《计算机辅助几何设计》第五次作业

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思考题 1

1.

证明. 由k阶非均匀B样条基函数的定义，

$$
\begin{aligned}
N_{i,3}(t) &= (t_{i+3}-t_i)\cdot[t_i,\cdots,t_{i+3}](\cdot-t)_+^2 \\
&= [t_{i+1},t_{i+2},t_{i+3}](\cdot-t)_+^2 - [t_i,t_{i+1},t_{i+2}](\cdot-t)_+^2 \\
&= \frac{[t_{i+2},t_{i+3}](\cdot-t)_+^2-[t_{i+1},t_{i+2}](\cdot-t)_+^2}{t_{i+3}-t_{i+1}} - \frac{[t_{i+1},t_{i+2}](\cdot-t)_+^2-[t_i,t_{i+1}](\cdot-t)_+^2}{t_{i+2}-t_i} \\
&= \frac{\frac{(t_{i+3}-t)_+^2-(t_{i+2}-t)_+^2}{t_{i+3}-t_{i+2}}-\frac{(t_{i+2}-t)_+^2-(t_{i+1}-t)_+^2}{t_{i+2}-t_{i+1}}}{t_{i+3}-t_{i+1}} - \frac{\frac{(t_{i+2}-t)_+^2-(t_{i+1}-t)_+^2}{t_{i+2}-t_{i+1}}-\frac{(t_{i+1}-t)_+^2-(t_i-t)_+^2}{t_{i+1}-t_i}}{t_{i+2}-t_i} \\
&= \begin{cases}
1-\dfrac{t_{i+2}-t_{i+1}-2t-\frac{(t_{i+1}-t)^2}{t_{i+1}-t_i}}{t_{i+2}-t_i}, \quad t_i \le t < t_{i+1} \\
\dfrac{t_{i+3}+t_{i+2}-2t-\frac{(t_{i+2}-t)^2}{t_{i+2}-t_{i+1}}}{t_{i+3}-t_{i+1}} - \dfrac{\frac{(t_{i+2}-t)^2}{t_{i+2}-t_{i+1}}}{t_{i+2}-t_i}, \quad t_{i+1} \le t < t_{i+2} \\
\dfrac{(t_{i+3}-t)^2}{(t_{i+3}-t_{i+2})(t_{i+3}-t_{i+1})}, \quad t_{i+2} \le t < t_{i+3} \\
0, \quad \text{otherwise}
\end{cases}
\end{aligned}
$$

QED

2.

证明. 由非均匀B样条基函数的局部支撑性以及B样条基函数的差商定义可得，

$$
\begin{aligned}
\frac{1}{t_{i+k}-t_i}\int_{-\infty}^{\infty} N_{i,k}(x)dx &= \frac{1}{t_{i+k}-t_i}\int_{t_i}^{t_{i+k}} N_{i,k}(x)dx \\
&= \int_{t_i}^{t_{i+k}} [t_i,\cdots,t_{i+k}](t-x)_+^{k-1}dx \\
&= [t_i,\cdots,t_{i+k}]\int_{t_i}^{t_{i+k}} (t-x)_+^{k-1}dx \\
&= [t_i,\cdots,t_{i+k}]\frac{(t-t_i)^k}{k} \\
&= \frac{1}{k}.
\end{aligned} \tag{1}
$$

QED

思考题 2

1.

证明. 这里我们设k为次数，当$t \in [l, l+1]$时，只需考虑基函数$N_{j-k}(t), j = l, \dots, l+k$。根据均匀B样条的性质，$N_{i,k}(t) = N_{0,k}(t-i)$，因此问题转化为：$\forall i = 0, \dots, k$，计算系数$s_{i,j}^{(k)}, i = 0, \dots, k$，

$$\forall j = 0, \dots, k, \quad N_{j-k,k}(t) = \sum_{i=0}^{k} s_{i,j}^{(k)} B_{i,k}(t), \quad t \in [0,1]. \tag{2}$$

将上式写成矩阵形式，

$$[N_{-k,k}(t) \cdots N_{0,k}(t)] = [B_{0,k}(t) \cdots B_{k,k}(t)] S^{(k)}, \tag{3}$$

