arrow[dr, "h"'] & R_pU \times_S T \arrow[l, "e"'] \arrow[r, "q"] \arrow[d, "g"] & R_pU \arrow[d, "f"] \\
& T \arrow[r, "p"] & S,
\end{tikzcd}$$

when p is finite étale and h is smooth quasi-projective, the ambidexterity morphism $\alpha_f : f_\sharp \to f_* \Sigma^{\mathrm{T}_f}$ and $\alpha_h : h_\sharp \to h_* \Sigma^{\mathrm{T}_h}$, together with distributivity, satisfy this commutative diagram

$$\begin{tikzcd}
f_\sharp q_\otimes e^* \arrow[r, "\alpha_f"] \arrow[d, "\mathrm{Dis}_{\sharp\otimes}"'] & f_* \Sigma^{\mathrm{T}_f} q_\otimes e^* \arrow[r, "\sim"] & f_* \Sigma^{q_* e^* \mathrm{T}_h} q_\otimes e^* \arrow[d, "\sim"] \\
p_\otimes h_\sharp \arrow[r, "\alpha_h"] & p_\otimes h_* \Sigma^{\mathrm{T}_h} \arrow[r, "\mathrm{Dis}_{\sharp *}"] & f_* q_\otimes e^* \Sigma^{\mathrm{T}_h},
\end{tikzcd}$$

where the letter T *means tangent bundle.*

Proof. By inspection of the HUGE diagrams and Proposition 3.15: first, relationships among schemes follow from

$$\begin{tikzcd}
U & \bullet \arrow[l, "e"'] \arrow[dr, "\theta'"] & \bullet \arrow[l, "d'"'] \arrow[dr, "\theta''"] & \bullet \arrow[l, "c"'] \arrow[r, "q"] \arrow[d, "\zeta"] \arrow[ll, bend right, "\mathrm{id}"'] & R_pU \arrow[d, "\delta"] \\
\arrow[from=1-1, to=2-2, "\theta"] & U \times_T U \arrow[dr, "\sigma_2"] & \bullet \arrow[l, "e'"'] \arrow[dr, "\sigma_2'"] & \bullet \arrow[l, "d"'] \arrow[r, "r"] \arrow[d, "\rho_2"] & R_pU \times_S R_pU \arrow[d, "\pi_2"] \\
 & & U \arrow[dr, "h"] & \bullet \arrow[l, "e"'] \arrow[r, "q"] \arrow[d, "g"] & R_pU \arrow[d, "f"] \\
 & & & T \arrow[r, "p"] & S,
\end{tikzcd}$$

where each black point means pullbacks and $d'c \simeq \mathrm{id}$ is the cause of triangular identity. Then the diagram we want to show can be extended as

$$\begin{tikzcd}[column sep=tiny]
 & & p_\otimes h_* h^* h_\sharp \arrow[dl] \arrow[d] & p_\otimes h_\sharp \arrow[l] \arrow[dl] & p_\otimes g_\sharp e^* \arrow[l] \arrow[dl] \arrow[d] & f_\sharp q_\otimes e^* \arrow[l] \arrow[d] \\
 & p_\otimes h_* \sigma_{2\sharp}\sigma_1^* \arrow[d] & p_\otimes g_* g^* h_\sharp \arrow[dl] & p_\otimes g_* g^* g_\sharp e^* \arrow[l] \arrow[dl] \arrow[d] & f_* f^* p_\otimes g_\sharp e^* \arrow[dl] & f_* f^* p_\otimes g_\sharp e^* \arrow[l] \arrow[d] \\
 & p_\otimes g_* e^* \sigma_{2\sharp}\sigma_1^* \arrow[d] & p_\otimes g_* \rho_{2\sharp}\rho_1^* e^* \arrow[l] \arrow[d] & f_* q_\otimes g^* g_\sharp e^* \arrow[dl] & & f_* \pi_{2\sharp}\pi_1^* q_\otimes e^* \arrow[d] \\
 & f_* q_\otimes e^* \sigma_{2\sharp}\sigma_1^* & f_* q_\otimes \rho_{2\sharp}\rho_1^* e^* \arrow[l] & f_* \pi_{2\sharp} r_\otimes \rho_1^* e^* \arrow[l] \arrow[dl] \arrow[dr] \arrow[ddr] \arrow[ddd] & & f_* \pi_{2\sharp} r_\otimes \rho_1^* e^* \arrow[ll] \arrow[d] \\
 & & f_* q_\otimes \rho_{2\sharp}\rho_1^* e^* \arrow[dl, "\sim"'] \arrow[d] & & f_* \pi_{2\sharp}\delta_*\delta^* r_\otimes \rho_1^* e^* \arrow[ddr] & f_* \pi_{2\sharp}\delta_* q_\otimes e^* \arrow[l] \arrow[dd, "\mathrm{id}"] \\
 & f_* q_\otimes \rho_{2\sharp} d^* e'^* \sigma_1^* \arrow[dl] \arrow[ddl] \arrow[d] & f_* q_\otimes \rho_{2\sharp}\theta''_* d'^* e^* & & f_* \pi_{2\sharp} r_\otimes \zeta_* e^* \arrow[dr] & \\
f_* q_\otimes \sigma_{2\sharp}\sigma_1^* \arrow[ddd, bend right] & f_* q_\otimes \rho_{2\sharp} d^* e'^* \theta_* \arrow[ur] \arrow[dd] & & f_* \pi_{2\sharp} r_\otimes \theta''_* d'^* e^* \arrow[ul] \arrow[ur] \arrow[rr, "\mathrm{Dis}_{\otimes *}"] & & f_* \pi_{2\sharp}\delta_* q_\otimes e^* \arrow[dl, "\Pi_{\pi_2}"] \\
f_* q_\otimes \sigma'_{2\sharp} e'^* \sigma_1^* \arrow[u] \arrow[dr] & & f_* \pi_{2\sharp} r_\otimes d^* e'^* \theta_* \arrow[ul] \arrow[ur] \arrow[dl, "\mathrm{Dis}_{\sharp\otimes}"'] & & f_* \Sigma^{\mathrm{T}_f} q_\otimes e^* \arrow[dl] & \\
 & f_* q_\otimes \sigma'_{2\sharp} e'^* \theta_* \arrow[dl] & & f_* \Sigma^{q_* e^* \mathrm{T}_h} q_\otimes e^* \arrow[dl] & & \\
f_* q_\otimes e^* \sigma_{2\sharp}\theta_* \arrow[rr, "\Pi_{\sigma_2}"] & & f_* q_\otimes e^* \Sigma^{\mathrm{T}_h}. & & &
\end{tikzcd}$$

Here, unlabeled maps are units, counits, and base changes. □

3.4. **Polynomial functors.** As a consequence of distributivity, we have the polynomial property of norms.

Proposition 3.17. *Let $p : T \to S$ be finite étale of degree $\leq n$. The functor $p_\otimes : \mathcal{SH}(T) \to \mathcal{SH}(S)$ is polynomial of degree $\leq n$.*

Proof. Using distributivity Proposition 3.14 for $U = T \sqcup T$, we know that for every two motivic spectra E and F in $\mathcal{SH}(T)$, $p_\otimes(E \oplus F) \simeq p_\otimes(E) \oplus p_\otimes(F) \oplus c_\sharp(q_{l\otimes} e_l^*(E) \otimes q_{r\otimes} e_r^*(F))$, in which we set $R_p(T \sqcup T) = S \sqcup C \sqcup S$, $C_T = L \sqcup R \xrightarrow{(e_l, e_r)} T \sqcup T$, $q_l : L \to C$, and $q_r : R \to C$. Note that we in fact also use the calculation of $\sharp$-pushforward for fold-maps. Then the property follows from inducti on since both q_l and q_r are of degree $\leq n - 1$. □

4. Normed Motivic Spectra

Given what we have done above, it is natural to generalize the notion of normed categories beyond $\mathcal{SH}^{\otimes}$ as follows.

Definition 4.1. Let S be a scheme. If $\mathcal{C}$ is a subcategory Sch_S that contains S and is closed under finite coproducts and finite étale extension, then we say a normed ∞-category over $\mathcal{C}$ is a functor

$$\mathcal{A} : \mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{fét}) \to \mathrm{Cat}_{\infty}, \quad (X \xleftarrow{f} Y \xrightarrow{p} Z) \mapsto p_{\otimes} f^*,$$

preserving finite products. We say $\mathcal{A}$ is presentably normed if:

(1) $\mathcal{A}(X)$ is always presentable for every object X;
(2) for every morphism f, f^* preserves all colimits;
(3) for every finite étale morphism h, h^* admits a left adjoint $h_{\sharp}$, and $p_{\otimes}$ preserves sifted colimits;
(4) for every Cartesian diagram

$$\begin{tikzcd} T' \arrow[r, "g"] \arrow[d, "q"'] & T \arrow[d, "p"] \\ S' \arrow[r, "f"] & S, \end{tikzcd}$$

with p finite étale, we have base change equivalences $q_{\sharp} g^* \simeq f^* p_{\sharp}$;
(5) for every diagram

$$\begin{tikzcd} U \arrow[drr, "h"'] & & R_pU \times_S T \arrow[ll, "e"'] \arrow[r, "q"] \arrow[d, "g"] & R_pU \arrow[d, "f"] \\ & & T \arrow[r, "p"] & S, \end{tikzcd}$$

with p and h finite étale, we have distributivity equivalences $\mathrm{Dis}_{\sharp\otimes} : f_{\sharp} q_{\otimes} e^* \simeq p_{\otimes} h_{\sharp}$.

Similarly, one can define the notion of nonunital normed ∞-categories by restricting to finite étale *surjective* maps.

Example 4.2. In Subsection 3.2, the functor $\mathcal{SH}^{\otimes}$ we constructed exhibits it as a normed ∞-category over Sch or $\mathrm{Sch}_{\mathrm{Spec}\,\mathbb{Z}}$ if one prefers.

As has been explained, $\mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{fét})$ is supposed to be referred as Fin_*, and the normed category $\mathcal{SH}^{\otimes}$ is thus a counterpart of the operad $\mathrm{Sp}^{\otimes}$. Hence, the following definition of normed spectra absolutely generalizes the basic notion in stable homotopy theory, $\mathbb{E}_{\infty}$-ring spectra.

Definition 4.3 (Normed spectra)**.** Consider that S is a scheme, and $\mathcal{C}$ is a subcategory Sch_S that contains S and is closed under finite coproducts and finite étale extension. A normed spectrum (respectively, an incoherent normed spectrum) over $\mathcal{C}$ is a section of $\mathcal{SH}^{\otimes}$ (respectively, $\mathrm{h}\mathcal{SH}^{\otimes}$), regarded as an ∞-category after straightening, over $\mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{fét})$, which is cocartesian over $\mathcal{C}^{\mathrm{op}} \subset \mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{fét})$. All of the normed spectra form a new ∞-category, denoted by $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$.

When $\mathcal{C}$ is the common category in algebraic geometry like Sm_S, Sch_S, and $\mathrm{FÉt}_S$, we prefer the notation $\mathrm{NAlg}_{\mathrm{Sm}}(\mathcal{SH}(S))$, for instance, instead.

