

# 《计算机辅助几何设计》第四次作业

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## 思考题 1

1.

• a.

证明. 由于  $C_n^i = C_{n-1}^i + C_{n-1}^{i-1}$ , 因此

$$M_{i,n}(t) = C_n^i t^i = (C_{n-1}^i + C_{n-1}^{i-1})t^i = M_{i,n-1}(t) + tM_{i-1,n-1}(t).$$

QED

• b.

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**Algorithm 1** 类 de Casteljau 算法

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**Input:**  $c_i, i = 0, 1, \dots, n, \quad t \in [0, 1]$

**Output:**  $c_0^n$

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1: for  $i = 0, 1, \dots, n$  do  
2:    $c_i^0 = c_i$   
3: end for  
4: for  $k = 1, 2, \dots, n$  do  
5:   for  $i = 0, 1, \dots, n - k$  do  
6:      $c_i^k = c_i^{k-1} + tc_{i+1}^{k-1}$   
7:   end for  
8: end for  
9: return  $c_0^n$ 
```

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2.

证明. 令

$$B'_{i,n}(t) = C_n^i (1-t)^{n-i-1} t^{i-1} (i-nt) = 0,$$

可得  $t = \frac{i}{n}$ , 因此  $B_{i,n}(t)$  在  $t = \frac{i}{n}$  处达到极值, 并且该极值是最大值。

QED

3.

证明. 设随机变量 $\xi$ 服从二项分布, 即

$$\xi \sim B(n, t).$$

则其分布列为

$$P(\xi = i) = B_{i,n}(t), \quad i = 0, 1, \dots, n.$$

于是,  $\xi$ 的期望为

$$E\xi = \sum_{i=0}^n i B_{i,n}(t).$$

根据二项分布对参数 $n$ 的可加性, 以及两点分布 $\eta$ 的期望为 $t$ , 可知

$$E\xi = nE\eta = nt.$$

因此, 我们有

$$\sum_{i=0}^n i B_{i,n}(t) = nt.$$

QED

## 思考题 2

1.

证明.

$$\begin{aligned} \int_0^1 \|P'(t)\| dt &= n \int_0^1 \left\| \sum_{i=0}^{n-1} B_{i,n-1}(t)(P_{i+1} - P_i) \right\| dt \\ &\leq n \int_0^1 \sum_{i=0}^{n-1} \|B_{i,n-1}(t)(P_{i+1} - P_i)\| dt \\ &= n \sum_{i=0}^{n-1} \|P_{i+1} - P_i\| \int_0^1 B_{i,n-1}(t) dt \\ &= \sum_{i=0}^{n-1} \|P_{i+1} - P_i\|. \end{aligned}$$

QED

2.

证明. • 由升阶公式可知,  $Q_i = \frac{i}{n+1}P_{i-1} + (1 - \frac{i}{n+1})P_i, i = 0, 1, \dots, n+1$ , 其中 $P_{-1} = P_{n+1} = 0$ 。于是,  $\forall i = 1, \dots, n-1$ ,

$$\begin{aligned} \|Q_{i+1} - Q_i\| &= \left\| \frac{i+1}{n+1}P_i + (1 - \frac{i+1}{n+1})P_{i+1} - \frac{i}{n+1}P_{i-1} - (1 - \frac{i}{n+1})P_i \right\| \\ &= \left\| \frac{i}{n+1}(P_i - P_{i-1}) + (1 - \frac{i+1}{n+1})(P_{i+1} - P_i) \right\| \\ &\leq \frac{i}{n+1}\|P_i - P_{i-1}\| + (1 - \frac{i+1}{n+1})\|P_{i+1} - P_i\| \\ &\leq \frac{i}{n+1}s_{\max}(P) + (1 - \frac{i+1}{n+1})s_{\max}(P) \\ &= \frac{n}{n+1}s_{\max}(P). \end{aligned}$$

当 $i = 0$ 时,

$$\|Q_1 - Q_0\| = \left\| \frac{1}{n+1}P_0 + \left(1 - \frac{1}{n+1}\right)P_1 - P_0 \right\| = \left\| \frac{n}{n+1}(P_1 - P_0) \right\| \leq \frac{n}{n+1}s_{\max}(P).$$

类似地, 当 $i = n$ 时,  $\|Q_{n+1} - Q_n\| \leq \frac{n}{n+1}s_{\max}(P)$ .

综上,  $s_{\max}(Q) \leq \frac{n}{n+1}s_{\max}(P)$ 。

- 由 $a$ 的结论可知, 升阶 $r$ 次后控制多边形最长边

$$s_{\max}(P_r) \leq \left(\frac{n}{n+1}\right)^r s_{\max}(P) \rightarrow 0, \quad r \rightarrow \infty.$$

QED

### 思考题 3

1.

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**Algorithm 2** Bézier 曲线交点求解算法

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**Input:**  $C_1(t), C_2(t)$

**Output:**  $C_1(t) \cap C_2(t)$

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1: hull1 ← ComputeConvexHull( $C_1$ .control_points)
2: hull2 ← ComputeConvexHull( $C_2$ .control_points)
3: if not ConvexHullsIntersect(hull1, hull2) then
4:   return  $\emptyset$ 
5: end if
6: if ShouldSplit( $C_1, C_2$ ) then
7:   ( $C1\_left, C1\_right$ ) ← SplitCurve( $C_1$ )
8:   ( $C2\_left, C2\_right$ ) ← SplitCurve( $C_2$ )
9:   return BezierIntersection( $C1\_left, C2\_left$ )
      + BezierIntersection( $C1\_left, C2\_right$ )
      + BezierIntersection( $C1\_right, C2\_left$ )
      + BezierIntersection( $C1\_right, C2\_right$ )
10: end if
11: return ApproximateIntersection( $C_1, C_2$ )

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2.

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**Algorithm 3**  $n$ 次多项式在 $[0, 1]$ 上的求根算法

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**Input:**  $f(x) = \sum_{i=0}^n a_i x^i$

**Output:**  $\{x_i \in [0, 1] | f(x_i) = 0\}$

