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Seminar on Chromatic Homotopy Theory

— The Latest Breakthroughs since 2020.

One conjecture that remained unsolved for 40 years is Telescope Conj. It's very hard because of people's lack of enough knowledge on some structual information for $Sp_{K(n)}$ & $Sp_{T(n)}$.

As a result, we'll pay more attention to some basic players at the beginning of this seminar.

More concretely,

Prop: For a ring spectrum,

$$K(n)\text{-acyclic} \Leftrightarrow T(n)\text{-acyclic}.$$

Thm (Hahn, 2016):

For an $\mathbb{E}_\infty$-ring A, ~~(maybe $n\geq 2$)~~

$$K(n)\text{-acyclic} \Rightarrow K(n+1)\text{-acyclic}.$$

Hence the following defn is reasonable:

Def: A ($\mathbb{E}_\infty$-)ring sp

$$ht(A) := \max\{n \mid n \in \operatorname{Supp} A\} \quad \left(\text{In particular } ht = -1 \text{ if } \operatorname{Supp} A = \phi\right).$$

$$\operatorname{Supp} A := \{n \in \mathbb{N} \mid A \otimes T(n) \neq 0\}$$

Thm (Blueshift Phenomenon, Kuhn, 2004) (It isn't the strongest form we expect.)

~~$ht X = n$ (X not necessarily $\mathbb{E}_\infty$) with trivial C_p action $\Rightarrow ht X^{tC_p} \leq n-1$~~

X $T(n)$-local $\Rightarrow L_{T(n)} X^{tC_p} = 0$.

Thm (Redshift Thm, C-M-N-N-BSY, 2022).

$ht R \geq 0$ $\mathbb{E}_\infty$-ring.

$$\Rightarrow ht(K(R)) = ht R + 1.$$

So we can get a large number of well-behaved examples in $Sp_{T(n)}$ as long as we have enough inputs.

A mysterious input comes from the ambidexterity and cyclotomic extns of $\mathbb{S}$ (~~and~~ part ~~is later to~~ will be discussed later).

In particular, cyclotomic complete spectra play an important role in the final disproof.

Today I want to show some purity propsitions leading to Redshift Thm. These results will last for the following several talks.

xt:

DGM Theorem:

$B \to A$ morphism b/w connective $\mathbb{E}_1$-ring,

s.t $\pi_0(B) \to \pi_0(A)$ surj with nil Kernel.

Then
$$\begin{tikzcd} K(B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner h"] & TC(B) \arrow[d] \\ K(A) \arrow[r] & TC(A) \end{tikzcd}$$

(2): HA 7.4.1.26:

A connective $\mathbb{E}_k$-alg.

Then Postnikov Tower

$$\cdots \to \tau_{\leq 3} A \to \tau_{\leq 2} A \to \tau_{\leq 1} A \to \tau_{\leq 0} A$$

is a successed square-zero extn,

Hence, DGM Theorem could be used inductively for connective rings.

Target of today: and next talk:

Purity Thm:

$n \geq 1$

$A \to L_{T(n-1) \oplus T(n)} A$ induces an equiv in $L_{T(n)} K(-)$.

We'll first prove a weaker version.

Thm: (1) $n \geq 1$. $A \to L_{T(0) \oplus \cdots \oplus T(n)} A$ induces equiv in $L_{T(n)} K(-)$.

(2) $n \geq 2$ $A \to L_{T(1) \oplus \cdots T(n)} A$ induces equiv in $L_{T(n)} K(-)$.

Cor: $\mathrm{ht}\, K(R) \leq \mathrm{ht}\, R + 1$

Pf: Immediately from Purity Thm.

Lem:

(1): V_n type n finite sp. with V_n-self map v_n.

X $T(n)$-acyclic

$\Leftrightarrow v_n^{-1} \pi_* \mathrm{Map}_*(V_n, X) = 0$.

(2): $\tau_{\geq k} X \to X$ are $T(n)$-equiv maps.

Pf: (1):

$$\mathrm{Map}_*(V_n, X) \simeq DV_n \otimes X$$

and the corresponding V_n-map comes from the v_n of DV_n.

$\therefore T(n) \otimes X \simeq DV_n \otimes X [DV_n^{-1}]$

$\simeq \mathrm{Map}_*(V_n, X)[V_n^{-1}]$.

(2): The cofiber is $\tau_{\leq k}$.

whose V_n-cohomology vanishes obviously

Lem: R ring sp

then $K(n)$-acyclic $\Leftrightarrow$ $T(n)$-acyclic.

Pf: $\Leftarrow$ $\surd$

$\Rightarrow$: 不妨设 V_n ring sp.

and then $T(n)$ is ring sp.

$\Rightarrow$ $T(n) \otimes R$ is a ring.

WTS $T(n) \otimes R = 0$

It suffices to check that $1 \in \pi_0(T(n) \otimes R)$ nil

Thus we could use Nil thm by tensoring $K(m)$

- $m \neq n$ $K(m) \otimes T(n) = 0$ unwinding periodicity thm
- $m = n$ since R $K(n)$-acyclic.

Notation:

$L_n^f := L_{H\mathbb{Q}\oplus T(1)\oplus\cdots\oplus T(n)}$.

$L_n^{p.f} := L_{T(0)\oplus\cdots\oplus T(n)}$ $\quad T(0) := \mathbb{S}[\frac{1}{p}]$.

(smashing localization).

Then

$$\begin{tikzcd} L_n^{p.f}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\, h"] & L_{T(m+1)\oplus\cdots\oplus T(n)}X \arrow[d] \\ L_m^{p.f}X \arrow[r] & L_m^{p.f} L_{T(m+1)\oplus\cdots\oplus T(n)}X \end{tikzcd}$$

Prop: (1) $\Sigma^\infty\Omega^\infty : Sp \to Sp$ detects $T(n)$-local equiv $n\geq 1$

(2) $\Sigma^\infty\Omega^\infty$ preserves m-connective $T(n)$-local equiv for $m \gg 0$ and n fixed.

Pf:

(1): Recall: $\exists$ Bousfield-Kuhn functor

$\Phi_i \circ \Omega^\infty \simeq L_{T(i)}$.

Hence, applying it to $\Omega^\infty \to \Omega^\infty\Sigma^\infty\Omega^\infty \to \Omega^\infty$, (zig-zag diagram).

we get $L_{T(i)} \to L_{T(i)}\Sigma^\infty\Omega^\infty \leftarrow L_{T(i)}$.

which implies the detection property.

(2): Claim: $\exists m\in\mathbb{N}$ s.t $\forall F$ m-connective $\boxed{\text{space}}$ $(m = n+1)$

with trivial $V_n^{-1}\pi_* \mathrm{map}(V_n, F)$, it must be $T(n)$-acyclic.

By induction on Portnikov Tower,

it suffices to show that $T(i)\otimes K(\pi, r) = 0$ $\forall r>i$ π finite gp. (odim generates all gp)

Then the rest relies on [CSY22, Thm E] since $T(i)\otimes K(r) = 0$ $\forall r>i$.

Use this claim to the fiber of $\Omega^\infty X \to \Omega^\infty Y$ to conclude it. $\square$

Prop: $n\geq 1$ $\exists N \gg 0$ s.t

$A \to B$ N-connective $L_n^{p.f}$-equiv

b/w connective ring sp

$K(A) \to K(B)$ is again $L_n^{p.f}$-equiv.

Pf: $(N = m-1 = n)$

$K(A) \to K(B)$ $T(0)$-equiv (Exercise).

WTS $\tau_{\geq 1}K(A) \to \tau_{\geq 1}K(B)$ $T(i)$-equiv $\forall 1\leq i\leq$

By plus construction

$\Sigma^\infty BGL \simeq \Sigma^\infty\Omega^\infty\tau_{\geq 1}K$,

it suffices to show that

$BGL(A) \to BGL(B)$ m-c

& induce

equiv b/c $V_i^{-1}\pi_*(-)$

Due to B just shifts all data, we may only consider GL.

Then $GL(\pi_0) \simeq \pi_0 GL$,

We may reduce it to $\tau_{\geq 1}GL \simeq \tau_{\geq 1}\underline{M}$ by LR decomposition $\infty\times\infty$ m

and then $Mat \simeq \mathrm{colim}\ \Omega^\infty\Lambda^r$

All necessary claims are right in this case.

Def: A localizing invariant is

$E : \mathrm{Cat}_\infty^{perf} \longrightarrow Sp$ making exact seq to fiber seq.

(e.g. $K(-)$).

We say E truncating on M-acyclic ring spectra if $E(R) \xrightarrow{\sim} E(\pi_0 R)$ $\forall$ connective M-acyclic R.

Lem. E localizing invariant.

If $\exists k \geq 0$ s.t $E(R) \xrightarrow{\sim} E(\tau_{\leq k} R)$ $\forall$ M-acyclic connective R, then E is truncating.

Cor: $L_{T(n)} K(-)$ is $L_n^{p.f}$-acyclic. (by prop above).

Cor $L_{T(n)} K(\mathbb{Z}/p^i) = 0$ $\forall i \geq 1$ $n \geq 1$

Pf: $K(\mathbb{F}_p)^\wedge_p = H\mathbb{Z}_p$ doesn't have any higher chromatic information.

Proof of Lem:

It suffices to show $E(\tau_{\leq k} R) \xrightarrow{\sim} E(\tau_{\leq k-1} R)$. $\forall$ --.

In this case

$$\begin{tikzcd} \tau_{\leq k} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & H\pi_0 R \arrow[d] \\ \tau_{\leq k-1} R \arrow[r] & H\pi_0 R \oplus H\pi_k R[k+1] \end{tikzcd}$$

induces a motivic pullback and thus

$$\begin{tikzcd} E(\tau_{\leq k} R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E(H\pi_0 R) \arrow[d] \\ E(\tau_{\leq k-1} R) \arrow[r] & E\left(\tau_{\leq k-1} R \odot^{H\pi_0 R \oplus H\pi_k R[k+1]}_{\tau_{\leq k} R} H\pi_0 R\right) \end{tikzcd}$$

The inner map induces an equiv on $\tau_{\leq k}$ and hence on $E(-)$ $\square$

Refer to [LT19].

Lem: A ring sp, then

$\mathcal{C}_{>n} \otimes \mathrm{Perf}(A) \longrightarrow \mathrm{Perf}(A) \longrightarrow \mathrm{Perf}(L_n^{p.f} A)$ exact

whose endomorphism ring of each obj is $L_n^{p.f}$-acyclic.

Pf: Recall the classical version.

$\mathcal{C}_{>n} \longrightarrow \mathrm{Perf}(\mathbb{S}) \longrightarrow \mathrm{Perf}(L_n^{p.f} \mathbb{S})$

Tensor it with $\mathrm{Perf}(A)$ and use smashing prop we obtain the first exact seq.

For the second claim, note that $\mathcal{C}_{>n} \otimes \mathrm{Perf}(A)$ is gen by $A \otimes F$ for $F \in \mathcal{C}_{>n}$ whose endor ring is $DF \otimes F \otimes A$ are $L_n^{p.f}$-acyclic as well $\square$.

Prop: $\mathcal{C}$ stable with $L_n^{p.f}$-acyclic endor ring for each $X \in \mathcal{C}$.

Then $L_{T(i)} K(\mathcal{C}) = 0$ $\forall$ $1 \leq i \leq n$.

Pf: $(\tau_{\geq 0} K(\mathcal{C}))[1] \simeq |K^{add}(S.\mathcal{C})|$.

comes from a colim of $\tau_{\geq 0} K(\hom_{\mathcal{C}}(X, X))$ $L_n^{p.f}$-acyclic.

~~Whose~~

Its π_0 are p-power torsion and thus $L_{T(i)}(-)$ vanishes by Cor above.

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Proof of ~~Ma~~ Our Target Today:

WTS $K(\mathcal{C}_{>n} \otimes \mathrm{Perf}(A)) = 0$

It's true since endo rings are $L_n^{p.f}$-acyclic and prop above.

For the stronger claim,

$$\begin{tikzcd} L_n^{p.f} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner h"] & L_{T(1)\oplus\cdots\oplus T(n)} A \arrow[d] \\ A[\frac{1}{p}] \arrow[r] & (L_{T(1)\oplus\cdots\oplus T(n)} A)[\frac{1}{p}]. \end{tikzcd}$$

[T19] tells us that

$$\begin{tikzcd} K(L_n^{p.f} A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner h"] & K(L_{T(1)\oplus\cdots\oplus T(n)} A) \arrow[d] \\ K(A[\frac{1}{p}]) \arrow[r] & K((L_{T(1)\oplus\cdots\oplus T(n)} A)[\frac{1}{p}]) \end{tikzcd}$$

By the first part,

we only need to show that the top map is $T(i)$-equiv for $i \geq 2$.

Thus we could reduce it to the bottom map.

They are algebras over $K(\mathbb{S}[\frac{1}{p}]) \underset{p\text{-adically}}{\simeq} K(\mathbb{Z}[\frac{1}{p}])$.

~~Then~~ The right one vanishes after $L_{T(i)}$ for Mitchell's Redshift for discrete rings. $\square$

Cor: $A \to B$ $T(1)\oplus\cdots\oplus T(n)$-equiv b/w connective ring sp. induces a surj on π_0 with nil Ker.

$\Rightarrow K(A) \to K(B)$ still $T(1)\oplus\cdots\oplus T(n)$-equiv

Pf:

$$\begin{tikzcd} A \arrow[r] & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner h"] & B \arrow[d] \\ & A[\frac{1}{p}] \arrow[r] & B[\frac{1}{p}] \end{tikzcd}$$

...

[[T19] helps us obtain a same homotopy pullback after taking K.

Then $K(\mathbb{S}[\frac{1}{p}]) \to K(A[\frac{1}{p}]) \xrightarrow{p\text{-adic equiv}} K(B[\frac{1}{p}])$

since p-adic K truncating on $\mathbb{S}[\frac{1}{p}]$ & nil-invariant.

$\Rightarrow K(P) \to K(B)$ $T(i)$-equiv $1 \leq i \leq n$.

Besides, $A \to P$ is $T(0)\oplus\cdots\oplus T(n)$-equiv

$\Rightarrow$ Theorem implies $K(A) \to K(P)$ $T(i)$-equiv $1 \leq i \leq n$.

Combing them ---- $\square$

Cor: The same as above, then $TC(A) \to TC(B)$ $T(i)$-equiv $1 \leq i \leq n$.

Pf:

$$\begin{tikzcd} K(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & TC(A) \arrow[d] \\ K(B) \arrow[r] & TC(B) \end{tikzcd}$$

$\square$

Conclusion:

Highly connectively $\overset{\text{reduce to}}{\Longrightarrow}$ Verify π_0 $\overset{\text{reduce to}}{\Longrightarrow}$ Classical version.

Cor: $n \geq 2$

$$L_{T(n)} K(A) \simeq L_{T(n)} K(\tau_{\geq 0} A) \simeq L_{T(n)} TC(\tau_{\geq 0} A).$$

Pf: Obvious. $\tau_{\geq 0} A \longrightarrow A$ $K(1)$-equiv.

Then by Lurie's zero-square tower and DGM thm we could reduce it to A discrete.

In which case $L_{T(n)} K(\mathbb{Z}) = 0$ is the base ring. $\square$

Rmk: The most explicit feature of this part is that there're large quantities of vanishment and algebra structure.
It's insightful but a little bit ridiculous for beginners when they first witness this procedure.

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Talk2 - Descent Theory.

Target (Inductive Vanishing):

R $\mathbb{E}_\infty$-ring $n \geq 1$

Then (A) $\Rightarrow$ (B) $\Rightarrow$ (C):

(A): $L_{T(n)} R = 0$ & $L_{T(n)} K(R^{tG}) = 0$.

(B): $\forall$ G finite gp $\circlearrowleft$ $\mathcal{C}$ R-linear idempotent-complete stable ∞-gp. Thus we have motivic pullback.

the following descent thm holds

$$L_{T(n)} K(\mathcal{C}^{hG}) \xrightarrow{\sim} \left(L_{T(n)} K(\mathcal{C})\right)^{hG}$$

(C): $L_{T(i)} K(R) = 0$ $\forall i \geq n+1$.

Rmk: Next week, we'll enhance this descent to cases of <u>π-finite</u> spaces.

Cor: $n \geq 0$ $\mathcal{C}$ $L_n^{p.f}$-local stable ∞-cat.

$\Rightarrow L_{T(m)} K(\mathcal{C}) = 0$ $\forall m \geq n+2$

& $\forall$ finite p-gp $G \circlearrowleft \mathcal{C}$

$$L_{T(n+1)} K(\mathcal{C}^{hG}) \simeq \left(L_{T(n+1)} K(\mathcal{C})\right)^{hG}$$

Pf: It suffices to show that for $R = L_n^{p.f} \mathbb{S}$

$$L_{T(n+1)} R = 0 \quad \& \quad \underline{L_{T(n+1)} K(R^{tG})} = 0.$$

From defn

Kuhn: Blueshift property implies that R^{tG} is $L_{n-1}^{p.f}$-local.

$\Rightarrow K(R^{tG})$ is thus $K(L_{n-1}^{p.f} \mathbb{S})$-mod which has trivial $T(n+1)$-cohomology.

use induction

Cor (Purity Thm):

A $\mathbb{E}_1$-ring $n \geq 1$

$A \longrightarrow L_{T(n-1) \oplus T(n)} A$ induces an equiv on $L_{T(n)} K(-)$.

Pf: Because of the main thm of LNNT, we may assume A is $L_n^{p.f}$-local.

Thus we have motivic pullback.

$$\begin{tikzcd} K(A) \arrow[r] \arrow[d, "\Gamma_h"'] & K(L_{T(n-1)\oplus T(n)} A) \arrow[d] \\ K(L_{n-2}^{p.f} A) \arrow[r] & K(L_{n-2}^{p.f}(L_{T(n-1)\oplus T(n)} A)) \end{tikzcd}$$

Both two items are $T(n)$-acyclic since $\mathrm{Perf}(-)$ are $L_{n-2}^{p.f}$-local.

Cor (General Descent):

R $\mathbb{E}_\infty$-ring. If $L_{T(n)}(R^{tG}) = 0$, then $\forall$ $\mathcal{C}$ R-linear idem-complete stable ∞-cat with an R-linear finite p-gp G-action, and $\forall$ E localizing invariant.

$$L_{T(n+1)}(E(\mathcal{C})_{hG}) \simeq L_{T(n+1)} E(\mathcal{C}_{hG}) \simeq L_{T(n+1)} E(\mathcal{C}^{hG})$$
$$\simeq L_{T(n+1)}(E(\mathcal{C})^{hG}) \simeq \left(L_{T(n+1)} E(\mathcal{C})\right)^{hG}.$$

Pf: There's a universal object

$\mathcal{U}: \mathrm{Cat}_\infty^{\mathrm{Perf},R} \longrightarrow \underline{\mathrm{Mot}_R}$ initial

for this kind of data, whose underling sp is $K(-)$. If the initial one satisfies this descent, then so do its modules by formal proof.

$\boxed{\text{We'll prove this in the following paragraphs}}$ $\square$

Cor: $A \longrightarrow B$ $\underline{T(n)\text{-local Galois ext'n}}$ b/w $\mathbb{E}_\infty$-rings with Galois gp G finite p-gp

i.e. $\begin{cases} B \otimes_A B \longrightarrow \prod_G B & T(n)\text{-equiv.} \\ A \xrightarrow{\sim} B^{hG} \end{cases}$

$\Rightarrow L_{T(n+1)} K(A) \xrightarrow{\sim} L_{T(n+1)}(K(B)^{hG}) \xrightarrow{\sim} (L_{T(n+1)} K(B))^{hG}$.

Pf: $K'(A) := K(\{T(n)\text{-local dualizable } A\text{-mod}\}) =: \mathcal{D}(A)$.

A $T(n)$-local $\mathbb{E}_\infty$-ring

$\therefore \mathrm{Perf}(A) \subseteq \mathcal{D}(A) \rightsquigarrow K(A) \longrightarrow K'(A)$.

and K' satisfies similar property due to descent

and $L_{T(n)} \mathrm{Mod}_A \simeq (L_{T(n)} \mathrm{Mod}_B)^{hG}$.

Besides fiber of $K(A) \longrightarrow K'(A)$ is $\underbrace{K(L_{n-1}^{p.f} \mathbb{S})\text{-mod}}_{\text{no } T(n+1)\text{-data}}$.

i.e. $\mathcal{D}(A)/\mathrm{Perf}(A)$ is $L_{n-1}^{p.f}$-local.

i.e. $\forall F$ type n finite complex $M \in \mathcal{D}(A)$

$M \otimes F \in \mathrm{Perf}(A)$.

b/c: M dualizable $\Rightarrow \mathrm{Hom}_{L_{T(n)}\mathrm{Mod}_A}(M, -)$ commutes with colim

$\xrightarrow{\text{Atv}} - \otimes F$, $M \otimes F$ cpt since $\otimes F$ preserves colim.

$\Rightarrow M \otimes F$ belongs to the thick subcat gen by $A \otimes F$ (since $L_{T(n)}\mathrm{Mod}_A$ cpt gen by $A \otimes F$)

$\Rightarrow M \otimes F$ perfect A-mod. $\square$

Fact (Lazy to prove it. Resolution: Mr. Yueshi Hou and I will organize a seminar on Matthew's descent theory): $\forall R$ $\mathbb{E}_\infty$-ring

(1) $L_{T(n)}(\Phi^G \mathrm{Mod}_{\mathbb{S}_p}(R)) = 0$

$\Leftrightarrow$ (2) $L_{T(n)} \mathrm{Mod}_G(R)$ Borel complete.

i.e. $M(G/H) \simeq M(G)^{hH}$.

It's not hard, but tedious, especially for readers unfamiliar with it

Prop: TFAE:

(1): ...

(2): ...

(3): $\forall R$-linear idem-complete stable co-cat $\mathcal{C}$ with G-act, G finite p-gp. $\forall$ localizing invariant Ξ with values in ~~[illegible]~~ sp.