Example 4.4. Sphere spectrum $\mathbf{1}_S$ is normed over Sch_S because norm functors and $*$-pullback functors are symmetric monoidal. In the sequel, we will show the normed structure of Voevodsky's Eilenberg-MacLane spectrum $\mathrm{H}\mathbb{Z}_S$ (see Corollary 6.3), the homotopy K-theory spectrum KGL_S (see Corollary 6.8), and the algebraic cobordism spectrum MGL_S (see Theorem 6.16).

I will study some basic properties of normed motivic spectra in the next subsections, especially the norm-pullback-pushforward adjuntions among $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$. Simultaneously, we will also consider localization and completion theory of normed motivic spectra. Many of them will not be directly used when proving our main theorems, so it's harmless for readers to omit the proof here.

4.1. **First property of normed spectra.** Unwinding definition, a normed spectrum (resp. an incoherent normed spectrum) over $\mathcal{C}$ consists of the following data with (or respectively, without) higher coherence compatibility:

- a series of objects $E_V \in \mathcal{SH}(V)$ for $V \in \mathcal{C}$;
- certain equivalences $f^* E_Y \simeq E_X$ for $f : X \to Y \in \mathcal{C}$;
- norm maps $\mu_p : p_\otimes E_V \to E_U$ where $p : V \to U$ is finite and étale in $\mathcal{C}$;
- for $q : W \to V$ and $p : V \to U$ finite and étale maps in $\mathcal{C}$, a square

$$\begin{tikzcd} p_\otimes q_\otimes E_W \arrow[r, "p_\otimes \mu_q"] \arrow[d, "\sim"'] & p_\otimes E_V \arrow[d, "\mu_p"] \\ (pq)_\otimes E_W \arrow[r, "\mu_{pq}"] & E_U; \end{tikzcd}$$

- for a cartesian diagram in $\mathcal{C}$

$$\begin{tikzcd} V' \arrow[r, "g"] \arrow[d, "q"'] & V \arrow[d, "p"] \\ U' \arrow[r, "f"] & U, \end{tikzcd}$$

  a pentagon

$$\begin{tikzcd} f^* p_\otimes E_V \arrow[r, "f^*\mu_p"] \arrow[d, "\sim"'] & f^* E_U \arrow[dd, "\sim"] \\ q_\otimes g^* E_V \arrow[d, "\sim"'] & \\ q_\otimes E_{V'} \arrow[r, "\mu_q"] & E_{U'}. \end{tikzcd}$$

Hence, when $V \to U$ is a finite étale Galois morphism, the fourth condition induces an equivariant action of $\mathrm{Aut}(V/U)$ (up to homotopy when incoherence), and thus a natural morphism

$$(p_\otimes E_V)_{h\,\mathrm{Aut}(V/U)} \to E_U.$$

Remark 4.5. Finite coproducts help us construct a restriction from $\mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{fét})$ to $\mathrm{Span}\,(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \simeq \mathrm{Fin}_*$, exhibiting the commutative algebra structure of each E_V. That's why we say normed spectra are the generalization of $\mathbb{E}_\infty$-rings. The cocartesian property of normed spectra suggests that if we replace finite étale morphisms in the span category by fold maps, the corresponding structure is the same as an object in $\mathrm{CAlg}(\mathcal{SH}(S))$. See [BH20, Corollary C.8].

Remark 4.6. Given a finite étale map $p : T \to S$, and a normed spectrum E over some subcategory of Sch_S, for $A \in \mathcal{SH}(T)$, we have a transfer map

$$\nu_p : \mathrm{Map}(A, E_T) \xrightarrow{p_\otimes} \mathrm{Map}(p_\otimes A, p_\otimes E_T) \xrightarrow{\mu_p} \mathrm{Map}(p_\otimes A, E_S),$$

which is readily seen to be a multiplicative $\mathbb{E}_\infty$-maps of $\mathbb{E}_\infty$-monoids.

Theorem 4.7 (Categorical features of normed spectra)**.** *Consider that S is a scheme, and $\mathcal{C}$ is a subcategory Sch_S that contains S and is closed under finite coproducts and finite étale extension.*

1. *$\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$ has colimits and finite limits, and is also presentable if $\mathcal{C}$ is small.*
2. *The forgetful functor $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}) \to \mathcal{SH}(S)$ is conservative and preserves sifted colimits and finite limits. If $\mathcal{C}$ is contained in Sm_S, then it preserves limits and hence monadic.*
3. *The forgetful functor $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}) \to \mathrm{CAlg}(\mathcal{SH}(S))$ is conservative and preserves colimits and finite limits. Still, if $\mathcal{C}$ is contained in Sm_S, then it preserves limits and hence both monadic and comonadic.*

(4) *For $A \in \mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$, the symmetric monoidal categories $\mathrm{Mod}_{A(-)}(\mathcal{SH}(-))$ can be promoted to a normed ∞-category over $\mathcal{C}$, denoted by $\mathrm{Mod}_A(\mathcal{SH})^{\otimes}$.*

(5) *If $\mathcal{C}_0$ is a subcategory of $\mathcal{C}$ satisfying the same closed property, whose value of $\mathcal{SH}$ determines the whole functor $\mathcal{SH} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ via right Kan extension. Then the natural functor $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}) \to \mathrm{NAlg}_{\mathcal{C}_0}(\mathcal{SH})$ is an equivalence.*

(6) *If all objects in $\mathcal{C}$ are finitely presented over S, and $\mathcal{C}'$ is a subcategory of Sch_S containing $\mathcal{C}$ and satisfying that any object in it is a cofiltered limit of objects in $\mathcal{C}$ and affine transition maps, then the inclusion induces an equivalence $\mathrm{NAlg}_{\mathcal{C}'}(\mathcal{SH}) \to \mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$.*

(7) *If there's another map $f : S' \to S$ and a subcategory $\mathcal{C}'$ of $\mathrm{Sch}_{S'}$ satisfying that $f_\sharp(\mathcal{C}') \subset \mathcal{C}$, then f^*, i.e., the restriction to $\mathrm{Span}(\mathcal{C}', \mathrm{all}, \mathrm{fet})$, preserves the structure of normed spectra.*

(8) *If there's another map $f : S' \to S$ and a subcategory $\mathcal{C}'$ of $\mathrm{Sch}_{S'}$ satisfying that $f^*(\mathcal{C}) \subset \mathcal{C}'$ and that $f_* : \mathcal{SH}(S') \to \mathcal{SH}(S)$ is compatible with any base change in $\mathcal{C}$, then f_* preserves the structure of normed spectra, that is, we have a functor*

$$f_* : \mathrm{NAlg}_{\mathcal{C}'}(\mathcal{SH}) \to \mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}) \quad A \mapsto (X \mapsto f_*A(X \times_S S')).$$

Proof. (1)-(3) All assertions about conservativity are obvious. Since $\mathcal{SH}^{\otimes}$ lands in those sifted cocomplete categories, we know that $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$, as a subcategory of the section category, admits sifted colimits that are preserved by the forgetful functor according to [Lur17b, Proposition 5.4.7.11]. It follows from the calculation of limits and colimits of section categories in [Lur17b, 5.1.2.2] that $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$ is closed for finite limits in the larger section category, where to use Lurie's proposition, we should notice that f^* preserves finite limits. When $\mathcal{C} \subset \mathrm{Sm}_S$, Poincaré duality implies that f^* preserves all limits and that so does the forgetful functor. In order to deal with the finite coproduct of commutative algebra objects, we note that $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}) \cong \mathrm{CAlg}(\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}))$ because of our discussion above. Then the forgetful functor is exactly induced by applying $\mathrm{CAlg}(-)$ to the forgetful functor in (2). [Lur17a, Proposition 3.2.4.7] states that any ∞-category of the form $\mathrm{CAlg}(-)$ admits finite coproducts, and that these coproducts are preserved by functors that are images of $\mathrm{CAlg}(-)$. Considering this, we've done.

(4) Recall that the construction of module categories $\mathrm{Mod}_R(\mathcal{D})$, in which $\mathcal{D}$ is a symmetric monoidal ∞-category and $R \in \mathrm{CAlg}(\mathcal{D})$, is functorial with respect to $\mathcal{D}$ and R. Then the functoriality is inherited from A.

(5) See [BH20, Corollary C.19] where they describe the right Kan extension process in detail.

Let's reserve (6) until the last paragraph and turn to (7) and (8) now. In terms of (7), likewise, we should show that the pullback section is cartesian over $\mathcal{C}'^{\mathrm{op}}$. Unwinding definition, f^* is induced by shrinking the span category of $\mathcal{C}$ to that of $\mathcal{C}'$, so the cocartesian edges are not changed. For (8), we also have to verify the cocartesian property of the section $(X \in \mathcal{C}) \mapsto (f_*A(X \times_S S') \in \mathcal{SH}(X))$ where A is a given normed spectrum over $\mathcal{C}'$. That is, for any morphism $g : Y \to X$ in

$$\begin{tikzcd} Y \times_S S' \arrow[r, "g'"] \arrow[d, "f'"] & X \times_S S' \arrow[d, "f"] \\ Y \arrow[r, "g"] & X, \end{tikzcd}$$

we have

$$g^* f_* A(X \times_S S') \simeq f'_* g'^* A(X \times_S S') \simeq f'_* A(Y \times_S S').$$

Here, the first equivalence is the compatibility, and the second is the cocartesian property of A itself.

(6) First, the restriction functor is supposed to be conservative. For every X in $\mathcal{C}'$, the category $\mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{fét})_{/X}$ is sifted obviously. Then by the relative left Kan extension theory [Lur17b,

Corollary 4.3.1.11], the restriction functor

$$\mathrm{Sect}(\mathcal{SH}^{\otimes}|\,\mathrm{Span}(\mathcal{C}',\mathrm{all},\mathrm{fét}))\to\mathrm{Sect}(\mathcal{SH}^{\otimes}|\,\mathrm{Span}(\mathcal{C},\mathrm{all},\mathrm{fét}))$$

admits a fully faithful left adjoint L. It remains to show that the left adjoint preserves norm structure. Unwinding the concrete formula of Kan extension, for $X\in\mathcal{C}'$ and E a section over the span of $\mathcal{C}$,

$$L(E)_X=\operatorname*{colim}_{Z\xleftarrow{f}Y\xrightarrow{p}X} p_{\otimes}f^*E_Z,$$

in which p is finite étale. I claim that the full subcategory $\mathcal{C}^{\mathrm{op}}_{/X}$ is cofinal, so that when E is a normed spectrum, this colimit diagram is constant and $L(E)$ is thus normed. By Quillen's Theorem A in [Lur17b, Theorem 4.1.3.1], it suffices to show that the categorical pullback

$$\mathcal{C}^{\mathrm{op}}\times_{\mathrm{Span}(\mathcal{C},\mathrm{all},\mathrm{fét})_{/X}}(\mathrm{Span}(\mathcal{C},\mathrm{all},\mathrm{fét})_{/X})_{(Z\xleftarrow{f}Y\xrightarrow{p}X)/}$$

is weakly contractible. It is actually filtered as shown in [Gro66]. □

Remark 4.8. The compatibility condition is always true if $\mathcal{C}\subset\mathrm{Sm}_S$ or f is proper due to smooth and proper base change.

4.2. **Norm-pullback-pushforward adjunctions.** In this subsection we will construct the norm-pullback-pushforward adjunctions in the categories of normed spectra. Due to that we are approaching adjunction pairs, it's convenient to take advantage of the language of $(\infty,2)$-categories. Indeed, the proof below is totally formal.

