Given that $y = x\sin x$, find $\frac{d^2y}{dx^2}$ and $\frac{d^4y}{dx^4}$, simplifying your results as far as possible, and show that

$$\frac{d^6y}{dx^6} = -x\sin x + 6\cos x. \qquad [3]$$

$y = x\sin x$

$\Rightarrow \frac{dy}{dx} = x\cos x + \sin x$

$\Rightarrow \frac{d^2y}{dx^2} = -x\sin x + \cos x + \cos x$

$= -x\sin x + 2\cos x$

$\Rightarrow \frac{d^3y}{dx^3} = -x\cos x - \sin x - 2\sin x$

$= -x\cos x - 3\sin x$

$\Rightarrow \frac{d^4y}{dx^4} = x\sin x - \cos x - 3\cos x$

$= x\sin x - 4\cos x$

$\Rightarrow \frac{d^5y}{dx^5} = x\cos x + \sin x + 4\sin x$

$\Rightarrow \frac{d^6y}{dx^6} = -x\sin x + \cos x + 5\cos x$

$= -x\sin x + 6\cos x$

Use induction to establish an expression for $\frac{d^{2n}y}{dx^{2n}}$, where n is a positive integer. [5]

let P_n be the statement that for some positive integer n,

$$\frac{d^{2n}y}{dx^{2n}} = (-1)^n(x\sin x - 2n\cos x)$$

For the base case $n=1$,

$$\frac{d^2y}{dx^2} = -x\sin x + 2\cos x$$

$$= (-1)^1(x\sin x - 2\cos x) \therefore P_1 \text{ is true}$$

Assume that for some positive integer $n=k$,

$$\frac{d^{2k}y}{dx^{2k}} = (-1)^k(x\sin x - 2k\cos x)$$

Then, for $n=k+1$,

$$\frac{d^{2k+1}y}{dx^{2k+1}} = (-1)^k(x\cos x + \sin x + 2k\sin x)$$

$$= (-1)^k(x\cos x + (2k+1)\sin x)$$

$$\Rightarrow \frac{d^{2(k+1)}y}{dx^{2(k+1)}} = (-1)^k(-x\sin x + \cos x + (2k+1)\cos x)$$

$$= (-1)^{k+1}(x\sin x - (2(k+1))\cos x)$$

$$\therefore P_k \Rightarrow P_{k+1}$$

$\therefore$ As P_1 is true, and $P_k \Rightarrow P_{k+1}$, by mathematical induction for all positive integers n, $\frac{d^{2n}y}{dx^{2n}} = (-1)^n(x\sin x - 2n\cos x)$