$L_{T(n)} \Xi(\mathcal{C}^{hG}) \underset{\text{assum}}{\simeq} (L_{T(n)} \Xi(\mathcal{C}))^{hG}$.

(4) The same as (3).

$L_{T(n)}(E(\mathcal{C})_{hG}) \simeq L_{T(n)} E(\mathcal{C}_{hG})$

$\simeq L_{T(n)} E(?^{hG})$

$\simeq L_{T(n)}(E(\mathcal{C})^{hG})$

$\simeq (L_{T(n)} E(\mathcal{C}))^{hG}$.

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Pf: (1) $\Leftrightarrow$ (2) by fact

(3) $\Rightarrow$ (2) ✓ for $E = L_{T(n)}K(-)$
$G = C_p$.

(4) $\Rightarrow$ (3) ✓

(1) $\Rightarrow$ (3) Dévissage: Assume $G = C_p$.

Then $L_{T(n)}E_{C_p}(\mathcal{C})$ is module over $L_{T(n)}K_{C_p}(R)$.

Then the fact implies the case of E since Φ^{C_p} sym monoidal.

(1) $\Rightarrow$ (4).

It comes from

$$E(\mathcal{C}_{\infty Bor})_{\infty Bor} \xrightarrow{\sim} E(\mathcal{C}_{\infty Bor}) \longrightarrow E(\mathcal{C}^{Bor})$$

$$\longrightarrow E(\mathcal{C}^{Bor})^{Bor} \longrightarrow (L_{T(n)}E(\mathcal{C}^{Bor}))^{Bor}$$

After $L_{T(n)}$, they become modules over $L_{T(n)}K_G(R)$ whose Borel-completeness implies that of E's.

$\xrightarrow[\text{obj}]{\text{underlying}}$ $L_{T(n)}$-localization induces equiv on G-fixed pts are the same

Proof of Target:

i.e.

(1) $L_{T(i)}R = 0$ & $L_{T(i)}K(R^{tC_p}) = 0$

$\Rightarrow$ (2) $L_{T(i)}\Phi^{C_p}(K_{C_p}(R)) = 0$

$\Rightarrow$ (3) $L_{T(j)}K(R) = 0 \quad \forall j \geq i+1$.

(2) $\Rightarrow$ (1) From $\exists\ \Phi^{C_p}K_{C_p}(R) \to K(R)^{tC_p}$

$L_{T(i)}\Phi^{C_p}(K_{C_p}(R)) = 0$.

we know $L_{T(i)}(K(R)^{tC_p}) = 0$

$\Rightarrow L_{T(i)}K(R) = 0 \quad \forall j \geq i+1$ by [BSY22]

(1) $\Rightarrow$ (2):

It's reasonable to assume R connective.

(Depends on a lot of lemma. some even in [BS18])

Then $\exists\ K(\mathrm{Perf}(R)_{\infty Bor})$

(the existence of $\mathcal{U}$) $\longrightarrow K_{C_p}(R) \longrightarrow K(\mathrm{Perf}(R)^{Bor}/\mathrm{Perf}_{\infty Bor})$

∞-fib seq.

linear on R^{tC_p}

Hence the latter term vanishes.

So it suffices to show

$$L_{T(i)}\Phi^{C_p}K(\mathrm{Perf}(R)_{\infty Bor}) = 0$$

DGM Theorem implies that

$$\begin{tikzcd} K(\mathrm{Perf}(R)_{\infty Bor}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & TC(\text{—}) \arrow[d] \\ K(\mathrm{Perf}(\tau_0 R)_{\infty Bor}) \arrow[r] & TC(\text{—}\tau_0) \end{tikzcd}$$

where the ~~right~~ left Bottom one is 0 after $L_{T(i)}\Phi^{C_p}$ due to ~~Mitchell's~~ CMNN Lem 4.3. and the right top one ---.

Rmk: Mackey functors due to [HN19, Theorem 4.1] is a way to understand equivariant spectra and so on. But I believe Lurie's tempered cohomology will become the mainstream one day.

Here every difficult / rare construction are ~~functors~~ K-theory from the cat of Mackey functors

Recall:

Def: G finite gp.

$\mathrm{Burn}_G^{\mathrm{eff}}$ effective Burnsfield ∞-cat

i.e. $N_\bullet^{\mathrm{st}}$ of the following (weak) $(\infty,1)$-cat

- Ob: finite G-set S
- Mor:

$$S \leftarrow U \rightarrow T.$$

- 2-Mor: isom b/w spans.

Def: $\mathcal{C}$ ∞-seniaddive presentable.

$\mathrm{Mack}_G(\mathcal{C}) :=$ full subcat of $\mathcal{C}$-valued presheaves sending finite coproduct to product.

denote $M^H := M(G/H)$.

Rmk: Nonabelian ~~derived~~ ∞-cat is the case of $\mathcal{C} = \mathcal{S}$.

Moreover $\mathrm{Mack}_G(\mathcal{C}) = \mathcal{P}_\Sigma(\mathrm{Burn}_G^{\mathrm{eff}}) \underline{\otimes} \mathcal{C}$ (in Pr.)

symm mono cat when $\mathcal{C}$. -- (Day convolution)

Def:

Borel-complete means

$$M(G)^{hH} \simeq M(G/H).$$

CoBorel-complete means for M

$$(M \otimes \underbrace{\mathcal{K}(G)}_{\text{(Yoneda embedding)}})_{hG} \xrightarrow{\sim} M$$

Besides $\exists$ corresponding (co)Borelization functor $(\ )_{\mathrm{Bor}}$ and $(\ )_{\mathrm{coBor}}$.

with natural map $(\ )_{\mathrm{coBor}} \longrightarrow (\ )_{\mathrm{Bor}}$.

Rmk: When taking value in Mot,

we get the so-called universal Mackey functor for localizing invariants.

which induce $K_G : \mathrm{Fun}(BG, \mathrm{Cat}_\infty^{\mathrm{perf}}) \to \mathrm{Mack}_G$

$$A \mapsto (H \mapsto K(A^{hH}))$$

So descent data are translated into the language of Borelness of K_G.

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Main Theorem of this talk:

Higher (Galois) Descent for $T(\cdot)$-localized K-theory.

Thm: $\mathcal{C}$ L_n^f-local ∞-cat acted by a π-finite p-~~sp~~ p-space G,

then $L_{T(n+1)}K(\mathcal{C}^{hG}) \simeq L_{T(n+1)}K(\mathcal{C})^{hG}$

$L_{T(n+1)}K(\mathcal{C}_{hG}) \simeq L_{T(n+1)}K(\mathcal{C})_{hG}$.

Cor: $R \to S$ $T(n)$-local G-Galois extn for which G n-finite p-gp. then

$L_{T(n+1)}K(R) \to L_{T(n+1)}K(S)$ $T(n+1)$-local G-Galois extn.

Outline:

$\mathrm{Cat}_{L_n^f}$:= stable idem-complete L_n^f-local ∞-cats.

$L_{T(n+1)}K: \mathrm{Cat}_{L_n^f} \longrightarrow \mathrm{Sp}_{T(n+1)}$

~~Recall: Both two sides are ∞-semiadditive~~

WTS this functor preserves all π-finite p-space lim & colims.

All we have known is that this functor indeed preserves colim indexed by discrete p-space (CMNN). Then a natural way is to do induction and study monochromatic layer.

Recall: $X \in \mathrm{Sp}$ is called n-monochromatic if X is L_n^f-local but L_{n-1}^f-acyclic.

So there's $M_n^f \to L_n^f \to L_{n-1}^f$ a fiber sequence (from Sp to Sp).

Analogously, $\mathcal{C} \in \mathrm{Cat}_{perf}$ is called n-monochromatic if L_n^f-local but L_{n-1}^f-acyclic and all these cats forms a full subcat $\mathrm{Cat}_{M_n^f}$.

According to simple calculation, we could get a monochromatization functor

$M_n^f: \mathrm{Cat}_{L_n^f} \longrightarrow \mathrm{Cat}_{M_n^f}$

fitting into a Verdier seq

$M_n^f(\mathcal{C}) \to \mathcal{C} \to L_{n-1}^f(\mathcal{C})$.

Claim:

(1): M_n^f preserves all lim & colim.

(2) Purity isom: $\forall \mathcal{C} \in \mathrm{Cat}_{L_n^f}$

$L_{T(n+1)}K(M_n^f(\mathcal{C})) \xrightarrow{\sim} L_{T(n+1)}K(\mathcal{C})$

(3): $\mathrm{Cat}_{M_n^f}$ is ∞-semiadditive.

(4): $\forall$ n-monochromatic ring sp R, $\forall A \simeq K(G,m)$ (G p-gp)

$L_{T(n+1)}K(R)[A] \xrightarrow{\sim} L_{T(n+1)}K(R[\Omega A])$

Proof of Main Thm (by virtue of claims)

By Claim (1) (2) (3), it suffices to show that

$L_{T(n+1)}K: \mathrm{Cat}_{M_n^f} \longrightarrow \mathrm{Sp}_{T(n+1)}$

preserves π-finite p-gp indexed colim.

By [CS22] (fundamental facts about six-functor formalism on anima)

we only need to show that ~~prove to finite~~ this functor preserves constant colim indexed by $A \simeq K(G,m)$.

i.e. a similar isom for general cats instead of a rig

However, all cats $\mathcal{C}$ can be written as a filtered colim of Mod_R^{perf}. $\square$

Recall the abstract nonsense diagram about devissage.

$F: \mathcal{C} \to \mathcal{D}$ functor b/w p-typically co-semi cats

(1) If F preserves constant m-finite p-space indexed colim then F is p-typically m-semiadditive.

(2) If ~~else F pres~~ else F preserves all $(m-1)$-finite p-space indexed colim, then we only need to verify the case of $K(G,m)$ for the condition of (1).

Proof: (1) A general m-finite p-space

$$\begin{tikzcd} FA_! \arrow[r, "\sim"] & FA_* \\ A_!F \arrow[u, "\beta_!"] \arrow[r, "\sim"] & A_*F \arrow[u, "\beta_*"'] \end{tikzcd}$$

By do induction we could assume F $(m-1)$-semiadd. Then postinifortower implies it.

(Vertical maps are base change, A_* means $\mathcal{C}^A \to \mathcal{C}$ or $\mathcal{D}^A \to \mathcal{D}$)

$\Rightarrow$ $\beta_!$ admits a retract.

Simultaneously $A^* \dashv A_* \simeq A_!$ implies that

$$\begin{tikzcd} & A_!FA^*A_! \arrow[r] \arrow[d, "\sim", "\rho_!"'] & A_!F \arrow[d, "\beta_!"] \\ FA_! \arrow[r] \arrow[rr, bend right, "id"'] & FA_!A^*A_! \arrow[r] & FA_! \end{tikzcd}$$

$\Rightarrow$ $\beta_!$ admits a section

$\Rightarrow$ $\beta_!$ (thus β_*) isom i.e. preserves all m-finite colims.

(2) Similarly.

Claim (1):

$L: \mathrm{Cat}_{\mathcal{M}_n^f} \rightleftarrows \mathrm{Cat}_{\mathcal{L}_n^f} : \mathcal{M}_n^f$ (preserves all lim)

and $L \circ \mathcal{M}_n^f : \mathrm{Cat}_{\mathcal{L}_n^f} \longrightarrow \mathrm{Cat}_{\mathcal{L}_n^f}$ is given by $\underline{- \otimes \mathcal{M}_n^f(\mathrm{Perf}(\mathcal{L}_n^f \mathbb{S}))}$

(We have proved then in [MMT])

which preserves all colim as well $\square$

Claim (2):

$$L_{T(n+1)} K(\mathcal{M}_n^f(\mathcal{C})) \longrightarrow L_{T(n+1)} K(\mathcal{C}) \longrightarrow L_{T(n+1)} K(\mathcal{L}_{n-1}^f \mathcal{C})$$

but the third term has no $T(n) \oplus T(n+1)$-data. fiber seq.

So $L_{T(n+1)} K(\mathcal{M}_n^f(\mathcal{C})) \simeq L_{T(n+1)} K(\mathcal{C})$ by Parity thm.

Claim (3):

Fact: $\mathcal{M}_n^f \mathrm{Sp} \simeq \mathrm{Sp}_{T(n)}$.

Cor: $\mathrm{Ind}: \mathrm{Cat}_{\mathcal{M}_n^f} \rightleftarrows \mathrm{Pr}^L_{\underline{T(n),\omega}} : (-)^\omega$ is an adjoint equiv pair. compactly generated

Pf: Unwinding defn,

$\mathcal{C} \in \mathrm{Cat}_{\mathcal{L}_n^f}$ is n-mono

iff $\mathrm{Ind}(\mathcal{C}) \otimes \mathcal{M}_n^f \mathrm{Sp} \subset \mathrm{Ind}(\mathcal{C})$

At same time $\mathcal{M}_n^f \mathrm{Sp} \otimes \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}) \to \mathrm{Sp}_{T(n)} \otimes \mathrm{Ind}(\mathcal{C})$ in Pr^L

the composition is equiv

Hence. $\mathcal{C}$ n-mono $\Leftrightarrow$ $\mathcal{C}$ $T(n)$-local

Thus the equiv in HA

$\mathrm{Cat}_{perf} \overset{\sim}{\rightleftarrows} \mathrm{Pr}^L_\omega$ restricts to this one. $\square$

Proof of Claim (3):

Replace $\mathrm{Cat}_{\mathcal{M}_n^f}$ by $\mathrm{Pr}^L_{T(n),\omega}$.

Do induction and use [CSY22]

it suffices to show that the canonical pairing

$\mathrm{Sp}_{T(n)}[A] \otimes \mathrm{Sp}_{T(n)}[A] \longrightarrow \mathrm{Sp}_{T(n)}$ $\forall A$ m-finite space

exhibits $\mathrm{Sp}_{T(n)}[A]$ as a self-dual obj.

The only possible coev comes from $\Delta: A \to A \times A$

is an $(m-1)$-finite map. $\square$

Rmk: In [Abdx and Finite Spans]

the key input of the universal property of S_n^m is the "formal" dual of Δ.

Claim (4):

Prop: $\mathrm{Perf}(R[\Omega A]) \simeq \mathrm{Perf}(R)[A] \in \mathrm{Cat}_{\mathrm{perf}}$.

Pf: It suffices to show that

$$\mathrm{LMod}_{R[\Omega A]}(\mathrm{Sp}) \simeq \mathrm{LMod}_R(\mathrm{Sp})[A].$$

See [HA, Prop 4.8.5.11].

Proof of Claim:

Induction. We may assume $A \simeq K(G,m)$ for $m \geq 2$.

Then $A \simeq \varinjlim_{\Delta^{op}} (\Omega A)^k =: \varinjlim A_k$

($n=0$ exercise)

For $n \geq 1$, we have

$$\begin{tikzcd}
\varinjlim L_{T(n+1)}K(R)[A_k] \arrow[r, "②"] \arrow[d, "①"'] & \varinjlim L_{T(n+1)}K(R[\Omega A_k]) \arrow[d, "③"] \\
L_{T(n+1)}K(R)[\varinjlim A_k] \arrow[r, "⑤"] & L_{T(n+1)}K(\varinjlim R[\Omega A_k]) \arrow[d, "④"] \\
 & L_{T(n+1)}K(R[\Omega \varinjlim A_k])
\end{tikzcd}$$

WTS ⑤ is equiv. We'll prove that ①~④ equiv.

①: $L_{T(n+1)}K(R)[-]$ preserves all colims

②: Induction Hypothesis (w.r.t m)

③: $L_{T(n+1)}K$ preserves sifted colim. [LMMT 4.31] (Recall convert it to TC).

④: Ω, $R[-]$ preserves sifted colim. $\square$

Proof of Higher Galois Descent.

$$\mathrm{Mod}_R(\mathrm{Sp}_{T(n)}) \simeq \mathrm{Mod}_S(\mathrm{Sp}_{T(n)})^{hG}$$

(exercise, or refer to prop 4.10).

$$\Rightarrow L_{T(n+1)}K(R)$$

$$\simeq L_{T(n+1)}K(\mathrm{Mod}_R(\mathrm{Sp}_{T(n)})^{dbl})$$

$$\simeq L_{T(n+1)}K((\mathrm{Mod}_S(\mathrm{Sp}_{T(n)})^{dbl})^{hG})$$

$$\simeq L_{T(n+1)}K(\mathrm{Mod}_S(\mathrm{Sp}_{T(n)})^{dbl})^{hG}$$

$$\simeq L_{T(n+1)}K(S)^{hG}.$$ (See last talk

(The first condition of Galois extn)

The second one is automatically satisfied. $\square$

Higher Semiadditive Algebraic K-Theory.

Recall: $\mathcal{S}_n$ mode of presentable p-typically co-semiadd of semi ht n.

Schlank once believed that $\mathcal{S}_n \simeq Sp_{T(n)}$.

But he was challenged soon by Allen Yuan.

Def: $\mathcal{S}_m^{(p)} \subset \mathcal{S}$ full subcat of m-finite p-spaces for $-2 \le m \le \infty$.

$$PCMon_m^{(p)}(\mathcal{C}) := Fun(Span(\mathcal{S}_m^{(p)})^{op}, \mathcal{C})$$

(p-typical) pre-m-commutative monoids

$\forall X \in PCMon_m^{(p)}(\mathcal{C})$ $\underline{X}$, the underlying obj, is defined as $X(*)$.

Def: $X \in PCMon_m^{(p)}(\mathcal{C})$ is called (p-typical) m-commutative monoid if it satisfies m-Segal condition, i.e. $X(A) \xrightarrow{\sim} \lim_{A} \underline{X}$.

All these monoids form a full subcat denoted by $CMon_m^{(p)}(\mathcal{C})$.

Fact: $CMon_m^{(p)}(\mathcal{C})$ presentable for $\mathcal{C} \subset Pr^L$.

and $\exists\, L^{seg}: PCMon \longrightarrow CMon$ the left adj of the inclusion.

Fact: $CMon_m^{(p)}(Sp_{(p)})$ is one model of the mode classifying p-local stable & p-typically m-sems. (denoted by $\mathcal{S}^{[m]}$)

Thm (See [Abbr and Finite Span]):

$$Cat_{m\text{-fin}} \xleftrightarrow{\sim} CMon_m^{(p)}(Cat_{m\text{-fin}})$$

admitting all m-finite colims.

the inverse map $(-)^{\sqcup}$ is called m-cartesian str.

Construction.

$$K^{[m]}: Cat_{m\text{-fin}}^{st} \xrightarrow{(-)^{\sqcup}} CMon_m^{(p)}(Cat_{m\text{-fin}}^{st})$$

m-semiadditive algebraic K-theory.

$\downarrow$ Waldhausen S-construction 8.

$$CMon_m^{(p)}(S)^{\Delta^{op}}$$

$\downarrow (-)^{gp}$ group completion

$$CMon_m^{(p)}(Sp)^{\Delta^{op}} \simeq (\mathcal{S}^{[m]})^{\Delta^{op}}$$

$\downarrow |\cdot|$

$$\mathcal{S}^{[m]}$$

$\downarrow \Omega$

$$\mathcal{S}^{[m]}$$

Unwinding defn, it's the m-segal condition of $A \mapsto K(\mathcal{C}^A)$

Conjecture:

Recall: $m > n$.

$$L_{T(n+1)}^{\mathcal{S}^{[m]}}: \mathcal{S}^{[m]} \rightleftarrows Sp_{T(n+1)} : L$$

$\forall R \in Alg(Sp_{T(n)})$.

So there's a comparison map

$$L_{T(n+1)} K(R) \longrightarrow L_{T(n+1)}^{\mathcal{S}^{[m]}} K^{[m]}(R)$$

Conj (1): $K^{[m]}(R)$ is automatically $T(n+1)$-localized.

Conj (2): The comparison map is equiv.

If both of them are true, we can get a simpler way to understand chromatically localized K-th (i.e. $K^{[m]}(R)$)

(Levy thought this idea is of few interests.)

Conj (1) is very hard.

But we're ready to prove Conj (2).

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Proof:

It suffices to show that

$L_{T(n+1)}K$ satisfies the following segal condition.

$\forall$ A π-finite p-space $\forall \mathcal{C} \in \mathrm{Cat}_{L,f}$ admitting m-finite colims.

$$L_{T(n+1)}K(\mathcal{C}^A) \simeq L_{T(n+1)}K(\mathcal{C})^A.$$

this is just our main thm. $\square$

Rmk: We have similar redshift conj for $K^{[m]}$.

But Conj(2) gives us an affirmative answer to it.

Chromatic Fourier Transform.

For lack of time, we have to ignore many details. Our main goal is to define a general notion of cyclotomic extn and then prove the shifting property of alg K-thy.

Def: Brown–Comenetz dual (Pontryagin dual in Sp).

$I_{\mathbb{Q}_p/\mathbb{Z}_p}$:= the spectrum which represents $\mathrm{Hom}_{Ab}(\pi_*(-), \mathbb{Q}_p/\mathbb{Z}_p)$.

$I_p^{(n)} M := \tau_{\geq 0}\, \mathcal{H}om(M, \Sigma^n I_{\mathbb{Q}_p/\mathbb{Z}_p})$ for $M \in Sp^{cn}$.

n-shifted Brown–Comenetz dual.

Def: $\mathcal{C} \in CAlg(Cat_\infty)$

$\mathcal{R} \in CAlg(Sp^{cn}_{(p)})$

$\forall S \in CAlg(\mathcal{C})$, we could define the $\mathcal{R}$-preorientation of ht n of S (or, of $\mathcal{C}$, if $S = \mathbb{1}_{\mathcal{C}}$) as <u>every</u> $w: I_p^{(n)}\mathcal{R} \longrightarrow S^\times$ in Sp^{cn}

Let $PO_{\mathcal{R}}^{(n)}(S; \mathcal{C}) := \mathrm{Map}_{Sp^{cn}}(I_p^{(n)}\mathcal{R}, S^\times)$.

Rmk: Due to adj $\mathbb{1}[-] : Sp^{cn} \rightleftharpoons CAlg(\mathcal{C}) : (-)^\times$,

$\mathcal{R}$-preori^w could be regarded as $\mathbb{1}[I_p^{(n)}\mathcal{R}] \xrightarrow{w} S$.

i.e. $\mathbb{1}[I_p^{(n)}\mathcal{R}]$ orepresents $PO_{\mathcal{R}}^{(n)}(-; \mathcal{C})$

and is thus the classifier of preori.

