The Master Theorem

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the non-negative integers by the recurrence:

$$T(n) = aT(n/b) + f(n)$$

where we interpret n/b to be either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ can be bounded asymptotically as follows:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$

3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Note: There are other forms of the Master Theorem

Matrix Multiplication

Given two n-by-n matrices, A and B, compute $C = AB$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$
\begin{bmatrix}
c_{11} & c_{12} & \cdots & c_{1n} \\
c_{21} & c_{22} & \cdots & c_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \cdots & c_{nn}
\end{bmatrix}
=
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\times
\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1n} \\
b_{21} & b_{22} & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{nn}
\end{bmatrix}
$$

Brute Force

$\Theta(n^3)$ arithmetic operations

Fundamental Question

Can we improve upon brute force?

Divide and Conquer: First Attempt

Divide-and-conquer.

- Divide: partition A and B into ½n-by-½n blocks.
- Conquer: multiply 8 ½n-by-½n recursively.
- Combine: add appropriate products using 4 matrix additions.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$
\begin{aligned}
C_{11} &= (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\
C_{12} &= (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\
C_{21} &= (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\
C_{22} &= (A_{21} \times B_{12}) + (A_{22} \times B_{22})
\end{aligned}
$$

Divide and Conquer: First Attempt

- Divide: partition A and B into $\frac{1}{2}n$ by $\frac{1}{2}n$ blocks

- Conquer: multiply 8 $\frac{1}{2}n$ by $\frac{1}{2}n$ recursively

- Combine: add appropriate products using 4 matrix additions

$$
\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}
$$

$$
\begin{aligned}
C_{11} &= \left(A_{11} \times B_{11} \right) + \left(A_{12} \times B_{21} \right) \\
C_{12} &= \left(A_{11} \times B_{12} \right) + \left(A_{12} \times B_{22} \right) \\
C_{21} &= \left(A_{21} \times B_{11} \right) + \left(A_{22} \times B_{21} \right) \\
C_{22} &= \left(A_{21} \times B_{12} \right) + \left(A_{22} \times B_{22} \right)
\end{aligned}
$$

$$
\mathrm{T}(n) = \underbrace{8T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \Rightarrow \mathrm{T}(n) = \Theta(n^3)
$$

Divide and Conquer: Strassen (1969)

Key Idea

Multiply 2-by-2 block matrices with only 7 multiplications (7 multiplications and 18 additions/subtractions)

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$
\begin{aligned}
C_{11} &= P_5 + P_4 - P_2 + P_6 \\
C_{12} &= P_1 + P_2 \\
C_{21} &= P_3 + P_4 \\
C_{22} &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$

$$
\begin{aligned}
P_1 &= A_{11} \times (B_{12} - B_{22}) \\
P_2 &= (A_{11} + A_{12}) \times B_{22} \\
P_3 &= (A_{21} + A_{22}) \times B_{11} \\
P_4 &= A_{22} \times (B_{21} - B_{11}) \\
P_5 &= (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\
P_6 &= (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\
P_7 &= (A_{11} - A_{21}) \times (B_{11} + B_{12})
\end{aligned}
$$

Divide and Conquer: Strassen (1969)

Key Idea

Multiply 2-by-2 block matrices with only 7 multiplications (7 multiplications and 18 additions/subtractions)

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

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\begin{aligned}
P_1 &= A_{11} \times (B_{12} - B_{22}) \\
P_2 &= (A_{11} + A_{12}) \times B_{22} \\
P_3 &= (A_{21} + A_{22}) \times B_{11} \\
P_4 &= A_{22} \times (B_{21} - B_{11}) \\
P_5 &= (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\
P_6 &= (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\
P_7 &= (A_{11} - A_{21}) \times (B_{11} + B_{12})
\end{aligned}
$$

$$
\begin{aligned}
C_{11} &= P_5 + P_4 - P_2 + P_6 \\
C_{12} &= P_1 + P_2 \\
C_{21} &= P_3 + P_4 \\
C_{22} &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$

$$T(n) = \underbrace{7T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \Rightarrow T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$

```
Px := sort points by x  //to split left/right
Py := sort points by y  //to get Sy sorted by y

ClosestPair(Px,Py)
  if only 2 points, return their distance
  δ_L := ClosestPair(left-half(Px,Py))
  δ_R := ClosestPair(right-half(Px,Py))
  δ  := min(δ_L,δ_R)

  Sy := points in Py within δ of dividing line
  for i = 1 to |Sy|
    for j = i+1 to i+8
      δ := min(δ, distance(Sy[i], Sy[j]))

  return δ
```

```
left/right-half(Px,Py):
 Px' := left/right half of Px
 Py' := go through every point and choose it or
 not based on comparing its x coordinate with
 the middle point of Px
```