Theorem 4.9. *Let $f:S'\to S$ be a finite étale morphism and let $\mathcal{C}$ be a subcategory of Sch_S that contains S and is closed under finite coproducts and finite étale extensions. Set $\mathcal{C}'=\mathcal{C}_{/S'}$. Then we have an adjunction pair*

$$f_{\otimes}:\mathrm{NAlg}_{\mathcal{C}'}(\mathcal{SH})\rightleftharpoons\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}):f^*$$

in which f^* is the same as that in Theorem 4.7(7) and $f_{\otimes}$ lifts the norm functor between stable motivic spectra.

Theorem 4.10. *Let $f:S'\to S$ be an arbitrary morphism. For $\mathcal{C}$ a subcategory of Sch_S that contains S and is closed under finite coproducts and finite étale extensions, and $\mathcal{C}'\subset\mathrm{Sch}_{S'}$ satisfying similar conditions, if $f_\sharp(\mathcal{C}')\subset\mathcal{C}$, $f^*(\mathcal{C})\subset\mathcal{C}'$, and f_* between stable motivic spectra is compatible with any base change in $\mathcal{C}$, then there is an adjunction*

$$f^*:\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})\rightleftharpoons\mathrm{NAlg}_{\mathcal{C}'}(\mathcal{SH}):f_*$$

where these two functors are well-defined by Theorem 4.7.

The key observation is that even if in our construction of $\mathcal{SH}^{\otimes}$ we never consider the $(\infty,2)$-structure of span categories, noninvertible 2-morphisms still play a profound role between certain section categories. More precisely, we have:

Theorem 4.11 ([BH20, Theorem 8.5 and 8.16])**.**

(1) *Let $\mathcal{C}$ be an ∞-category and let* right $\subset$ left *be classes of morphisms in $\mathcal{C}$ that contains all equivalences and are closed under composition arbitrart pullback. Suppose further that if f and fg are right morphisms, then so is g. For $X\in\mathcal{C}$, denote by $\mathcal{C}_X\subset\mathcal{C}_{/X}$ the full subcategory of $\mathcal{C}_{/X}$ spanned by those left morphisms over X. Given a functor*

$$\mathcal{A}:\mathrm{Span}(\mathcal{C},\mathrm{left},\mathrm{right})\to\mathrm{Cat}_\infty,\quad (X\xleftarrow{f}Y\xrightarrow{p}Z)\mapsto p_{\otimes}f^*,$$

let $\mathcal{A}_X$ denote the restriction of $\mathcal{A}$ to $\mathrm{Span}(\mathcal{C}_X,\mathrm{left},\mathrm{right})$, and then there is an $(\infty,2)$-functor

$$\mathbf{Span}(\mathcal{C},\mathrm{left},\mathrm{right})\to\mathbf{Cat}_\infty,$$

where the 2-morphisms over the former category are of the form

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] \arrow[dd, "\text{right}"] & \\ X & & Z \\ & Y', \arrow[ul] \arrow[ur] & \end{tikzcd}$$

sending $(X \xleftarrow{f} Y \xrightarrow{p} Z)$ *to* $\mathrm{Sect}(\mathcal{A}_X) \xrightarrow{f^*} \mathrm{Sect}(\mathcal{A}_Y) \xrightarrow{p_\otimes} \mathrm{Sect}(\mathcal{A}_Z)$*. Here,* f^* *is the restriction,* $p_\otimes$ *is induced from* $\mathcal{A}(Y) \xrightarrow{p_\otimes} \mathcal{A}(Z)$*, and both of them preserves sections that are cocartesian over backward morphisms.*

(2) *Let* $p : \mathcal{X} \to \mathcal{C}$ *be a cocartesian fibration and let* $f : \mathcal{C} \rightleftharpoons \mathcal{D} : g$ *be an adjunction with unit* $\eta : \mathrm{id} \to gf$*. Suppose that for every* $c \in \mathcal{C}$*, the functor* $\eta(c)_* : \mathcal{X}_c \to \mathcal{X}_{gf(c)}$ *admits a right adjoint* $\eta(c)^!$*, compatible to each other in the sense that it gives rise to a relative adjunction* $\eta_*\mathcal{X} \rightleftharpoons f^*g^*\mathcal{X} : \eta^!$ *over* $\mathcal{C}$*. Then there is an adjunction*

$$g^* : \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}, \mathcal{X}) \rightleftharpoons \mathrm{Fun}_{\mathcal{C}}(\mathcal{D}, \mathcal{X}) : \eta^! f^*.$$

Given that the whole contents on $(\infty, 2)$-categories are far away from our main task, and that the addition 2-morphisms burden us much, I suggest reader ignore the techniques used here and recognize this theorem.

Proof of Theorem 4.9 and 4.10. First, let me prove Theorem 4.9. In (1), take left = all, right = fét, and $\mathcal{A} = \mathcal{SH}^\otimes$. The assumptions in (1) above hold for finite étale morphisms, so we have an $(\infty, 2)$-functor by (1). Consider the adjoint 1-morphisms in $\mathbf{Span}(\mathcal{C}, \mathrm{left}, \mathrm{right})$, for a right morphism f,

$$Y \xleftarrow{\mathrm{id}} Y \xrightarrow{f} X \quad \text{and} \quad X \xleftarrow{f} Y \xrightarrow{\mathrm{id}} Y,$$

the property of $(\infty, 2)$-functors implies that we have the adjunctions between section categories, as well as the adjunctions between the full subcategories of sections that are cocartesian over backward morphisms by (1), that is,

$$f_\otimes : \mathrm{NAlg}_{\mathcal{C}'}(\mathcal{SH}) \rightleftharpoons \mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}) : f^*.$$

Still, for Theorem 4.10, checking it by hand, we know that the adjunction $f_\sharp : \mathcal{C}' \rightleftharpoons \mathcal{C} : f^*$ induces an adjunction

$$f^* : \mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{f\acute{e}t}) \rightleftharpoons \mathrm{Span}(\mathcal{C}', \mathrm{all}, \mathrm{f\acute{e}t}) : f_\sharp.$$

Then $\mathcal{SH}^\otimes$ over $\mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{f\acute{e}t})$, if we combine it with (2) above, induces an adjunction between section categories. Using Theorem 4.7, we know that such an adjunction can be restricted to the full subcategories of sections that are cocartesian over backward morphisms, i.e., normed spectra. $\square$

Remark 4.12. In Theorem 4.10, if we assume f is finite étale, then $\mathcal{C}'$ has the only choice $\mathcal{C}_{/S'}$. First and foremost, $f_\sharp(\mathcal{C}') \subset \mathcal{C}$ implies that $\mathcal{C}' \subset \mathcal{C}_{/S'}$ anytime. A property of algebraic geometry states that when f is finite and étale, the unit map $X \to f^* f_\sharp X$ is always finite and étale for each S'-scheme X. It immediately follows from the definition that the converse direction is right.

4.3. **Localization of normed spectra.** In traditional commutative algebra theory, one have to pay enough attention to localizations, whose reason are related to many mathematical aspects. Analogous to what we did in $\mathbb{E}_\infty$-ring theory, localization of normed motivic spectra, under some mild conditions, is realized as a sequential colimits. That's what we want to develop in this subsection. First, let's recall the definition of localization first.

Definition 4.13. For an abstract nonsense symmetric monoidal ∞-category $\mathcal{D}$, an invertible object L, and a morphism $\alpha : \mathbf{1} \to L$, we say an object $E \in \mathcal{D}$ is α-local if $\mathrm{id}_E \otimes \alpha : E \to E \otimes L$ is an equivalence. We denote by $L_\alpha \mathcal{D}$ the full subcategory of α-local objects. We say a morphism $E \to E'$

exhibits E' as the α-localization of E if this morphism is initial from E to objects in $L_\alpha\mathcal{D}$. If such E' exists, we will denote it by $L_\alpha E$.

Obviously, if $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^L)$, by adjoint functor theorem, we could easily define this localization for each object, endow $L_\alpha\mathcal{D}$ with a canonical symmetric monoidal structure, and lift localization to a symmetric monoidal functor. However, it's still difficult to find a concrete answer to the object $L_\alpha E$.

Consider the following diagram where every morphism is naturally induced by α

$$E \to E \otimes L \to E \otimes L^{\otimes 2} \to \cdots .$$

If the colimit of this diagram exists, we shall denote it by $E[\alpha^{-1}]$. Unwinding definition, we know that every morphism from E to any α-local object will factor through $E[\alpha^{-1}]$. Although $E[\alpha^{-1}]$ isn't necessarily α-local, it's still reasonable to determine when it is equivalent to $L_\alpha E$.

Proposition 4.14. *For a compactly generated stable symmetric monoidal ∞-category $\mathcal{D}$, an invertible object L, a morphism $\alpha : \mathbf{1} \to L$, and an object $E \in \mathcal{D}$, if $E[\alpha^{-1}]$ exists, then in this case, $E \to E[\alpha^{-1}]$ exhibits $E[\alpha^{-1}]$ as the α-localization of E.*

Proof. In such case, one can find a 1-category $\mathcal{E}$ and a conservative functor $\pi : \mathcal{D} \to \mathcal{E}$ preserving sequential colimits. As a result, it remains to show that $E[\alpha^{-1}] \to E[\alpha^{-1}] \otimes L$ is an isomorphism after applying π. Then we need to find a lift in the diagram

$$\begin{tikzcd}
\pi(E \otimes L^{\otimes n}) \arrow[r, "\alpha^2"] \arrow[d, "\alpha"'] & \pi(E \otimes L^{\otimes n+2}) \arrow[d, "\alpha"] \\
\pi(E \otimes L^{\otimes n} \otimes L) \arrow[r, "\alpha^2 \otimes L"] \arrow[ur, dashed, "\alpha"] & \pi(E \otimes L^{\otimes n+2} \otimes L).
\end{tikzcd}$$

Since $\mathcal{E}$ is a 1-category, it suffices to find homotopies in $\mathcal{D}$, which directly follows from the definition and also appears as an easy lemma [Dug14, Lemma 4.17]. $\square$

Remark 4.15. This proposition helps us construct α-localization in $\mathcal{SH}(S)$ and even $\mathrm{Mod}_E(\mathcal{SH}(S))$ for a commutative algebra object E. Then let $f : S' \to S$ be an arbitrary morphism between schemes, we can claim that f^* and f_* preserve localization functors and so does $f_\sharp$ if f is also smooth. This is because they preserve colimits and the projection formula holds when L is invertible.

Moreover, we can define Picard-graded homotopy groups, i.e., a right-lax symmetric monoidal functor,

$$\pi_\star : \mathrm{Mod}_E(\mathcal{SH}(S)) \to \mathrm{Fun}(\mathrm{Pic}(\mathrm{Mod}_E(\mathcal{SH}(S))), \mathrm{Ab}), \quad M \mapsto (L \mapsto [L, M]_E).$$

In particular, if M is an incoherent commutative E-algebra, $\pi_\star(M)$ is also a commutative algebra object.

Now let's turn to the question of localizing normed spectra. To stress that we're dealing with normed objects, we'll use $L_\alpha^\otimes \mathrm{NAlg}_-(-)$ for the corresponding full subcategory.