Example: $I_p^{(n)}\mathbb{Z}/p^r \simeq \Sigma^n \mathbb{Z}/p^r$ (preori $\longleftrightarrow$ root of unity)

This observation recovers all theorems in Higher Cyclotomic Extn.

Def:

$$\begin{tikzcd} pt \arrow[r, "\mathbb{1}"] \arrow[d, "I_p^{(n)}\mathcal{R}"'] \arrow[dr, phantom, "\Downarrow \mathrm{RKE}"] & CAlg(\mathcal{C}) \\ Mod_{\mathcal{R}}^{[0,n]} \arrow[ur, dashed, "\mathbb{1}^{[I_p^{(n)}(-)]}"'] & \end{tikzcd}$$

$\mathcal{R}$-mod concentrating in $[0,n]$

Hence: $\mathbb{1}[-] \longrightarrow \mathbb{1}^{[I_p^{(n)}(-)]}$

$\rightsquigarrow \underline{\mathbb{1}[I_p^{(n)}\mathcal{R}] \longrightarrow \mathbb{1}}$

i.e. $PO_{\mathcal{R}}^{(n)}(\mathbb{1}; \mathcal{C})$

or $PO_{\mathcal{R}}^{(n)}(\mathcal{C})$

So we define the associated Fourier transform of $w: \mathbb{1}[I_p^{(n)}\mathcal{R}] \longrightarrow \mathbb{1} \in PO_{\mathcal{R}}^{(n)}(\mathcal{C})$ to be the corresponding natural trans,

$\mathcal{F}_w : \mathbb{1}[-] \longrightarrow \mathbb{1}^{[I_p^{(n)}(-)]}$ underlying space

Def: A preori^w of $\mathcal{C}$ is called an orientation if $\mathcal{F}_w$ is equiv for $\forall M \in Mod_{\mathcal{R}}^{[0,n]}$.

All the summands of ori form a subspace denoted by $Or_{\mathcal{R}}^{(n)}\mathcal{C}$.

~~Rmk.~~

Example: ~~The cyclotomic extn~~

As above, orientation is the same as (higher) primitive roots of unity.

Rmk: The orientation in EC2 & EC3 is equiv to this defn.

Now we are going to constract cyclotomic extns as orientation classifiers.

Prop: $\mathcal{C} \in CAlg(Pr)$ n-semiadd.
$R \in CAlg(Sp^{cn}_{(p)})$

$POr^{(n)}_{R}(-) : CAlg(Mod_{\mathcal{C}}(Pr)) \longrightarrow S$
is corep by $Mod_{\mathbb{1}[I_p^{(n)}R]}(\mathcal{C})$.
(Categorified Version)

Pf: $Map(Mod_{\mathbb{1}[I_p^{(n)}R]}(\mathcal{C}), \mathcal{D})$
$\simeq Map(\mathbb{1}_{\mathcal{C}}[I_p^{(n)}R], End(\mathbb{1}_{\mathcal{D}}))$
$\simeq Map(I_p^{(n)}R, End(\mathbb{1}_{\mathcal{D}})^{\times})$
$\simeq Map(I_p^{(n)}R, \mathbb{1}_{\mathcal{D}}^{\times}) = POr^{(n)}_{R}(\mathcal{D})$

Def: $\mathcal{C}^{por}_{R,n} := Mod_{\mathbb{1}[I_p^{(n)}R]}(\mathcal{C})$

Let $u_{trant} : I_p^{(n)}R \longrightarrow (\mathbb{1}_{\mathcal{C}^{por}_{R,n}})^{\times} = \mathbb{1}[I_p^{(n)}R]^{\times}$
be the trivial preori.

Rmk:
Hence, every preori comes from the classifying functor and u_{trant}.

Def: $\mathcal{C}^{or}_{R,n} :=$ full subcat $X \in \mathcal{C}^{por}_{R,n}$
s.t $F^{X}_{u_{trant}} : hom(\mathbb{1}^{[I_p^{(n)}M]}, X) \xrightarrow{\sim} hom(\mathbb{1}[M], X)$ $\forall M \in$ fin

Prop: $\exists$ localization
$\mathcal{C}^{por}_{R,n} \xrightarrow{L^{or}} \mathcal{C}^{or}_{R,n}$ s.t $\mathcal{C}^{or}_{R,n}$ corep $Or^{(n)}_{R}(-)$
and L gives the fully faithful embedding ($\otimes$-ideal)
(L^{or} is chosen as the localization w.r.t $Id_{\mathcal{C}^{por}_{R,n}} \otimes F_{u_{trant}}$)

Pf: Then WTS: $F(u_{trant})$ ori $\Longleftrightarrow$ $F : \mathcal{C}^{por}_{R,n} \longrightarrow \mathcal{D}$ factors through $\mathcal{C}^{or}_{R,n}$.
$\Leftarrow$: $F_{L^{or}(u_{trant})} \simeq L^{or}(F_{u_{trant}})$ unwinding defn. — .
$\Rightarrow$: Take the right adj G of F, and then show that the essential image of G lies in $\mathcal{C}^{or}_{R,n}$.

i.e. WTS $hom(\mathbb{1}^{[I_p^{(n)}M]}, G(X)) \simeq hom(\mathbb{1}[M], G(X))$ $\forall X \in \mathcal{D}$.

It can be reduced to
$hom(F(\mathbb{1}^{[I_p^{(n)}M]}), X) \simeq hom(F(\mathbb{1}[M]), X)$

This is true since u is an orientation.
$= F(u_{trant})$

Fact: $\mathcal{C}^{or}_{R,n} \simeq Mod_{\underbrace{\mathbb{1}[w^{(n)}_{R}]}_{\text{idempotent}}}(\mathcal{C})$ if $\mathcal{C} \in CAlg(Pr^{\otimes-n})$, $R \in CAlg(Sp^{cn}_{(p)})$, $\mathbb{1}[w^{(n)}_{R}] \in \mathbb{1}[I_p^{(n)}R]$ comm alg

Proof: It suffices to show that [RS22, Thm].
$\mathcal{C}^{or}_{R,n} \subseteq \mathcal{C}^{por}_{R,n}$ is closed under all ① lim, all ② colim, ③ $-\otimes X$, ④ $hom(X, -)$ $\forall X \in \mathcal{C}^{por}$.

Note that $hom(-, \lim) \simeq \lim hom(-,-)$
and $hom(Z, hom(Y,X)) \simeq hom(Y, hom(Z,X))$
① and ④ ✓
② and ③ come from the duality. (n-semiadd!) $\square$

Def: In this case, $\mathbb{1}[w^{(n)}_{R}]$ is called R-cyclotomic extn of ht n.

Recall: A Galois extn $R \to S$ is faithful if $- \otimes S$ is conservative.

$\mathbb{S}_{T(n)}[w_{p^\infty}^{(n)}]$ ~~[illegible]~~ hasn't been verified its faithfulness based on the theory [we] proved. So we have:

Def: $\widehat{Sp_{T(n)}}$ or $Sp^{cyc}_{T(n)}$ ..

$:= (Sp_{T(n)})_{\mathbb{S}_{T(n)}[w_{p^\infty}^{(n)}]}$ is the (universal) smashing symm monoidal localization (virtually $\mathbb{Z}_{(p)}$-orientable) whose unit is given by $\left(\mathbb{S}_{T(n)}[w_{p^\infty}^{(n)}]\right)^{hG}$ (see Prop 6.19)

$G := T_p \times \mathbb{Z} \subset \mathbb{Z}_p^\times$. $\left(T_p := \begin{cases} \mathbb{Z}/p-1 & p \text{ odd} \\ \mathbb{Z}/2 & p=2 \end{cases}\right)$

$\therefore Sp_{K(n)} \subset \widehat{Sp_{T(n)}} \subset Sp_{T(n)}$.

Pf: WTS: $R := \left(\mathbb{S}_{T(n)}[w_{p^\infty}^{(n)}]\right)^{hG}$ is an idem alg Bous equiv to $\mathbb{S}_{T(n)}[w_{p^\infty}^{(n)}]$.

- In $Sp_{T(n)}$, $(-)^{hT_p} \simeq (-)_{hT_p}$
  $(-)^{h\mathbb{Z}} \simeq \Sigma^{-1}(-)_{h\mathbb{Z}}$ $\Rightarrow (-)^{hT_p}$ and $(-)^{h\mathbb{Z}}$ commutes with colim + tensor products.

So $R \otimes X \simeq \left(\mathbb{S}_{T(n)}[w_{p^\infty}^{(n)}] \otimes X\right)^{hG}$ ~~since~~ since $G = T_p \times \mathbb{Z}$

$\Rightarrow$ (R-acyclic $\Rightarrow$ ($\mathbb{S}_{T(n)}[w_{p^\infty}^{(n)}]$-acyclic))

Besides $\exists$ $R \to \mathbb{S}_{T(n)}[w_{p^\infty}^{(n)}]$ not map

$\Rightarrow$ ($\mathbb{S}_{T(n)}[w_{p^\infty}^{(n)}]$-acyclic $\Rightarrow$ R-acyclic)

- $R \otimes R \simeq \left(R[w_{p^\infty}^{(n)}]\right)^{hG}$
  $\simeq \left(\left(\mathbb{S}_{T(n)}[w_{p^\infty}^{(n)}] \otimes \mathbb{S}_{T(n)}[w_{p^\infty}^{(n)}]\right)^{hG}\right)^{hG}$
  $\simeq \left(C(\mathbb{Z}_p^\times : \mathbb{S}_{T(n)}[w_{p^\infty}^{(n)}])^{hG}\right)^{hG}$
  $\simeq C(\mathbb{Z}_p^\times : R)^{hG} \simeq C(\mathbb{Z}_p^\times / T_p : R)^{h\mathbb{Z}} \simeq C(\mathbb{Z}_p ; R)^{h\mathbb{Z}}$ ...

Fact: The defn of higher cyclotomic extn in CSY & BCSY are the same. (This is the hard part).

Thm: $L_{T(n+1)} K(R)[w_{p^\infty}^{(n+1)}]$
$\simeq L_{T(n+1)} K(R[w_{p^\infty}^{(n)}]) \in Sp_{T(n+1)}$.

Pf: Note that $I_p^{(n+1)} M \simeq \Sigma I_p^{(n)} M$ $\forall M$ torsion only p-gp ($\Leftrightarrow$ filtered colim of finite p-gp)

So we'll show that:

~~$\forall$ a finite p-gp~~ $M = \Sigma M$

$L_{T(n+1)} K(R)[M]$
$\simeq L_{T(n+1)} K(R[\Omega M])$

Reflection: and reflects the decomposition of primitive part or nonprimitive part

i.e. $M = \Sigma^{n+1} \mathbb{F}_p$

Note that $R = \mathbb{S}_{T(n)}$

$L_{T(n+1)} K(\mathbb{S}_{T(n)})[\Sigma^{n+1}\mathbb{F}_p] \xrightarrow{\sim} L_{T(n+1)} K(\mathbb{S}_{T(n)})[\Sigma^{n+1}\mathbb{F}_p]$

$\downarrow$ $\qquad\qquad \downarrow$

$L_{T(n+1)} K(\mathbb{S}_{T(n)}) \xrightarrow{\sim} L_{T(n+1)} K(\mathbb{S}_{T(n)}/R_{T(n)})$

Isom: Directly from last talk.

WTS: $C(\mathbb{Z}_p : R)^{h\mathbb{Z}} \simeq R$ $\forall$ p-complete ring sp.

~~Assume~~ Base change to $\mathbb{S}_{(p)}$ and then $\mathbb{F}_p$.

i.e. $0 \to \mathbb{F}_p \to C(\mathbb{Z}_p ; \mathbb{F}_p) \xrightarrow{\text{id}-\text{Frob}} C(\mathbb{Z}_p : \mathbb{F}_p) \to 0$

$\square$

i.e. $R \simeq R_1 \times R_2$

$\wr\wr$

$\overline{R} \simeq \overline{R_1} \times \overline{R_2}$

s.t

$$\begin{tikzcd} R \arrow[r, "\sim"] \arrow[d] & \overline{R_1} \arrow[d] \\ R_1 \arrow[r, "\sim"] & R_2 \end{tikzcd}$$

$\Rightarrow R_2 \simeq \overline{R_2}$ (Exercise!)

$S = R[w_{p^r}^{(n)}]$ $1 \le r < \infty$

WTS $L_{T(n+1)} K(S^{hH}) \xrightarrow{\sim} L_{T(n+1)} K(S$

$$\begin{tikzcd} L_{T(n+1)} K(S^{hH})[w_{p^r}^{(n+1)}] \arrow[r] \arrow[d, "\sim", "①"'] & L_{T(n+1)} K(S)^{hH}[w \\ L_{T(n+1)} K(S^{hH}[w]) \arrow[r] \arrow[d, "\sim", "③"'] & L_{T(n+1)}(S)[w]^{hH} \arrow[d, "\sim", "④"'] \\ L_{T(n+1)} K(S[w]^{hH}) \arrow[r, "\sim", "⑤"'] & L_{T(n+1)} K(S[w] \end{tikzcd}$$

after $\otimes_{S_{T(n+1)}} [w_{p^r}^{(n+1)}]$

①④ : Cyclotomic Redshift

②⑤ : Semi-additivity

③ : Galois :

$S[w] \simeq \prod_{(\mathbb{Z}/p^r)^\times} S$.

Similarly, we have, H finite subgp of Gal gp.

$$L_{T(n+1)} K(R[w_{p^\infty}^{(n)}]^{hH}) \simeq \left(L_{T(n+1)} K(R)[w_{p^\infty}^{(n+1)}]\right)^{hH}.$$

(b/c: ~~Do induc~~ Reduce it to $R[w_{p^r}^{(n)}]$ $r < \infty$

then take use of $\otimes_{T(n+1)} [w_{p^r}^{(n+1)}]$ is faithful.

and Galois extn).

Prop: $R_{cyc}^{\wedge} := R[w_{p^\infty}^{(n)}]^{h(T_p \times \mathbb{Z})}$.

then $L_{T(n+1)} K(R)_{cyc}^{\wedge} \simeq L_{T(n+1)} K(R[w_{p^\infty}^{(n)}]^{h(T_p \times \mathbb{Z})})$

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Telescope Conjecture: Cyclotomic Spectra and THH.

Recall:

Denote by $Sp^{B\mathbb{T}}$ the p-cplt sp with $\mathbb{T}$-action and by $\mathbb{T}(w)$ the circle with $\mathbb{T}$-action induced by $\mathbb{T} \to \mathbb{T}/C_w$, $w \in \mathbb{N}_+$.

$\Rightarrow \mathbb{T}(w) \to *$ induces

$\mathbb{S}[\mathbb{T}(w)] \to \mathbb{S}$ whose cofib is denoted by $\mathbb{S}^{(w)}$.

Unwinding defn, $\mathbb{S}^{(w)} \in Sp^{B\mathbb{T}}$ is of underlying sp $\mathbb{S}^{2}$ and hence tensor-invertible.

$\mathbb{S}^{k(w)} := (\mathbb{S}^{(w)})^{\otimes k}$ $\Sigma^{k(w)} := -\otimes(\mathbb{S}^{(w)})^{\otimes k}$ and $\mathbb{S} \to \mathbb{S}^{(w)}$ induces

$\mathbb{S}^{-(w)} \xrightarrow{a_{(w)}} \mathbb{S}$, ~~called~~ a certain type of Euler classes.

So $\to \mathbb{S}^{-n(1)} \xrightarrow{a_{(1)}} \mathbb{S}^{-(n-1)(1)} \to \cdots \to \mathbb{S}$ is a filtration

called $a_{(1)}$ (universal) - Borkstein filtration and taking $(-)^{h\mathbb{T}}$.

(After tensoring it with X, the associated exact couple induces a SS $\Rightarrow \pi_* X^{h\mathbb{T}}$)

Rmk: $Sp^{B\mathbb{T}} \to Sp^{BC_{p^k}}$, a natural forgetful functor.

So it's reasonable to consider $\mathbb{S}^{(w)} \cdots$ in $Sp^{BC_{p^k}}$.

Besides $Sp^{BC_{p^k}}$ is regarded as $Sp^{B\mathbb{T}}$. (p-cplt).

Lem: $X \in Sp^{B\mathbb{T}}$, $1 \le j \le \infty$.

$$(\mathbb{S}^{\mathbb{T}} \otimes X)^{tC_{p^j}} \cong \sum_+^\infty \mathbb{T}^{tC_{p^j}} \otimes X \cong 0$$

Thus $(a_{(1)})^{tC_{p^j}} : X^{tC_{p^j}} \to (\Sigma^{(1)} X)^{tC_{p^j}}$ is equiv.

Pf: $\mathbb{T}$ has finite equivariant decomp by free C_{p^j}-cells.

This implies the vanishment.

So the equiv following from the (co)fib. ($\mathbb{T}(1) =$ ~~trivial~~ ordinary $\mathbb{T}$-action) $\square$

Recall:

Any $\mathbb{T}$-sp X admits a Connes operator $\sigma : \Sigma X \to X$.

(As an $\mathbb{S}[\mathbb{T}]$-mod which owns a natural class $\sigma \in \pi_1(\mathbb{S}[\mathbb{T}])$ from $S^1 \xrightarrow{id} \mathbb{T}$,) ~~will be trans~~

σ also induces $\sigma : \pi_i X \to \pi_{i+1} X$.

Recall:

$$CycSp := \text{Lax Equalizer}\left(Sp^{B\mathbb{T}} \underset{(-)^{tC_p}}{\overset{id}{\rightrightarrows}} Sp^{B\mathbb{T}}\right)$$

$CycSp_+ :=$ Bounded below obj.

i.e. $X \in CycSp$ is determined by a pair

$$\left(\begin{array}{l} X \in Sp^{B\mathbb{T}}, \\ X \in CycSp. \end{array} \quad \varphi : X \to X^{tC_p}\right).$$

Def: $TC^-(X) := X^{h\mathbb{T}}$

$TP(X) := X^{t\mathbb{T}}$

$TC(X) := \mathrm{map}_{CycSp}(\mathbb{S}, X)$.

$TR^{k+1}(X) := X^{hC_{p^k}} \times_{(X^{hC_{p^{k-1}}})^{tC_p}} \cdots \times_{X^{tC_p}} X$

$TR(X) := \lim_k TR^k(X)$. Rmk: By Tate fixed pt lem.:

$X \in CycSp_+$

$X^{tC_{p^{k+1}}} \xrightarrow{\sim} (X^{tC_p})^{hC_{p^k}}$ simplies TR.

Recall: From [NS18] we know

$$TC(X) \to TC^-(X) \underset{can}{\overset{\varphi}{\rightrightarrows}} (X^{tC_p})^{h\mathbb{T}}$$

is an equalizer diagram.

Def: $L_{\langle p^\infty\rangle}\mathbb{S} \in \mathrm{CycSp}$.

$\begin{cases} \text{Underlying sp } \bigoplus_{j\geq 0} \mathbb{S}[\mathbb{T}/C_{p^j}] \\ \text{Frob: } \mathbb{S}[\mathbb{T}/C_{p^j}] \to (\mathbb{S}[\mathbb{T}/C_{p^{j+1}}])^{hC_p} \end{cases}$

$$\downarrow$$

$$(\mathbb{S}[\mathbb{T}/C_{p^{j+1}}])^{tC_p}$$

(This is equiv by Segal Conj.)

Lem:

TR is corepresented by $L_{\langle p^\infty\rangle}\mathbb{S}$. in CycSp_+

Pf: Unwinding defn,

$$\mathrm{map}_{\mathrm{CycSp}_+}(L_{\langle p^\infty\rangle}\mathbb{S}, X) \simeq \mathrm{Eq}\left(\mathrm{map}_{\mathbb{S}^{B\mathbb{T}}}(L\mathbb{S}, X) \underset{\varphi\circ(-)}{\overset{(-)^{tC_p}\circ\varphi}{\rightrightarrows}} \mathrm{map}_{\mathbb{S}^{B\mathbb{T}}}(L\mathbb{S}, X^{tC_p})\right)$$

While $\mathrm{map}_{\mathbb{S}^{B\mathbb{T}}}(L\mathbb{S}, Y) \simeq \prod_{j\geq 0} \mathrm{map}_{\mathbb{S}^{B\mathbb{T}}}(\mathbb{S}[\mathbb{T}/C_{p^j}], Y)$

$$\simeq \prod_{j\geq 0} Y^{hC_{p^j}} \quad \forall Y \in \mathrm{Sp}^{\mathbb{T}}.$$

So the eq is identified with

$$\mathrm{Eq}\left(\prod_{j\geq 0} X^{hC_{p^j}} \underset{\varphi\circ(-)}{\overset{(-)^{tC_p}\circ\varphi}{\rightrightarrows}} \prod_{j\geq 0} (X^{tC_p})^{hC_{p^j}}\right)$$

By Tate fixed pt lem, this is exactly $\mathrm{TR}(X)$. $\square$

Lem (Recognition Principle for $L_{\langle p^\infty\rangle}\mathbb{S}$).

R bounded below, comm dg

s.t $R \xrightarrow{\sim} R^{tC_p}$. X cyclo sp, also an R-mod, s.t

(1) $\exists$ splitting $X \simeq \bigoplus_{k\geq 0} X_k$ in $\mathrm{Mod}(R)^{B\mathbb{T}}$.

(2) Each piece $X_k \simeq R \otimes \mathbb{S}[\mathbb{T}/C_{p^k}]$ in $\mathrm{Mod}(R)^{B\mathbb{T}}$

(3) $\exists\, \varphi_k : X_k \simeq X_{k+1}^{tC_p}$ and they are the Frob in CycSp

$\Rightarrow X \simeq R \otimes L_{\langle p^\infty\rangle}\mathbb{S}$.

Pf: Exercise!

