Homework #1

You do not need to turn in these problems. The goal is to be ready for the in-class quiz that will cover the same or similar problems.

Problem 1: Relations

This problem is meant to test your ability to understand and reason precisely about formal definitions. Recall the following types of relations:

- A relation R on a set A is **reflexive** if $\forall a \in A$, $(a, a) \in R$.
- A relation R on a set A is **symmetric** if (a, b) implies (b, a) for $\forall a, b \in A$
- A relation R on a set A is **transitive** if (a, b) and (b, c) implies (a, c) for $\forall a, b, c \in A$

Consider the following claim and proof:
Claim: If a relation R is symmetric and transitive, then it is also reflexive.
Proof: By symmetry, $(a, b) \in R$ implies $(b, a) \in R$. Transitivity therefore implies $(a, a) \in R$.

Is this proof correct? If not, give a counter-example.

Problem 2: Sets and Counterexamples

Show that for arbitrary sets A, B, and C, taken from the universe $\{1, 2, 3, 4, 5\}$ that the following two claims are not always true by using a simple counter example for each:

(a) if $A \cap B \subseteq C$, then $C \subseteq A \cup B$

(b) if $C \subseteq A \cup B$, then $A \cap B \subseteq C$

Problem 3: Classic Proofs by Contradiction

Prove each of the following.

(a) If n^2 is even, n is even.

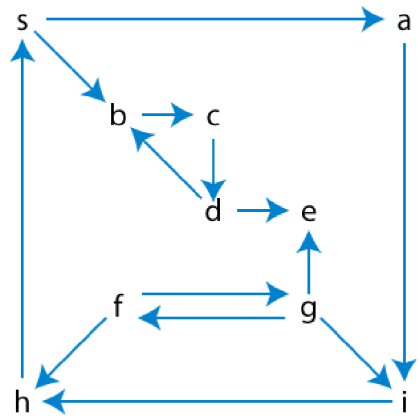
(b) $\sqrt{2}$ is irrational.

Problem 4: Proof by Induction

Prove the following by induction: Any postage that is a positive integer number of cents greater than 7 cents can be formed using just 3-cent stamps and 5-cent stamps.

Problem 5: Connected components

For the following directed graph, list the strongly connected components.



Problem 6: Trees

a. Prove that if $G = (V, E)$ is a tree, then $|E| = |V| - 1$.

b. Recall that in a rooted tree, the degree of a node is how many children it has. Consider any rooted tree where all nodes have degree at most two (i.e., binary tree). Show by induction that the number of degree-2 nodes nodes is 1 fewer than the number of leaves.

Problem 7: Colorings

Given an undirected graph $G = (V, E)$, a **k-coloring** of G is a function $c : V \to \{0, 1, \ldots, k-1\}$ such that $c(u) \neq c(v)$ for every edge $\{u, v\} \in E$. In other words, the numbers $0, 1, \ldots, k-1$ represent the k colors, and adjacent vertices must have different colors.

a. Show that any tree is 2-colorable.

b. Let d be the maximum degree of any vertex in graph G. Prove that we can color G with $d + 1$ colors.

c. Show that the following are equivalent:

1. G is bipartite.
2. G is 2-colorable.
3. G has no cycles of odd length.