Question 10 - 9231/May/June/21/13 Q3

(a) Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n}(5r^4+r^2)=\tfrac{1}{2}n^2(n+1)^2(2n+1).$$ [6]

let P_n be the statement that for some positive integer $n=k$, it is true that

$$\sum_{r=1}^{n}(5r^4+r^2)=\frac{1}{2}n^2(n+1)^2(2n+1)$$

For the base case, $n=1$,

$$\sum_{r=1}^{1}(5r^4+r^2)=6$$

$$\sum_{r=1}^{1}(5r^4+r^2)=\frac{1}{2}(1)(2)^2(3)=6$$

Assume that for some positive integer $n=k$, P_k is true

$$\sum_{r=1}^{k}(5r^4+r^2)=\frac{1}{2}k^2(k+1)^2(2k+1)$$

Then, for $n=k+1$,

$$\sum_{r=1}^{k+1}(5r^4+r^2)=\frac{1}{2}k^2(k+1)^2(2k+1)+5(k+1)^4+(k+1)^2$$

$$=(k+1)^2\left(\frac{1}{2}k^2(2k+1)+5(k+1)^2+1\right)$$

$$=(k+1)^2\left(k^3+\frac{1}{2}k^2+5k^2+10k+5+1\right)$$

$$=(k+1)^2\left(k+\frac{3}{2}\right)(k+2)^2$$

$$=\frac{1}{2}(k+1)^2((k+1)+1)^2(2(k+1)+1)$$

$$\therefore P_k \Rightarrow P_{k+1}$$

$\therefore$ As P_1 is true and $P_k \Rightarrow P_{k+1}$, by mathematical induction, $\displaystyle\sum_{r=1}^{n}(5r^4+r^2) = \frac{1}{2}n^2(n+1)^2(2n+1)$ for all positive integers n

(b) Use the result given in part **(a)** together with the List of formulae (MF19) to find $\displaystyle\sum_{r=1}^{n} r^4$ in terms of n, fully factorising your answer. [3]

$$\sum_{r=1}^{n}(5r^4+r^2) = \frac{1}{2}n^2(n+1)^2(2n+1)$$

$$\Rightarrow \sum_{r=1}^{n} r^4 = \frac{1}{5}\left(\frac{1}{2}n^2(n+1)^2(2n+1) - \sum_{r=1}^{n} r^2\right)$$

$$= \frac{1}{30}\left(3n^2(n+1)^2(2n+1) - n(n+1)(2n+1)\right)$$

$$= \frac{1}{30}n(n+1)(2n+1)\left(3n(n+1)-1\right)$$

$$= \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$