Notation:

$\mathrm{THH}: \mathrm{Alg}(\mathrm{Sp}) \to \mathrm{CycSp}$ as in [NS18]

so we could ~~thus~~ consider

$\mathrm{TC}^-\ \mathrm{TC}\ \mathrm{TR}\ \mathrm{TP}$ for a ring sp

by precomposing them with THH.

Given a p-cplt ring sp with a $\mathbb{Z}$-action R (especially $\mathbb{S}^{B^p\mathbb{Z}}$),

we'll now study the tower

$$\mathrm{THH}(R^{h\mathbb{Z}}) \to \mathrm{THH}(R^{hp\mathbb{Z}}) \to \cdots \to \mathrm{THH}(R^{hp^k\mathbb{Z}})$$

Recall:

$$W(-): \mathrm{Perf} \underset{\text{forgetful}}{\overset{\text{fully faithful}}{\rightleftarrows}} \mathrm{CAlg}(\mathrm{Sp}) : \pi_0^b(-)$$

Construction:

$\mathcal{D}$ cat with finite lim,

$G \in \mathrm{Mon}(\mathcal{D})^{gp}$. Then:

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d, equal] & G\times G \arrow[d, "(a,b)\mapsto(a,ba^{-1})"] & G \arrow[l, "\Delta"'] \arrow[d, equal] \\ G \arrow[r, "a\mapsto(a,1)"] & G\times G & G \arrow[l, "a\mapsto(a,1)"'] \end{tikzcd}$$

induces

$$\begin{tikzcd} \mathcal{L}G \arrow[rr, "\simeq"] \arrow[dr, hook] \arrow[dr, shift right] & & G\times\Omega_e G \arrow[dl, "\mathrm{proj}_1"] \\ & G & \end{tikzcd}$$

Taking pullback of last diag

res to the base pt of $\mathbb{T}$.

Lem: $\mathrm{Latt}_{\mathbb{Z}_p} :=$ ~~discrete~~ ordinary finite proj $\mathbb{Z}_p$-mod.

$C^0(A) :=$ continuous $\mathbb{F}_p$-valued fcts from $A \in \mathrm{Latt}_{\mathbb{Z}_p}$

Then $\mathbb{W}\pi_0^b \to \mathrm{id}$ induces a nat diagram

$$\begin{tikzcd} \mathbb{W}C^\circ(A) \arrow[r, "\iota"] \arrow[d, "\sim"'] & \mathbb{W}C^\circ(A^\delta) \arrow[d, "\sim"] \\ \mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S} \arrow[r, "\text{natural assembly map}"'] & \mathbb{S}^{\Omega_e BA} \end{tikzcd}$$

where $A \in \mathrm{Latt}_{\mathbb{Z}_p}$ and A^δ is the discrete top of A.

Pf: $\underbrace{\mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S}}_{\text{See [Lev22]}}$ and $\underbrace{\mathbb{S}^{\Omega_e BA}}_{\text{trivial}}$ bounded below

So

$$\begin{tikzcd} \operatorname{colim}_K \mathbb{F}_p \otimes_{\mathbb{F}_p^{BA/p^k}} \mathbb{F}_p \arrow[r, "\sim", "①"'] \arrow[d, "\sim"', "③"] & \mathbb{F}_p \otimes_{\mathbb{F}_p^{BA}} \mathbb{F}_p \arrow[d, "\sim"] & (\mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S}) \otimes \mathbb{F}_p \arrow[l, "\sim", "②"'] \arrow[d] \\ \operatorname{colim} \mathbb{F}_p^{\Omega_e BA/p^k} \arrow[r, "⑤"] \arrow[d, "\sim"', "⑧"] & \mathbb{F}_p^{\Omega_e BA} \arrow[d, "\sim", "⑥"] & \mathbb{S}^{\Omega_e BA} \otimes \mathbb{F}_p \arrow[l, "\sim", "④"'] \\ C^\circ(A) \arrow[r] & C^\circ(A^\delta) & \end{tikzcd}$$

All squares are commutative by naturality.

① is equiv b/c $H_c^*(A, \mathbb{F}_p) \cong H^*(A, \mathbb{F}_p)$

② ④ $\mathbb{F}_p \otimes \mathbb{S}^{BG} \simeq \mathbb{F}_p^{BG}$. (We need $\mathbb{S}$ bounded below (property of $\mathbb{S}$))

③: See Lurie Rational and p-adic homtp thy.

⑤ ⑥: $\Omega_e BA \simeq A^\delta$

$C^\circ(A)$ is locally constant fct. □

Then the lem follows from $\mathbb{W}(-)$ exactness image.

($R \cong \mathbb{W}(R \otimes \mathbb{F}_p)$ if $\mathbb{F}_p \otimes R$ is ordinary perf-algebra.)

Cor:

$$\begin{tikzcd} \mathbb{S}^{BA} \otimes \mathbb{W}C^\circ(A) \arrow[r] \arrow[d, "\sim"] & \mathbb{S}^{BA} \otimes \mathbb{W}C^\circ(A^\delta) \arrow[d, "\sim"] \\ \mathrm{THH}(\mathbb{S}^{BA}) \arrow[r] & \mathbb{S}^{\mathcal{L}BA} \end{tikzcd}$$

Pf: Note that

$$\mathbb{S}^{BA} \otimes (\mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S}) \cong \mathrm{THH}(\mathbb{S}^{BA})$$

and $\mathcal{L}G \simeq G \times \Omega_e G$ □

Cor:

$$\begin{tikzcd} \mathbb{W}C^\circ(A) \arrow[r, "\text{res at } 0 \in A"] \arrow[d] & \mathbb{W}\mathbb{F}_p \arrow[d] \\ \mathrm{THH}(\mathbb{S}^{BA}) \arrow[r] \arrow[d, equal] & \mathbb{S}^{BA} \arrow[d, equal] \\ \operatorname{colim}_{\mathbb{T}} \mathbb{S}^{BA} \arrow[r, "\text{res at } * \in \mathbb{T}"] & \mathbb{S}^{BA} \end{tikzcd}$$

Pf:

$$\begin{tikzcd} \mathbb{W}C^\circ(A) \arrow[r, "\text{res at } 0"] \arrow[d] & \mathbb{W}\mathbb{F}_p \arrow[d] \\ \mathbb{S}^{BA} \otimes \mathbb{W}C^\circ(A) \arrow[r, "\mathrm{id} \otimes \text{res at } 0"] & \mathbb{S}^{BA} \otimes \mathbb{W}\mathbb{F}_p \end{tikzcd}$$

□

Construction:

$\varepsilon \in \pi_1 \mathbb{S}^{B\mathbb{Z}}$ corr to $1 \in H^1(B\mathbb{Z}, \mathbb{Z})$

$\zeta \in \pi_1 \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$ the image of ε under $\mathbb{S}^{B\mathbb{Z}} \to \mathrm{THH}(\mathbb{S}^{B\mathbb{Z}})$

Def: $w \in \mathbb{Z}_p$

$B\mathbb{Z}_p(w)$ denote $B\mathbb{Z}_p$ with $B\mathbb{Z}$-action by left mult ~~with~~ via $w: B\mathbb{Z} \to B\mathbb{Z}_p$.

So $\forall R \in CAlg(S_p)$

$$R^{B\mathbb{Z}_p(w)} \in CAlg(S_p)^{B\mathbb{T}}_{R/-}$$

Besides when $w \in \mathbb{Z} \subseteq \mathbb{Z}_p$

$$R^{B\mathbb{Z}_p(w)} \cong R^{\mathbb{T}/C_w}$$

Rmk: $$\mathcal{L}B\mathbb{Z}_p \cong \bigsqcup_{w \in \mathbb{Z}_p} B\mathbb{Z}_p(w) \in CAlg(S_p)^{B\mathbb{T}}$$

Lem: For σ Connes operator,

$$\sigma(\zeta) = (1+\eta\zeta)\cdot Id_{\mathbb{Z}_p} \text{ in } \pi_0(THH(\mathbb{S}^{B\mathbb{Z}}))$$

$$\simeq \pi_0(\mathbb{S}^{B\mathbb{Z}} \otimes WC^0(\mathbb{F}_p))$$

$$\cap$$

$$Id_{\mathbb{Z}_p}.$$

Lem: $R \in CAlg(S_p)$

$p: \mathbb{T} \to \mathbb{T}/C_p$ induces

$\psi_p: THH(R) \to p^* THH(R)$ in $CAlg(S_p)^{B\mathbb{T}}$

which refines to a $\mathbb{T}$-equivariant equiv

$$THH(\mathbb{S}^{B\mathbb{Z}_p})|_{p\mathbb{Z}_p} \simeq p^* THH(\mathbb{S}^{B\mathbb{Z}})$$

and induces $C^0(p\mathbb{Z}_p) \to C^0(p\mathbb{Z}_p)$ in π_0^b for $THH(\mathbb{S}^{B\mathbb{Z}}) \to THH(\mathbb{S}^{B\mathbb{Z}})|_{p\mathbb{Z}_p}$

Pf: Since $THH(\mathbb{S}^{B\mathbb{Z}_p}) \to \mathbb{S}^{\mathcal{L}B\mathbb{Z}_p}$ induces inj on π_*.

We may assume the similar prop for $\mathbb{S}^{\mathcal{L}B\mathbb{Z}_p}$.

Then the lem is trivial. $\square$

Cor: There's an isom of $WC^0(p^k\mathbb{Z}_p^\times)$-mod in $S_p^{B\mathbb{T}}$ $k \geq 0$.

$$THH(\mathbb{S}^{B\mathbb{Z}_p})|_{p^k\mathbb{Z}_p^\times} \cong WC^0(p^k\mathbb{Z}_p^\times) \otimes \Sigma^{-1}\mathbb{S}[\mathbb{T}/C_{p^k}].$$

In particular

$$\forall M \ THH(\mathbb{S}^{B\mathbb{Z}_p})|_{\mathbb{Z}_p^\times}\text{-mod}, \ M^{tC_{p^j}} = 0 \ \forall\, 1 \leq j \leq \infty.$$

Pf: By the lemma above,

it suffices to prove it when $k=0$.

ζ restricted to $\pi_{-1}THH(\mathbb{S}^{B\mathbb{Z}_p})|_{\mathbb{Z}_p^\times}$

induces a $\mathbb{T}$-equiv map b/c $WC^0(\mathbb{Z}_p^\times)$-mod

$$z: WC^0(\mathbb{Z}_p^\times) \otimes \Sigma^{-1}\mathbb{S}[\mathbb{T}] \to THH(\mathbb{S}^{B\mathbb{Z}_p})|_{\mathbb{Z}_p^\times}$$

On htpy gps it's the map.

$$z: \pi_* WC^0(\mathbb{Z}_p^\times)\{[*],[\mathbb{T}]\} \to C^0(\mathbb{Z}_p^\times)\{1,\zeta\}$$

where $[*] \mapsto [\zeta]$ (Recall the identification above)

$$[\mathbb{T}] \mapsto z(\sigma[*])$$
$$= id(1+\eta\zeta)$$

where id is an unit in RHS.

$\Rightarrow$ isom on π_* $\square$

Lem: $R \in CAlg(S_p)^{BC_p}$ bounded below

$\Rightarrow$ In $\pi_0^b R \leftarrow \pi_0^b R^{hC_p} \to \pi_0^b R^{tC_p}$,

the left map is equiv if the act on $\pi_0 R$ trivial

the right is equiv if C_p-action itself is trivial.

Left: Trivial

Right: $\mathbb{S} \to \mathbb{Z}_p$ posti tower

$(-)^{hC_p}$ $(-)^{tC_p}$ $\pi_0\mathbb{S}$ exact & commu with uniformally bounded below (lim -

nil equiv

We may assume R is of the form $\mathbb{Z}_p \otimes R$.

Then you could calculate the comp.

(Note that $(-)^{hC_p} \simeq (--)[[t]]$

where $|t| = -2$ $\rightsquigarrow$ $(-)^{tC_p} \simeq (--)((t))$

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Rmk: When C_p-action extends to a $\mathbb{T}$-action, it must act on $\pi_0 R$ trivially.

Now we have:

Conclusion:

$$\begin{tikzcd}
\pi_0^b THH(\mathbb{S}^{B\mathbb{Z}}) \arrow[d, "\sim"] & \pi_0^b THH(\mathbb{S}^{B\mathbb{Z}})^{hC_p} \arrow[l, "\mathbb{T}\text{-action}", "\sum"'] \arrow[r] \arrow[d, "\sim"] & \pi_0^b THH(\mathbb{S}^{B\mathbb{Z}})^{tC_p} \arrow[d, "\sim"] \\
C^\circ(\mathbb{Z}_p) \arrow[r, equal] & C^\circ(\mathbb{Z}_p) \arrow[r, "res"] & C^\circ(p\mathbb{Z}_p)
\end{tikzcd}$$

$$\mathbb{S}^{B\mathbb{Z}_p} \otimes WC^\circ(\mathbb{Z}_p) \simeq THH(\mathbb{S}^{B\mathbb{Z}_p}) \simeq THH(\mathbb{S}^{B\mathbb{Z}_p}).$$

$THH|_{\mathbb{Z}_p^\times}$ doesn't have nontrivial $(-)^{tC_p}$ while $(-)|_{p\mathbb{Z}_p}$ induces $C^\circ(p\mathbb{Z}_p)$.
($C_p \curvearrowright \mathbb{F}_p$ is trivial on 0)

Prop:

The Frob

$\varphi: THH(\mathbb{S}^{B\mathbb{Z}_p}) \to THH(\mathbb{S}^{B\mathbb{Z}_p})^{tC_p}$ is equiv

and induces the map from $p\mathbb{Z}_p \xrightarrow{x \frac{1}{p}} \mathbb{Z}_p$ on π_0^b

Pf: Recall $\mathrm{can}^{-1} \circ \Delta_p$ is id on $\mathbb{S}$. Then

$$\begin{tikzcd}
\mathbb{S}^{B\mathbb{Z}_p} \arrow[r, "\Delta_p"] \arrow[d, "THH"'] & ((\mathbb{S}^{B\mathbb{Z}_p})^{\otimes p})^{tC_p} \arrow[r, "n"] \arrow[d] & (\mathbb{S}^{B\mathbb{Z}_p})^{tC_p} \arrow[d] & \mathbb{S}^{B\mathbb{Z}_p} \arrow[l, "\sim", "\mathrm{can}"'] \arrow[d, "THH"] \\
THH \arrow[r, "\varphi"] & THH^{tC_p} \arrow[r, "\varphi^{tC_p}", "\sim"'] & (p^* THH)^{tC_p} & THH \arrow[l, "\simeq"']
\end{tikzcd}$$

by universal prop the map of the lower line is equiv.

Prop: $W(C^\circ(\mathbb{Z}_p^\times)) \otimes \Sigma^{-1} L_{\langle p^\infty \rangle} \mathbb{S} \to THH(\mathbb{S}^{B\mathbb{Z}}) \to THH(\mathbb{S}^{B\mathbb{Z}})$ fib seq.

Pf: Denote the fiber by F.

$$\begin{cases} F \simeq \bigoplus_{k \geq 0} F|_{p^k \mathbb{Z}_p^\times} & (\text{since the decomp of } WC^\circ(\mathbb{Z}_p^\times)) \\ F|_{p^k \mathbb{Z}_p} \simeq WC^\circ(\mathbb{Z}_p^\times) \otimes \Sigma^{-1} \mathbb{S}[T/C_{p^k}] & \Rightarrow F \simeq WC^\circ(\mathbb{Z}_p^\times) \otimes \Sigma^{-1} L_{\langle p^\infty \rangle} \mathbb{S} \\ F|_{p^k \mathbb{Z}_p} \simeq F|_{p^{k-1} \mathbb{Z}_p}^{tC_p} & \text{prop above} \end{cases}$$

$\square$

Cor: $THH(\mathbb{S}^{B\mathbb{Z}})$ has t-amplitude $[-1, 3]$ in $CycSp_+$.

Cor: $R \in Alg(Sp)$ connective.

$\Rightarrow WC^\circ(\mathbb{Z}_p^\times) \otimes TC(R)$ is in the thick subcat gen by $fib(TC(R^{B\mathbb{Z}}) \to TC(R^{B\mathbb{Z}}))$.

Pf: $TC \to TR \xrightarrow{1-F} TR$ fib by [NS18]

$\Rightarrow L_{\langle p^\infty \rangle} \mathbb{S} \to L_{\langle p^\infty \rangle} \mathbb{S} \to \mathbb{S}$ fib in $CycSp$

$\xrightarrow[\text{above}]{\text{+prop}} W(C^\circ(\mathbb{Z}_p^\times))$ belongs to the thick subcat of the $fib(THH(\mathbb{S}^{B\mathbb{Z}}) \to THH(\mathbb{S}^{B\mathbb{Z}}))$ i.e. $W(C^\circ(\mathbb{Z}_p^\times)) \otimes L_{\langle p^\infty \rangle} \mathbb{S}$ (closed under cofib).

For general case, note that $WC^\circ(\mathbb{Z}_p^\times)$ is a direct sum of $\mathbb{S}_p$ and THH is symm mon. $\mathbb{T}C$ preserves for connective colim rings. $\square$

Rmk:

$R \in \mathrm{Alg}(Sp_{T(n)}) \quad n \geq 1$

$X \in Sp$

$F(R) := L_{T(n+1)}(X \otimes K(R)) \neq 0.$

$\Rightarrow F(R^{B\mathbb{Z}}) \not\xrightarrow{\sim} F(R)^{B\mathbb{Z}}$.

This's one main feature of our disproof.

Section 2 Other Prelimaries.

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Fact:

In $Sp^{BC_{p^k}}$ there's a levelwise t-structure and Antieau–Nikolaus define the cyclotomic t-structure on $CycSp$.

One essential feature is that

① $X \in CycSp$ connective $\Leftrightarrow$ X connective as a Sp.

② A filtered colim of cyc bounded in $[a,b]$ is bounded in $[a, b+3]$.

Def: $X \in Sp^{B\mathbb{T}}$ bounded below

(1) X weak canonical vanishing with parameter b (or say X satisfies $WCV(\leq b)$)

if $\forall\, 1 \leq j \leq \infty$ $\forall\, m > b$

$$\pi_m X^{hC_{p^j}} \xrightarrow{0} \pi_m X^{tC_{p^j}}.$$

(2): X strong canonical vanishing with parameter b ($SCV(\leq b)$).

if $\exists\, d \geq 0$ s.t

$$\tau_{>b} X \to X \xrightarrow{a_{(1)}^{d}} \Sigma^{d(1)} X \quad \text{is } \mathbb{T}\text{-equiv null}.$$

Lem: (1) $SCV(\leq b) \Rightarrow WCV(\leq b)$.

pf:

$$\begin{tikzcd}
\tau_{>b}(X^{hC_{p^j}}) \arrow[r] \arrow[d] & (\tau_{>b}X)^{hC_{p^j}} \arrow[r] \arrow[d] & (\tau_{>b}X)^{tC_{p^j}} \arrow[dr, "0"] & \\
X^{hC_{p^j}} \arrow[rr] & & X^{tC_{p^j}} \arrow[r, "\sim"] & \left(\Sigma^{d(1)} X\right)^{tC_{p^j}}
\end{tikzcd}$$

See lemma in last talk

(2): $SCV(\leq b) \Rightarrow SCV(\leq b+1)$

$WCV(\leq b) \Rightarrow WCV(\leq b+1)$.

$\square$

Def (Exponent of Nilpotence):

$\mathcal{C}$ stable symm mono

$B \in Alg(\mathcal{C})$

$I_B := fib(\mathbb{1} \to B)$

$\forall X \in \mathcal{C}$ $\quad exp_B(X) := \min_{m} \{ m \mid I_B^{\otimes m} \otimes X \xrightarrow{0} \mathbb{1}^{\otimes m} \otimes X \}$

called exponent of nilpotence of X.

Lem: $X \to Y \to Z$ cofib seq

$\Rightarrow exp_B Y \leq exp_B X + exp_B Z$.

Lem (1): $R, B \in Alg(\mathcal{C})$ $\quad M \in Mod_R(\mathcal{C})$

$exp_B(M) \leq exp_B(R)$

(2): $F: \mathcal{C} \to \mathcal{D}$ exact lax sym

$A \in Alg(\mathcal{C})$ $\quad B \in Alg(\mathcal{D})$

$\forall M \in \mathcal{C}$

$exp_B(F(M)) \leq exp_A(M) \cdot exp_B(F(A))$

(3): $\mathcal{C}$ also presentable with compatible t-str.

$L: Sp \to \mathcal{C}$ the str map.

$X \in \mathcal{C}_{[c,b]}$ and $p^m: X \to X$ vanishes.

$\Rightarrow exp_{\mathbb{H}_p}(X) \leq (b-c+1)\, m$.

Proof

(1): M is a retract of $M \otimes R$.

$\therefore R \otimes (I_B^{\otimes m} \to \mathbb{1})$ null

$\Rightarrow M \otimes R \otimes (I_B^{\otimes m} \to \mathbb{1})$ null

$\Rightarrow M \otimes (I_B^{\otimes m} \to \mathbb{1})$ null

(2):

$I_B^{\otimes m} \otimes FA \to FA$ null

$I_A^{\otimes n} \otimes M \to M$ null

$\Rightarrow F(I_A)^{\otimes n} \otimes F(M) \to F(M)$ null.

Simultaneously

$F(I_A) \to \mathbb{1}_B \to FA$ fib seq.

$\Rightarrow \mathbb{1}_B \to F(I_A)^{\otimes m} \otimes F(M)$

Use resolution.

(3): Resolve it by t-str to assume $X \in \mathcal{C}^{\heartsuit}$.

$\Rightarrow X$ is $\iota(\mathbb{Z}/p^m)$-mod $\Rightarrow$ assume $X = \iota(\mathbb{Z}/p^m)$

$\Rightarrow \exp_{\iota\mathbb{F}_p}(\iota(\mathbb{Z}/p^m)) \leq \exp_{\mathbb{F}_p}(\mathbb{Z}/p^m) \cdot \exp_{\iota\mathbb{F}_p}(\iota\mathbb{F}_p) \leq m$

Def: $X \in Sp^{BT}$ bounded below.

