Question 16 - 9231/May/June/20/13 Q2

The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for $n \geqslant 1$.

**(a)** Prove by induction that $u_n = 2^n - 1$ for all positive integers $n$. [5]

Let $P_n$ be the statement that, for some positive integer $n=k$,

$$u_n = 2^n - 1$$

For the base case $n=1$,

$$u_1 = 2^1 - 1 = 1 \therefore P_1 \text{ is true}$$

Assume that for some positive integer $n=k$,

$$u_k = 2^k - 1$$

Then, for $n=k+1$,

$$\begin{aligned} u_{k+1} &= 2u_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \therefore P_k \Rightarrow P_{k+1} \end{aligned}$$

$\therefore$ As $P_1$ is true, and $P_k \Rightarrow P_{k+1}$, by mathematical induction, it is true that

$$u_n = 2^n - 1$$

for all positive integers $n$