Question 1 - 9231/May/June/22/12 Q3

The sequence of positive numbers $u_1, u_2, u_3, \dots$ is such that $u_1 > 4$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{u_n^2 + u_n + 12}{2u_n}.$$

(a) By considering $u_{n+1} - 4$, or otherwise, prove by mathematical induction that $u_n > 4$ for all positive integers n. [5]

let P_k be the statement that for an integer $n = k$, $u_k > 4$

$$u_{k+1} = \frac{u_k^2 + u_k + 12}{2u_k} = \frac{u_k}{2} + \frac{1}{2} + \frac{6}{u_k}$$

$$\Rightarrow u_{k+1} - 4 = \frac{u_k}{2} + \frac{1}{2} + \frac{6}{u_k} - 4$$

$$= \frac{u_k}{2} + \frac{6}{u_k} - \frac{7}{2}$$

$u_1 > 4$ $\therefore$ P_1 is true. | Assume that, for some integer k such that $k \geq 1$, P_k is true

For u_{k+1},

as $\frac{u_k}{2} > 2$, $0 < \frac{6}{u_k} < \frac{3}{2}$; $\frac{u_k}{2} + \frac{6}{u_k} > \frac{7}{2}$,

$\therefore \frac{u_k}{2} + \frac{6}{u_k} - \frac{7}{2} > 0$

$\therefore$ This implies $u_{k+1} - 4 > 0$

$$\Rightarrow u_{k+1} > 4$$

$\therefore$ As P_1 is true and $P_k \Rightarrow P_{k+1}$, $u_n > 4$ for $n \geq 1$