(1) X satisfies Tate nilpotence of exp d if $\exp_{\mathbb{S}^T}(X^{tC_p}) \leq d$.

(2) X satisfies $\mathbb{F}_p$ nilpotence of exp d if $\exp_{\mathbb{F}_p}(X) \leq d$.

Def: $X \in CycSp$ $b \in \mathbb{Z}$

X satisfies Segal condition with para b (Segal($\leq b$)) if $\mathrm{fib}(X \xrightarrow{\varphi} X^{tC_p})$ is b-truncated.

Lem: X bounded below $\in CycSp$

then Segal($\leq b$) + $\exp_{\mathbb{S}^T}(X^{tC_p}) \leq d$

$\Rightarrow$ SCV($\leq b+2d$)

$$\begin{tikzcd}
\tau_{>b+2d}X \arrow[r] & X \arrow[r, "a^d_{(1)}"] \arrow[d, "\varphi"] & \Sigma^{d(1)} X \arrow[d] \\
& X^{tC_p} \arrow[r, "a^d_{(1)}"] & \Sigma^{d(1)} X^{tC_p}
\end{tikzcd}$$

induced by Segal

$\Sigma^{d(1)}\mathrm{fib} \to \Sigma^{1(1)} \simeq -\otimes S$

The bottom horizon map is null due to exp $\leq d$.

since $\mathbb{S}^{-(1)} \xrightarrow{a_{(1)}} \mathbb{S} \to \mathbb{S}^T$ fib seq. $\square$

Lem: $X \in CycSp_+$

then $X \in$ WCV($\leq b$) + Segal($\leq b$)

$\Rightarrow X \in CycSp_{\leq b}$ $\square$

Lem (Trivial)

Cor: $X \in CycSp_{[c,b]}$

p^m acts on it by zero

$\Rightarrow \exp_{\mathbb{F}_p}(X) \leq (b-c+1) \cdot m$. $\square$

Prop (See Antieau-Nikolaus):

$X \in CycSp_{\leq b} \Rightarrow$ Segal($\leq b$).

Prop / Conclusion:

(1) SCV($\leq b$) $\Rightarrow$ WCV($\leq b$)

(2) WCV($\leq b$) + Segal($\leq b$) $\Rightarrow CycSp_{\leq b}$

(3) $CycSp_{\leq b} \Rightarrow$ Segal($\leq b$)

(4) $X \in CycSp_{[c,b]}$ + p^m acts by 0

$\Rightarrow$ SCV($\leq b + 2(b-c+1)m$)

(5). $\exp_{\mathbb{S}^T}(X^{tC_p}) \leq \exp_{\mathbb{F}_p}(X)$. $\square$

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Cor: $c \le b$

$m \ge m_p^{A_2}$

R any hA_2 ring in $CycSp_{[c,b]}$ with $p^m = 0$

$\Rightarrow \exists$ Bökstedt class $u \in \pi_{2p^k} R$ for $k = (b-2c+2)m$

Pf: ① Devissage to $Mod(CycSp, \tau^{cyc}_{\le b-2c} \mathbb{S})$

② Construct a map $\mathbb{S}/p^{2m} \otimes \tau^{cyc}_{\le c} \mathbb{S} \xrightarrow{i} R$.

③ The Bökstedt class gives that of R via i.

Lem: R p-nil hA_2-alg in $CycSp$ with $u \in \pi_k R$ s.t.

(1) R/u b-truncated

(2) $u \in Im(\Sigma R_{hT} \to R)$

$\Rightarrow R$ $SCV(\le b)$

Pf:

$$\begin{tikzcd}
\operatorname{colim}_d \left(\Sigma^{-1+d(1)} \mathbb{S}/a_{(1)}^d \otimes R\right)^{hT} \arrow[r] \arrow[d, equal] & R^{hT} \arrow[r] \arrow[d, equal] & (\mathbb{S}^T \otimes R)^{hT} \arrow[d, equal] \\
\Sigma R_{hT} \arrow[r, "Nm"] & R_{hT}^{hT} \arrow[r] & R
\end{tikzcd}$$

Due to p-nil.

colim and π_k comm.

$\Rightarrow \exists d$ $a_{(1)}^d \cdot u \equiv 0$

$$\begin{tikzcd}
\tau_{\le b} R \arrow[d] \arrow[dr, "0"] & \\
\Sigma^{2p^k} R \arrow[r, "u\cdot -"] \arrow[dr, "0"'] & R \arrow[r] \arrow[d, "a_{(1)}^d"] & R/u \\
& \Sigma^{d(1)} R &
\end{tikzcd}$$

implies

$\tau_{\le b} R \to R \to \Sigma^{d(1)} R$

factors through $\Sigma^{2p^k} R \xrightarrow{0} \Sigma^{d(1)} R$

$\square$

Def (底紧性):

$\mathcal{C}$ st pre cat with t-str

$X \in \mathcal{C}$ bounded below an almost cpt obj

if $\forall F: K \to \mathcal{C}$ filtered diagram with uniformly bounded range in $[c,b]$

$\operatorname{colim}_K Map(X, F(k)) \simeq Map(X, \operatorname{colim}_K)$

Fact: X bounded below p-nil cyclo.

Underlying sp almost cpt

$\Rightarrow X$ almost cpt

Conversely, X almost cpt

$\Rightarrow$ Underlying sp almost cpt.

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Def $h\mathbb{E}_n$-ring

(1): R $h\mathbb{A}_\infty$-ring means an $\begin{cases} \mathbb{A}_\infty\text{-alg in } h\mathcal{C} \\ \mathbb{E}_n\text{-alg} \end{cases}$

(2): R $h\mathbb{A}_2$-ring in $\mathcal{C}$ sym mono

$z: S \to R$ is called hcentral

if

$$\begin{tikzcd} S\otimes R \arrow[d, bend right, "z\cdot"'] \arrow[d, bend left, "\cdot z"] \\ R \end{tikzcd} \qquad \begin{tikzcd} S\otimes R\otimes R \arrow[r, "S\otimes m"] \arrow[d, "(z\cdot)\otimes R"'] & S\otimes R \arrow[d, "z\cdot"] \\ R\otimes R \arrow[r, "m"] & R \end{tikzcd} \qquad \begin{tikzcd} S\otimes R\otimes R \arrow[r] \arrow[d] & S\otimes R \arrow[d, "\cdot z"] \\ R\otimes R \arrow[r] & R \end{tikzcd}$$

commute. (Similarly, an element in $\pi_* R$ could be called hcentral when regarded as a map from $\mathbb{1}$)

(3) R $h\mathbb{A}_2$-ring in CycSp and $\mathrm{Segal}(\le b)$

$u \in \pi_{2p^k} R$ $(k \ge 0)$ is called a Bökstedt class

if ① u and $\varphi(u)$ hcentral

② $u \in \mathrm{Im}(\Sigma R_{h\mathbb{T}} \to R)$

③ $\varphi(u) \in \pi_{2p^k}(R^{tC_p})$ a unit.

Lem: $i: R \to S$ hcentral map of $h\mathbb{A}_2$-rings in CycSp and both stisfy $\mathrm{Segal}(\le b)$.

Then i preserves Bökstedt class

Pf: Noturality!

Lem: R $h\mathbb{A}_2$ in CycSp & $\mathrm{Segal}(\le b)$.

$u \in \pi_{2p^k} R$ Bödkast class.

$\Rightarrow$ (1) R/u $(b+1+2p^k)$-truncated

(2) $\varphi: R[u^{-1}] \xrightarrow{\sim} R^{tC_p}$

Pf: (1) $\varphi(u)$ unit.

$\Rightarrow$ $\mathrm{fib}(\varphi)/u \cong R/u$.

$$\begin{tikzcd} \mathrm{fib}(\varphi) \arrow[r] \arrow[d, "u"] & R \arrow[r, "\varphi"] \arrow[d, "u"] & R^{tC_p} \arrow[d, "\varphi(u)"] \\ \mathrm{fib}(\varphi) \arrow[r] & R \arrow[r, "\varphi"] & R^{tC_p} \end{tikzcd}$$

$\Rightarrow$ Truncated

(Recall: $\mathrm{Segal}(\le b)$ i.e. $\mathrm{fib}(\varphi)$ b-truncated)

(2) $\mathrm{fib}(\varphi)[u^{-1}] = 0$

since it's truncated and $\deg u > 0$. $\square$

Lem: $b \ge 0$

$m \ge 2m_p^{\mathbb{A}_2}$ (i.e. m s.t. $\mathbb{S}/p^m$ See Burklund. $\mathbb{A}_2$)

Then $R := \mathbb{S}/p^m \otimes \tau^{cx}_{\le b} \mathbb{S}$

admits hcring str

$\mathrm{Segal}(\le b+1)$

has Bökstedt class $u \in \pi_{2p^k} R$.

where $k = \left(\left\lfloor \frac{b}{2} \right\rfloor + 1\right) m$.

It'll induce a Bödkast class.

Pf: R is always a hcring. Then consider $\mathbb{T}$-tate SS

p^m acts on R by zero

and $R \in \mathrm{CycSp}_{[0,b+1]}$.

$\Rightarrow$ R $\mathrm{Segal}(\le b+1)$

$\mathrm{WCV}(\le b+1+2(b+2)m)$

$\Rightarrow$ $\mathrm{TC}(R)$ bounded in $[-1, b+1]$

$\pi_* \mathrm{TC}(R)[t^{\pm 1}]$

$\Rightarrow \pi_* \mathrm{TC}(R)^{t\mathbb{T}}$

There's a permanent cycle $t^{-p^{b'm}}$

$b' := \left\lfloor \frac{b}{2} \right\rfloor + 1$

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Target of this talk:

Cyclotomic Asymptotic Constancy:

Def (fp-type n)

$n \geq -1$ X p-cplt bounded below

is called of fp-type n if

$\forall U$ finite spectrum of type $n+1$,

$U \otimes X$ is π-finite. ($\Leftrightarrow \exists U$ finite sp of exact type $n+1$).

Def (ht n Lichtenbaum-Quillen property):

$\mathbb{E}_1$-ring R is ~~called~~ of ht n LQ property

if $THH(R)$ bounded below

& $\forall V$ finite sp of type $n+2$ ($\Leftrightarrow \exists \cdots$)

$V \otimes THH(R)$ is bounded.

Thm: $R \in Alg_{\mathbb{E}_1 \otimes A_2}(Sp^{B\mathbb{Z}, u})$ connective.

locally unipotent (gen by colims of obj with trivial action)

fp-type $n \geq -1$

ht n LQ

$\Rightarrow \forall k \gg 0$ $R^{hp^k\mathbb{Z}}$ also ht n LQ.

And $\forall V$ finite sp of type $n+2$

$\forall k \gg 0$, $V \otimes THH(R^{hp^k\mathbb{Z}}) \simeq V \otimes THH(R^{Bp^k\mathbb{Z}})$

Rmk: fp-type and LQ are finiteness conditions.

Cons:

$$\mathbb{S}^{B\mathbb{Z}} \to \mathbb{S}^{Bp\mathbb{Z}} \to \mathbb{S}^{Bp^2\mathbb{Z}} \to \cdots \to \mathbb{S}$$

$W_K := THH(\mathbb{S}^{Bp^k\mathbb{Z}})$ $W := W_0$

$\rightsquigarrow$

$$W \to W_1 \to \cdots \to W_\infty \simeq \mathbb{S}.$$

$R \in Alg(Sp^{B\mathbb{Z}, u})$. then $\exists$ nat W_K-mod str on $THH(R^{hp^k\mathbb{Z}})$

and for W_K-mod X

$X|_0 := X \otimes_{W_K} \mathbb{S}^{Bp^k\mathbb{Z}}$ since it's a mod by res to 0

Lem: $R \in Alg(Sp^{B\mathbb{Z}, u})$.

$\Rightarrow THH(R)$ locally uni

$\Rightarrow TC(R)$ locally uni if R connective

Pf:

Lem: $R \in Alg(Sp^{B\mathbb{Z}, u})$

$W_K \otimes_W THH(R^{h\mathbb{Z}})$

$\simeq THH(R^{hp^k\mathbb{Z}})$.

Pf: Due to THH symm mono

it suffices to show that

$R^{h\mathbb{Z}} \otimes_{\mathbb{S}^{B\mathbb{Z}}} \mathbb{S}^{Bp^k\mathbb{Z}} \simeq R^{hp^k\mathbb{Z}}$.

See A.21

$\square$

~~A weak version~~:

我们继续做一点有限性的准备工作

Lem: M p-nil W-mod in CycSp

$M|_{p^k\mathbb{Z}_p}$ WCV($\leq b$)

M Segal ($\leq b$)

$\Rightarrow M \in \tau^{cyc}_{\leq b}$ CycSp.

Pf: $U_{ij} := \{a \in \mathbb{Z}_p \mid i \leq v_p(a) \leq j\}$

$$X_{ij} := Eq\left(\prod_{r\geq 0}(M|_{U_{ij}})^{hC_{p^r}} \underset{\varphi}{\overset{can}{\rightrightarrows}} \prod_{r\geq 1}(M|_{U_{i+1,j+1}})^{tC_{p^r}}\right)$$

Unwinding defn,

$\exists\ X_{01} \to X_{02} \to \cdots X_{0k} \to TR(X)$.

with graded piece $X_{01}, X_{12} \cdots X_{k-1,k}\ X_{k,0}$.

WTS $TR(X)$ b-truncated thus M b-truncated

So it suffices to show each piece b-trun.

For example, $i \leq k$. 从 Eq 变成 fib 则上 3 WCV($\leq b$).

$$X_{r-1,i} \simeq \mathrm{fib}\left(\prod_{j\geq 0}(\varphi|_{U_{i-1,i}})^{hC_{p^i}}\right)$$

By Segal($\leq b$), it b-truncated. $\square$

Prop: $m \geq m_p^{A_2}$ R hA_2-ring in W-mod of CycSp.

$p^m = 0$

(1) $R|_0$ cyclotomic bounded in $[c, b]$

(2) $\pi_* R \cong \pi_0 WC^0(\mathbb{Z}_p) \otimes \pi_*(R|_0)$.

(3) R Segal ($\leq b'$).

$\Rightarrow R$ cyc bounded in $[c, e]$ $e := b' + 1 + 2p^{(b-2c+2)m}$.

Pf: $k := (b-2c+2)m$

(1) + (2) $\Rightarrow$ R underlying obj c-connective.

Due to $R|_0$ bounded with $p^m = 0$

$\exists$ Bökstedt element $u \in \pi_{2p^k} R|_0$

So $\exists$ $1 \otimes u \in \pi_{2p^k}(K)$

u Bökstedt

$\Rightarrow (R|_0)/u$ e-truncated (see see 2)

$\Rightarrow \pi_*(\Sigma^{2p^k} R|_0) \xrightarrow{u \cdot -} \pi_*(R|_0)$ is $\begin{cases} \text{iso in } * > \\ \text{inj } * = \end{cases}$

$\Rightarrow 1 \otimes u$ induces the same property

i.e. $\mathrm{coof}(1 \otimes u)$ bounded in $[c, e]$.

Simultaneously, $R|_0 \simeq \mathrm{colim}\, R|_{p^j\mathbb{Z}_p}$,

we may lift $1 \otimes u$ to $R|_{p^j\mathbb{Z}_p}$, for some j

and by Lem 2.36, 2-30(1)

we know $R|_{p^j\mathbb{Z}_p}$ WCV($\leq e$)

Using the lem above, we finish this proof $\square$

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Prop (Bootstrap Trivialization):

X p-nil W-mod in CycSp

$X_k := X \otimes_W W_k$.

(1) X_k uniformly cyc bounded

(2) $X \simeq W \otimes X_\infty$

(3) X_∞ almost cpt in CycSp.

$\Rightarrow \forall k \gg 0 \quad \exists\, X_k \simeq W_k \otimes X_\infty$ as W_k-mod.

Pf: B/C uniformly bounded + almost cpt

$\exists\, k_1 \gg 0$ and

$$\begin{tikzcd} & X_{k_1} \arrow[dr] & \\ X_\infty \arrow[ur, dashed] \arrow[rr, "id"] & & X_\infty \end{tikzcd}$$

$\rightsquigarrow \forall k \geq k_1$ we have $\psi_k : W_k \otimes X_\infty \to X_k$

recovering the $X_\infty \xrightarrow{id} X_\infty$ when $k \to \infty$. linearization.

WTS: $\forall k \gg 0 \quad \psi_k$ is equiv.

b/c: X_k $W_k \otimes X_\infty$ uniformly bounded, ($THH(\mathbb{S}^{BZ})$ has t-am $[-1,3]$ by Antieau-Nikolaus)

we can choose to prove that

$\forall s \in \mathbb{Z} \ \exists\, k \gg 0 \quad \pi_s \psi_k$ isom.

$\cancel{\pi_* X_k \simeq \pi_*(W_k \otimes X_\infty) \simeq \pi_0(WC^0(p^k \mathbb{Z}_p))[\zeta] \otimes \pi_* X_\infty}$

Let $N_{k,s} = \pi_0(WC^0(p^k \mathbb{Z}_p))[\zeta] \otimes \pi_s X_\infty$ 保留.

$\psi_{ks} = N_{k,s} \to N_{k,s}$ induced by $\pi_s \psi_k$

$\psi_{\infty,s}$ is isom by cdim defn.

By formal argument, if considering $\overline{N_{k,s}} := N_{ks}/\zeta$, $\overline{\psi_{ks}} := \psi_{ks}/\zeta$,

one can show that

inorder to know ψ_{ks}, $\longleftrightarrow$ comm (事情好操作)

it suffices to know $\overline{\psi_{ks}}$ (since $\mathrm{Tor}^1_{\mathbb{Z}_p[\zeta]}(\mathbb{Z}_p, N_{k,s}) = 0$). If following from [Lem A.15] $\square$

Lem: X finite sp of type w

$q \geq 0$

$\exists$ connective $\mathbb{E}_k$-alg V with finite underlying spectrum of type n

and X is a retract of $V \otimes X$.

Pf: The key input is [Bus22].

Sketch of Proof of Main Thm.

Recollection:

$R \in \mathrm{Alg}_{\mathbb{E}_1 \otimes \mathbb{A}_2}(\mathrm{Sp}^{BZ,u})$ connective fp-type $n \geq -1$

X finite sp of type $n+2$

$X \otimes THH(R)$ bounded in $[c,b]$

$\Rightarrow \exists\, k(R,X,n)$ s.t. $\forall k \geq k(R,X,n)$

$X \otimes THH(R^{hp^k\mathbb{Z}})$ bounded in $[c-1,b+1]$

& as W_k-mod in CycSp

$X \otimes THH(R^{hp^k\mathbb{Z}}) \simeq X \otimes W_k \otimes THH(R)$.

By lem above,

$\exists V$ type $n+2$

with $\mathbb{E}_1 \otimes \mathbb{A}_2$-alg str

s.t. X retract of $X \otimes V$.

We'll mainly consider V from then on.

(0) $V \otimes THH(R)$

$X \otimes THH(R)$ almost cpt

(直接计算, 注意到 R fp-type n $\Rightarrow R/p$ finite mtp gps)

(1a): $p^k\mathbb{Z}$-act on $THH(R)$ uni

(1b): $THH(R^{hp^k\mathbb{Z}}) \simeq W_k \otimes_W THH(R^{h\mathbb{Z}})$ } 已证.

(1c):

$$THH(R^{hp^k\mathbb{Z}})|_0 \simeq THH(R)^{hp^k\mathbb{Z}}.$$

b/c: Using Lem A.21 ($Mod(\mathbb{S}^{B\mathbb{Z}}) \simeq Sp^{B\mathbb{Z}}$, $X \mapsto \mathbb{S} \otimes_{\mathbb{S}^{B\mathbb{Z}}} X$ $\quad Y \mapsto Y^{h\mathbb{Z}}$)

it suffices to show that equiv after tensoring $\mathbb{S}$.

$$LHS = \mathbb{S} \otimes_{\mathbb{S}^{B\mathbb{Z}}} THH(R^{h\mathbb{Z}})|_0$$

$$\simeq \mathbb{S} \otimes_{\mathbb{S}^{B\mathbb{Z}}} THH(R^{h\mathbb{Z}}) \otimes_{WC^\circ(\mathbb{Z}_p)} WC^\circ(\mathbb{F}_p).$$

$$\simeq W_\infty \otimes_W THH(R^{h\mathbb{Z}}) \quad (W = THH(\mathbb{S}^{B\mathbb{Z}}) \simeq \mathbb{S}^{B\mathbb{Z}} \otimes WC^\circ(\mathbb{Z}_p).)$$

$$\simeq THH(R) \quad (\text{见上})$$

$$\simeq \mathbb{S} \otimes_{\mathbb{S}^{B\mathbb{Z}}} THH(R)^{h\mathbb{Z}}. \quad (THH(R) \text{ is uni}).$$

(1d): $p^k\mathbb{Z} \curvearrowright V \otimes R \in Alg_{\mathbb{E}_1 \otimes A_1}(Sp^{B\mathbb{Z},u})$ is trivial for $k \gg 0$.

since connective + π-finite.

locally uni really means trivial for $k \gg 0$. (A.43)

(1e): $p^k\mathbb{Z} \curvearrowright V \otimes THH(R) \in Alg_{A_2}(CycSp^{B\mathbb{Z},u})$ trivial for $k \gg 0$.

Similarly.

(2a): $V \otimes THH(R^{hp^k\mathbb{Z}}) \simeq V \otimes WC^\circ(p^k\mathbb{Z}_p) \otimes THH(R)^{hp^k\mathbb{Z}}$

The latter isom follows from trivial action.

$\simeq V \otimes W_k \otimes THH(R)$. $k \gg 0$.

so $V \otimes WC^\circ(p^k\mathbb{Z}_p) \otimes THH^{hp^k\mathbb{Z}}$

$\simeq V \otimes WC^\circ(p^k\mathbb{Z}_p) \otimes \mathbb{S}^{Bp^k\mathbb{Z}} \otimes THH(R)$.

The first one is Dehn Trivialization.

(2b): $V \otimes THH(R^{hp^k\mathbb{Z}})$ Segal ($\le b$) $\forall k \gg 0$

See also Dehn --.

b/c: V/v^a plays the role of X.

