

Prove by induction that every positive integer power of 5 can be written as the sum of squares of two distinct positive integers.

Let P_n be the statement that for some positive integer n , $5^n = x^2 + y^2$, where x and y are positive integers

For the base case, $n=1$,

$$5^1 = 2^2 + 1^2 \therefore P_1 \text{ is true}$$

For $n=2$,

$$5^2 = 4^2 + 3^2 \therefore P_2 \text{ is true}$$

Assume that, for some positive integer $n=k$,

$$5^k = x^2 + y^2$$

Then, for $n=k+2$,

$$\begin{aligned} 5^{k+2} &= 5^2(x^2 + y^2) \\ &= (5x)^2 + (5y)^2 \therefore P_k \Rightarrow P_{k+2} \end{aligned}$$

\therefore As P_1 is true, P_2 is true, and $P_k \Rightarrow P_{k+2}$,
by mathematical induction,

$$5^n = x^2 + y^2$$

For any positive integer n , where x and y are integers.