其中，$S^{(k)}$是B-spline到Bezier表示的k次转换矩阵。由于B-spline基函数和Bernstein基函数都具有权性，从而 $S^{(k)}$的每一行元素之和均为1。因此，将B-spline曲线转化为Bezier形式后，每个Bezier控制顶点都是B-spline控制顶点的凸线性组合。

将Cox-deBoor公式应用到式(2)，可得

$$\sum_{i=0}^{k} s_{i,j}^{(k)} B_{i,k}(t) = \sum_{i=0}^{k-1} \left(\frac{t-j+k}{k} s_{i,j-1}^{(k-1)} + \frac{j+1-t}{k} s_{i,j}^{(k-1)}\right) B_{i,k-1}(t). \tag{4}$$

根据Bernstein基函数的递推式，再将$B_{i,n}(t)$替换掉，可得$\forall j = 0, \dots, n, i = 0, \dots, n-1$，

$$t s_{i+1,j}^{(k)} + (1-t) s_{i,j}^{(k)} = \frac{t-j+k}{k} s_{i,j-1}^{(k-1)} + \frac{j+1-t}{k} s_{i,j}^{(k-1)}, \quad t \in [0,1], \tag{5}$$

其中，$\forall i = 0, \dots, k-1, s_{i,-1}^{(k-1)} = s_{i,k}^{(k-1)} = 0$。因此，在$t = 0$处计算式(4)，我们有

$$s_{i,j}^{(k)} = \frac{k-j}{k} s_{i,j-1}^{(k-1)} + \frac{j+1}{k} s_{i,j}^{(k-1)}, \quad \forall j = 0, \dots, k, i = 0, \dots, k-1, \tag{6}$$

而在$t = 1$处计算式(5)，我们有

$$s_{i+1,j}^{(k)} = \frac{k+1-j}{k} s_{i,j-1}^{(k-1)} + \frac{j}{k} s_{i,j}^{(k-1)}, \quad \forall j = 0, \dots, k, i = 0, \dots, k-1. \tag{7}$$

因此，从1×1矩阵$S^{(0)} = s_{0,0}^{(0)} = 1$开始（这是因为$N_{0,0}(t) = B_{0,0}(t) = 1, \forall t \in [0,1]$），$(k+1) \times (k+1)$ 转换矩阵$S^{(k)} = (s_{i,j}^{(k)})_{i,j=0,\dots,k} (k \geq 1)$可以被递归定义为：$\forall j = 0, \dots, k$，

$$\begin{aligned} s_{i,j}^{(k)} &= \frac{k-j}{k} s_{i,j-1}^{(k-1)} + \frac{j+1}{k} s_{i,j}^{(k-1)}, \quad \forall i = 0, \dots, k-1, \\ s_{k,j}^{(k)} &= \frac{k+1-j}{k} s_{k-1,j-1}^{(k-1)} + \frac{j}{k} s_{k-1,j}^{(k-1)}. \end{aligned} \tag{8}$$

通过式(8)，可以很容易计算出转换矩阵$S^{(k)}$，即我们有

$$[N_{l-k,k}(t) \cdots N_{l,k}(t)] = [B_{0,k}(t-l) \cdots B_{k,k}(t-l)] S^{(k)}, \quad t \in [l, l+1]. \tag{9}$$

QED

思考题 3

1.