Then 逆图.

~~Cor: $V_* TC(R^{hp^k\mathbb{Z}}) \to V_* TC(R)^{hp^k\mathbb{Z}}$~~

$$\begin{tikzcd} V_* TC(R^{hp^k\mathbb{Z}}) \arrow[r] \arrow[d, "\sim"] & V_* TC(R)^{hp^k\mathbb{Z}} \arrow[d, "\sim"] \\ V_* TC(R^{B\mathbb{Z}}) \arrow[r] & V_* TC(R)^{B\mathbb{Z}} \end{tikzcd}$$

as $\mathbb{Z}[v]$-mod here v means Hun-self map. V type $\#1$

三: Bootstrapp.

(3a) $\exists k' \gg 0$ s.t $V \otimes THH(R^{hp^{k'}\mathbb{Z}})$

I W_k uniformly bounded by prop 4.27 uniformly bounded for $k \gg$

II Use prop 4.17 to deal with $V \otimes THH$.

Hypo (1): 见 (1c)

Hypo (2): (1a) + (2a)

Hypo (3): (2b)

(3b): $X \otimes THH(R^{hp^k\mathbb{Z}})$

$\simeq X \otimes W_k \otimes THH(R)$

$\forall k \gg 0$.

Retract! 其分 bound 见 $THH(\mathbb{S}^{B\mathbb{Z}})$ amplitude $\square$

(3b): $V \otimes THH(R^{hp^k\mathbb{Z}})$

$\simeq V \otimes W_k \otimes THH(R)$ $\forall k \gg 0$.

Use prop 4.28 for $X := V \otimes THH(R^{B\mathbb{Z}})$.

$X_\infty = V \otimes THH(R)$.

$X_k = W_k \otimes_W X \otimes V$.

$= V \otimes THH(R^{hp^k\mathbb{Z}})$.

Hypo (1): (3a) + (1b).

Hypo (2): (2a) + (1b)

Hypo (3): uni 性.

Rmk: Dehn Trivialization 是纯纯抽象话. 核心输入已证.

Cor: 同上 $\forall k \gg 0$

$$\begin{tikzcd} X \otimes THH(R^{hp^k\mathbb{Z}}) \arrow[r] \arrow[d, "\sim"] & X \otimes THH(R)^{h\mathbb{Z}} \arrow[d, "\sim"] \\ X \otimes THH(R^{B\mathbb{Z}}) \arrow[r] & X \otimes THH(R)^{B\mathbb{Z}} \end{tikzcd}$$

b/c: $X \otimes THH(R^{hp^k\mathbb{Z}}) \simeq X \otimes W_k \otimes THH(R)$

$THH(R)^{hp^k\mathbb{Z}} \simeq THH(R^{hp^k\mathbb{Z}})|_0$ $\square$

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Adams Operation on $BP\langle n\rangle$.

The most important input to disprove conj.

Notation:

- $l := m_p^{\mathbb{E}_1}$
- $BP\langle n\rangle$ with $\mathbb{E}_3$-$MU_{(p)}$-alg str.
- $E_n := E_n(\overline{\mathbb{F}}_p)$
- $\mathbb{G}_n$ ht n Morava st gp.

  $\mathbb{G}_n \circlearrowright E_n$ and

  $1 \to \mathcal{O}_D^\times \to \mathbb{G}_n \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 1$

  where $\mathcal{O}_D$ means the division alg over $\mathbb{Q}_p$ of Hasse inv $\frac{1}{n}$.

Recall: $\exists\, \psi^l \in \mathbb{G}_n$ be the comm alg auto $\psi^l : E_n \to E_n$ from $\mathbb{Z}_p^\times \circlearrowright$ the formal gp law.

More concretely, $\psi^l \circlearrowright \pi_{2k} E_n$ by l^k.

So $E_n^\psi \in \mathrm{CAlg}(S_{p,K(n)}^{B\mathbb{Z}})$ means the alg E_n with ψ_l-action.

Recall: Stable Adams Conj gives us

$$\begin{tikzcd} BU_{(p)} \arrow[r, "\psi^l"] \arrow[dr, "J"'] & BU_{(p)} \arrow[d, "J"] \\ & BSL_1(S_{(p)}) \end{tikzcd} \quad \text{in } Sp^{cn}.$$

$\overset{\text{Thomifying}}{\rightsquigarrow}$ $\Psi^l : MU_{(p)} \to MU_{(p)}$

called Adams operation on $MU_{(p)}$.

(We will fix one since it depends on a certain nonstdy)

$MU_{(p)}^\psi$ means the special $\mathbb{Z}$-act $\in \mathrm{CAlg}(S_p^{B\mathbb{Z}})$.

Thm: The $\mathbb{E}_1 \otimes A_2$-$MU_{(p)}$-alg underlying the $\mathbb{E}_3$-$MU_{(p)}$-alg $BP\langle n\rangle$ admits a lift

$BP\langle n\rangle^\psi \in \mathrm{Alg}_{\mathbb{E}_1\otimes A_2}(\mathrm{Mod}(S_p^{B\mathbb{Z}} : MU_{(p)}^\psi))$

s.t

(1) $\exists\, l : BP\langle n\rangle^\psi \to E_n^\psi$ in $\mathrm{Alg}_{\mathbb{E}_1}$

(2).
$$\begin{tikzcd} L_{T(n)} BP\langle n\rangle^\psi \arrow[r, "l"] \arrow[d, "\sim"] & E_n^\psi \arrow[d, "\sim"] \\ (E_n^\psi)^{h(\mu_{p^n-1}\times\hat{\mathbb{Z}})} \arrow[r] & E_n^\psi \end{tikzcd}$$
in $\mathrm{Alg}_{\mathbb{E}_1}(S_p^{B\mathbb{Z}})$

where $\mu_{p^n-1}\times\hat{\mathbb{Z}} \subset \mathbb{G}_n$ from

$$\begin{tikzcd} 1 \arrow[r] & \mu_{p^n-1} \arrow[r] \arrow[d, hook] & \mu_{p^n-1}\times\hat{\mathbb{Z}} \arrow[r] \arrow[d, hook] & \hat{\mathbb{Z}} \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \mathcal{O}_D^\times \arrow[r] & \mathbb{G}_n \arrow[r] & \mathbb{Z} \arrow[r] & 1 \end{tikzcd}$$

(3): ψ is a locally unipotent action after p-completion.

Recall:

$BP\langle n\rangle$ admits a $\mathbb{E}_3$-$MU_{(p)}[y]$-alg str, where $\deg y = 2(p^n-1)$ acts by v_n.

Lem: $W(\mathbb{F}_p)\otimes MU_{(p)}[y^{\pm 1}] \longrightarrow \operatorname{colim}_k W(\mathbb{F}_{p^k})\otimes MU_{(p)}[y^{\pm\frac{1}{(p^n-1)}}]$ is a $\mu_{p^n-1}\rtimes\hat{\mathbb{Z}}$ pro-Galois extn.

Pf: $A := W(\mathbb{F}_p)\otimes MU_{(p)}[y^{\pm 1}]$

$B :=$ RHS,

Since $\pi_0 W(\mathbb{F}_{p^n})$ has a primitive (p^n-1)-root of unity,

$\pi_* A \longrightarrow \pi_* B$ is a graded $\mu_{p^n-1}\rtimes\hat{\mathbb{Z}}$-pro Galois extn.

By Rognes's Thm, this can be lifted to Sp. $\square$

Prop: $\exists$ Identification as follows (for $\mathbb{E}_3$-alg)

$$\begin{tikzcd} L_{T(n)}BP\langle n\rangle \arrow[r] \arrow[d, "\sim"] & L_{T(n)}\left(W\overline{\mathbb{F}_p}\otimes \underbrace{BP\langle n\rangle[v_n^{\frac{1}{p^n-1}}]}_{MU_{(p)}[y^{\frac{1}{p^n-1}}]\otimes_{MU_{(p)}[y]} BP\langle n\rangle}\right) \arrow[d, "\sim"] \\ E_n^{h(\mu_{p^n-1}\rtimes\hat{\mathbb{Z}})} \arrow[r] & E_n \end{tikzcd}$$

Pf: $A :=$ Left Top

$B :=$ Right Top

$$\pi_* A \simeq \left(\mathbb{Z}_{(p)}[v_1,\dots,v_{n-1},v_n^{\pm 1}]\right)^\wedge_{(p,\dots,v_{n-1})}$$

$$\Rightarrow \pi_* B \simeq \left(\pi_* A[\underbrace{v_n^{\frac{1}{p^n-1}}}_{\deg=2}]\otimes W(\overline{\mathbb{F}_p})\right)^\wedge_{(p,\dots,v_{n-1})}.$$

$$\simeq \pi_* E_n.$$

By some calculation of obstruction classes, it would induce an $\mathbb{E}_3$-equiv to E_n.

By Recall + Lem above,

$$A \cong B^{h(\mu_{p^n-1}\rtimes\hat{\mathbb{Z}})}$$

this could be enhanced as an $\mathbb{E}_3$-map due to [Ram23]

Cor: $\exists$ $\mathbb{E}_3$-$MU_{(p)}$-alg map and E_n could be regarded as an $\mathbb{E}_3$-$MU_{(p)}$-alg

$$\iota: BP\langle n\rangle \longrightarrow E_n.$$

Prop (Fra13): $m\geq 2$.

Natural $\mathbb{E}_m$-alg map

$\underset{\text{centre}}{Z_{\mathbb{E}_m}}(E_n)\longrightarrow E_n$ is equiv.

and the $\mathbb{E}_{m+1}$-alg str of $Z_{\mathbb{E}_m}(E_n)$ agree with the restriction of the $\mathbb{E}_\infty$-str of E_n.

Cor: $\mathbb{E}_3$-$MU_{(p)}$-alg str on E_n arises from an $\mathbb{E}_4$-alg map $MU_{(p)}\to E_n$

i.e. $\mathbb{E}_4$-complex orientation.

Next we'll show that this underlying $\mathbb{E}_3$-map is compatible with Adams operation.

Prop: The $\mathbb{E}_3$-map $MU_{(p)}\longrightarrow E_n$ can be refined to $MU_{(p)}\longrightarrow E_n^{\psi}$.

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Sketch:

Unwinding defn,

the space of $\mathbb{E}_4$-alg maps $MU_{(p)} \to E_n$

is exactly the space of nullhomotopies of

$$BU_{(p)} \xrightarrow{J} BSL_1(\mathbb{S}_{(p)}) \to BSL_1(E_n)$$

in the cat of 4-fold loop spaces.

i.e. nullhomotopies of pointed spaces

$$B^4 BU_{(p)} \to B^5 SL_1(\mathbb{S}_{(p)}) \to B^5 SL_1(E_n).$$

Claim: Each homotopy can be refined to a $\mathbb{Z}$-equiv nullhomotopy

$$B^{4,[l^2]} BU_{(p)} \to B^{5,[l^2]} SL_1(\mathbb{S}_{(p)}) \to B^{5,[l^2]} SL_1(E_n^{\psi}). \quad (*)$$

(Dual of $S^{n,[k]} \otimes -$, $S^{n,[k]}$: deg k self map of S^n.)

Stronger version:

$$\mathrm{Map}_{Sp_{\mathbb{Z}}}(B^{4,[l^2]} BU_{(p)}, B^{5,[l^2]} SL_1(E_n^{\psi}))$$

torsion free, in odd degrees, ψ^l acting by l^k on π_{2k+1},

embed into

$$\mathrm{Map}(B^{4,[l^2]} BU_{(p)}, B^{5,[l^2]} SL_1(E_n^{\psi})_{\mathbb{Q}}).$$

Proof of Claim:

① Composite of $(*)$ is $\mathbb{Z}$-equiv nullhom.

$$X := \mathrm{Map}_{Sp_{\mathbb{Z}}}(B^{4,[l^2]} BU_{(p)}, B^{5,[l^2]} SL_1(E_n^{\psi}))$$

$$Y := \mathrm{Map}_{Sp_{\mathbb{Z}}}(B^{4,[l^2]} BU_{(p)}, B^{5,[l^2]} SL_1(E_n^{\psi})_{\mathbb{Q}})$$

$$\Rightarrow \mathrm{Map}_{Sp_{\mathbb{Z}}^{B\mathbb{Z}}}(B^{4,[l^2]} BU_{(p)}, B^{5,[l^2]} SL_1(E_n^{\psi})) \simeq X^{h\mathbb{Z}}.$$

$\Rightarrow$ Obstruction of $(*)$ is a class in $\pi_0(X^{h\mathbb{Z}})$

By Stronger version, we can reduce it to $\pi_0 Y^{h\mathbb{Z}}$

But $B^{5,[l^2]} SL_1(\mathbb{S}_{(p)})_{\mathbb{Q}}$ is contractible

i.e.

$$B^{4,[l^2]} BU_{(p)} \to B^{5,l^2} SL_1(\mathbb{S}_{(p)}) \to B^{5,[l^2]} SL_1(\mathbb{S}_{(p)})_{\mathbb{Q}} \to B^{5,[l^2]} SL_1(E_n^{\psi})_{\mathbb{Q}}.$$

②: Similarly, analyzise torsors of $\pi_1(X^{h\mathbb{Z}})$ and $\pi_1(X)$.

but $\pi_2(X) = 0$

$\pi_1(X)$ trivial action $\square$

We can then do induction to $B(2n)^{\psi} \to E_n^{\psi}$

Machinery: Gluing $\mathbb{E}_1$-cells.

Cor: The $\mathbb{E}_3$-$MU_{(p)}$-alg str on E_n refines to a lift of E_n^{ψ} to $\mathbb{Z}$-equiv $\mathbb{E}_2$-$MU_{(p)}$-alg E_n^{ψ}.

Pf: $\mathbb{E}_4$-alg $MU_{(p)} \to E_n$

$\rightsquigarrow \mathbb{E}_3$-alg $MU_{(p)}^{\psi} \to E_n^{\psi}$

$\rightsquigarrow \mathbb{E}_2$-$MU_{(p)}^{\psi}$-alg E_n^{ψ}. $\square$

$\mathrm{rk} = 0$

Recall:

$A \in \mathrm{Alg}(\mathcal{C})$ $\quad \mathcal{C}$ sym mono pre st with t-str.

$\mathrm{Sp}(\mathrm{Alg}(\mathcal{C})_{/A})$

is just sending $B \to A$

to A-bimod $\mathrm{fib}(A \otimes A \to A \otimes_B A)$

with right adj trivial square zero extn.

Def: $A \in \mathrm{Alg}(\mathcal{C})$ connected if

it's connective & $\pi_0 \mathbb{1} \to \pi_0 A$.

All connected alg ~~called~~ denoted by $\mathrm{Alg}(\mathcal{C})^{\geq 1}$.

(closed under colim. $\exists$ canonical truncation.

$A \otimes \pi_0 \mathbb{1} \to \pi_0 \mathbb{1}$).

Construction:

$$\mathbb{H}: \mathrm{Alg}(\mathcal{C})^{\geq 1} \xrightarrow{\otimes \pi_0 \mathbb{1}} \mathrm{Alg}(\mathrm{Mod}(\mathcal{C}; \pi_0 \mathbb{1}))_{-/\pi_0 \mathbb{1}}$$

$$\xrightarrow[\text{stabilization}]{\Sigma^\infty} \mathrm{Mod}(\mathcal{C}; \pi_0 \mathbb{1})$$

i.e. $\mathrm{fib}(\pi_0 \mathbb{1} \to \underset{A \otimes \pi_0 \mathbb{1}}{\pi_0 \mathbb{1} \otimes \pi_0 \mathbb{1}})$.

and $A \to A \otimes \pi_0 \mathbb{1} \to \pi_0 \mathbb{1} \oplus \mathbb{H}(A)$. by stablization adj.

E.g: If $\mathbb{1} = \pi_0 \mathbb{1}$.

$\mathbb{H}(\mathbb{1}\{X\}) = X$

and $\mathbb{1}\{X\} \to \mathbb{1} \oplus \mathbb{H}(\mathbb{1}\{X\})$ is exactly

$\bigoplus X^{\otimes i} \to \mathbb{1} \oplus X$.

E.g: $\mathcal{C} = \mathrm{Mod}(MU_{(p)})$

$\Rightarrow \pi_* \mathbb{H}(BP\langle n \rangle)$ torsion free

b/c: Tor SS for

$\mathbb{Z}_{(p)} \otimes_{BP\langle n \rangle \otimes MU_{(p)}} \mathbb{Z}_{(p)} \simeq \mathbb{Z}_{(p)} \oplus \mathbb{H}(BP\langle n \rangle)$. $\square$

Lem: $f: A \to B$ in $\mathrm{Alg}(\mathcal{C})^{\geq 1}$ $\quad k \geq 1$

(1) $\mathrm{fib}(f)$ k-connective

$\Leftrightarrow \mathrm{fib}(\mathbb{H}(f))$ k-connective.

(2): $\mathrm{fib}(f)$ k-connective

$\Rightarrow \mathrm{fib}(f) \to \mathrm{fib}(\mathbb{H}(f))$ $(k+1)$-connective.

Pf: (2) $\Rightarrow$ (1) by induction ✓

(2): Using simplicial resolution,

we'll assume $f: \mathbb{1}\{X\} \to \mathbb{1}\{Y\}$.

induced by k-connective

$g: X \to Y$.

for X, Y 1-connective.

the fib seq

So $\mathrm{fib}\, F \to \mathrm{fib}(f) \to \mathrm{fib}(\mathbb{H}(f))$

is exactly

$\bigoplus_{j \geq 2} \mathrm{fib}(g^{\otimes j}) \to \bigoplus_{j \geq 1} \mathrm{fib}(g^{\otimes j}) \to \mathrm{fib}(g)$

and $\mathrm{fib}(g^{\otimes j})$ is $(k-1+j)$-connective

since X, Y 1-con. $\square$

Lem (Burkland Cellular Filtration):

$f: A \to B$ in $\mathrm{Alg}(\mathcal{C})^{\geq 1}$

filtration

$\mathbb{H}(A) = M_0 \xrightarrow{m_1} M_1 \to \cdots \to M_\infty = \mathbb{H}(B)$

s.t (1) m_i k_i-connective for $k_i \geq 0$, $k_i \uparrow$

(2) $\forall i \geq 1$ $\exists X_i \in \mathcal{C}_{\geq k_i}$

s.t $X_i \otimes \pi_0 \mathbb{1} \simeq \mathrm{fib}(m_i)$

& $[X_i, Y] = 0$ $\forall$ (k_i+2)-con Y

Then

(The lifting of "ideal" to $\mathbb{1}$-alg)

$\exists$ a filtration in $\mathrm{Alg}(\mathcal{C})^{\geq 1}$

$$A = R_0 \xrightarrow{r_1} R_1 \to \cdots \to R_\infty \to B$$

s.t

(1) $R_i \to B$ k_{i+1}-connective

(2) $\mathbb{H}(-)$ sends $R_\cdot$ to $M_\cdot$

(3)
$$\begin{tikzcd} \exists\ \mathbb{1}\{X_i\} \arrow[r] \arrow[d] & \mathbb{1} \arrow[d] \\ R_{i-1} \arrow[r, "r_i"] \arrow[ur, phantom, "\lrcorner"] & R_i \end{tikzcd}$$

(4): $R_\infty \to B$ ∞-connective.

Pf: Induction:

$$\begin{tikzcd} A \arrow[r] & R_{i-1} \arrow[r] & B \arrow[r, squiggly] & \mathbb{H}(A) \arrow[r] \arrow[d, "\sim"] & \mathbb{H}(R_{i-1}) \arrow[r] \arrow[d, "\sim"] & \mathbb{H}(B) \arrow[d, "\sim"] \\ & & & M_0 \arrow[r] & M_{i-1} \arrow[r] & M_\infty \end{tikzcd}$$

define R_i = pushout as above.

Note that

$$\begin{tikzcd} X_i \arrow[r, dashed, "g_i"] \arrow[d] & \mathrm{fib}(R_{i-1} \to B) \arrow[d, "(k_i+1)\text{-connective last by (1) and lem above}"] \\ \mathrm{fib}(m_i) \arrow[r, "\text{induced from } M_{i-1}\to M_i \to M_\infty"] & \mathrm{fib}(\mathbb{H}(R_{i-1}) \to \mathbb{H}(B)) \end{tikzcd}$$

$\Rightarrow \exists\ X_i \to \mathrm{fib}(R_{i-1} \to B)$ since $[X_i, Y] = 0$ $\forall$ (k_i+2)-con Y.

fix a lift g_i and thus we have $\bar{g}_i : \mathbb{1}\{X_i\} \to R_{i-1}$.

$$\begin{tikzcd} \mathbb{1}\{X_i\} \arrow[r] \arrow[d, "\bar{g}_i"] & \mathbb{1} \arrow[r] \arrow[d] & \mathbb{1} \arrow[d] \\ R_{i-1} \arrow[r] & R_i \arrow[r] \arrow[ur, phantom, "\lrcorner"] & B \end{tikzcd}$$

$(*)$

It means R_i connected.

Applying $\mathrm{fib}(\mathbb{H}(-))$ to the rows of $(*)$, we have

$$\begin{tikzcd} X_i \arrow[r] \arrow[dr, "g_i"] & \mathrm{fib}(\mathbb{1}\{X_i\} \to \mathbb{1}) \arrow[r] \arrow[d] & X_i \otimes \pi_0 \mathbb{1} \arrow[d, "u_i"] \\ & \mathrm{fib}(R_{i-1} \to B) \arrow[r] & \mathrm{fib}(\mathbb{H}(R_{i-1}) \to \mathbb{H}(B)) \end{tikzcd}$$

Since

$X_i \otimes \pi_0 \mathbb{1} \simeq \mathrm{fib}(m_i)$.

$\mathbb{H}(-)$ sends R_i to M_i. (2) ✓

So $\mathbb{H}(R_\infty) \xrightarrow{\sim} \mathbb{H}(B)$

$\Rightarrow R_\infty \to B$ ∞-connective by last lem.