```

1:  $f(x) = \sum_{i=0}^n a_i x^i = \sum_{i=0}^n b_i B_{i,n}(x) = p(x)$ 
2: return BezierIntersection( $p(x), y = 0|_{x \in [0,1]}$ )

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## 思考题 4

1.

- 2维:  $p_0 = (-1, 0)'$ ,  $p_1 = (0, 1)'$ ,  $p_2 = (2, 2)'$ ,  $p_3 = (1, 0)'$ ,  $p_4 = (0, -1)'$ .

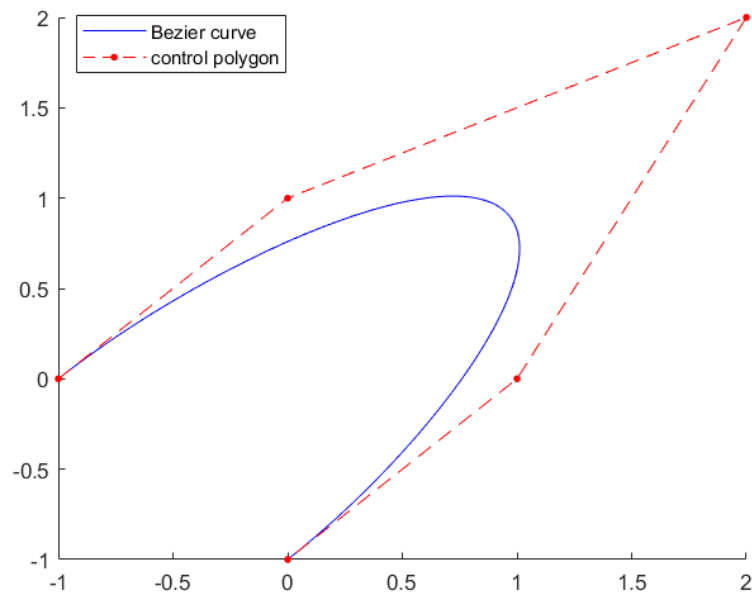


图 1: 2维Bézier曲线

- 3维:  $p_0 = (-1, 0, 0)'$ ,  $p_1 = (0, 2, 1)'$ ,  $p_2 = (2, 2, 3)'$ ,  $p_3 = (1, 0, 2)'$ ,  $p_4 = (0, -1, 0)'$ .

