# ECON 409 Project Proposal

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## Model

As suggested by Molodtsova and Papell, the model is outlined as follows:

$$\Delta s_{t+1} = \beta_0 + \beta_1(\tilde{\pi}_t - \pi_t) + \beta_2(\tilde{y}_t - y_t) + \epsilon_t, \quad \epsilon_t \sim_{i.i.d.} N(0, \sigma^2)$$

$$\Delta s_{t+1} = s_{t+1} - s_t$$

$$s_t = \ln S_t$$

This model is delineated as symmetric, non-smoothed, and homogeneous, with:

- $S_t$ representing the price of 1 pound sterling in US dollars.
- $\tilde{\pi}_t$ being the UK inflation rate.
- $\pi_t$ being the US inflation rate.
- $\tilde{y}_t$ indicating the UK output gap.
- $y_t$ indicating the US output gap.

The output gaps are determined through:

$$\ln Y_t = y_t + \bar{y}_t$$

$$\bar{y}_t = \alpha_0 + \bar{y}_{t-1} + \nu_t, \quad \nu_t \sim_{i.i.d.} N(0, \sigma_\nu^2)$$

$$y_t = \alpha_1 y_{t-1} + \eta_t, \quad \eta_t \sim_{i.i.d.} N(0, \sigma_\eta^2)$$

Introducing a stochastic trend for potential output and an autoregressive model for the output gap. It assumes that $\ln Y_t$ and $\bar{y}_t$ are I(1) variables, and $y_t$ is I(0).

For the UK output gap, we have:

$$\ln \tilde{Y}_t = \tilde{y}_t + \bar{\tilde{y}}_t$$

$$\bar{\tilde{y}}_t = \gamma_0 + \bar{\tilde{y}}_{t-1} + \tilde{\nu}_t, \quad \tilde{\nu}_t \sim_{i.i.d.} N(0, \sigma^2_{\tilde{\nu}})$$

$$\tilde{y}_t = \gamma_1 \tilde{y}_{t-1} + \tilde{\eta}_t, \quad \tilde{\eta}_t \sim_{i.i.d.} N(0, \sigma^2_{\tilde{\eta}})$$

This approach assumes $\ln \tilde{Y}_t$ and $\bar{\tilde{y}}_t$ are I(1) variables and $\tilde{y}_t$ is I(0).

## Data

The model utilizes monthly data, with updates occurring on the last business day of each month:

- $\pi_t$: US CPI index, released with a one-month delay.
- $\tilde{\pi}_t$: UK CPI index, released with a one-month delay.
- $Y_t$: US Industrial Production Index, released with a one-month delay.
- $\tilde{Y}_t$: UK Industrial Production Index, released with a two-month delay.

Given the inherent lags within the data, we propose the following autoregressive models for inflation rates, treating them as random walks:

$$\pi_t = \pi_{t-1} + e_t^{\pi}, \quad e_t^{\pi} \sim_{i.i.d.} N(0, \sigma^2_{e^{\pi}})$$
$$\tilde{\pi}_t = \tilde{\pi}_{t-1} + e_t^{\tilde{\pi}}, \quad e_t^{\tilde{\pi}} \sim_{i.i.d.} N(0, \sigma^2_{e^{\tilde{\pi}}})$$

Both models encapsulate the stochastic nature of inflation rates in their respective economies.

To account for the delays in data availability, we adjust our original exchange rate model as follows:

$$\Delta s_{t+1} = \beta_0 + \beta_1(\mathbb{E}_{t-1}[\tilde{\pi}_t] - \mathbb{E}_{t-1}[\pi_t]) + \beta_2(\mathbb{E}_{t-2}[\tilde{y}_t] - \mathbb{E}_{t-1}[y_t]) + \epsilon_t$$

$$\Delta s_{t+1} = s_{t+1} - s_t$$

$$s_t = \ln S_t$$

The expectations are determined based on the information available at time $t-1$ and $t-2$:

$$\mathbb{E}_{t-1}[\pi_t] = \pi_{t-1}$$

$$\mathbb{E}_{t-1}[\tilde{\pi}_t] = \tilde{\pi}_{t-1}$$

$$\mathbb{E}_{t-1}[y_t] = \alpha_1 y_{t-1}$$

$$\mathbb{E}_{t-2}[\tilde{y}_t] = \gamma_1^2 \tilde{y}_{t-2}$$

Integrating these expectations, the revised model becomes:

$$\Delta s_{t+1} = \beta_0 + \beta_1(\tilde{\pi}_{t-1} - \pi_{t-1}) + \beta_2(\gamma_1^2 \tilde{y}_{t-2} - \alpha_1 y_{t-1}) + \epsilon_t$$

## Ridge Regression

We define the loss function as:

$$L(\widehat{\Delta s}_{t+1}|\beta_0, \beta_1, \beta_2) = MSE(\widehat{\Delta s}_{t+1}) + \lambda \sum_{i=0}^{2} \beta_i^2$$

where:

$$\beta^* = (\beta_0^*, \beta_1^*, \beta_2^*)' = \text{arg min}_{\beta \in \mathbb{R}^3} L(\widehat{\Delta s}_{t+1}|\beta)$$

## Tunning Parameters

1. $\lambda$: Regularization term in ***Ridge*** regression.
2. $h$ : Rolling window size for model estimation and the output gaps, to avoid including structural breaks that could affect forecasts.

## Estimating $y_t$ and $\tilde{y}_t$

To estimate $\Delta \ln Y_t$:

$$\Delta \ln Y_t = \Delta y_t + \Delta \bar{y}_t$$

Knowing $y_t$ is $I(0)$:

$$\Delta y_t = y_t - y_{t-1}$$
$$\mathbb{E}[\Delta y_t] = \mathbb{E}[y_t - y_{t-1}]$$
$$\mathbb{E}[\Delta y_t] = 0$$

Since:

$$\mathbb{E}(y_{t+s}) = 0, \forall s \in \mathbb{Z}$$

For $\Delta\bar{y}_t$:

$$\Delta\bar{y}_t = \bar{y}_t - \bar{y}_{t-1}$$

$$\Delta\bar{y}_t = \alpha_0 + \bar{y}_{t-1} + \nu_{t-1} - \bar{y}_{t-1}$$

$$\mathbb{E}[\Delta\bar{y}_t] = \alpha_0 + \mathbb{E}[\bar{y}_{t-1}] + \mathbb{E}[\nu_{t-1}] - \mathbb{E}[\bar{y}_{t-1}]$$

$$\mathbb{E}[\Delta\bar{y}_t] = \alpha_0$$

Thus:

$$\mathbb{E}[\Delta \ln Y_t] = \mathbb{E}[\Delta y_t] + \mathbb{E}[\Delta \bar{y}_t]$$
$$\mathbb{E}[\Delta \ln Y_t] = \alpha_0$$

We estimate the drift $\alpha_0$ of $\bar{y}_{t-1}$ as:

$$\hat{\alpha}_0 = \frac{1}{h} \sum_{i=t-1-h}^{t-1} \Delta \ln Y_i$$

$h$ is the previously mentioned second tuning parameter.

Similarly, for $\tilde{y}_{t-2}$:

$$\hat{\gamma}_0 = \frac{1}{h} \sum_{i=t-2-h}^{t-2} \Delta \ln \tilde{Y}_i$$