Homework #1

You do not need to turn in these problems. The goal is to be ready for the in-class quiz that will cover the same or similar problems.

Problem 1: Relations

This problem is meant to test your ability to understand and reason precisely about formal definitions. Recall the following types of relations:

- A relation R on a set A is **reflexive** if $\forall a \in A$: $(a, a) \in R$.
- A relation R on a set A is **symmetric** if for $\forall a, b \in A$: $(a, b) \in R$ implies $(b, a) \in R$.
- A relation R on a set A is **transitive** if for $\forall a, b, c \in A$: $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$.

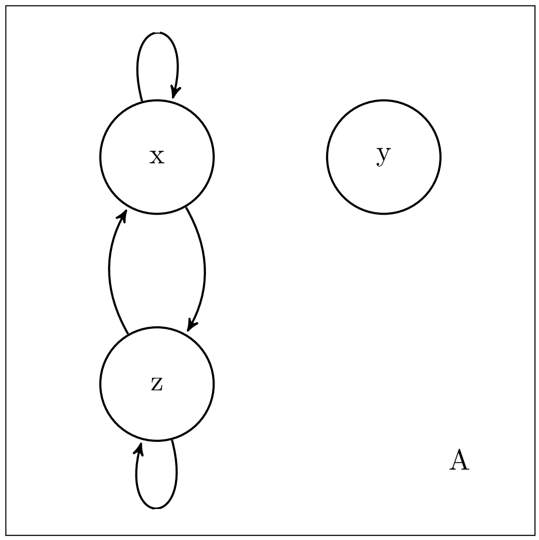
Consider the following claim and proof:
Claim: If a relation R is symmetric and transitive, then it is also reflexive.
Proof: By symmetry, $(a, b) \in R$ implies $(b, a) \in R$. Transitivity therefore implies $(a, a) \in R$.

Is this proof correct? If not, give a counter-example.

Solution

The proof is incorrect. The key point is that definitions of symmetric and transitive can be satisfied even if the relation is never true for some $a \in A$. There are many counterexamples, but consider the following:

Let $A = \{x, y, z\}$ and $R = \{(x, z), (z, x), (z, z), (x, x)\}$, as shown in the diagram below.



Note that y is in A but the relation is never true for y (i.e., $\nexists a \in A$ such that $(a, y) \in R$ or $(y, a) \in R$).

R is symmetric: the statement $\forall a, b \in A$, **if** $(a, b) \in R$, then $(b, a) \in R$ holds true. R is transitive: $\forall a, b \in A$, **if** $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ also holds true. However, for R to be reflexive, $\forall a \in A$, (a, a) must be in R. Since $y \in A$ but $(y, y) \notin R$, R is not reflexive.

Problem 2: Sets and Counterexamples

Show that for arbitrary sets A, B, and C, taken from the universe $\{1, 2, 3, 4, 5\}$ that the following two claims are not always true by using a simple counter example for each:

(a) if $A \cap B \subseteq C$, then $C \subseteq A \cup B$

Solution

Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$ $A \cap B = \emptyset$ so $A \cap B \subseteq C$ is automatically true for this example, but clearly $3 \in C$ while $3 \notin A \cup B$. Thus $C \subseteq A \cup B$ is false for this example.

(b) if $C \subseteq A \cup B$, then $A \cap B \subseteq C$

Solution

Let $A = \{1, 2\}$, $B = \{1, 2\}$, and $C = \{1\}$. In this example, $C \subseteq A \cup B$ is true since $1 \in A$. But $A \cap B = \{1, 2\}$ in this example and since $2 \in A \cap B$ but $2 \notin C$, it follows that $A \cap B \subseteq C$ is false for this example.

Problem 3: Classic Proofs by Contradiction

Prove each of the following.

(a) If n^2 is even, n is even.

Solution

Towards contradiction[a], assume that if n^2 is even, then n is odd. Let n^2 be even. Then $n = 2k + 1$ for some integer k by definition of an odd number. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$. Let $j = 2k^2 + 2k$, which is an integer since k is an integer. Then $n^2 = 2j + 1$, which is odd by definition of an odd number, which contradicts the assumption that n^2 is even.