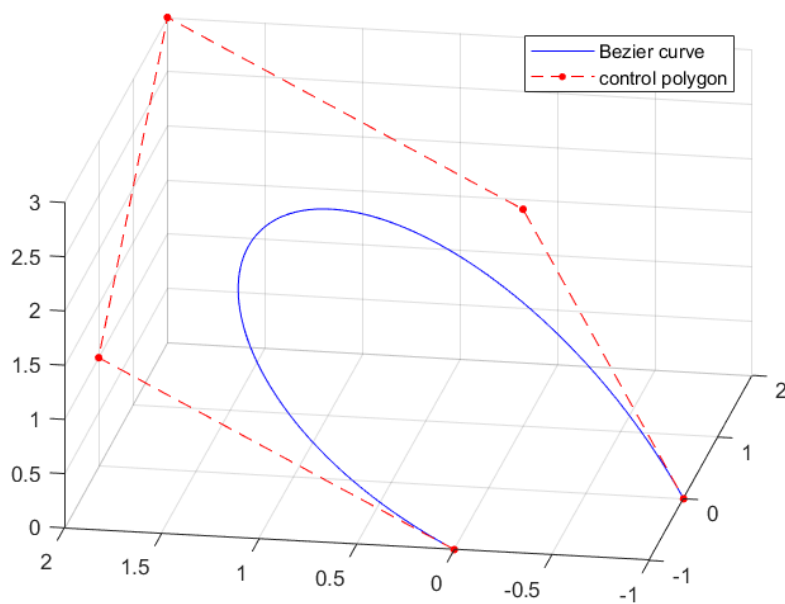


图 2: 3维Bézier曲线

## 2.

采用不同的参数化方法实现Bezier曲线拟合，分别为均匀参数化、累积弦长参数化和切比雪夫累积弦长参数化<sup>1</sup>，并比较拟合效果。

- 2维:  $p_0 = (-1, 0)'$ ,  $p_1 = (0, 1)'$ ,  $p_2 = (3, 3)'$ ,  $p_3 = (1, 0)'$ ,  $p_4 = (0, -1)'$ .

