Question 13 - 9231/May/June/20/11 Q6

Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

**(a)** The transformation in the $x$-$y$ plane represented by $\mathbf{A}^{-1}$ transforms a triangle of area $30\,\text{cm}^2$ into a triangle of area $d\,\text{cm}^2$.

Find the value of $d$. [3]

$$A^{-1} = \frac{1}{2-0}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$$

$\therefore \det(A^{-1}) = \frac{1}{2}$

$\therefore \text{Ans} = 30 \times \frac{1}{2} = 15$

**(b)** Prove by mathematical induction that, for all positive integers $n$,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}.$$ [5]

let $P_n$ be the statement that for some positive integers $n$,

$$A^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$$

For the base case $n=1$,

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^1 & 0 \\ 2^1 - 1 & 1 \end{pmatrix} \therefore P_1 \text{ is true}$$

Assume that for some positive integer $n=k$,

$$A^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$$

Then, for $n=k+1$

$$A^{k+1} = \begin{pmatrix} 2^k & 0 \\ 2^k-1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1}-2+1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1}-1 & 1 \end{pmatrix}$$

$\therefore P_k \Rightarrow P_{k+1}$

$\therefore$ As $P_1$ is true and $P_k \Rightarrow P_{k+1}$, by mathematical induction, it is true that

$$A^n = \begin{pmatrix} 2^n & 0 \\ 2^n-1 & 1 \end{pmatrix}$$

for all positive integers $n$.