Question 7 - 9231/Oct/Nov/21/13 Q3

The sequence of real numbers $a_1, a_2, a_3, \dots$ is such that $a_1 = 1$ and

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^3.$$

(a) Prove by mathematical induction that $\ln a_n \geqslant 3^{n-1} \ln 2$ for all integers $n \geqslant 2$. [6]

[You may use the fact that $\ln\left(x + \frac{1}{x}\right) > \ln x$ for $x > 0$.]

let P_n be the statement that for some integer $n \geq 2$, $\ln a_n \geq 3^{n-1} \ln 2$

For the base case $n=2$,

$$a_2 = \left(1 + \frac{1}{1}\right)^3$$

$$\Rightarrow \ln a_2 = 3\ln 2 \geq 3^{2-1} \ln 2$$

$\therefore P_0$ is true

Assume that for some integer $n=k$ such that $k \geq 2$, P_k is true

Then, for $n = k+1$,

$$a_{k+1} = \left(a_k + \frac{1}{a_k}\right)^3$$

$$\Rightarrow \ln a_{k+1} = 3\ln\left(a_k + \frac{1}{a_k}\right)$$

$\therefore$ as $\ln\left(a_k + \frac{1}{a_k}\right) > \ln a_k$, and $\ln a_k \geq 3^{k-1} \ln 2$

$\ln a_{k+1} > 3(3^{k-1} \ln 2) \Rightarrow \ln a_{k+1} > 3^k \ln 2$

$\therefore \ln a_{k+1} \geq 3^k \ln 2 \therefore P_k \Rightarrow P_{k+1}$

$\therefore$ As P_0 is true and $P_k \Rightarrow P_{k+1}$, by mathematical induction, it is true that $\ln a_n \geq 3^{n-1} \ln 2$ for all integers $n \geq 2$

(b) Show that $\ln a_{n+1} - \ln a_n > 3^{n-1} \ln 4$ for $n \geqslant 2$. [2]

$$\ln a_{n+1} - \ln a_n$$

$$= 3 \ln\left(a_n + \frac{1}{a_n}\right) - \ln a_n$$

$$\ln\left(a_n + \frac{1}{a_n}\right) > \ln a_n$$

$$\therefore \ln a_{n+1} - \ln a_n > 3 \ln a_n - \ln a_n$$

$$\Rightarrow \ln a_{n+1} - \ln a_n > 2 \ln a_n$$

$$\because \ln a_n \geq 3^{n-1} \ln 2$$

$$\therefore \ln a_{n+1} - \ln a_n > 2(3^{n-1} \ln 2)$$

$$\Rightarrow \ln a_{n+1} - \ln a_n > 3^{n-1} \ln 4$$