[a] For a proof by contradiction, to prove P, we first assume $\neg P$, then show that $\neg P \Rightarrow (C \wedge \neg C)$. In this case, P is the proposition "if n^2 is even, then n is even", $\neg P$ is the proposition "if n^2 is even, then n is odd", C is the proposition "n^2 is even", and $\neg C$ is the proposition "n^2 is odd".

(b) $\sqrt{2}$ is irrational.

Solution

Towards contradiction, suppose $\sqrt{2}$ is rational. Then $\sqrt{2} = a/b$ where a and b are positive integers by definition of a rational number. Assume that a/b is simplified to its lowest terms. Then $\sqrt{2}^2 = 2 = a^2/b^2$, so $a^2 = 2b^2$. Then a^2 is an even number since b^2 is an integer. Then a is also even (recall the previous proof for **(a)** above). So $a = 2k$ for some integer k. Then

$$\begin{aligned} 2 &= (2k)^2/b^2 \\ 2 &= 4k^2/b^2 \\ 2b^2 &= 4k^2 \\ b^2 &= 2k^2 \end{aligned}$$

So b^2 is even, and so b is even. Since both a and b are even, a/b is not simplified to lowest terms, contradicts the assumption that a/b is simplified to its lowest terms.

Problem 4: Proof by Induction

Prove the following.

> Any postage that is a positive integer number of cents greater than 7 cents can be formed using just 3-cent stamps and 5-cent stamps.

Solution

Let $P(n)$ be the proposition that a postage of n cents can be formed using 3-cent and 5-cent stamps. We will show that $P(n)$ is true for all $n \geq 8$.
Base Case: $P(8)$ is true because an 8 cents postage can be formed from a single 3-cent stamp and a single 5-cent stamp.
Inductive Step: Let $k \geq 8$. Assume that $P(k)$ is true (that is, assume a postage of k cents can be formed using just 3-cent stamps and 5-cent stamps).
We will show that if $P(k)$ is true, then $P(k+1)$ is true. Since we assume $P(k)$ is true, we know there exist integers $x, y \geq 0$ such that $k = 3x + 5y$. Consider the two possible cases: a 5-cent stamp is used ($y > 0$) or a 5-cent stamp is not used ($y = 0$).

Case 1:
If at least one 5-cent stamp is used, one can remove the 5-cent stamp and replace it with two 3-cent stamps for the correct postage.
Therefore, $P(k+1)$ is true for case 1.

Case 2:
Since $y = 0$, $k + 1 = 3x + 1$.
$n \geq 8$, so $3x \geq 8$, so $x > 3$.
Thus, if no 5-cent stamps are used, one can remove three 3-cent stamps (since the postage will be at least 9 cents) and replace them with two five cent stamps.
Therefore, $P(k+1)$ is true for case 2.

Solution

The following is an alternative proof using **strong induction** (look it up if you don't know what strong induction is).
Let $P(n)$ be the proposition that a postage of n cents can be formed using 3-cent and 5-cent stamps.
Base Cases: $P(8)$ is true since $5 + 3 = 8$, $P(9)$ is true since $3 + 3 + 3 = 9$, $P(10)$ is true since $5 + 5 = 10$.
Inductive Step: Assume $P(n)$ is true for all n such that $7 < n < k$. Then we will show $P(k)$ is true. Since $P(k-3)$ is true by our inductive hypothesis, we know there exists a collection of stamps which sum to $k-3$. Then we can add one 3-cent stamp to this collection to get k, thus $P(k)$ is true.
Note: $P(11)$ is the first case in which we can apply this argument, so $P(8), P(9)$ and $P(10)$ must be proven explicitly.
(Note that this proof implies that you never need more than two five-cent coins.)

Problem 5: Connected Components

For the following directed graph, list the strongly connected components.

Solution

The strongly connected components are: $\{e\}$, $\{a, h, i, s\}$, $\{b, c, d\}$, and $\{f, g\}$.

Problem 6: Trees

(a) Prove that if $G = (V, E)$ is a tree, then $|E| = |V| - 1$.

Solution

We will prove the following proposition $P(n)$: "let G be an arbitrary tree with n vertices, then $|E| = n - 1$". We will prove $P(n)$ for all n by induction on n.