Chasing

$R_{i-1} \to R_i \to B$ we can get (1)

$\square$

$$\Rightarrow \begin{tikzcd} \mathbb{1}\{X_i\} \arrow[r] \arrow[d, "\bar{g}_i"] & \mathbb{1} \arrow[r] \arrow[d] & \mathbb{1} \arrow[d] \\ R_{i-1} \arrow[r] & R_i \arrow[r] \arrow[ur, phantom, "\lrcorner"] & B \end{tikzcd}$$

$(*)$

Using $\tau_{\leq 0}$ we have

$$\begin{tikzcd} \tau_{\leq 0}\mathbb{1}\{X_i\} \arrow[r] \arrow[d] & \mathbb{1} \arrow[d] \\ \mathbb{1} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \tau_{\leq 0} R_i = \mathbb{1}\{\Sigma X_i\} \end{tikzcd}$$

by inducing.

$\mathbb{E}_1$-case (Weak version) of Main Thm:

Prop: Unfolding $\mathbb{E}_1$-$MU_{(p)}$-alg of $BP\langle n\rangle$ lifts to $BP\langle n\rangle^{\psi} \in Alg(Mod(Sp^{B\mathbb{Z}}; MU_{(p)}^{\psi}))$.

s.t $\exists\ \iota : BP\langle n\rangle^{\psi} \to E_n^{\psi}$ $\mathbb{Z}$-equiv $\mathbb{E}_1$-$MU_{(p)}^{\psi}$-alg map.

By defn,

Pf: $Mod(Sp^{B\mathbb{Z}}; MU_{(p)}^{\psi}) \cong Mod(MU_{(p)})^{h\mathbb{Z}}$

where ψ acts on $MU_{(p)}^{\psi}$ induce a $\mathbb{Z}$-act on $Mod(MU_{(p)})$.

In order to prove this prop we have to construct a square of $\mathbb{E}_1$-$MU_{(p)}$-alg

$$\begin{tikzcd} BP\langle n\rangle \arrow[r] \arrow[d, "\iota"] & (\psi^2)_* BP\langle n\rangle \arrow[d, "(\psi^2)_*\iota"] \\ E_n \arrow[r, "\psi^2"] & (\psi^2)_* E_n \end{tikzcd}$$

$\mathbb{E}_2$-$MU_{(p)}^{\psi}$-str of E_n.

from prop 5.9 an identification of $L_{T(n)} BP\langle n\rangle$.

Due to Burklund cells, we'll construct this square by induction.

Note that

$$\mathbb{H}(MU_{(p)}) = 0 \to \Sigma^1 \pi_1(\mathbb{H}(BP\langle n\rangle)) \to \Sigma^1\pi_1 \oplus \Sigma^3\pi_3 \to \cdots \to \bigoplus_{i \text{ odd}} \Sigma^i \pi_i \simeq \mathbb{H}(BP\langle n\rangle)$$

(The Postnikov tower + SS we know only odd terms left).

Each $\pi_{2i-1}(\mathbb{H}(BP\langle n\rangle))$ is finite copies of $\mathbb{Z}_{(p)}$.

So all of the conditions to construct cellular decompositions are satisfied.

(Take $X_i = \Sigma^{-2i-2} MU_{(p)}$)

$\rightsquigarrow \exists\ MU_{(p)} = R_0 \xrightarrow{r_1} R_1 \xrightarrow{r_2} R_2 \to \cdots \to \underline{R_\infty \simeq BP\langle n\rangle}$

s.t

$$\begin{tikzcd} MU_{(p)}\{\bigoplus^N \Sigma^{-2i-2} MU_{(p)}\} \arrow[r] \arrow[d] & MU_{(p)} \arrow[d] \\ R_{i-1} \arrow[r] & R_i \end{tikzcd}$$

$\therefore$ We have to consider

$$\begin{tikzcd} \bigoplus^N \Sigma^{-2i-2} MU_{(p)} \arrow[r] \arrow[d] & MU_{(p)}\{\bigoplus^N \Sigma^{-2i-2} MU_{(p)}\} \arrow[r] \arrow[d] & R_{i-1} \arrow[d, "r_i"] \arrow[dd, bend left, "f_{i-1}"] \\ MU_{(p)} \arrow[r] & R_i \arrow[dl] \arrow[dr, dashed, "f_i"] & \\ BP\langle n\rangle \arrow[r] \arrow[d, "\iota"] & (\psi^2)_* BP\langle n\rangle \arrow[d, "(\psi^2)_*\iota"] & \\ E_n \arrow[r, "\psi^2"] & (\psi^2)_* E_n & \end{tikzcd}$$

χ

So, to construct f_i, we should show that for $\chi : \bigoplus^N \Sigma^{-2i-2} MU_{(p)} \to MU_{(p)}\{\bigoplus \Sigma MU\} \to R_{i-1}$.

$\cancel{(f_{i-1}) \circ f \circ \chi}$

$(\psi^2)_*\iota \circ f_{i-1} \circ \chi \sim 0$.

(since $(\psi^2)_*\iota$ induce inj over π_*)

But in this case, it facts through the augment ideal of $\mathbb{H}(MU_{(p)}) = 0$.

One last thing to verify is that f_i makes the lowest part commute

Which is equiv to every null homotopy of

$$\bigoplus \Sigma^{2i-2} MU_{(p)} \xrightarrow{x} R_{i-1} \longrightarrow (\psi^l)_* E_n$$

is compatible to each other.

Note that $\pi_{2i-1} E_n = 0$, so all such nullhomotopies are connected.

$\square$

Proof of Main Thm:

$E_1 \otimes A_2^{-(MU_{(p)})}$ str of $BP\langle n\rangle^{\psi}$ could be constructed similarly. (A_2 - is much simpler).

$\iota: BP\langle n\rangle^{\psi} \longrightarrow E_n^{\psi}$ is exactly from weak version.

Then comm square

$$\begin{tikzcd} L_{T(n)} BP\langle n\rangle^{\psi} \arrow[r, "\iota"] \arrow[d, "\sim"] & E_n^{\psi} \arrow[d, "\sim"] \\ (E_n^{\psi})^{h(\mathbb{F}_p^\times \ltimes \hat{\mathbb{Z}})} \arrow[r] & E_n^{\psi} \end{tikzcd}$$

follows from the fact that
- nonequivariant case ✓
- $\psi^l \in \mathbb{G}_n$ central $\Rightarrow$ commutes with $\mathbb{F}_p^\times \ltimes \hat{\mathbb{Z}}$.

Locally unipotent because $\pi_* BP\langle n\rangle$ concentrated in $2k$ and is acted by ψ^l via l^k on π_{2k}.

(Then calculate some easy things) $\square$

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Note that $L_{\cdot\cdot}$ is right adj.
a good cover would imply the closed prop.

Disproving Telescope Conj:

What we'll prove is that

$$(Sp_{T(n+1)})^{\wedge}_{cyc} \neq Sp_{T(n+1)}.$$

In last talk we've constructed an Adams operation on $BP\langle n\rangle$.

So today we want to deduce it by showing that

Thm: $L_{T(n+1)} K(BP\langle n\rangle^{hp^k\mathbb{Z}})$ isn't cyclotomically cpt for $n\geq 1$ and $k\gg 0$.

First we have to compute its completion.

Conclusion:

$$L_{T(n+1)} K(BP\langle n\rangle^{hp^k\mathbb{Z}})^{\wedge}_{cyc} \simeq L_{T(n+1)} K(BP\langle n\rangle)^{hp^k\mathbb{Z}}.$$

and the coassembly map is indeed the natural map. $\forall n\geq 1$, $k\geq 0$.

Recollection (Descent and Covers).

$\mathcal{C}$ with finite lim & colim

finite coproducts are disjoint & universal

$F: \mathcal{C}^{op} \to D$ product preserving fct.

(e.g: $\mathcal{C} = CAlg(Sp_{T(n)})^{op}$, $F = L_{T(n+1)} K$.)

Def: An F-cover in $\mathcal{C}$ in sense of [LZ12a] is $f: A\to B$ in $\mathcal{C}$ s.t $\forall$ base change g F satisfies descent for Čech nerve of g.

Easy

Facts: F-covers is closed under composition, base change.

$f\circ g$ cover $\Rightarrow$ so is f.

Prop: F-covers are exactly covers for universal F-descent topology.

Pf: Exercise.

Prop (1): $R\in Alg(Sp)$, S profinite

$\Rightarrow K(WC^0(S)\otimes R) \simeq WC^0(S)\otimes K(R)$

(2) $n\geq 1$, $R\in Alg(Sp_{T(n)})$, S profinite

$$L_{T(n+1)} K(L_{T(n)}(WC^0(S)\otimes R)) \simeq L_{T(n+1)}(WC^0(S)\otimes K(R)).$$

Pf: (2) (Assume (1))

$WC^0(S)\otimes R \to L_{T(n)}(WC^0(S)\otimes R)$ is $T(n)\oplus T(n+1)$-local isom. since R $T(n)$-local.

So

$$LHS \simeq L_{T(n+1)} K(WC^0(S)\otimes R) \underset{(1)}{\simeq} L_{T(n+1)}(WC^0(S)\otimes K(R))$$

Parity Thm.

(1): $K(-)$, $W(-)$, $-\otimes R$ comm with filtered colim + finite product

Then reduce it to $S=pt$. $\square$

Cor: $n \geq 1$

$R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)}^{B\mathbb{Z}(n)})$ ✡

$F := L_{T(n+1)} K$ (and thus $L_{T(m+1)} \hat{K}_{cyc}$)

then

$$\begin{tikzcd} & F(R^{h\mathbb{Z}}) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\circlearrowright"] & \\ F(R)^{h\mathbb{Z}} \arrow[rr, "\sim"] & & \underbrace{\lim_{\Delta} F(R^{\otimes_{R^{h\mathbb{Z}}} \bullet+1})}_{\text{from } R^{h\mathbb{Z}} \to R.} \end{tikzcd}$$

In particular,

$R^{h\mathbb{Z}} \to R$ satisfies F-descent

$\Leftrightarrow$ coassembly map is equiv

pf:

$$\begin{tikzcd} F(R^{h\mathbb{Z}}) \arrow[r] \arrow[d] & \lim_{\Delta} F(R^{h\mathbb{Z}} \otimes_{R^{h\mathbb{Z}}} R^{\otimes_{R^{h\mathbb{Z}}} \bullet}) \arrow[d] \\ F(R)^{h\mathbb{Z}} \arrow[r, "\sim", "\wr"'] & \left(\lim_{\Delta} F(R \otimes_{R^{h\mathbb{Z}}} R^{\otimes_{R^{h\mathbb{Z}}} \bullet})\right)^{h\mathbb{Z}} \end{tikzcd}$$

since $R \to R \otimes_{R^{h\mathbb{Z}}} R$ admits rectraction.

Claim $\forall$ R-alg A,

$$F(R^{h\mathbb{Z}} \otimes_{R^{h\mathbb{Z}}} R \otimes_R A) \xrightarrow{\sim} F(R \otimes_{R^{h\mathbb{Z}}} R \otimes_R A)^{h\mathbb{Z}}$$

Recall our first talk,

we proved $WC^{\circ}(\mathbb{Z}_p)^{\otimes R} \simeq R \otimes_{R^{h\mathbb{Z}}} R$.

with equivariant $\mathbb{Z}$-action

$+1 : \mathbb{Z}_p \to \mathbb{Z}_p$

$\mathbb{Z}$-act $\quad$ R-act

Then the following conclusion follows from lem above + A.35(3) (the adj of unipotent alg)

$$\begin{tikzcd} S^{B\mathbb{Z}} \arrow[r] \arrow[d] \arrow[dr, "R^{h\mathbb{Z}}"] & \mathbb{S} \arrow[d] \arrow[drr] & & \\ \mathbb{S} \arrow[r] \arrow[dr] & WC^{\circ}(\mathbb{Z}_p) \arrow[r] \arrow[d] & R \arrow[d] & \\ & R \arrow[r] & R \otimes_{R^{h\mathbb{Z}}} R & \end{tikzcd}$$

Next WTS:

For $R = L_{T(n)} BP\langle n \rangle$. (Purity implies that nothing changed after $L_{T(n)}$)

① $R^{hp^k\mathbb{Z}} \to R$ is $L_{T(n+1)} \hat{K}_{cyc}$-cover

② $L_{T(n+1)} K(R)$ is cyclotomically complete.

So by last Cor. (unipotent is true !!!).

$$\cancel{L_{T(n+1)}} K(R^{hp^k\mathbb{Z}}) \to L_{T(n+1)} K(R)^{hp^k\mathbb{Z}}$$

is equiv after cyc cplt

and the latter one is already cyc cplt.

i.e. This is completion map.

Claim 1:

$\mathbb{S}_{T(n)} \to \mathbb{S}_{T(n)}[\omega_{p^\infty}^{(n)}]$

is a $L_{T(n+1)} \hat{K}_{cyc}$-cover

Lem: $G = \Omega X$ loop space

if $a \in \mathcal{C}^{BG}$ s.t

$F(-) \xrightarrow{\sim} F(- \otimes a)^{hG}$

then $a \to *$ is an F-cover.

$\square$

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Proof: $\overset{F}{\mathcal{C}^{\circ} \xrightarrow{a\otimes -} \mathcal{C}^{BG} \xrightarrow{F} \mathcal{D}^{BG} \xrightarrow{(-)^{hG}} \mathcal{D}}$

~~$\mathcal{C} \to \mathcal{D} \xrightarrow{(-)} \mathcal{D}$~~ It's easy to understand that

So it suffices to verify f is an $F(a\otimes -)$-cover.

i.e $a\otimes f$ is an F-cover.

This's definitely true since $a\otimes a \xrightarrow{\Delta} a$. $\square$

Proof of Claim 1:

By lem, we need to show

$$L_{T(n+1)}K(R)^{\wedge}_{cyc} \simeq \left(L_{T(n+1)}K(\mathbb{S}_{T(n)}[w^{(n)}_{p^\infty}]\otimes R)^{\wedge}_{cyc}\right)^{h(T_p\times\mathbb{Z})}.$$

This's just Higher Descent. $\square$

Claim 2: $n\geq 1$

$\mathbb{S}_{T(n)} \longrightarrow W(\overline{\mathbb{F}}_p)\otimes \mathbb{S}_{T(n)}$ $\quad L_{T(n+1)}K$-cover

$\mathbb{S}_{T(n)} \longrightarrow \mathbb{S}^{ab}_{T(n)}$ $\quad L_{T(n+1)}K^{\wedge}_{cyc}$-cover.

Pf: Recall $\mathbb{S}^{ab}_{T(n)} \simeq W(\overline{\mathbb{F}}_p)\otimes \mathbb{S}_{T(n)}[w^{(n)}_{p^\infty}]$.

so the first ~~claim~~ Prop + Claim 1 $\Rightarrow$ second prop.

($L_{T(n+1)}K$-cover is auto $L_{T(n+1)}K^{\wedge}_{cyc}$-cover).

To prove 1st prop,

we can use the lem again. where $a = W(\overline{\mathbb{F}}_p)$

with $\mathbb{Z}$-action of Frob

i.e. In the following digram, we need to show that

it's equiv

$$\begin{tikzcd} L_{T(n+1)}K(R) \arrow[r, "\simeq"] \arrow[d] & L_{T(n+1)}TC(\tau_{\geq 0}R) \arrow[d, "\simeq"] \\ L_{T(n+1)}K(L_{T(n)}(W(\overline{\mathbb{F}}_p)\otimes R))^{h\mathbb{Z}} \arrow[r, "\simeq"] & L_{T(n+1)}TC(W(\overline{\mathbb{F}}_p)\otimes \tau_{\geq 0}R)^{h\mathbb{Z}} \end{tikzcd}$$

The horizontal maps are equiv by purity.

The right vertical map is equiv b/c:

$$\begin{aligned} THH(R) &\simeq W(\overline{\mathbb{F}}_p)^{h\mathbb{Z}}\otimes THH(R) \\ &\simeq THH(W(\overline{\mathbb{F}}_p))^{h\mathbb{Z}}\otimes THH(R) \\ &\simeq \left(THH(W(\overline{\mathbb{F}}_p))\otimes THH(R)\right)^{h\mathbb{Z}} \\ &\simeq THH(W(\overline{\mathbb{F}}_p)\otimes R)^{h\mathbb{Z}} \end{aligned}$$ $\square$

Cor: $n\geq 1$ $R\in CAlg(Sp_{T(n)})$

with structral map $\mathbb{S}_{T(n)}[w^{(n)}_{p^\infty}]^{hT_p} \to$

then $L_{T(n+1)}K(R)$ $\quad L_{T(n)}(W\overline{\mathbb{F}}_p\otimes R)$

is cyclotomically cplt.

Pf: Due to $\mathbb{S}_{T(n)} \to \mathbb{S}_{T(n)}\otimes W\overline{\mathbb{F}}_p$

is $L_{T(n+1)}K$-cover,

it suffices to show that

$R' := L_{T(n)}(W\overline{\mathbb{F}}_p\otimes R)$

s.t $L_{T(n+1)}K(R')$ cyc cplt.

i.e. $L_{T(n+1)}K(R')$

$\to L_{T(n+1)}K(\mathbb{S}_{T(n)}[w^{(n)}_{p^\infty}]^{hT_p}\otimes R')$

has a retraction

which ~~exa~~ is the outcome of hypothesis.

Recall cplt is $L_{T(n+1)}K(\mathbb{S}_{T(n)}[w^{(n)}_{p^\infty}]^{hT_p}\otimes R')^{h\mathbb{Z}}$.

$\square$

Basic Fact:

$$A \xrightarrow[\text{Galois}]{\text{finite}} B \xrightarrow{\text{pro-Gal}} C \to D$$

$\Rightarrow B \otimes_A D \to C \otimes_A D$ has a retraction.

Pf: Assume $C = D$. (Otherwise base change)

$G := \mathrm{Gal}(C/A)$

$H := \mathrm{Gal}(B/A)$.

$$\begin{array}{ccc} B \otimes_A C & \to & C \otimes_A C \\ \wr\| & & \wr\| \\ C^{G/H} & \longrightarrow & C^G \end{array}$$

from $G \twoheadrightarrow G/H$.

but G profinite so it has a section $\square$

Proof of ① & ②: $R = L_{T(n)} BP\langle n\rangle$

②: Using Cor1:

we want to construct $\mathbb{S}_{T(n)}[W_{p^\infty}^{(n)}]^{hT_p} \to L_{T(n)}(W\bar{\mathbb{F}}_p \otimes R)$.

Recall: $L_{T(n)} BP\langle n\rangle \simeq E_n^{h(\mu_{p-1} \rtimes \hat{\mathbb{Z}})}$

so after base change to $W\bar{\mathbb{F}}_p$.

$R^{hp^k\mathbb{Z}} \xrightarrow{f_K} R$ becomes $(E_n^{h\mu_{p-1}})^{hp^k\mathbb{Z}} \xrightarrow{\tilde{f}_K} E_n^{hC}$ for

Claim: $\exists$

$$\begin{array}{ccc} \mathbb{S}_{T(n)}[W_{p^k}^{(n)}]^{hT_p} & \longrightarrow & \mathbb{S}_{T(n)}[W_{p^\infty}^{(n)}]^{hT_p} \\ \downarrow & \lrcorner & \downarrow \\ (E_n^{h\mu_{p-1}})^{hp^k\mathbb{Z}} & \longrightarrow & E_n^{h\mu_{p-1}} \end{array}$$

so ② is true.

① This pushout holds

because $G_n \xrightarrow{cyc} \mathbb{Z}_p^\times$ maps ψ^l to l^n

then

$$\begin{array}{ccc} G_n/\mu_{p^n-1} & \longrightarrow & G_n/\mu_{p^n-1}/\psi^{l^{p^k}} \\ \downarrow \mathbb{Z}_p^\times \ \Gamma & & \downarrow \\ \mathbb{Z}_p & \longrightarrow & \mathbb{Z}_p/p^k \end{array}$$

implies the pushout by Galois corr.

① : $\mathbb{S}_{T(n)} \to \left(E_n^{h\mu_{p-1}}\right)^{hp^k\mathbb{Z}} \xrightarrow{\tilde{f}_K} E_n^{h\mu_{p-1}} \to \mathbb{S}[W_{p^\infty}^{(n)}]$

implies $\tilde{f}_K \otimes \mathbb{S}[W_{p^\infty}^{(n)}]$

$= f_K \otimes \mathbb{S}_{T(n)}^{ab}$ admits retract

$\Rightarrow L_{T(n+1)} K_{cyc}^\wedge$-cover.

i.e.

$$\begin{array}{ccccc} & & R^{hp^k\mathbb{Z}} \otimes \mathbb{S}_{T(n)} & \longrightarrow & R \otimes \mathbb{S}_{T(n)} \\ & \nearrow & \downarrow \text{cover} & \lrcorner & \downarrow \\ \mathbb{S}_{T(n)} & & & & \\ \text{cover}\downarrow & & \lrcorner & & \\ \mathbb{S}_{T(n)}^{ab} & \longrightarrow & R^{hp^k\mathbb{Z}} \otimes \mathbb{S}_{T(n)}^{ab} & \xrightarrow[\text{cover}]{} & R \otimes \mathbb{S}_{T(n)}^{ab} \end{array}$$

$\mathbb{S}_{T(n)} \xrightarrow{\text{cover}} \mathbb{S}_{T(n)}^{ab}$

$\rightsquigarrow$ implie top map is a cover $\square$

Final Disproof:

Claim: For general R with $\mathbb{Z}$-acyl.

$$\begin{array}{ccccc} L_{T(n+1)} K(L_{T(n)} R^{h\mathbb{Z}}) & \xleftarrow{\sim} & L_{T(n+1)} K(R) & \xrightarrow{\sim} & L_{T(n+1)} TC(R)^{h\mathbb{Z}} \\ \downarrow & & \downarrow & & \downarrow \\ L_{T(n+1)} K(L_{T(n)} R)^{h\mathbb{Z}} & \xleftarrow{\sim} & L_{T(n+1)} K(R)^{h\mathbb{Z}} & \xrightarrow{\sim} & L_{T(n+1)} TC(R)^{h\mathbb{Z}} \end{array}$$

with R $T(n+1)$-acyclic connective $\mathbb{E}_1$

The lower horizontal maps are equiv by Purity thm.

upper left

The right top. see [Lev22].

