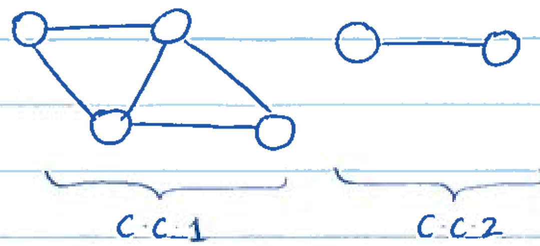
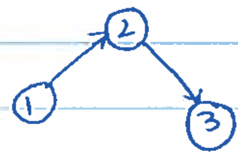
9/6/18

- 

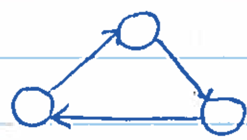
$\Rightarrow$ Each piece is a connected component.

- within each component you can reach all vertices.

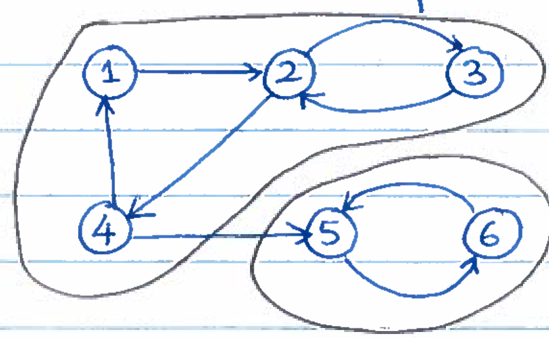
$\longrightarrow$ For a directed graph,



Not connected as can't reach from 3 to 1

But  is connected

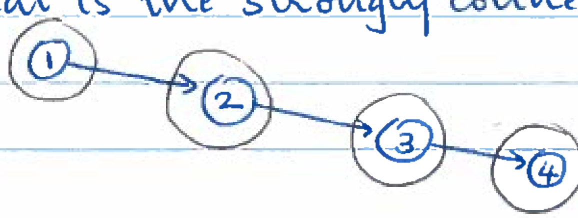
- Strongly Connected Components containing vertex u is a set of all vertices v such that u is reachable from v and v is reachable from u.



strongly connected component 1

S.C.C 2

?
- What is the strongly connected component in



Each vertex is a S.C.C.

- Graph is strongly connected if it has one strongly connected component

⟹ **PROOF TECHNIQUES:**

⇓ Try to be comfortable with proof as this will come up a lot later on.

1. Proof By Contradiction
2. Proof By Example
3. Proof By Induction
4. Proof By Contrapositive : $A \Rightarrow B$ if $\neg B \Rightarrow \neg A$
5. Direct Proof

→ Example:

- Claim: $\overbrace{\text{If an undirected graph } G \text{ is a tree,}}^{A}$ then $\underbrace{\text{there is exactly one simple path between any two vertices}}_{B}$

connected, acyclic

simple path is one that doesn't repeat vertices

- Proof:

-Use proof by contrapositive.

- If $\exists$ pair of vertices without simple path between them, then not connected $\Rightarrow$ not tree.

Suppose there are at least 2 simple paths P_1, P_2 from u to v, first vertex when paths are same after a

first vertex befor paths are different

The following path forms a cycle:

starts at a, go by P_1 to b, go backward by $P2$ to a

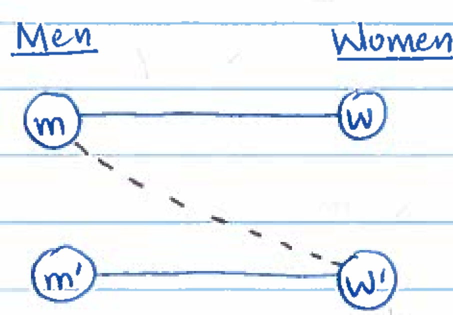
Since a tree is acyclic, G is not a tree.

QED

⇒ GALE-SHAPLEY ALGORITHM:

- Optimally matching
  - applicants to open positions
  - med school graduates to residency programs
  - eligible males to marry eligible females.

- Matching can be unstable.



If m prefers w' to w and w prefers m to m'.

↓

unstable so have an incentive to switch.

Stable Marriage Problem.

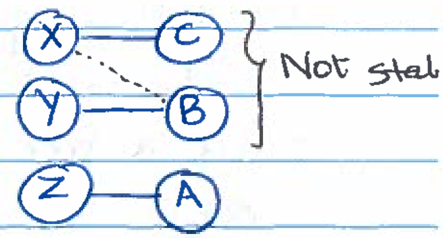
- Example

Preference List

Men	most favourite		least favourite
X	A	B	C
Y	B	A	C
Z	A	B	C

Women			
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

? Is this Stable?



X prefers B over C and B prefers X over Y. So unstable.

→ Given n men and n women with their preference lists,

- find a stable matching.

[Refer to slides for algorithm]

X	A	B	C
Y	B	A	C
→ Z	~~A~~	~~B~~	C

A	Y	X	~~Z~~
B	X	Y	Z
C	X	Y	Z

- Benefits men over women.
  - ↳ beneficial when you consider applicants and med school where applicants get benefits

Facts:

① Once a women is proposed to, she is always engaged from that point on.

② Quality of a woman's match increases over time.

③ Quality of a men's match decreases over time.

? Does this algorithm ever terminate?

- Have an upper bound on the number of times the while loop can occur?
  - Yes, but what is the upper bound?
  - What is a good measure of progress?
    - # free men? No, as this may increase or decrease over time.
    - Number of engagements? No, might make or break.
    - Number of proposals? Yes!
      Max number of proposals = n^2
      (every man proposes to every woman)

- Hence, algorithm terminates after $\leq n^2$ iterations of while loop.

? Are that any unmatched men or women at the end of the algorithm?

- No! (If 'n' men and 'n' women)
- Why? Since #men = #women, unpaired woman $\Rightarrow$ unpaired man.
- Proof By Contradiction:

  Suppose $\exists$ unpaired man. He must have proposed to every woman (otherwise the while loop wouldn't end)

  By fact ①, all women are engaged at the end.

  Contradicts assumption that $\exists$ unmatched man.

  QED

? Does the Gayle-Shapley algorithm result in a stable solution?

Claim: The Gayle-Shapley (GS) algorithm returns a stable matching.

Proof: Suppose $\exists$ instability: m — w

m prefers w' to w and m' — w'

w' prefers ~~w~~ m to m'

Last proposal of m must have been to w.

Had m proposed to w' at some earlier time?

- If NO, m prefers w to w' (otherwise would've proposed to w' earlier) $\Rightarrow\Leftarrow$ (contradiction)
- If YES, m was rejected by w' in favor of some other man but by fact ②, w' must prefer m' to m $\Rightarrow\Leftarrow$

<u>QED</u>