Theorem 4.16. *Consider that S is a scheme, and $\mathcal{C}$ is a subcategory Sch_S that contains S and is closed under finite coproducts and finite étale extension. Let $E \in \mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$, and let $\alpha : E \to L$ be an E-module map for some $L \in \mathrm{Pic}(\mathrm{Mod}_E)$.*

(1) *If $\mathcal{C} \subset \mathrm{Sm}_S$, then there exists a localization functor $L_\alpha^\otimes : \mathrm{NAlg}_{\mathcal{C}}(\mathrm{Mod}_E) \to L_\alpha^\otimes \mathrm{NAlg}_{\mathcal{C}}(\mathrm{Mod}_E)$ as the left adjoint of inclusion.*

(2) *The following are equivalent.*

   (a) *$L_\alpha^\otimes E$ exists and the canonical morphisms*

$$E_X[\alpha_X^{-1}] \to (L_\alpha^\otimes E)_X$$

   are equivalences for each scheme $X \in \mathcal{C}$.

(b) *The localization functor exists as the left adjoint of inclusion, and can be computed by objectively taking* $(-)_X[\alpha_X^{-1}]$.
(c) *For every finite and étale morphism* $f: X \to Y$ *in* $\mathcal{C}$, *the element* $\nu_f(\alpha_X)$ *is a unit in* $\pi_\star(E_Y)[\alpha_Y^{-1}]$. *Here,* ν_f *is the transfer map defined in Remark 4.6.*

Proof. (1) Unwinding definition, we have this cartesian diagram of ∞-categories

$$\begin{tikzcd} L_\alpha^\otimes \mathrm{NAlg}_{\mathcal{C}}(\mathrm{Mod}_E) \arrow[r, hook] \arrow[d] & \mathrm{NAlg}_{\mathcal{C}}(\mathrm{Mod}_E) \arrow[d] \\ L_{\alpha_S} \mathrm{Mod}_{E_S} \arrow[r, hook] & \mathrm{Mod}_{E_S}. \end{tikzcd}$$

The right two categories are accessible, and the left bottom category is a reflective full subcategory of the right bottom one and thus also accessible. Note that the bottom horizontal map preserves colimits and the right vertical map preserves sifted colimits (Theorem 4.7(2)). Using [Lur17b, Lemma 5.4.5.5 and Proposition 5.4.6.6], we know that $L_\alpha^\otimes \mathrm{NAlg}_{\mathcal{C}}(\mathrm{Mod}_E)$ is accessible, and the inclusion preserves sifted colimits as well. It is obvious that a tensor product of two α-local objects is still α-local. So $L_\alpha^\otimes \mathrm{NAlg}_{\mathcal{C}}(\mathrm{Mod}_E)$ admits all small colimits, and is hence presentable. The statement of (1) follows from adjoint functor theorem and Theorem 4.7(2) when $\mathcal{C} \subset \mathrm{Sm}_S$.

(2) (b) $\Longrightarrow$ (a) is trivial.

For (a) $\Longrightarrow$ (c), given $f: X \to Y$ a finite étale morphism in $\mathcal{C}$, there is a commutative square of $\mathrm{Pic}(\mathrm{Mod}_{E_X})$-graded rings

$$\begin{tikzcd} \pi_\star(E_X) \arrow[r] \arrow[d, "\nu_f"'] & \pi_\star((L_\alpha^\otimes E)_X) \arrow[d, "\nu_f"] \\ \pi_{f_\otimes(\star)}(E_Y) \arrow[r] & \pi_{f_\otimes(\star)}((L_\alpha^\otimes E)_Y), \end{tikzcd}$$

where the horizontal maps are ring homomorphisms and the vertical maps are multiplicative by the analysis in Remark 4.6. The element α_X will be sent to a unit in $\pi_\star((L_\alpha^\otimes E)_X)$ because $L_\alpha^\otimes E$ is α-local. As a result, $\nu_f(\alpha_X)$ is a unit in $\pi_{f_\otimes(\star)}((L_\alpha^\otimes E)_Y)$, which is isomorphic to $\pi_\star(E_Y)[\alpha_Y^{-1}]$ under the assumption of (a).

Last, to prove (c) $\Longrightarrow$ (b) is totally a problem on how to glue the localization functors of fibers and then get a global one, i.e., objective localization is compatible with $*$-pullback and $\otimes$-pushforward, and restriction to cocartesian sections doesn't disturb localization process. Here, reader unfamiliar with the ∞-categorical details could see [BH20, Corollary D.8], and we'll next show these points and conclude.

- Pullback functors preserve L_α-equivalences. This is due to localization functors are given by filtered colimits and thus commute with pullback functors.
- Norm functors preserve L_α-equivalences. Norm functors preserve filtered colimits as well, so $f_\otimes(M_X[\alpha_X^{-1}]) \simeq f_\otimes(M_X)[\nu_f(\alpha_X^{-1})]$ for any finite étale $f: X \to Y$. But the assumption of (c) implies that (note that any normed E-module is actually an E-algebra!) after inverting α_Y, the natural morphism $f_\otimes(M_X) \to f_\otimes(M_X)[\nu_f(\alpha_X^{-1})]$ will become an equivalence. That means this morphism is an L_α-equivalence, and so is $f_\otimes(M_X) \to f_\otimes(M_X[\alpha_X^{-1}])$.
- After objectively localization, a section cocartesian over $\mathcal{C}^{\mathrm{op}}$ is still cocartesian. It's obvious.

By the first two points, the full subcategory of objective α-local sections is reflective, with localization functor F that is already the global left adjoint functor. Then we can restrict it to the category of normed spectra to finish the proof. □

Corollary 4.17 (Inverting integers)**.** *Consider that* S *is a scheme, and* $\mathcal{C}$ *is a subcategory* Sch_S *that contains* S *and is closed under finite coproducts and finite étale extension. Let* $E \in \mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$, *then for any nonnegative integer n, regarded as the multiplication endormorphism of E itself, $L_n^{\otimes}E$ exists and is computed objectively. In particular, the rationalization of E is well-defined in the sense of normed spectra.*

Proof. By the theorem above, we want to show that for any $f : X \to Y$ finite and étale, $f_{\otimes}(n)$ is invertible in $\pi_{0,0}(\mathbf{1}_Y)[1/n]$. According to the compatibility in the proof of Theorem 5.9, it suffices to calculate the similar outcome of $\mathrm{A}_G(-)$, the initial G-Tambara functor, Burnside rings. Then all necessary details can be found in [BH20, Lemma 12.10]. □

4.4. **Completion of normed spectra.** Conclusions in this subsection can be seen as a duality of those about localizations. Still, we begin by recalling the definition of complete objects.

Definition 4.18. For an abstract nonsense symmetric monoidal ∞-category $\mathcal{D}$, an invertible object L, and a morphism $\alpha : \mathbf{1} \to L$, we say an object $E \in \mathcal{D}$ is α-complete if for every α-local object A, $\mathrm{Map}(A, E)$ is contractible. The full subcategory of α-complete objects is denoted by $\mathcal{D}_\alpha^\wedge$, and a map $E \to E'$ is said an α-completion of E if it is initial among all morphisms from E to $\mathcal{D}_\alpha^\wedge$ in which case E' is also denoted by $E_\alpha^\wedge$.

If $\mathcal{D}$ is presentable, then completion always exists and forms a symmetric monoidal left adjoint to the inclusion, with respect to a canonical symmetric monoidal structure of $\mathcal{D}_\alpha^\wedge$.

Lemma 4.19. *Given a stable symmetric monoidal ∞-category $\mathcal{D}$, an invertible object L, and a morphism $\alpha : \mathbf{1} \to L$, every object E belonging to $\mathcal{D}$ admits the α-completion $\lim_n E/\alpha^n$ whenever this limit exists.*

Proof. Each E/α^n is α-complete by definition, and then so do the limit $\lim_n E/\alpha^n$. Since our category is stable, we are free to take fiber of $E \to \lim_n E/\alpha^n$ and it remains to show that this fiber is α-local. That is the equivalence between the limits of

$$\begin{tikzcd}
E \otimes L^{-1} \arrow[d, "\alpha"'] & E \otimes L^{-1} \otimes L^{-1} \arrow[l, "\alpha\otimes L^{-1}"'] \arrow[d, "\alpha"'] & E \otimes L^{-2} \otimes L^{-1} \arrow[l, "\alpha\otimes L^{-1}"'] \arrow[d, "\alpha"'] & \cdots \arrow[l, "\alpha\otimes L^{-1}"'] \\
E & E \otimes L^{-1} \arrow[l, "\alpha"'] & E \otimes L^{-2} \arrow[l, "\alpha"'] & \cdots . \arrow[l, "\alpha"']
\end{tikzcd}$$

By virtue of Yoneda's theorem and stable condition, we can apply $[X, -]$ to it and use Milnor exact sequence and 5-lemma. The other terms in Milnor exact sequence are isomorphic because the functor $[X, -]$ induces an equivalence between pro-abelian groups like what we did in the proof of Proposition 4.14. □

Theorem 4.20. *Consider that S is a scheme, and $\mathcal{C}$ is a subcategory Sm_S that contains S and is closed under finite coproducts and finite étale extension. Let $E \in \mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$, and let $\alpha : E \to L$ be an E-module map for some $L \in \mathrm{Pic}(\mathrm{Mod}_E)$. Suppose that for every surjective finite étale map $f : X \to Y$ in $\mathcal{C}$, α_Y is a unit in the $\mathrm{Pic}(\mathrm{Mod}_{E_Y})$-graded ring $\pi_\star(E_Y)[\nu_f(\alpha_X^{-1})]$. (Be careful with thess conditions and those in* Theorem 4.16(2)(c).*) Then completion functor exists for normed E-module, as a left adjoint, and is computed objectwise.*

Proof. The following three conclusions are all right.

- $f : X \to Y$ makes that f^* sends α_Y-local objects to $\alpha_X \simeq f^*(\alpha_Y)$-local objects. See Remark 4.15.
- The surjective finite étale morphism $f : X \to Y$ induces the norm functor sending α_X-local objects to α_Y-local objects. Due to the same remark as above, $f_{\otimes}$ sends α_X-local objects to $\nu_f(\alpha_X)$-local ones, whence α_Y-local by assumption.
- Each f induces the functor f^* preserving α-complete objects. This is because when f is smooth, $f_\sharp$ exists and preserves local objects by Remark 4.15.

So the objective completion functor satisfies the abstract principles in [BH20, Lemmas D.3(1) and D.6], inducing an adjunction between section categories and thus restricting to norm spectra. We don't need to additionally check its complete property since Lemma 4.19. □

Corollary 4.21 (Completion with respect to integers)**.** *Consider that S is a scheme, and $\mathcal{C}$ is a subcategory Sm_S that contains S and is closed under finite coproducts and finite étale extension. Let $E \in \mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$, then for any nonnegative integer n, regarded as the multiplication endormorphism of E itself, the n-completion exists and can be computed objectwise.*

Proof. Similar to the proof of Corollary 4.17, combine the theorem above and [BH20, Lemma 12.10]. □

5. Norms and Galois Theory

Now I want to develop a counterpart of norms in a delicately defined equivariant homotopy theory $\mathcal{SH}(-)$ for a groupoid, so as to study the relationship of norms between this equivariant stable homotopy of étale fundamental groupoid of some scheme X and $\mathcal{SH}(X)$. By a profinite groupoid, we mean a pro-object in the 2-category FinGpd of groupoids with finite π_0 and π_1, and the category of profinite groupoids is denoted by Pro(FinGpd).