Base Case: $n = 1$. There is only one vertex and thus no edges (when we defined undirected graphs, we disallowed self-loops).

Inductive Step: Assume $P(k-1)$ is true. We will show that $P(k)$ is true. Let G be an arbitrary tree with k vertices. Choose an arbitrary root r for the tree. Let v be an arbitrary leaf of G rooted at r; such a leaf exists since $k \geq 2$. By definition of a leaf, v has only one incident edge. Remove v and its incident edge to get G'. G' has $k - 1$ vertices, therefore since $P(k-1)$ is true, G' has $k - 2$ edges. We removed exactly one vertex and exactly one edge from G to get G'. Thus G has k vertices and $k - 1$ edges, therefore $P(k)$ is true.

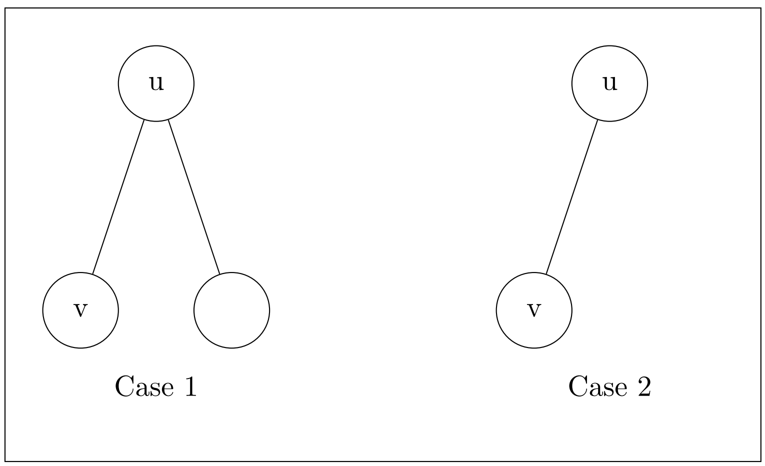
(b) Recall that in a rooted tree, the degree of a node is how many children it has. Consider any rooted tree where all nodes have degree at most two (i.e., binary tree). Show by induction that the number of degree-2 nodes is 1 fewer than the number of leaves.

Solution

We want to show that the proposition *"any rooted tree T with n nodes has the number of degree-2 nodes 1 fewer than the number of leaves"* is true for all $n \geq 0$.

We will prove this by induction on n, the number of nodes in the tree. **Base Case:** Consider a binary tree with one node. It has zero degree-2 nodes and one leaf.

Inductive Step: Assume the proposition is true when $n = k$ (that is, an arbitrary binary tree with k nodes has the number of degree-2 nodes 1 fewer than the number of leaves). Let T_{big} be an arbitrary rooted binary tree with $k+1$ nodes and m degree-2 nodes. Let v, u be nodes in T_{big} with v any leaf node and u its parent. u can have either one child or two children.



Consider the tree T_{small} that is created after deleting v from T_{big}.

T_{big} has $k + 1$ nodes, so T_{small} has k nodes, so we can apply the inductive hypothesis. However, there are two cases for the number of leaves in T_{small}.

Case 1:

If u has two children, removing v makes u a degree-1 node. Then, T_{small} must have $m - 1$ degree-2 nodes (since it has one less than T_{big}). By the inductive hypothesis, it follows that T_{small} has m leaves. Since v was a leaf, removing it from T_{big} means that T_{small} has one fewer leaf. So, T_{big} has $m + 1$ leaves and m degree-2 nodes.

Case 2:

If u has no other child than v, removing v does not reduce the number of degree-2 nodes, so T_{small} has m degree-2 nodes. By the inductive hypothesis, T_{small} has $m + 1$ leaves. Reinserting v neither increases the number of degree-2 nodes (since u is degree-1 with v) nor increases the number of leaves, since without v, u is a leaf, and with v, u is not a leaf but v is a leaf. Then T_{big} also has m degree-2 nodes and $m + 1$ leaves.

In both cases, the number of degree-2 nodes in T_{big} is one fewer than the number of leaves in T_{big}.

Solution

The following is an alternative solution. We will prove the proposition $P(n)$ "if T is a rooted tree with n degree-2 nodes, then T has $n+1$ leaves". We will prove this by induction on n, the number of degree-2 nodes.