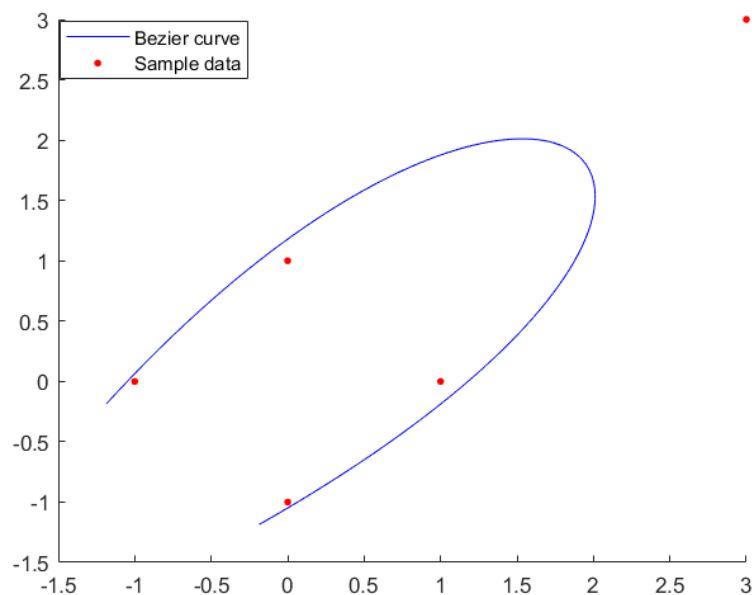


图 3: 均匀参数化

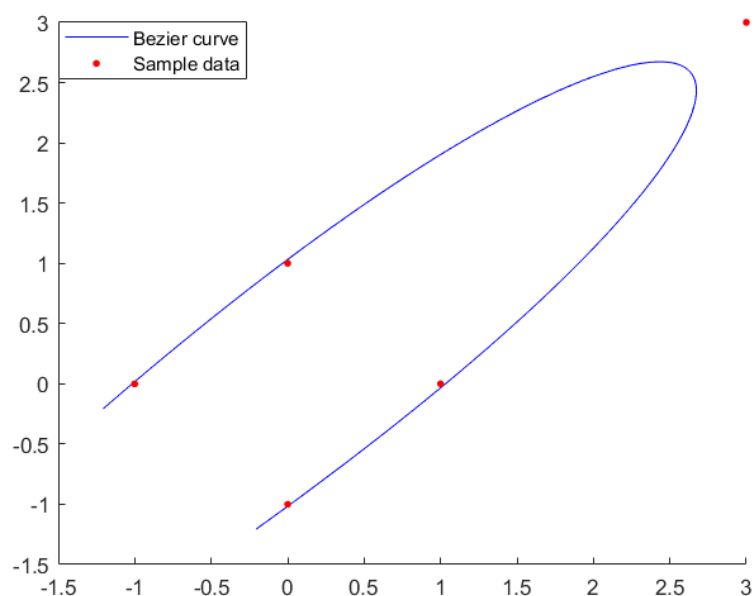


图 4: 累积弦长参数化

<sup>1</sup>参考《数据插值中的参数化新方法》

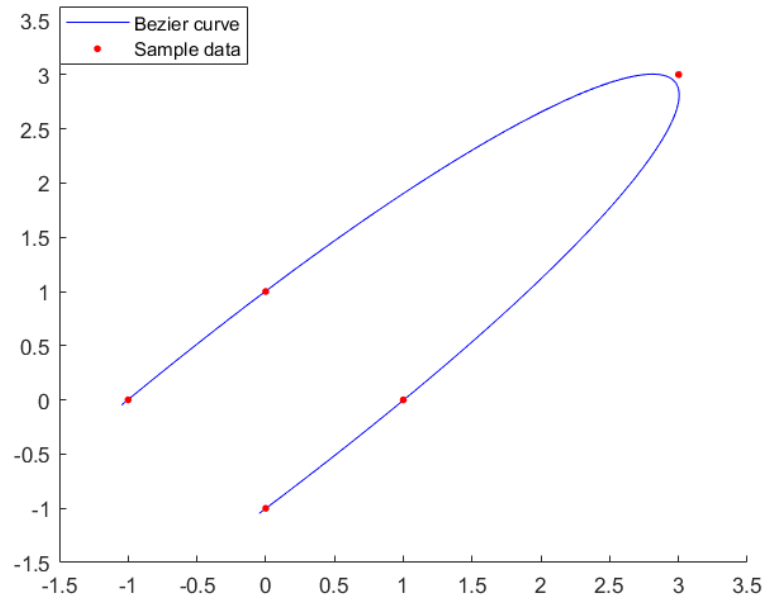


图 5: 切比雪夫累积弦长参数化

- 3维:  $p_0 = (-1, 0, 0)'$ ,  $p_1 = (0, 2, 1)'$ ,  $p_2 = (3, 3, 3)'$ ,  $p_3 = (1, 0, 2)'$ ,  $p_4 = (0, -1, 0)'$ .