5.1. **Norms in stable equivariant homotopy theory.** The following construction of $\mathcal{SH}(X)$ for a profinite groupoid can be regarded as a version of Mackey functors.

Definition 5.1. A morphism in $\mathrm{Pro}(\mathcal{S})$ is called finitely presented if it is the pullback of one morphism in $\mathcal{S}$.

Lemma 5.2. *The functor $\mathrm{Fun}(-,\mathcal{S}) : \mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ admits an extension to $\mathrm{Pro}(\mathcal{S})$. Let $p : Y \to X$ be finitely presented map in $\mathrm{Pro}(\mathcal{S})$.*

1. *The restriction functor p^* admits a left adjoint $p_\sharp$ and a right adjoint p_*.*
2. *Abbreviate $\mathrm{Span}(\mathrm{Fun}(X,\mathcal{S}),\mathrm{all},\mathrm{all})$ as $\mathrm{Span}(\mathrm{Fun}(X,\mathcal{S}))$. Note that*

$$\mathrm{Fun}(X,\mathcal{S}) \xrightarrow{(-)_+} \mathrm{Fun}(X,\mathcal{S})_+ \simeq \mathrm{Span}(\mathrm{Fun}(X,\mathcal{S}),\mathrm{mono},\mathrm{all}) \to \mathrm{Span}(\mathrm{Fun}(X,\mathcal{S})).$$

Then $\mathrm{Span}(p_*)$ can be restricted to $p_\otimes : \mathrm{Fun}(Y,\mathcal{S})_+ \to \mathrm{Fun}(X,\mathcal{S})_+$.

Proof. The two adjoint functors are respectively left Kan extension and right Kan extension. Since right Kan extension p_* is a right adjoint, it preserves monomorphisms and thus can be restricted. □

So for X and $Y \in \mathrm{Pro}(\mathrm{FinGpd})$, if we set $\mathrm{Fin}_X = \mathrm{Fun}(X,\mathrm{Fin}) \subset \mathrm{Fun}(X,\mathcal{S})$, the 1-category of *finite X-sets*, the functors p^* induces a functor between Fin_X and Fin_Y. In order to restrict the other functors, we need to consider the homotopy (co)limits of discrete sets, which implies that p_*, as well as, $p_\otimes$, exists for finitely presented morphisms while $p_\sharp$ only exists for a so-called finite covering morphism, i.e., a finitely presentable morphism with finite sets as fibers. See Lemma 9.3 and below in [BH20] for it. Hence, all finite X-sets can be identified with finite coverings over X by straightening.

Now we can construct $\mathcal{SH}^\otimes : \mathrm{Span}(\mathrm{Pro}(\mathrm{FinGpd}),\mathrm{all},\mathrm{fp}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$ using an analogous way like that in Subsection 3.3.

- Check by hand that we have a functor

$$\mathrm{Fin}^\otimes : \mathrm{Span}(\mathrm{Pro}(\mathrm{FinGpd}),\mathrm{all},\mathrm{fp}) \to \mathrm{Cat}_1, \quad X \mapsto \mathrm{Fin}_X, \quad (U \xleftarrow{f} T \xrightarrow{p} S) \mapsto p_* f^*.$$

- Apply Span(-) to it and restrict the functor we obtain to pointed finite sets.
- Take nonabelian derived categories, then there will be a functor

$$\mathcal{H}_\bullet^\otimes : \mathrm{Span}(\mathrm{Pro}(\mathrm{FinGpd}),\mathrm{all},\mathrm{fp}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty).$$

- Invert all $p_\otimes(\mathrm{S}^1)$ for $p : Y \to X$ finite covering maps. Then it ends.

Remark 5.3. Analogously, one can also define the notion of normed X-spectra. The norm functor between $\mathcal{SH}(X)$ is exactly $p_\otimes$ induced from unstable cases.

Remark 5.4. By directly comparing our model here with Mackey functors, we know that $\mathcal{SH}(BG)$ is equivalent to the stable ∞-category of genuine G-spectra if G is a finite group. In this case, for $H \subset G$ finite subgroups, the covering map $p : BH \to BG$ induces $p_\otimes$ coinciding with Hill-Hopkins-Ravenel's norm functor N_H^G. Besides, if $N \subset G$ is a normal subgroup, then the quotient map $q : G \to G/N$ will induce geometric fixed point functor Φ^N via our norm construction.

Remark 5.5. One can study higher versions of equivariant homotopy theory in which we consider n-finite X-anima instead of finite sets. For the basic 2-finite case, please refer to [Sch18].

5.2. **Comparison between scheme-theoretic and groupoid-theoretic norms.** Let's begin with a number-theoretic theorem, of which Bachmann-Hoyois's new proof still works for general ∞-topos. I recommend Lei Fu's book [Fu11] for fundamental facts in étale cohomology theory.

Theorem 5.6. *Let S be an arbitrary scheme. Then there is a natural equivalence*

$$\mathrm{FEt}_S \simeq \mathrm{Fun}(\Pi_1^{\acute{e}t}(S), \mathrm{Fin}),$$

where $\Pi_1^{\acute{e}t}$ is the shape of the étale ∞-topos. When S is quasi-compact as we assume throughout the notes, we have

$$\mathrm{FEt}_S \simeq \mathrm{Fun}(\widehat{\Pi}_1^{\acute{e}t}(S), \mathrm{Fin}),$$

in which $\widehat{\Pi}_1^{\acute{e}t}$ means the profinite completion. If further S is connected, $\widehat{\Pi}_1^{\acute{e}t}(S) \simeq B\widehat{\pi}_1^{\acute{e}t}(S, x)$ for any geometric point x of S.

Now we want to bridge the gap between

$$\mathcal{SH}^\otimes : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{f\acute{e}t}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

constructed in Subsection 3.3 and

$$\mathcal{SH}^\otimes : \mathrm{Span}(\mathrm{Pro}(\mathrm{FinGpd}), \mathrm{all}, \mathrm{fp}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

constructed above. According to simple algebro-geometric observation, the profinite-complete étale fundamental groupoid construction can be lifted as a functor

$$\widehat{\Pi}_1^{\mathrm{\acute{e}t}} : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{f\acute{e}t}) \to \mathrm{Span}(\mathrm{Pro}(\mathrm{FinGpd}), \mathrm{all}, \mathrm{fcov}) \to \mathrm{Span}(\mathrm{Pro}(\mathrm{FinGpd}), \mathrm{all}, \mathrm{fp}).$$

Then we shall construct a natural transformation

$$c : \mathcal{SH}^\otimes \circ \widehat{\Pi}_1^{\mathrm{\acute{e}t}} \to \mathcal{SH}^\otimes,$$

which can be decomposed into the following steps as in Subsection 3.3.

- By [GR17], $\mathrm{FEt}_S \simeq \mathrm{Fun}(\widehat{\Pi}_1^{\acute{e}t}(S), \mathrm{Fin})$ in Theorem 5.6 can be lifted as a natural equivalence
$$\mathrm{Fin}_+^\otimes \circ \widehat{\Pi}_1^{\mathrm{\acute{e}t}} \simeq \mathrm{FEt}_+^\otimes : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{f\acute{e}t}) \to \mathrm{CAlg}(\mathrm{Cat}_1).$$
- Compose it with the natural inclusion
$$\mathrm{FEt}_+^\otimes \hookrightarrow \mathrm{SmQP}_+^\otimes : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{f\acute{e}t}) \to \mathrm{CAlg}(\mathrm{Cat}_1),$$
and we will obtain $\mathrm{Fin}_+^\otimes \circ \widehat{\Pi}_1^{\mathrm{\acute{e}t}} \hookrightarrow \mathrm{SmQP}_+^\otimes$.
- Take nonabelian derived categories and invert all motivic equivalences in the right-hand side. So there's a natural transformation
$$c : \mathcal{H}_\bullet^\otimes \circ \widehat{\Pi}_1^{\mathrm{\acute{e}t}} \to \mathcal{H}_\bullet^\otimes : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{f\acute{e}t}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty).$$

- Note that given a scheme S, $\Sigma^\infty c_S(p_\otimes(\mathrm{S}^1)) \simeq \Sigma^\infty q_\otimes(\mathrm{S}^1) \simeq q_\otimes(\Sigma^\infty \mathrm{S}^1)$ is invertible for every finite covering $p : Y \to \widehat{\Pi}_1^{\text{ét}}(S)$ corresponding to the finite étale map $q : T \to S$. Hence, the transformation above induces what we want, the comparison c between $\mathcal{SH}^\otimes \circ \widehat{\Pi}_1^{\text{ét}}$ and $\mathcal{SH}^\otimes$.

Remark 5.7. Such a comparison exists between normed spectra over étale fundamental groupoid and normed spectra over FEt as showed in [BH20, Proposition 10.8].

Remark 5.8. One can similarly construct a comparison transformation Re_B between functors from $\mathrm{Span}(\mathrm{Sch}^{\mathrm{fp}}_{\mathbb{R}}, \mathrm{all}, \mathrm{f\acute{e}t})$ to $\mathrm{CAlg}(\mathrm{Cat}_\infty)$ called Betti realization, generalizing the work in [HO16] in the sense that the C_2-equivariant Betti realization functor is compatible with norms. Instead of étale fundamental groupoid, we use GAGA theorem and CW-theory of algebraic varieties for this comparison. Here, our idea is that the complex point of $\mathbb{R}$-schemes admit a Galois action of C_2. See [Ill78].

5.3. **The Rost norm on Grothendieck–Witt groups.** Based on Morel's computation, we will identify the Rost norm N_f, defined in [Ros03], with our norm for the Grothendieck-Witt group.

Theorem 5.9. *Let S be a regular semilocal scheme over a field k of characteristic not equal to 2. Then we have:*

(1) *in $\mathcal{SH}(S)$, $[\mathbf{1}_S, \mathbf{1}_S] \simeq \mathrm{GW}(S)$;*
(2) *for $f : T \to S$ a finite étale morphism, $\nu_f : \mathrm{GW}(T) \simeq [\mathbf{1}_T, \mathbf{1}_T] \xrightarrow{f_\otimes} [\mathbf{1}_S, \mathbf{1}_S] \simeq \mathrm{GW}(S)$ is exactly N_f.*

Proof of Sketch. Here, we only show the case of local rings, in order to reflect the essence better. The complete proof is in [BH20, Theorem 10.12].

For (1), due to Popescu's desingularization theorem [Sta, Tag 07GC], [Hoy15, Lemma A.7(1)], and [Bac18, Lemma 49], we may assume S is the localization of a smooth affine k-scheme.

