curlyF: n=1 Poisson(θ). θ is our unknown parameter of interest. It is not the same θ in the binomial. Let's try to find the conjugate prior for this parametric model (likelihood).

$$P(\theta|x) \propto P(x|\theta)P(\theta) = \frac{e^{-\theta}\theta^x}{x!}P(\theta) \propto e^{-\theta}\theta^x k(\theta) \overset{\text{Pattern Matching}}{=} \underbrace{e^{-\theta}\theta^x}\, e^{-\theta b}\theta^a = \underbrace{e^{-\theta(b+1)}\theta^{x+a}}_{\text{Kernel for same rv}}$$

We want to find P(θ) of the same distribution as P(θ | X).

Let's figure out P(θ) from k(θ).

$$\int_0^\infty k(\theta)\,d\theta = \int_0^\infty e^{-\theta b}\theta^a\,d\theta = \int_0^\infty \theta^{a+1-1}e^{-b\theta}\,d\theta \overset{u\text{-subst}}{=} \cdots = \frac{\Gamma(a+1)}{\beta^{a+1}}$$

$$\Rightarrow P(\theta) = \frac{b^{a+1}}{\Gamma(a+1)}\theta^{a+1-1}e^{-b\theta} = \text{Gamma}(a+1, b)$$

Back to probability class...

$$Y \sim \text{Gamma}(\alpha,\beta) = \overbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-\beta y}}^{f(y)} \propto y^{\alpha-1}e^{-\beta y} = k(y)$$

$$\text{Supp}[Y] = (0,\infty), \quad \text{Param. Space}: \alpha > 0, \beta > 0$$

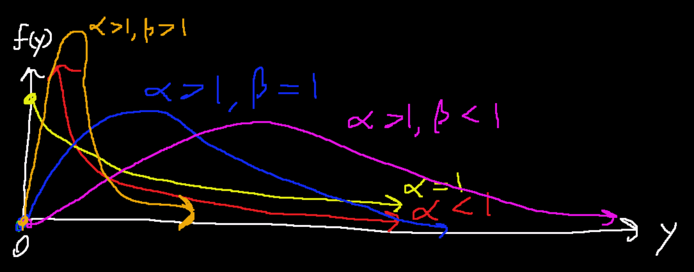
$$E[Y] = \int_0^\infty y f(y)\,dy \overset{u\text{-subst}}{=} \cdots = \frac{\alpha}{\beta}$$

$$\text{Mode}[Y] \overset{\text{calc}}{=} \cdots = \frac{\alpha-1}{\beta} \text{ if } \alpha \geq 1$$

$$\text{Med}[Y] = q \text{ s.t. } \int_0^q f(y)\,dy = \frac{1}{2} \ldots$$

no closed form expression is possible so we use a computer to do numerical integration.

$$= \text{qgamma}(0.5, \alpha, \beta)$$



Let's go back to inference for θ in the Poisson model. Now we consider $\mathcal{F}$: iid Poisson(θ)

$$P(\theta|x) \propto P(x|\theta)P(\theta) = P(\theta)\prod_{i=1}^n \frac{e^{-\theta}\theta^{x_i}}{x_i!} = P(\theta)\frac{e^{-n\theta}\theta^{\sum x_i}}{\prod x_i!} \propto e^{-n\theta}\theta^{\sum x_i}k(\theta)$$

$$\overset{\text{pattern matching}}{=} e^{-n\theta}\theta^{\sum x_i}\underbrace{\theta^{\alpha-1}e^{-\beta\theta}}_{\text{gamma kernel}} = \underbrace{\theta^{\alpha+\sum x_i - 1}e^{-(\beta+n)\theta}}_{\text{gamma kernel}}$$

$$\propto \text{Gamma}(\alpha + \sum x_i, \beta + n)$$

x_0, # of pseudoevents

# of events in the data

# of observations in data

n_0, # of pseudoobserv.

$$\Rightarrow \hat{\theta}_{MMSE} = \frac{\alpha+\sum x_i}{\beta+n}, \quad \hat{\theta}_{MAP} = \frac{\alpha+\sum x_i - 1}{\beta+n} \text{ only if } \alpha+\sum x_i \geq 1,$$

$$\hat{\theta}_{MMAE} = \text{qgamma}(0.5, \alpha+\sum x_i, \beta+n),$$

$$CR_{\theta,1-\alpha_0} = \left[\text{qgamma}\left(\frac{\alpha_0}{2}, \alpha+\sum x_i, \beta+n\right), \text{qgamma}\left(1-\frac{\alpha_0}{2}, \alpha+\sum x_i, \beta+n\right)\right]$$

$$H_1: \theta > \theta_0 \text{ vs. } H_0: \theta \leq \theta_0, \quad p_{val} = P(H_0|x) = P(\theta \leq \theta_0|x)$$

$$= \int_0^{\theta_0} P(\theta|x)\,d\theta = \text{pgamma}(\theta_0, \alpha+\sum x_i, \beta+n)$$

Let's derive the MLE:

$$\mathcal{L}(\theta;x) = \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod x_i!} \Rightarrow \ell(\theta;x) = -n\theta + \sum x_i \ln(\theta) - \ln\left(\prod x_i!\right)$$

$$\Rightarrow \ell'(\theta;x) = -n + \frac{\sum x_i}{\theta} \overset{\text{set}}{=} 0 \Rightarrow \frac{\sum x_i}{\theta} = n \Rightarrow \hat{\theta}_{MLE} = \frac{\sum x_i}{n} = \bar{x}$$

Let's prove that thetahatMMSE is a shrinkage estimator and let's find the value of ρ.

$$\hat{\theta}_{MMSE} = \frac{\alpha+\sum x_i}{\beta+n} = \frac{\alpha}{\beta+n}\cdot\frac{\beta}{\beta} + \frac{\sum x_i}{\beta+n}\cdot\frac{n}{n} = \underbrace{\frac{\beta}{\beta+n}}_{\rho}\underbrace{\frac{\alpha}{\beta}}_{E[\theta]} + \underbrace{\frac{n}{\beta+n}}_{1-\rho}\underbrace{\frac{\sum x_i}{n}}_{\hat{\theta}_{MLE}}$$

$$\rho = \frac{n_0}{n+n_0}$$

$$\lim_{n\to\infty}\rho = 0$$