Algorithms Notes 9-6

Cameron Chalk

1 Basics of graph connectivity continued.

In directed graphs, a *strongly connected component* (SCC) containing vertex u is the set of all vertices v such that u is reachable from v and v is reachable from u. (Example (1,2) (2,3) (3,4) has four different strongly connected components.) Any graph can be partitioned into its SCCs[1].

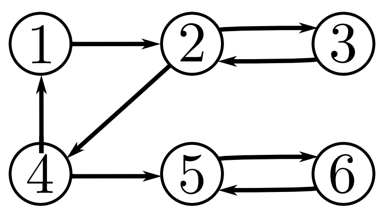

Figure 1: A directed graph. The two SCCs containing vertex 1 is $\{1, 2, 3, 4\}$.

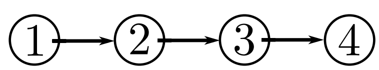

Figure 2: A directed graph with four distinct SCCs: $\{1\}, \{2\}, \{3\}$, and $\{4\}$.

2 Proof techniques review.

- Proof by contradiction
- Proof by example
- Proof by induction
- Proof by contrapositive
- Direct proof

[1]This can be proven by showing that v is in the the SCC containing u if and only if u is in the SCC containing v.

2.1 Example of proof by contrapositive.

Claim 2.1. *If an undirected graph G is a tree, then there is exactly one simple path between any two vertices.*

Proof. We will prove that if there is not exactly one simple path between any two vertices, then G is not a tree[2]. Case 1: If there is no simple path between a pair of vertices, then G is not connected and thus not a tree. Case 2: If there is more than one simple path between a pair of vertices, let two paths p_1 and p_2 from u to v. Let a be the last vertex before p_1 and p_2 differ; let b be the first vertex after a which p_1 and p_2 have in common. Then the following is a path and a cycle: begin at a, then go by p_1 to b, then go backward by p_2 to a. Therefore, since there is a cycle, G is not a tree—a tree by definition is acyclic. $\square$

3 Gale-Shapley algorithm.

3.1 The problem: stable matching.

We want some sense of an "optimal matching" for real-world cases like:

- Applicants to open positions
- Medical school graduates to residency programs
- Eligible males to marry eligible females

We will use the language of marrying males and females. Typically, one may think of solving matching by maximizing some utility function. However, this can still result in *unstable* matchings. For the marriage example, imagine the following example: say m and w are married, and m' and w' are married. Then consider the following:

- m prefers w' to w
- w' prefers m to m'.

This is called an *unstable* match, since m and w' both prefer each other to their current partners. Note that this matching is unstable independent of any sort of utility function, so this problem is independent of utility maximization.

Consider the following preferences list:

- man X prefers A over B over C
- man Y prefers B over A over C
- man Z prefers A over B over C

[2] Let A be "G is a tree" and let B be "there is exactly one simple path between any two vertices". Then the contrapositive, $\neg B \Rightarrow \neg A$, is the statement we will prove.

- woman A prefers Y over X over Z
- woman B prefers X over Y over z
- woman C prefers X over Y over Z

Consider matching X with C, Y with B, and Z with A. It is not obvious, but this matching is not stable; man X and woman B prefer each other over their current partners.

A question arises: how do we find a stable matching? Does one always exist? We will explore an algorithm which always finds a stable matching; the proof of the algorithm's correctness also implies that a stable matching *always* exists.

Observation 3.1. *Once a woman is proposed to, she always has an engagement.*

Observation 3.2. *The quality of a woman's match can only increase.*

Observation 3.3. *The quality of a man's match can only decrease.*

Claim 3.1. *The Gale-Shapley algorithm terminates.*

Proof. A man can propose to a woman at most once. This gives a loose upper bound of n^2 total proposals (where n = number of men = number of women). The algorithm terminates if no proposals can be made. □

Claim 3.2. *No man or woman is unmatched at termination.*

Proof. Since the number of men is equal to the number of women, then an unmatched woman implies an unmatched man. Towards contradiction, suppose there is an unmatched man at termination. He must have proposed to every woman, otherwise the *while* loop would not terminate. Since every woman has been proposed to at least once, then Observation 3.1 gives that all woman are engaged at the end. Then since the number of women is equal to the number of men, all women engaged implies that all men are engaged, which contradicts our assumption that there is an unmatched man. □

Claim 3.3. *Gale-Shapley returns a stable matching (i.e., there is no unstable match in the matching).*

Proof. Towards contradiction, assume there exists an unstable match in which m and w are married, m' and w' are married, yet m and w' prefer each other to their current partner. The last proposal made by m must have been made to w.

Case 1: m has not proposed to w'. Since men propose in order of their preference, then this means that m proposed to w before w', therefore m must prefer w to w', which contradicts the assumption that m prefers w'.

Case 2: m proposed to w' at some earlier time. Then m was rejected by w' in favor of someone else. By Observation 3.2, a woman can only move to more preferred matches. Then w' must prefer her current match over m, contradicting the assumption that w' prefers m over m'. □