Base Case: $P(0)$ is true since if T has zero degree-2 nodes, it is the root node followed by arbitrarily many degree-one nodes, the last of which is the only leaf.

Inductive Step: Assume $P(k)$ is true. We will show $P(k+1)$ is true. Let T be an arbitrary rooted tree with $k+1$ degree-2 nodes. Let d be the maximum depth of any degree-2 node of T. Let v be an arbitrary degree-2 node with depth d. Let T' be T with the left subtree of v removed. Since v has depth d and d is the maximum depth of any degree-2 node, we know that no nodes in v's left subtree have degree-2. Thus, the removal of v's left subtree changes v from a degree-2 node to a degree-1 node, and no other degree-2 nodes are removed. Therefore, since T has $k+1$ degree-2 nodes, T' has k degree-2 nodes. Since $P(k)$ is true and T' has k degree-2 nodes, T' has $k+1$ leaves.
Consider v's left subtree rooted at v's left child. Since v's left subtree has no degree-2 nodes, by the base case $P(0)$, we know it has only one leaf. Thus, by adding v's left subtree to T' as a child of node v to get T, we increase the number of leaves by one, and v becomes a degree-2 node again, increasing the number of degree-2 nodes by one. Since T' has k degree-2 nodes and $k+1$ leaves, T has $k+1$ degree-2 nodes and $k+2$ leaves. Since T is an arbitrary tree, $P(k+1)$ is true.

Problem 7: Colorings

Given an undirected graph $G = (V, E)$, a **k-coloring** of G is a function $c : V \to \{0, 1, \ldots, k-1\}$ such that $c(u) \neq c(v)$ for every edge $(u, v) \in E$. In other words, the numbers $0, 1, \ldots, k-1$ represent the k colors, and adjacent vertices must have different colors.

(a.) Show that any tree is 2-colorable.

Solution

We can prove that any tree is 2-colorable by specifying: "*a tree with height n is 2-colorable for all $n \geq 1$*". Recall the definition of height for trees: *the number of edges on the longest path between the root and a leaf.*
Base Case: Consider a tree of height 1. Color the node at depth 0 the first color and color the nodes at depth 1 the second color. The proposition is true for $n = 1$.
Inductive Step: Assume that any tree of height $n = k$ is 2-colorable. [This is the inductive hypothesis, and we will use it to show that a tree of height $k+1$ is also 2-colorable.] Let C be a tree of height $k+1$, with all the nodes at height $k+1$ the color different from the nodes at height k. Remove all of the nodes at height $k+1$ to make a new tree, say, C' of height k. By the inductive hypothesis, C' is 2-colorable. Since all of the nodes removed were one color and the color different from the nodes at k, C is also 2-colorable. Therefore, if any tree of height k is 2-colorable, then any tree of height $k+1$ is 2-colorable.
Therefore any tree with height n is 2-colorable for all $n \geq 1$.

Solution

Alternatively:

Every node in any tree must have a height that is odd or even. Since all integers are either odd or even, assigning one color to nodes of odd height and another to nodes of even height makes the tree 2-colored.

(b.) Let d be the maximum degree of any vertex in graph G. Prove that we can color G with $d+1$ colors.

Solution

Proof by induction on the number of nodes in the graph.

Base case: Consider a graph of size 1. The maximum degree of any vertex is 0; this graph can be colored with 1 color.

Inductive Hypothesis: Assume that every graph G with n nodes and maximum degree d s.t. $0 \leq d \leq n-1$ can be colored with $d+1$ colors

Inductive step: Let G be a graph of size $n+1$ in which the maximum degree of any vertex is d s.t. $0 \leq d \leq n$. We separate cases:

case 1: $d \leq n-1$.

We unplug arbitrarily any node u from our graph. The remaining graph can be colored with $d+1$ colors (Inductive Hypothesis). Plugging u back in we note that is has at most d neighbors. Thus our initial graph G can be colored with $d+1$.

case 2: $d = n$

Again we unplug arbitrarily any node u. We note that the remaining graph G' has n nodes and thus maximum degree at most $n-1$. Using the Inductive Hypothesis we get that G' can be colored with n colors. Plugging u back in, we note that it may have at most n neighbor and thus we paint u with a new color. Summing up we get that our initial graph G can be colored with $n+1 = d+1$ colors.