Let $\pi_{0,0}^{\mathrm{pre}}(\mathbf{1}_k)$ denote the presheaf $U \mapsto [\Sigma^\infty_+ U, \mathbf{1}_k]$ on Sm_k. Morel's results in [Mor12] imply that

$$L_{\mathrm{zar}}\pi_{0,0}^{\mathrm{pre}}(\mathbf{1}_k) \simeq L_{\mathrm{Nis}}\pi_{0,0}^{\mathrm{pre}}(\mathbf{1}_k) \simeq \underline{\mathrm{GW}}, \tag{1}$$

where the last term is unramified Grothendieck–Witt sheaf, then at least we have a zig-zag diagram

$$\pi_{0,0}^{\mathrm{pre}}(\mathbf{1}_k) \to \underline{\mathrm{GW}} \leftarrow \mathrm{GW}$$

of ring-valued presheaves on Sm_k. By adjunction property and the assumptions on S, if we denote the structure map of S by a, then $\pi_{0,0}^{\mathrm{pre}}(\mathbf{1}_k)(S) \simeq [\Sigma^\infty_+ S, \mathbf{1}_k]_{\mathcal{SH}(k)} \simeq [a_\sharp \mathbf{1}_S, \mathbf{1}_k]_{\mathcal{SH}(k)} \simeq [\mathbf{1}_S, a^*\mathbf{1}_k]_{\mathcal{SH}(S)} \simeq [\mathbf{1}_S, \mathbf{1}_S]_{\mathcal{SH}(S)}$. So it suffices to check the two maps are isomorphisms for the sections of S.

The isomorphism of $\mathrm{GW}(S) \simeq \underline{\mathrm{GW}}(S)$ follows from the analogous one for Witt groups [Bal05, Theorem 100] and the formula $\mathrm{GW}(S) \simeq \mathrm{W}(S) \times_{\mathbb{Z}/2} \mathbb{Z}$ [GR17, §1.4]. Besides, our assumption of locality, together with the equivalences in (1), leads to the first map being an isomorphism because, in terms of sheaves, the sections on local rings coincide with the stalks, and sheafification doesn't change stalks.

For (2), note that in this case $\mathrm{GW}(S) \hookrightarrow \Pi_x \mathrm{GW}(\kappa(x))$ is an injection, where x runs over all points of S. Then we may assume S is the spectrum of a field. Given $S' \to S$ a finite Galois extension with Galois group G, by [Nak13], there is an initial functor A_G, defined over the subcategory of FEt_S consisting of those finite étale S-schemes split by S', among all so-called G-*Tambara* functors. And the maps ν_f or N_f endow $\mathrm{GW}(-)$ with a G-Tambara structure. In order to show that $\nu_f = \mathrm{N}_f$, we only need to for each $\omega \in \mathrm{GW}(T)$ find a finite Galois extension $S' \to S$ splitting f such that ω lies in the image of $\mathrm{A}_G(T) \to \mathrm{GW}(T)$. Unwinding definition, $\mathrm{GW}(T)$ is generated by $\mathrm{Tr}_p(1)$ for every $p : T' \to T$ finite étale morphism, and with respect to these generators, any finite Galois extension $S' \to S$ splitting fp satisfies our target. $\square$

Corollary 5.10. *Let S be a regular semilocal scheme over a field of characteristic $\neq 2$, let $f : T \to S$ be a finite étale map, and let X be a smooth quasi-projective T-scheme such that $\Sigma^\infty_+ X \in \mathcal{SH}(T)$ is dualizable. Then*

$$\chi(\Sigma^\infty_+ R_f X) = \mathrm{N}_f(\chi(\Sigma^\infty_+ X)) \in \mathrm{GW}(S).$$

Here, R_f is Weil restriction and χ is Euler characteristics.

Proof. Combine Theorem 5.9 and the fact that $p_\otimes$ is symmetric monoidal as shown in Subsection 3.3. □

6. Examples of Normed Spectra

At the end of our notes, I will provide you with several examples of normed spectra. Actually, we take a roundabout approach here. To be more precise, certain symmetric monoidal functors encode the data of normed spectra. This is a powerful philosophy named after categorification.

6.1. $\mathbf{H}\mathbb{Z}$ and norms in slice filtration.

In this subsection we show that the norm functors preserve effective and very effective spectra, and explore some of the consequences of this statement.

Convention 6.1. For $n \in \mathbb{Z}$, we denote by $\mathcal{SH}^{\mathrm{eff}}(S)(n)$ the full subcategory generated by $\mathrm{S}^{2n,n} \otimes \Sigma^{-m}\Sigma^\infty_+ X$ for $m \geq 0$ and $X \in \mathrm{Sm}_S$ under colimits, whose objects are called n-effective spectra. The colimit-preserving inclusions

$$\cdots \subset \mathcal{SH}^{\mathrm{eff}}(S)(n+1) \subset \mathcal{SH}^{\mathrm{eff}}(S)(n) \subset \mathcal{SH}^{\mathrm{eff}}(S)(n-1) \subset \cdots \subset \mathcal{SH}(S)$$

gives rise to a functorial filtration for each spectrum $E \in \mathcal{SH}(S)$

$$\cdots \to f_{n+1}E \to f_n E \to f_{n-1}E \to \cdots \to E,$$

where the functor f_n is the right adjoint to the inclusion and the cofiber of $f_{n+1}E \to f_n E$ induces a functor denoted by s_n.

Likewise, we have a full subcategory of very n-effective objects, denoted by $\mathcal{SH}^{\mathrm{veff}}(S)(n)$ and generated by $\mathrm{S}^{2n,n} \otimes \Sigma^\infty_+ X$ for $X \in \mathrm{Sm}_S$ under colimits. At this time, we denote the right adjoint by $\tilde{f}_n$ and the cofiber by $\tilde{s}_n$. When $n = 0$, we may omit the index n, and call them (very) effective spectrum. The notation will also become $\mathcal{SH}^{\mathrm{eff}}(S)$ or $\mathcal{SH}^{\mathrm{veff}}(S)$. Note that $\mathcal{SH}^{\mathrm{veff}}(S)$ forms a nonnegative part of some t-structure of $\mathcal{SH}^{\mathrm{eff}}(S)$. Hence, we obtain another functor $\underline{\pi}_0^{\mathrm{eff}} : \mathcal{SH}^{\mathrm{veff}}(S) \to \mathcal{SH}^{\mathrm{eff}\heartsuit}(S)$.

Now we want to show that (very) effective spectra are compatible with norm structure and finally construct functors to describe the relationships coherently. To conclude, we have:

Theorem 6.2. *There exist subfunctors of $\mathcal{SH}^\otimes$*

$$\mathcal{SH}^{\mathrm{veff}\otimes} \subset \mathcal{SH}^{\mathrm{eff}\otimes} \subset \mathcal{SH}^\otimes : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{fét}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty),$$

and natural transformations

$$s_0 : \mathcal{SH}^{\mathrm{eff}\otimes} \to s_0\mathcal{SH}^{\mathrm{eff}\otimes} : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{fét}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty),$$

$$\tilde{s}_0 : \mathcal{SH}^{\mathrm{veff}\otimes} \to \tilde{s}_0\mathcal{SH}^{\mathrm{veff}\otimes} : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{fét}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty),$$

$$\underline{\pi}_0^{\mathrm{eff}} : \mathcal{SH}^{\mathrm{veff}\otimes} \to \mathcal{SH}^{\mathrm{eff}\otimes\heartsuit} : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{fét}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

taking its values pointwise.

Furthermore, for $\mathcal{C} \subset \mathrm{Sm}_S$ that contains S and is closed under finite coproducts and finite étale extension, there are adjunctions

$$\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}^{\mathrm{eff}}) \underset{f_0}{\overset{\longleftarrow}{\longrightarrow}} \mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$$

$$\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}^{\mathrm{veff}}) \underset{\tilde{f}_0}{\overset{\longleftarrow}{\longrightarrow}} \mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$$

$$\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}^{\mathrm{eff}}) \overset{s_0}{\underset{\longleftarrow}{\longrightarrow}} \mathrm{NAlg}_{\mathcal{C}}(s_0\mathcal{SH}^{\mathrm{eff}})$$

$$\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}^{\mathrm{veff}}) \overset{\tilde{s}_0}{\underset{\longleftarrow}{\longrightarrow}} \mathrm{NAlg}_{\mathcal{C}}(\tilde{s}_0\mathcal{SH}^{\mathrm{eff}})$$

$$\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}^{\mathrm{veff}}) \overset{\pi_0^{\mathrm{eff}}}{\underset{\longleftarrow}{\longrightarrow}} \mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}^{\mathrm{eff}\heartsuit}),$$

in which each functor is computed pointwise.

Proof. In terms of the first statement, by abstract categorical formalism, it suffices to show that (very) effective spectra are preserved under $*$-pullback, finite étale $\sharp$-pushforward, surjective finite étale $\otimes$-pushforward, and tensor products, that 1-effective spectra (resp. very 1-effective spectra) form an ideal in effective spectra (resp. very effective spectra), and that 1-connective effective spectra forms an ideal in very effective spectra, because they come from either full subcategories or quotient subcategories.

In other words, for $p : T \to S$ finite étale, we will prove:

(1) if $\xi \in K(X)$ is of rank n, then S^{ξ} is very n-effective;
(2) $p_{\otimes}(\mathrm{S}^{-1})$ is effective;
(3) when p is surjective, then $p_{\otimes}(\mathrm{S}^1)$ is 1-connectively effective;
(4) if p is of degree d and $\xi \in K(X)$ is of rank n, then $p_{\otimes}(\mathrm{S}^{\xi})$ is very dn-effective.

For (1), since our problem is Zariski local by smooth base change, we may assume ξ is trivial of rank n. In this case, this is obvious. Using the comparison c in Subsection 5.2, for (2), if we denote by $\tilde{p} = \widehat{\Pi}_1^{\text{ét}}(p)$ the finite covering map of profinite groupoids, we know that $p_{\otimes}(\mathrm{S}^{-1}) \simeq c_S(\tilde{p}_{\otimes}(\mathrm{S}^{-1}))$ is effective since c_S already factors through effective spectra by definition. Then (3) follows by similar process and direct calculation: over each connected component, if we denote the fiber (ignoring automorphisms) by A a nonempty finite set with a G-action, then $p_{\otimes}(\mathrm{S}^1)$ is a one-point compactification of the real G-representation $\mathbb{R}^A$, which is a suspension spectrum since $\mathbb{R}^A$ admits a trivial 1-dimensional summand at least. (4) is a direct corollary of Proposition 3.8 and (1).

Now let's turn to the second statement. By [BH20, Lemmas D.3(1) and D.6], there have already been adjunctions between section categories. So it remains to show that these adjunctions can be restricted to normed spectra, which follows from the fact that when f is smooth, $f_{\sharp}$ preserves effective, very effective, 1-effective, very 1-effective, and 1-connectively effective spectra. □

Corollary 6.3. *The zero slice of the motivic sphere spectrum $s_0(\mathbf{1})$ is normed over Sm_S for a scheme S. When S is essentially smooth over a field, $\mathrm{H}\mathbb{Z} \simeq s_0(\mathbf{1})$ (see [Hoy15]), Voevodsky's motivic cohomology spectrum, is normed.*

6.2. **KGL and norms of linear ∞-categories.** Now I want to explain the normed structure of KGL. As is known, KGL comes from algebraic K-theory of idempotent-complete stable small ∞-categories $\mathrm{Perf}(-)$ (or Effimov K-theory for dualizable presentable ∞-categories $\mathrm{QCoh}(-)$ if one prefers). Hence, we are supposed to believe that there is a analogous counterpart of $\mathcal{SH}^{\otimes}$ consisting of these information. This target directly leads to our noncommutative motives and the enhancement $\mathcal{SH}^{\otimes}_{\mathrm{nc}}$.