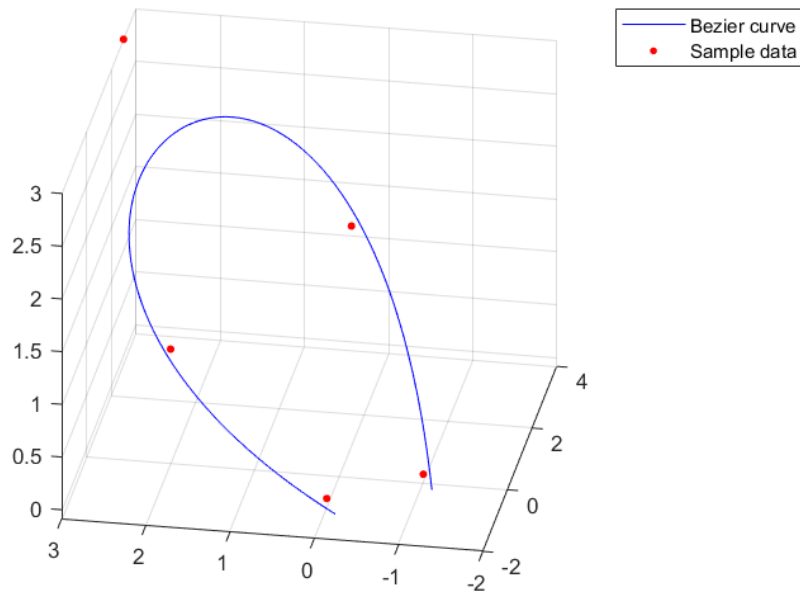


图 6: 均匀参数化

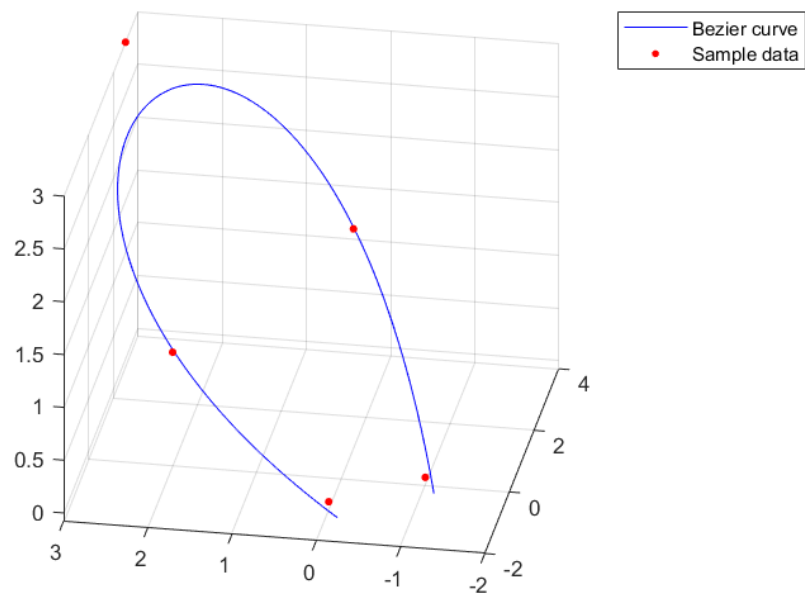


图 7: 累积弦长参数化

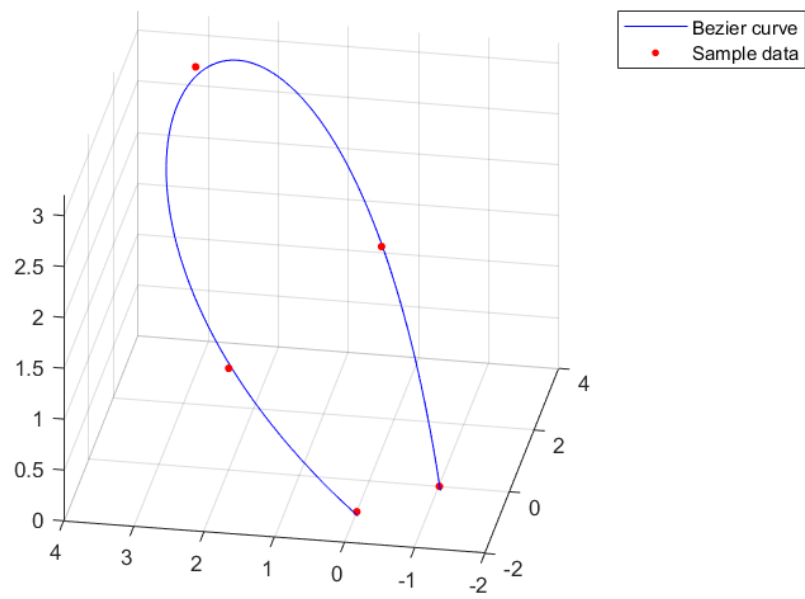


图 8: 切比雪夫累积弦长参数化

### 3.

对于不同数量的点  $n = 4, 8, 16, 32$ , 采用几何Hermite插值来逼近单位圆, 并比较逼近效果。