Hence:

let R be $BP\langle n\rangle$ with adams op.

$$\begin{tikzcd}
L_{T(n+1)}K(L_{T(n)}BP\langle n\rangle^{hp^k\mathbb{Z}}) & L_{T(n+1)}K(BP\langle n\rangle^{hp^k\mathbb{Z}}) \arrow[l, "\sim"'] \arrow[r, "\sim"] & L_{T(n+1)}TC(BP\langle n\rangle^{hp^k\mathbb{Z}}) \\
L_{T(n+1)}K(L_{T(n)}BP\langle n\rangle)^{hp^k\mathbb{Z}} \arrow[u] & L_{T(n+1)}K(BP\langle n\rangle)^{hp^k\mathbb{Z}} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u] & L_{T(n+1)}TC(BP\langle n\rangle)^{hp^k\mathbb{Z}} \arrow[u]
\end{tikzcd}$$

Using the calculation of $L_{T(n+1)}K(BP\langle n\rangle^{hp^k\mathbb{Z}})^{\wedge}_{cyc}$, we thus reduce it to show that the Verticle map isn't an equiv.

Actually we can assume $k \gg 0$ since cyc localizing is smashing and transition maps are E_∞.

$$\left(\begin{tikzcd}
A \arrow[r, "\sim"] \arrow[d] & A^{\wedge}_{cyc} \arrow[d] \\
B \arrow[r, "\sim"] & B^{\wedge}_{cyc}
\end{tikzcd} \quad \text{implies } K \vee \Rightarrow K+1 \vee \right)$$

But for $k \gg 0$, the unipotent action and all finiteness property of $BP\langle n\rangle$ help us reduce it to fct spaces, i.e.

$$\begin{tikzcd}
T(n+1)_*TC(BP\langle n\rangle^{hp^k\mathbb{Z}}) \arrow[r] \arrow[d, "\sim"] & T(n+1)_*TC(BP\langle n\rangle)^{hp^k\mathbb{Z}} \arrow[d, "\sim"] \\
T(n+1)_*TC(BP\langle n\rangle^{B\mathbb{Z}}) \arrow[r] & T(n+1)_*TC(BP\langle n\rangle)^{B\mathbb{Z}}
\end{tikzcd}$$

For $R = BP\langle n\rangle \in Alg_{E_1\otimes A_2}(Sp^{B\mathbb{Z}, u})$ connective, f_p-type $\geq n$ with ht n (Q-property) (See HW22).

where the fib of the lower map generates a thick subcat containing $T(n+1)_*(WC^0(\mathbb{Z}_p^\times) \otimes TC(BP\langle n\rangle)) \neq 0$.

since $T(n+1) \otimes TC(BP\langle n\rangle) \neq 0$. $\square$

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色同伦论讲演

今天我来介绍 Chromatic Homotopy Theory.
前置知识有无穷范畴，代数几何，稳定同伦论，由于我们时间有限，我将简要叙述其主要精神并陈述个别定理，最终，讨论色展零点定理.

众所周知，拓扑K理论，普通$\mathbb{Z}$-系数上同调是至关重要的研究对象.（再例如复配边谱 MU）.
Quillen (in 1970s) 发现，对一类充分好的谱（复可定向，特征是 $E^*(\mathbb{C}P^\infty)\simeq\pi_*E[[t]]$）.
其上线丛的 Chern 类会诱导出 formal group law.

i.e. $f(c_1(\mathcal{L}_1), c_1(\mathcal{L}_2)) = c_1(\mathcal{L}_1\otimes\mathcal{L}_2)$, a power series

in which f 满足

$$\begin{cases} f(x,0)=x \\ f(x,y)=f(y,x) \\ f(f(x,y),z)=f(x,f(y,z)) \end{cases}$$

亦对应了张量积的性质.

反之，Landweber 发现，只要 formal group 满足 flatness，我们便可以还原一个（复可定向）谱.（见 ppt）
这正是色同伦论的滋养.

借助各种几何世界的成例，我们得到了大量极富意义的谱，推进了 Adams 利用 image of J 手法得到的对 π_*^{st} 的逼近.

为了清楚说明这些东西，我们将介绍 height of formal gp law 与对应的 stratification.

Def: f f.g.l.

height of f w.r.t to the prime p

指 $[p](t) := f(\overbrace{t, f(t,t)}^{p-1}, \cdots)$

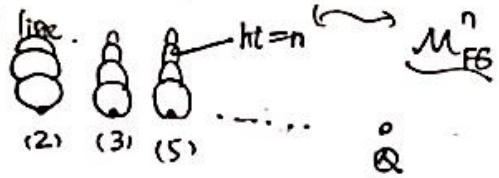
从 t^{p^n} 开始，且对应项系数是 unit.

e.g. 特征0的域上，

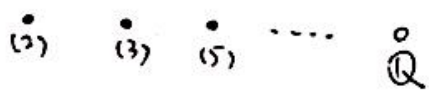
每个 prime 处 ht 都是 0.

借用 $\mathcal{M}_{FG}$, moduli stack of formal groups.

我们有如下分层：



这比起经典的 Spec$\mathbb{Z}$ 更加精细.



更关键的是，这个图景确实生活在谱世界中

Fact: $\exists$ 一系列称为 Morava K-thy 的谱 $K(n)$

$\approx$ 上图中 $\mathcal{M}_{FG}^n$ 的地位

(Balmer Spectral Space of Sp).

不同，因为它们生活在Sp

所以可以作实际的localization，请看下页ppt.
（颠覆了前人认识，更加精细的局部化）

言以蔽之，

代数几何的视角为同伦论打开了天地，

这是代数几何与同伦结合的第一次浪潮.

recall: Ravenel's Conj:

① Nilpotence Thm: K(n)构成了Sp中所有"素域".

② Periodicity Thm: ~~有有~~ 递归计算过程中会出现周期现象（简化计算）.

③ ~~Smashing~~ Thm
Chromatic Convergence: 计算每个K(n)-local信息再拼起来，的确能恢复p-cplt的全貌.

④ Telescope Conj:
If so K(n)-local ⟺ T(n)-local ✗
这会给出另一种计算方式.

~~因~~
读两页ppt. 介绍零点定理.
这预示了代数几何与同伦论结合的第二次浪潮.（SAG）.

Digression:
Witt向量上的形式幂级数环是universal deformation ring.

Thm: K ch=p perfect field.
上有f.g.l f height =n.
则在 $W(K)[[u_1 \cdots u_{n-1}]]$ 上有universal deformation of f
s.t $\mathrm{Hom}_K(W(K)[[u_1 \cdots u_{n-1}]], A) \simeq$ Deformation of f at A.
这个deformation易证flat,
于是有谱 $E_n(k)$ or $E(k)$ Morava E-thy or Lubin-Tate thy.

Lurie利用第二次浪潮中
由他发展的SAG理论，构造了Spherical Witt向量 f enhanced deformation
并证明了 提升E(k)作为$\mathbb{E}_\infty$-ring.

Thm:
$E(-): \mathrm{Perf}_k \rightleftarrows \mathrm{CAlg}_{E(k)}(Sp_{T(n)}) : \pi_0$

(1) E(-) is colocalization (i.e. f.f)

(2) ~~$\pi_0 E(A) \simeq W(A)$~~
E(-)本质像为偶周期且 π_0 / Landweber ideal perfect alg

他的学生Allan Yuan等人
观察到Witt向量上有δ-structure.
尝试推广到一般的E(k)上.
其部分工作如下.

Thm: K ch=p perfect
∃ $\mathrm{CRing}_{\pi_0 E(k)}$ 上 monad $\mathbb{T}$
s.t. $\pi_0: \mathrm{CAlg}_{E(k)}(Sp_{T(n)}) \longrightarrow \mathrm{CRing}_{\pi_0 E(k)}$
可提升到 $\mathrm{Alg}_{\mathbb{T}}$ (i.e. $\mathrm{CRing}_{\pi_0 E(k)}$ with some operators' action)
且 $(U_{\mathbb{T}}(-)/m)^{\#}: \mathrm{Alg}_{\mathbb{T}} \underset{f.f}{\rightleftarrows} \mathrm{Perf} : \pi_0 E(-)$
大量的伴随丰富了交换图表
与追图可得的结果.

零点定理现如今更多作为技术化工具而使用
一如经典版本.

随着chromactic thy的发展
我们坚信K(n) T(n)将深入
融入SAG的研究.
反馈给经典数论与代数几何.
i.e. 第三次浪潮！

Why do you need different models?

- 没有好的环
- 没有三角.

Target:

Nilpotence Thm:

MU在如下意义下探测幂零:

$\forall R$ 环谱. $\mathrm{Ker}(\pi_* R \to MU_* R) =$ 幂零元.

(与范畴版本一致).

Periodicity Thm: In p-local sp.

$\forall n$-type finite sp X,

$\exists\ v_n : \Sigma^N X \to X$ 诱导 $K(n)$-equiv

$K(i)$-nil $i \neq n$.

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Spherical Witt Vectors.

In [EC2], Lurie defined the spherical enhancement of Witt vectors by virtue of higher deformation theory.

One of his methods could be referred in my thesis.

Today we'll use the existence of it and its universal property.

Thm (Lurie):

$W(-)$ or $\mathbb{S}_{W(-)}$, the spherical witt vectors, s.t.

$$\mathbb{S}_{W(-)} : \mathrm{Perf} \rightleftarrows \mathrm{CAlg}(Sp_p) : \underline{\pi_0(-)^b}$$

means the inverse limit along Frobenius of π_0/p.

identifies Perf as a colocalization of $\mathrm{CAlg}(Sp_p)$, whose essential image consists of connective ring spectra R s.t $\mathbb{F}_p \otimes R$ discrete & perfect.

Inspired by classical result, Nikolaus and Yakerson raised and proved the following theorem about the cakulation of $\mathbb{S}_{W(R)}$.

Thm1 R perfect $\mathbb{F}_p$-alg. Let $I = \mathrm{fib}(\mathbb{S}[R] \to R)$.

$$\mathbb{S}_{W(R)} \simeq \mathbb{S}[R]^\wedge_I := \varprojlim_n \mathbb{S}[R]/I^n$$

$$\simeq \lim_{\Delta} \left(R \rightrightarrows R \otimes_{\mathbb{S}[R]} R \rightrightarrows \cdots \right)$$

Rmk: Since I is the fib b/w $\mathbb{E}_\infty$-rings. $\mathbb{S}[R]/I^n$ is still $\mathbb{E}_\infty$-ring.

Well, a good question is whether the completion with res. to R is the same as the Bousfied localization at R. (in p-complete $\mathbb{S}[R]$-mod).

Based on Nikolaus' observation, we get the following thm.

Thm2:

① $\mathrm{Mod}_{\mathbb{S}_{W(R)}}(Sp_p) \xrightarrow{f.f} \mathrm{Mod}_{\mathbb{S}[R]}(Sp)$

② $\forall M$ bounded below $\mathbb{S}[R]$-mod,

$$\mathbb{S}_{W(R)} \hat{\otimes}_{\mathbb{S}[R]} M \simeq M^\wedge_I$$

③ For the following property we have

$(I) \Leftrightarrow (V) \Leftrightarrow (VI) \Leftrightarrow (VII) \Leftrightarrow (VIII)$

$(II) \Rightarrow (III) \Rightarrow (I)$

for M p-cplt. $\Rightarrow (IV)$

When M bounded below. $(IV) \Rightarrow (II)$ holds

Thus these are equiv!

(I) M lies the essential image of embed.

(II) $M \xrightarrow{\sim} M^\wedge_I$

(III) M is R-local.

(IV) $R\Gamma_{\mathbb{S}}(H_*(M, \mathbb{F}_p))$ is also

(V) $\rho_r + \rho_s - \rho_{r+s} : M \to M$ additive is a map divisible by p. where $r, s \in R$

(VI) $\rho_r + \rho_s - \rho_{r+s} : \pi_* M \to \pi_* M$ is a map divisible by p.

(VII): $R\Gamma_{\mathbb{S}}(\pi_*(M)/p)$ and $\pi_*(M)[p]$

(VIII) $\pi_n(M)$ lies in the essential image of p-cplt $\mathbb{S}[R]$-mod.

Cor1: A p-cplt $\mathbb{E}_1$-sp

$$\Rightarrow \begin{tikzcd} \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_1}}(\mathbb{S}_{W(R)}, A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Map}_{\mathrm{Ring}}(R, \pi_0 A/p) \arrow[d] \\ \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}}((R,\cdot), (\Omega^\infty A, \cdot)) \arrow[r] & \mathrm{Map}_{\mathrm{Mon}}((R,\cdot), (\pi_0 A/p, \cdot)) \end{tikzcd}$$

Cor2: A p-cplt $\mathbb{E}_n$-ring $1 \le n \le \infty$

(For $n=1$, assume additionally $\pi_0 A$ lies in the centre of $\pi_1 A$)

$\Rightarrow$ The pullback above could be enhanced into

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_n}}(\mathbb{S}_{W(R)}, A) \simeq \mathrm{Map}_{\mathrm{CRing}}(R, \pi_0 A^b).$$

Key fact:

$$\mathbb{S}[R]^\wedge_p \to \mathbb{S}_{W(R)}$$

is p-cplt homo epi

Pf: After $-\otimes \mathbb{F}_p$,

we reduce it to

$\mathbb{F}_p[R] \to R$ which is a surj map

Proof of Thm1:

$$\mathbb{S}[R]^\wedge_I = \lim_\Delta \left(R \rightrightarrows R \hat{\otimes}_{\mathbb{S}[R]} R \substack{\to\\\to\\\to} \cdots \right)$$

(by R $\mathbb{F}_p$-alg)

$$\simeq \lim_\Delta \left(R \rightrightarrows R \hat{\otimes}_{\mathbb{S}[R]^\wedge_p} R \substack{\to\\\to\\\to} \cdots \right)$$

$$\simeq \lim_\Delta \left(R \rightrightarrows R \hat{\otimes}_{\mathbb{S}_{W(R)}} R \substack{\to\\\to\\\to} \cdots \right)$$

However $R \simeq \mathbb{F}_p \otimes \mathbb{S}_{W(R)}$.

$$\simeq \lim_\Delta \left(\mathbb{F}_p \otimes \mathbb{S}_{W(R)} \rightrightarrows \mathbb{F}_p \otimes \mathbb{F}_p \otimes \mathbb{S}_{W(R)} \substack{\to\\\to\\\to} \cdots \right)$$

$\simeq \mathbb{S}_{W(R)}$ by Adams Spectral sequence / or Adams tower

(Note that $\mathbb{S}_{W(R)}$ connective & p-cplt) $\square$

Proof of Thm2:

① ✓

② Similar to the proof above.

③ See Next Page.

Preliminary:

We say $A \to B$ is p-cplt homological epi if $B \hat{\otimes}_A B \xrightarrow{\sim} B$.

(1): $A \to B$ p-cplt homo epi

$\Leftrightarrow$ $A \otimes \mathbb{F}_p \to B \otimes \mathbb{F}_p$ ordinary homo epi.

(2): Surj map b/w perfect $\mathbb{F}_p$-alg is homo epi

(3): p-cplt homo epi $A \to B$

implies $\mathrm{Mod}_B(\mathrm{Sp}_p) \xrightarrow{\text{f.f}} \mathrm{Mod}_A(\mathrm{Sp}_p)$

(4): In such case

$$M \hat{\otimes}_A N \simeq M \hat{\otimes}_B N$$

Pf: (1) Nakayama.

(2) $B \otimes^L_A B$ is an animated perfect $\mathbb{F}_p$-alg (must discrete) On π_0, idempotency holds for direct calculation.

(3) (4) Idempotent !!!

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Continued

(I) ⇒ (V):

By defn,

$\exists\ \mathbb{S}[R] \to \mathbb{S}_{W(R)} \to \mathrm{map}(M, M)$ ring map.

Then $\mathbb{S}[R] \to \underline{\mathbb{S}_{W(R)}/p} \to \mathrm{map}(M, M)/p$.

whose π_0 is just R.

So the claim follows.

(V) ⇒ (VI) ⇒ (VII) ✓.

(VII) ⇒ (VIII):

By assumption, $\pi_n(M)/p^k \pi_n(M)[p]$ lies in the essential image.

⇒ $\pi_n M /\!/ p$ lies in essential image

⇒ so does $\pi_n M /\!/ p^k$ $\forall k$.

⇒ So does $\pi_n M$ since $\pi_n M$ derived p-cplt

$\pi_n M \simeq \lim \pi_n M /\!/ p^k$.

(VIII) ⇒ (I).

Postnikov tower.

(II) ⇒ (III)

Note that $M_I^\wedge$ is limit of $R \hat{\otimes}_{\mathbb{S}[R]} \cdots \hat{\otimes} M$.

with each term being R-local.

(III) ⇒ (I).

WTS M is $\mathbb{S}_{W(R)}$-mod.

Note the idempotency. Which means that this property ⇔ $\mathbb{S}_{W(R)}$-local.

But $\exists\ \mathbb{S}_{W(R)} \to R$.

So $\mathbb{S}_{W(R)}$-acyclic ⇒ R-acyclic i.e. R-local ⇒ $\mathbb{S}_{W(R)}$-local.

(I) ⇒ (IV):

$\mathbb{S}[R] \to \mathbb{S}_{W(R)}$.

Then base change to $\mathbb{F}_p$, M

and we'll get

$\mathbb{F}_p[R] \to R$ M.

i.e. $R \circlearrowright H_*(M, \mathbb{F}_p)$ also additive.

With additional boundedness.

(IV) ⇒ (I):

$M \otimes \mathbb{F}_p$ is thus R-mod by (4).

⇒ $M \otimes \mathbb{F}_p^{\otimes \cdots \otimes \mathbb{F}_p}$ is $\mathbb{S}_{W(R)}$-mod.

But with bounded below property

$M \simeq M_I^\wedge \simeq \lim(\mathbb{F}_p \otimes M \otimes \cdots \rightrightarrows$

$\in \mathrm{Mod}_{\mathbb{S}_{W(R)}}(\mathrm{Sp}_p)$

□

Proof of Cor 1:

$\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}_p)_{\mathbb{S}_{W(R)}/}$

$\simeq \mathrm{Alg}_{\mathbb{E}_1}({}_{\mathbb{S}_{W(R)}}\mathrm{Mod}^{(\mathbb{S}_p)}_{\mathbb{S}_{W(R)}})$

in which ${}_{\mathbb{S}_{W(R)}}\mathrm{Mod}_{\mathbb{S}_{W(R)}} \xrightarrow{f.f} {}_{\mathbb{S}[R]}\mathrm{Mod}_{\mathbb{S}[R]}$

with essential image

$\{ M \mid p\text{-cplt s.t } \pi_n M/p \quad \pi_n M[p]$ equipped with R-mod str $\}$

Note that

For $A \in \mathrm{Alg}_{\mathbb{E}_1}({}_{\mathbb{S}[R]}\mathrm{Mod}_{\mathbb{S}[R]})$ then multiplicative R-action on $\pi_n A/p \quad \pi_n A[p]$ is realized by $\mathbb{Z}[R] \to \pi_0 A \to \pi_0 A/p \subset \pi_n A/p \quad \pi_n A[p]$.

Combining all the details above, the cat of p-cplt $\mathbb{E}_1$-alg under $\mathbb{S}_{W(R)}$ is equiv to the cat of p-cplt $\mathbb{E}_1$-alg under $\mathbb{S}[R]$ s.t $\mathbb{Z}[R] \to \pi_0 A/p$ factors over R.

That's exactly what the pullback diagram means $\square$

Lem: A p-cplt $\mathbb{E}_n$-ring $n \geq 2$

$\tilde{A} \to A$ square zero extn by a p-cplt symm bimodule P.

$\Rightarrow$ Any $\mathbb{E}_1$-ring map admits a contractible lift to $\tilde{A}$

$$\begin{tikzcd} & \tilde{A} \arrow[d] \\ \mathbb{S}_{W(R)} \arrow[r] \arrow[ur, dashed] & A \end{tikzcd}$$

Pf: Calculate $L_{\mathbb{S}_{W(R)}/\mathbb{S}}$. $\square$

Proof of Cor 2:

Assume $n-1$ ✓ $(n \geq 2)$

$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_n}}(\mathbb{S}_{W(R)}, A)$

$\simeq \lim_{[k] \in \Delta} \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_{n-1}}}(\mathbb{S}_{W(R)}^{\hat{\otimes}(k+1)}, A)$ (Dunn's work)

$\simeq \lim \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_{n-1}}}(\mathbb{S}_{W(R^{\otimes_{\mathbb{F}_p}(k+1)})}, A)$

$\simeq \lim \mathrm{Map}_{\mathrm{Ring}}(R^{\otimes_{\mathbb{F}_p}(k+1)}, \pi_0 A^b)$ by hypothesis

$\simeq \mathrm{Map}_{\mathrm{Ring}}(R, \pi_0 A^b)$

If $n = \infty$, by take colim.

$n = 1$. Assume A connective

$\mathrm{Map}_{\mathrm{CRing}}(R, \pi_0 A^b)$

$\simeq \mathrm{Map}_{\mathrm{CRing}}(R, \pi_0 A/p)$ (Perfectness)

$\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_1}}(\mathbb{S}_{W(R)}, \pi_0 A/p)$

So WTS

$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_1}}(\mathbb{S}_{W(R)}, A) \to \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_1}}(\mathbb{S}_{W(R)}, \pi_0 A/p)$ (Universal property)

But

$\underbrace{A \to \pi_0 A}_{\text{Postnikov tower + Lem.}} \to \underbrace{\pi_0 A/p \to \pi_0 A/p}_{\text{Square zero extn by } \pi_0 A/p \text{ by } \Sigma \pi_0 A/p}$

$\pi_0 A = \lim \pi_0 A/p^m$ by p-cplt

each $\pi_0 A/p^{m+1}$ is square zero extn of $\pi_0 A/p^m$ by $\pi_0 A/p$.

+ Lem. $\square$