Notations 6.4. For R a commutative ring, denote by

$$\mathrm{Cat}_R^{\mathrm{st}} = \mathrm{Mod}_{\mathrm{Mod}_R(\mathrm{Sp})}(\mathrm{Pr}^L_{\mathrm{st}})$$

the ∞-category of stable presentable R-linear ∞-categories, and let

$$\mathrm{Cat}_R^{\mathrm{cg}} = \mathrm{Mod}_{\mathrm{Mod}_R(\mathrm{Sp})}(\mathrm{Pr}^{L,\omega}_{\mathrm{st}}),$$

the full subcategory of compactly generated ones. If X is an arbitrary R-scheme, we will write $\mathrm{QCoh}(X) \in \mathrm{Cat}_R^{\mathrm{st}}$ for the stable R-linear ∞-category of quasi-coherent sheaves on X. Note that under our assumption that X is qcqs, $\mathrm{QCoh}(X)$ is also compactly generated.

By [HSS17], $\mathrm{Cat}_R^{\mathrm{cg}}$ is compactly generated, whose compact objects form a full subcategory denoted by $\mathrm{Cat}_R^{\mathrm{fp}}$. In the *op. cit.*, they constructed two functors $\mathrm{Aff}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_{(\infty,2)})$ defined by $R \mapsto \mathrm{Cat}_R^{\mathrm{st}}$ and $R \mapsto \mathrm{Cat}_R^{\mathrm{cg}}$, in which for any ring homomorphism f, f^* preserves the subcategory $\mathrm{Cat}^{\mathrm{fp}}$ and admits a right adjoint f_* preserving $\mathrm{Cat}^{\mathrm{fp}}$ at least when f is smooth.

Theorem 6.5 ([Lur18, Appendix D])**.** *The three functors* $\mathrm{Aff}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_{(\infty,2)})$ *sending R to* $\mathrm{Cat}_R^{\mathrm{st}}$, $\mathrm{Cat}_R^{\mathrm{cg}}$, *and* $\mathrm{Cat}_R^{\mathrm{fp}}$, *are sheaves for fppf, étale, and finite flat topology respectively.*

So by the technique of right Kan extension in [BH20, Corollary C.13], we enhance these sheaves with respect to finite flat topology as functors $\mathrm{Span}(\mathrm{Aff}, \mathrm{all}, \mathrm{fét}) \to \mathrm{CAlg}(\mathrm{Cat}_{(\infty,2)})$, in which a finite étale morphism p will additionally induce a functor denoted by $p_\otimes$. Next, we will from them construct our noncommutative motives.

Construction 6.6. All these construction for general schemes can be obtained via right Kan extension from affine cases, so we assume the affineness throughout this paragraph. We set

$$\mathrm{SmNC}_R = \mathrm{Cat}_R^{\mathrm{fp,op}},$$

a small idempotent complete semiadditive ∞-category with finite limits. Recall that a excision square in $\mathrm{Cat}_R^{\mathrm{cg}}$ is a cartesian square in $\mathrm{Cat}_R^{\mathrm{st}}$

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{g} & \mathcal{D}, \end{array}$$

such that g, and hence f, admits a fully faithful right adjoint, and the inverse image of 0 has a single compact generator. Consider the following conditions on a presheaf $F \in \mathcal{P}(\mathrm{SmNC}_R)$:

- weak excision: F preserves finite products, and for every excision square of the form above satisfies that $\mathrm{fib}(F(f)) \xrightarrow{\sim} \mathrm{fib}(F)(g)$;
- excision: $F(0)$ is contractible and F take excision squares to pullbacks;
- $\mathbb{A}^1$-invariance: for every $\mathcal{A} \in \mathrm{Cat}_R^{\mathrm{fp}}$, the natural map $F(\mathcal{A}) \to F(\mathcal{A} \otimes_R \mathrm{QCoh}(\mathbb{A}^1_R))$ is an equivalence.

We denote by $\mathcal{P}_{\mathrm{wexc}}(\mathrm{SmNC}_R)$ (respectively, by $\mathcal{P}_{\mathrm{exc}}(\mathrm{SmNC}_R)$; by $\mathcal{H}_{\mathrm{wnc}}(\mathrm{Spec}\, R)$; by $\mathcal{H}_{\mathrm{nc}}(\mathrm{Spec}\, R)$) the reflective subcategory spanned by presheaves satisfying weak excision (resp. excision; weak excision and $\mathbb{A}^1$-invariance; excision and $\mathbb{A}^1$-invariance). Similarly, a morphism will be called a weakly excisive equivalence (resp. an excisive equivalence; a weakly motivic equivalence; a motivic equivalence) if its reflection is an equivalence in the corresponding category.

According to [Lur18, Corollary 9.4.2.3], QCoh induces the functor $\mathrm{QCoh} : \mathrm{SmAff}_R \to \mathrm{SmNC}_R$. Then we can take nonabelian derived categories and invert motivic equivalences, obtaining a functor

$$\mathcal{L} : \mathcal{H}_\bullet(\mathrm{Spec}(R)) \to \mathcal{H}_{\mathrm{nc}}(\mathrm{Spec}(R)).$$

Here, the right-hand side is already pointed due to the same property of SmNC_R by [Lur17b, Proposition 4.8.2.11]. Let's denote by $\mathcal{SH}_{\mathrm{nc}}(\mathrm{Spec}(R))$ the presentably symmetric monoidal ∞-category obtained from inverting $\mathcal{L}(\mathrm{S}^{\mathbb{A}^1})$ in $\mathcal{H}_{\mathrm{nc}}(\mathrm{Spec}(R))$, and then we have

$$\mathcal{L} : \mathcal{SH}(\mathrm{Spec}(R)) \to \mathcal{SH}_{\mathrm{nc}}(\mathrm{Spec}(R)).$$

If Lemma 6.9 holds, then one can enhance $\mathcal{SH}_{\mathrm{nc}}(S)$ as a global one as what we have done above. In other words, we have

Theorem 6.7. *There is a functor*

$$\mathcal{SH}_{\mathrm{nc}}^{\otimes} : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{fét}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

and the natural transformation $\mathcal{L} : \mathcal{SH}^{\otimes} \to \mathcal{SH}_{\mathrm{nc}}^{\otimes}$ *defined from* QCoh.

Corollary 6.8. *For every scheme S, the homotopy K-theory KGL_S is normed over Sch_S. In particular, the presheaf of homotopy K-theory spaces $\Omega^\infty \mathrm{KH} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathcal{S})$ extends to a functor from $\mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{fét})$ to $\mathrm{CAlg}(\mathcal{S})$.*

Proof. By [Rob15, Theorem 4.7], the fiberwise right adjoint of $\mathcal{L}$ exists and is KGL. Due to categorical formalism, they can be combined into a relative right adjoint whence it sends the norm structure of the unit to KGL. ☐

Now we turn to the key lemma.

Lemma 6.9.

(1) *For $p : R \to R'$ a finite étale morphism between rings, the functor $p_\otimes : \mathcal{P}_\Sigma(\mathrm{SmNC}_{R'}) \to \mathcal{P}_\Sigma(\mathrm{SmNC}_R)$ sends weakly excisive (resp. weakly motivic) equivalences to excisive (resp. motivic) equivalences.*

(2) *$\mathcal{SH}_{\mathrm{nc}}(\mathrm{Spec}\, R)$ is obtained from $\mathcal{H}_{\mathrm{wnc}}(\mathrm{Spec}\, R)$ by inverting $\mathcal{L}(\mathbb{A}^1/\mathbb{G}_m)$.*

Proof. (2) is trivial because $\mathcal{L}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathrm{S}^1 \otimes \mathcal{L}(\mathbb{G}_m, 1)$ in $\mathcal{H}_{\mathrm{wnc}}$ and $\mathrm{Sp}(\mathcal{H}_{\mathrm{wnc}}(\mathrm{Spec}\, R)) \simeq \mathrm{Sp}(\mathcal{H}_{\mathrm{nc}}(\mathrm{Spec}\, R)) \simeq \mathcal{SH}_{\mathrm{nc}}(\mathrm{Spec}\, R)$ by definition.

For (1), by the universal property of right Kan extension, we know that $p_\otimes$ preserves $\mathbb{A}^1$-equivalence. Using a standard reduction process, we only need to show that for an excision square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{g} & \mathcal{D} \end{array}$$

in $\mathrm{Cat}_{R'}^{\mathrm{fp}}$, the corresponding weakly excisive equivalence $\mathcal{C}/\mathcal{D} \to \mathcal{A}/\mathcal{B}$ in $\mathcal{P}_\Sigma(\mathrm{SmNC}_{R'})$ is sent to an excisive equivalence under $p_\otimes$.

Using similar notations in Lemma 3.1, we claim that the diagram

$$\begin{array}{ccc} p_\otimes \mathcal{A} & \longrightarrow & p_\otimes(\mathcal{A}|\mathcal{B}) \\ \downarrow & & \downarrow \\ p_\otimes \mathcal{C} & \longrightarrow & p_\otimes(\mathcal{C}|\mathcal{D}) \end{array}$$

is an excision square. Thus, it remains to produce a natural map

$$p_\otimes(\mathcal{A}/\mathcal{B}) \to p_\otimes(\mathcal{A})/p_\otimes(\mathcal{A}|\mathcal{B})$$

and prove that it is an excisive equivalence. The claim is because we can reduce to the case of fold maps by Theorem 6.7 in which case they can be expressed as explicit limits of tensor products.

At this time, Bar constructions help us write $p_\otimes(\mathcal{A})/p_\otimes(\mathcal{A}|\mathcal{B})$ as a colimit of a simplicial object of the form $p_\otimes\mathcal{A} \times p_\otimes(\mathcal{A}|\mathcal{B})^{\times n}$. Similarly, $p_\otimes(\mathcal{A}/\mathcal{B})$ is a colimit of the simplicial object

$$p_\otimes(\mathcal{A} \times \mathcal{B}^{\times n}) \simeq p_\otimes\mathcal{A} \times p_\otimes(\mathcal{A} \times \mathcal{B}|\mathcal{B}) \times \cdots \times p_\otimes(\mathcal{A} \times \mathcal{B}^{\times n}|\mathcal{B}).$$

Then the natural map is from sending the term $p_\otimes(\mathcal{A} \times \mathcal{B}^{\times i}|\mathcal{B})$ to the ith copy of $p_\otimes(\mathcal{A}|\mathcal{B})$.

Note that there is a diagram of simplicial objects which is degreewise a pushout in $\mathcal{P}_\Sigma(\mathrm{SmNC}_R)$

$$\begin{tikzcd}
p_\otimes(\mathcal{A} \times \mathcal{B}^{\times\bullet}|\mathcal{B}^{\times\bullet+1}) \arrow[r, hook] \arrow[d] & p_\otimes(\mathcal{A} \times \mathcal{B}^{\times\bullet}) \arrow[d] \\
p_\otimes(\mathcal{A}|\mathcal{B}) \times p_\otimes(\mathcal{A}|\mathcal{B})^{\times\bullet} \arrow[r, hook] & p_\otimes\mathcal{A} \times p_\otimes(\mathcal{A}|\mathcal{B})^{\times\bullet}.
\end{tikzcd}$$

The colimit of the left bottom corner is contractible since it splits. Hence, it suffices to show that the colimit of the left top corner is also contractible. This is a simple formal question, so we leave it to the readers. $\square$

6.3. **MGL and motivic Thom spectra.** Finally, we prove that Voevodsky's algebraic cobordism spectrum MGL is normed. It can be decomposed into two steps. First, MGL is a specialization to general motivic Thom spectrum relevant to J-homomorphism. Second, motivic Thom spectra construction is compatible with normed spectra in the sense of [BH20, Proposition 16.17].