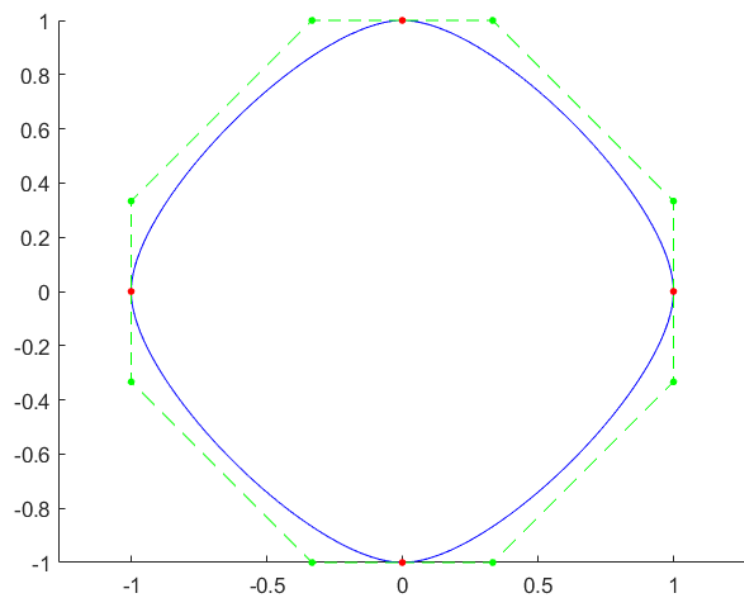


图 9:  $n = 4$

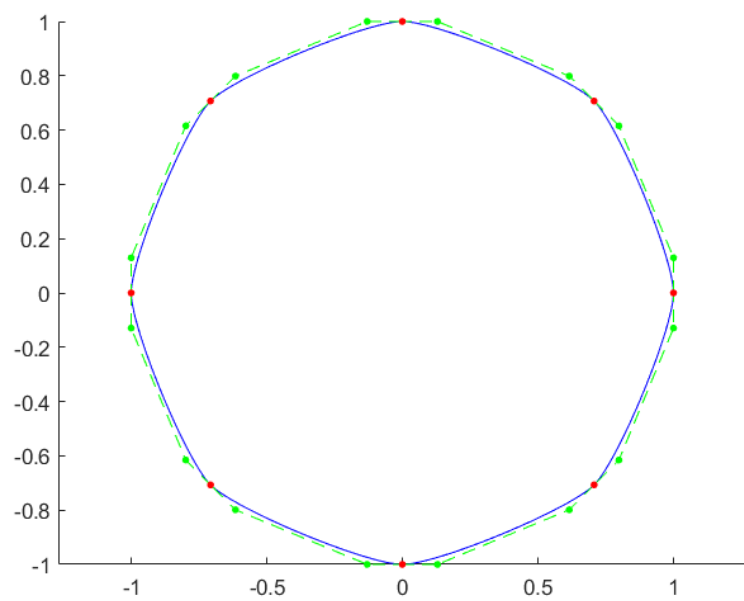


图 10:  $n = 8$



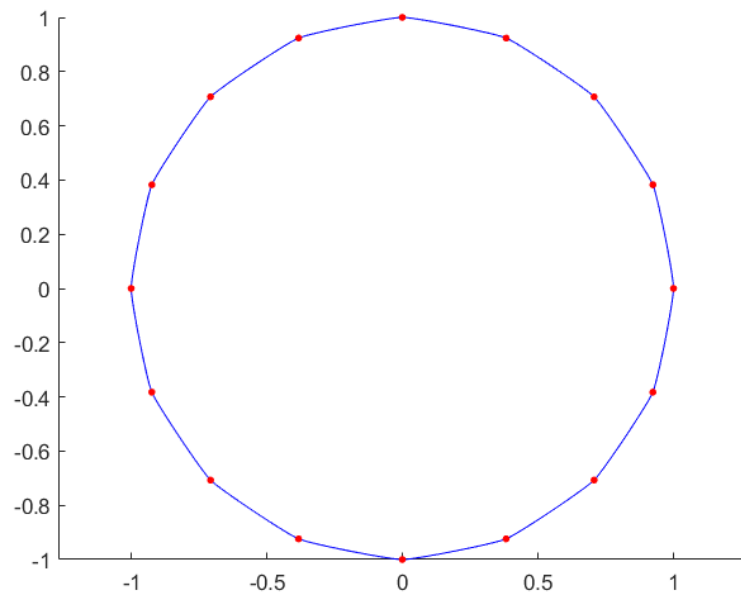


图 11:  $n = 16$

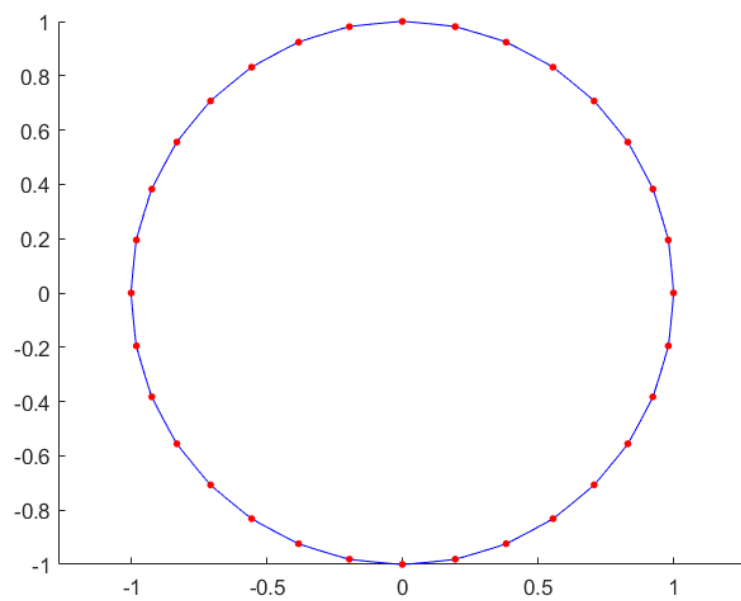


图 12:  $n = 32$