It is given that

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) = \frac{a_n \ln x + b_n}{x^{n+1}},$$

where a_n and b_n depend only on n.

(i) Find a_1, a_2 and a_3. [3]

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2} \qquad \therefore a_1 = -1$$

$$\frac{d^2}{dx^2}\left(\frac{\ln x}{x}\right) = \frac{-x-(1-\ln x)(2x)}{x^4}$$

$$= \frac{-1-2+2\ln x}{x^3}$$

$$= \frac{-3+2\ln x}{x^3} \qquad \therefore a_2 = 2$$

$$\frac{d^3}{dx^3}\left(\frac{\ln x}{x}\right) = \frac{2x^2-(-3+2\ln x)(3x^2)}{x^6}$$

$$= \frac{2+9-6\ln x}{x^4}$$

$$= \frac{11-6\ln x}{x^4} \qquad \therefore a_3 = -6$$

(ii) Use mathematical induction to establish a formula for a_n. [5]

Let P_n be the statement that for some positive integer n,

$$a_n = (-1)^n n!$$

For the base case $n=1$,

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = -1 = (-1)^1(1!) \therefore P_1 \text{ is true}$$

Assume that, for some positive integer $n=k$,

$$a_k = (-1)^k k!$$

Then, for $n=k+1$,

$$\frac{d^k}{dx^k}\left(\frac{\ln x}{x}\right) = \frac{b_k + (-1)^k k! \ln x}{x^{k+1}}$$

$$\Rightarrow \frac{d^{k+1}}{dx^{k+1}}\left(\frac{\ln x}{x}\right) = \frac{(-1)^k k! x^k - (b_k + (-1)^k k! \ln x)((k+1)x^k)}{x^{2k+2}}$$

$$= \frac{(-1)^k k! - (k+1)b_k + (-1)^{k+1}(k+1)(k!)\ln x}{x^{k+2}}$$

$$= \frac{(-1)^k k! - (k+1)b_k + (-1)^{k+1}(k+1)! \ln x}{x^{k+2}}$$

$$\therefore a_{k+1} = (-1)^{k+1}(k+1)! \therefore P_k \Rightarrow P_{k+1}$$

$\therefore$ As P_1 is true and $P_k \Rightarrow P_{k+1}$, by mathematical induction, for all positive integers n,

$$a_n = (-1)^n n!$$