Notations 6.10. We denote by

$$\mathrm{Sph}(S) = \mathrm{Pic}(\mathcal{SH}(S))$$

the Picard ∞-groupoid of $\mathcal{SH}(S)$, which is also an $\mathbb{E}_\infty$-group with respect to the smash product within motivic spectra. Since both pullbacks and norms preserve invertible objects, the construction is functorial and defined over $\mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{fét})$. Besides, it's also a Nisnevich sheaf as so is $\mathcal{SH}(-)$.

Let $(\mathrm{Sm}_S)_{//\mathcal{SH}} \to \mathrm{Sm}_S$ denote the cartesian fiberation classified by $\mathcal{SH}$, and let $\mathcal{P}((\mathrm{Sm}_S)_{//\mathcal{SH}})$ denote the ∞-category of presheaves which are *small* colimits of representables. Here, the cardinality and the choice of the universe are critical.

The motivic Thom spectrum is thus extended to the functor

$$M_S : \mathcal{P}((\mathrm{Sm}_S)_{//\mathcal{SH}}) \to \mathcal{SH}(S)$$

via colimits from

$$(\mathrm{Sm}_S)_{//\mathcal{SH}} \to \mathcal{SH}(S), \quad (f : X \to S, P \in \mathcal{SH}(X)) \mapsto f_\sharp P.$$

Example 6.11. The composition of motivic sheafification and stabilization $\Sigma^\infty_+ L_{\mathrm{mot}} : \mathcal{P}(\mathrm{Sm}_S) \to \mathcal{SH}(S)$ is given by, by definition, the composition

$$\mathcal{P}(\mathrm{Sm}_S) \xrightarrow{X \mapsto (X, \mathbf{1}_X)} \mathcal{P}((\mathrm{Sm}_S)_{//\mathcal{SH}}) \xrightarrow{M_S} \mathcal{SH}(S).$$

Lemma 6.12. *Let S be a scheme.*

(1) *The restricted functor $M_S : \mathcal{P}(\mathrm{Sm}_S)_{/\mathcal{SH}} \to \mathcal{SH}(S)$ inverts Nisnevich equivalences.*
(2) *For $A \in \mathcal{H}(S)_{/\mathcal{SH}}$, the restriction of M_S to $\mathcal{P}(\mathrm{Sm}_S)_{/A}$ inverts motivic equaivalences.*

Proof. It suffices to, by [BH20, Lemma 16.8], check (1) and (2) for the preimage, under forgetful functor, of a generating set of corresponding types in $\mathcal{P}(\mathrm{Sm}_S)$ because $\mathcal{SH}$ or A is a local object.

Hence, to prove (1), take a smooth S-scheme X, $\phi : X \to \mathcal{SH}$, and a Nisnevich sieve $\iota : U \hookrightarrow X$. We want to show that $M_S(\phi \circ \iota) \simeq M_S(\phi)$. This is because the left Kan extension provides an explicit formula of M_S and the Nisnevich sieve induces a cofinal subdiagram.

Similarly, for (2), it remains to show that if $\pi : X \times \mathbb{A}^1 \to X$ is the projection map, then $M_S(\phi \circ \pi) \simeq M_S(\phi)$. It follows from explicit formulas and $\mathbb{A}^1$-equivalence $\pi_\sharp\pi^* \simeq \mathrm{id}_{\mathcal{SH}(X)}$. $\square$

Construction 6.13 (Motivic J-homomorphism)**.** Recall that in Remark 3.9, what we did can be summarized as a functor $K^{\oplus} : \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{fét}) \to \mathrm{CAlg}(\mathcal{S}^{\mathrm{gp}})$ and a transformation $K^{\oplus} \to \mathrm{Sph}$. Note that Sph is right Kan extended from affine schemes, so this transformation factors through the right Kan extension of $K^{\oplus} := K(\mathrm{Vect}(-))$ restricted to affine schemes, which is equivalent to Thomason–Trobaugh K-theory $K(\mathrm{Perf}(-))$. We denote it by $j : K \to \mathrm{Sph}$.

We denote by $e : K^{\circ} \to K$ the rank 0 part of K and by $\gamma : \mathrm{Gr}_{\infty} \to K^{\circ}$ the tautological bundle minus the trivial bundle, which is a motivic equivalence as shown in [MV99].

It is obvious that MGL_S is defined as $M_S(j \circ e \circ \gamma)$. Actually, we have:

Theorem 6.14. *Let S be a scheme. Then $\gamma : \mathrm{Gr}_{\infty} \to K^{\circ}$ induces an equivalence*

$$\mathrm{MGL}_S \simeq M_S(j \circ e).$$

Proof. Assume that S is regular. Then K is $\mathbb{A}^1$-homotopy invariant Nisnevich sheaf. So by Lemma 6.12(2), $M_S(j \circ -)$ inverts motivic equivalences. Since γ is a motivic equivalence, the theorem holds in this case.

For an arbitrary scheme S, we may consider the structure map $f : S \to \operatorname{Spec} \mathbb{Z}$ and pullback the equivalence in the setting of $\operatorname{Spec} \mathbb{Z}$. Precisely,

$$\mathrm{MGL}_S \simeq f^* \mathrm{MGL}_{\operatorname{Spec} \mathbb{Z}} \simeq M_S(f^*(K^{\circ}|_{\mathrm{Sm}_{\mathbb{Z}}}) \to K^{\circ} \xrightarrow{j \circ e} \mathrm{Sph})$$

by direct diagram chasing, in which $f^*(K^{\circ}|_{\mathrm{Sm}_{\mathbb{Z}}}) \to K^{\circ}|_{\mathrm{Sm}_S}$ is known to be a Zariski equivalence. Hence, we can finish the reduction process by Lemma 6.12(1). $\square$

Repeating our standard process to enhance functoriality, Bachmann-Hoyois proved the following lemma and verified the normed property of motivic Thom spectra.

Lemma 6.15 ([BH20, Proposition 16.17])**.** *Let S be a scheme, $\mathcal{C}$ be a subcategory Sch_S that contains S and is closed under finite coproducts and finite étale extension, and $\mathcal{L}$ be a class of smooth morphisms in $\mathcal{C}$ closed under composition, pullback, and finite étale Weil restriction. Then there is a functor*

$$M|_{\mathcal{L}} : \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{fét}), \mathcal{S})_{//\mathcal{SH}} \to \mathrm{Sect}(\mathcal{SH}^{\otimes}|\,\mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{fét}))$$

sending $\psi : A \to \mathcal{SH}$ to $M_X(\psi|_{\mathcal{L}_X^{\mathrm{op}}})$. Moreover, it is also cocartesian over morphisms $f : Y \to X$ in $\mathcal{C}$ such that $f^*(A|_{\mathcal{L}_X^{\mathrm{op}}}) \to A|_{\mathcal{L}_Y^{\mathrm{op}}}$ is an M_Y-equivalence.

Apply this lemma to $\mathrm{MGL}_S \simeq M_S(j \circ e)$ and verify the conditions about cocartesian morphisms in the second statement of the lemma. Then we obtain:

Theorem 6.16. *For every scheme S, the algebraic cobordism spectrum MGL_S is normed spectra over Sch_S.*

References

[Bac18] Tom Bachmann. Some remarks on units in grothendieck–witt rings. *Journal of Algebra*, 499:229–271, 2018.

[Bal05] Paul Balmer. Witt groups. *Handbook of K-theory*, 1:2, 2005.

[BH20] Tom Bachmann and Marc Hoyois. Norms in motivic homotopy theory. *arXiv:1711.03061*, 2020.

[CD19] Denis-Charles Cisinski and Frédéric Déglise. *Triangulated categories of mixed motives*, volume 6. Springer, 2019.

[CGP15] Brian Conrad, Ofer Gabber, and Gopal Prasad. *Pseudo-reductive groups*, volume 26. Cambridge University Press, 2015.

[Del09] Pierre Deligne. Voevodsky's lectures on motivic cohomology 2000/2001. In *Algebraic Topology: The Abel Symposium 2007*, pages 355–409. Springer, 2009.

[Dug14] Daniel Dugger. Coherence for invertible objects and multigraded homotopy rings. *Algebraic & Geometric Topology*, 14(2):1055–1106, 2014.

[Fu11] Lei Fu. *Etale cohomology theory*, volume 13. world scientific, 2011.

[Gil19] Stefan Gille. On quadratic forms over semilocal rings. *Transactions of the American Mathematical Society*, 371(2):1063–1082, 2019.

[GR17] Dennis Gaitsgory and Nick Rozenblyum. A study in derived algebraic geometry: Volume ii: Deformations, lie theory and formal geometry. *American Mathematical Society*, 2017.

[Gro66] Alexandre Grothendieck. Éléments de géométrie algébrique: Iv. Étude locale des schémas et des morphismes de schémas, troisiéme partie. *Publ. Math. I.H.É.S.*, 28, 1966.

[HO16] Jeremiah Heller and Kyle Ormsby. Galois equivariance and stable motivic homotopy theory. *Transactions of the American Mathematical Society*, 368(11):8047–8077, 2016.

[Hoy15] Marc Hoyois. From algebraic cobordism to motivic cohomology. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2015(702):173–226, 2015.

[HSS17] Marc Hoyois, Sarah Scherotzke, and Nicolo Sibilla. Higher traces, noncommutative motives, and the categorified chern character. *Advances in Mathematics*, 309:97–154, 2017.

[Ill78] Sören Illman. Smooth equivariant triangulations of g-manifolds for g a finite group. *Mathematische Annalen*, 233(3):199–220, 1978.

[KK82] Manfred Knebusch and Manfred Kolster. Reduced wittrings. In *Wittrings*, pages 56–87. Springer, 1982.

[Lur17a] Jacob Lurie. Higher algebra, 2017. Available online.

[Lur17b] Jacob Lurie. Higher topos theory, 2017. Available online.

[Lur18] Jaocb Lurie. Spectral algebraic geometry, 2018. Available online.

[Mor12] Fabien Morel. $\mathbb{A}^1$-*algebraic topology over a field*, volume 2052. Springer, 2012.

[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *Publications Mathématiques de l'IHÉS*, 90:45–143, 1999.

[Nak13] Hiroyuki Nakaoka. A generalization of the dress construction for a tambara functor, and its relation to polynomial tambara functors. *Advances in Mathematics*, 235:237–260, 2013.

[Rob15] Marco Robalo. K-theory and the bridge from motives to noncommutative motives. *Advances in Mathematics*, 269:399–550, 2015.

[Ros03] Markus Rost. The multiplicative transfer for the grothendieck witt ring. *preprint*, 2003.

[Ryd11] David Rydh. Representability of hilbert schemes and hilbert stacks of points. *Communications in Algebra*, 39(7):2632–2646, 2011.

[Sch18] Stefan Schwede. *Global homotopy theory*, volume 34. Cambridge University Press, 2018.

[Sch22] Peter Scholze. Six-functor formalisms, 2022. Available online.

[Sta] The Stacks project authors. The stacks project. Available online.