Solution

Here we provide a second solution proving the existence of a coloring by giving an algorithm that creates it.

For any graph G with maximum degree d pick arbitrarily some node u you haven't painted before and paint it using a color (from the $d+1$ choices) that does not conflict with the already colored neighbors of u.

The algorithm terminates with a valid coloring after n steps as in every step a new node is colored. To see this it's enough to realize that every node will have at most d neighbors-restrictions when it is to be colored. Since there are $d+1$ choices, there will always be a color which creates no conflicts.

c. Show that the following are equivalent:

1. G is bipartite.
2. G is 2-colorable.
3. G has no cycles of odd length.

Solution

To prove that statements are "equivalent", we need to show that if one statement is true, all the statements are true, and if one is false, then all are false. To do this, we need to show $1 \Leftrightarrow 2$, $2 \Leftrightarrow 3$, and $1 \Leftrightarrow 3$. However, note that we don't need to prove both directions for each statement. For example, if we prove that $1 \Rightarrow 2$, $2 \Rightarrow 3$, and $3 \Rightarrow 1$, then we can prove $1 \Rightarrow 3$ easily using transitivity[a]: since $1 \Rightarrow 2$ and $2 \Rightarrow 3$, then $1 \Rightarrow 3$.

G is bipartite $\Rightarrow$ G is 2-colorable.
From the definition of bipartite, the nodes V in G can be partitioned into two sets V_1 and V_2 such that there are no edges between nodes in the same set. Color all of the nodes in V_1 one color and all of the nodes in V_2 the other color. Then there are no edges between nodes of the same color, since there are no edges between two nodes in the same partition.

G is 2-colorable $\Rightarrow$ G has no cycles of odd length (the proof of this is very similar to G is bipartite $\Rightarrow$ G has no cycles of odd length, if you chose to prove that instead).
Recall the definition of a cycle: informally it is a path from one node, through other nodes, ending at the original node. Any path in a (correctly) 2-colored graph must alternate colors. So a cycle p from u back to u in a 2-colorable graph must take the form $\langle u, v_1, u_1, v_2, u_2, \ldots, u_{k-1}, v_k, u\rangle$ where v nodes are one color and u nodes are the other. If a cycle had odd length, the last node before completing the cycle would be the same color as the start node, and thus the graph would not be 2-colorable; a contradiction.

G has no cycles of odd length $\Rightarrow$ G is bipartite.
We give a constructive proof that bipartitions G. The construction is proven to work in all cases unless G has an odd cycle; therefore, as long as G has no odd cycle, then G is bipartite.
For any connected component of G fix arbitrarily some node u and assign u to set V_1. Any node with even shortest-path distance from u is assigned to V_1 and any node with odd shortest-path distance from u is assigned to V_2. Next we argue that if this is not a bipartition, then G must have an odd cycle. We show that if two nodes with the same parity shortest-path distance from u (i.e., two nodes in the same set in our partition) have an edge between them, then G must have an odd cycle. Suppose that there is an edge e between the nodes v and v' where both have shortest-path distances from u with the same parity. Let p and p' be the shortest paths from u to v and v', respectively. Let w be the last vertex which p and p' have in common (note that they at least have u in common). Let d and d' be the shortest-path distances between w and v, and w and v', respectively. Note that since we assumed that the shortest-path distances from u to v and u to v' have the same parity, then d and d' have the same parity. Consider the following path: start at w, take the shortest path to v, take the edge e from v to v', then take the shortest path from v' back to w. This path is a cycle with length $d + 1 + d'$, which is odd since d and d' have the same parity.

[a] If you want to prove a large number of statements are equivalent, e.g. as in the Tree theorem we saw in lecture, you only need to do the following: let each statement be a vertex in a directed graph. Draw an edge (a, b) between statements a and b if you have a proof that $a \Rightarrow b$. Then, ensure that the graph you generate this way has exactly one strongly connected component, and you have ensured that the statements are equivalent. The intuition is that each statement is "reachable" from any other statement, so if some statement a is true, there is some path of implications (possibly through other statements) which transitively