- 次数$p = 3$，节点向量$u = [0,0,0,0,1/2,1,1,1,1]$

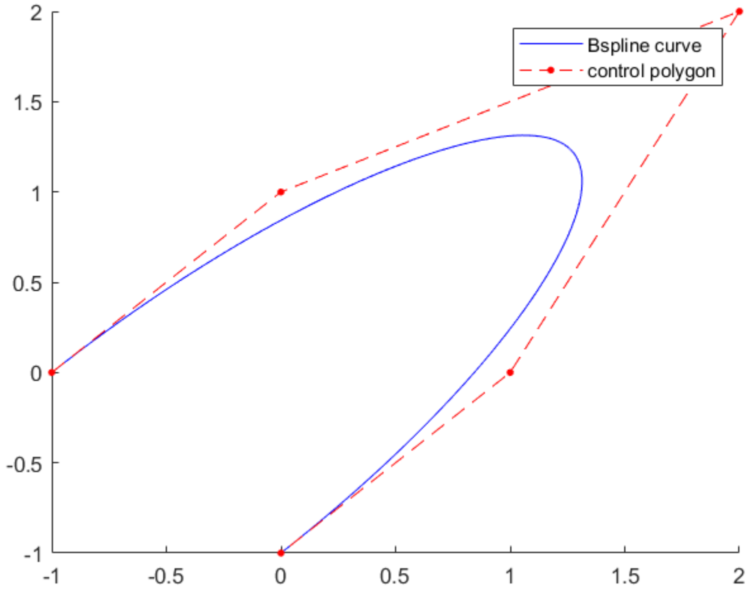

图 1: 控制顶点 $p_0 = (-1,0)', p_1 = (0,1)', p_2 = (2,2)', p_3 = (1,0)', p_4 = (0,-1)'$

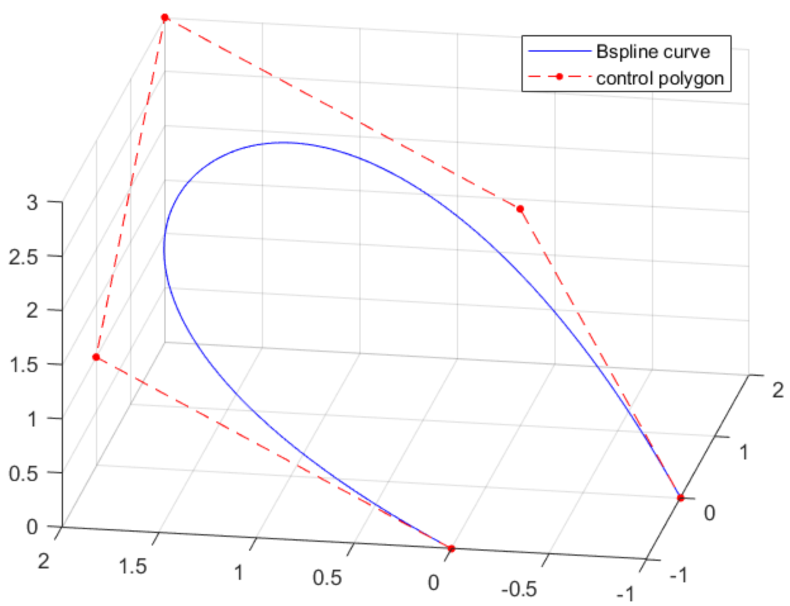

图 2: 控制顶点 $p_0 = (-1,0,0)', p_1 = (0,2,1)', p_2 = (2,2,3)', p_3 = (1,0,2)', p_4 = (0,-1,0)'$

- 次数$p = 2$，节点向量$u = [0,0,0,1/3,2/3,1,1,1]$

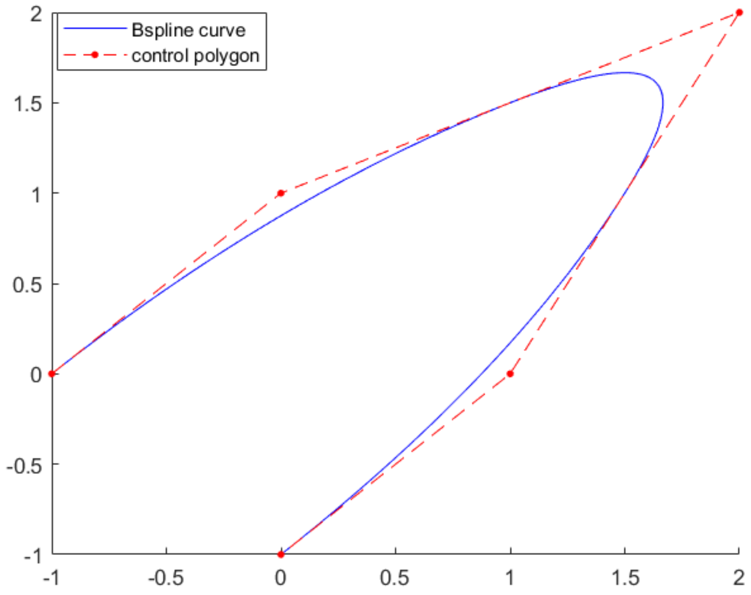

图 3: 控制顶点 $p_0 = (-1,0)', p_1 = (0,1)', p_2 = (2,2)', p_3 = (1,0)', p_4 = (0,-1)'$

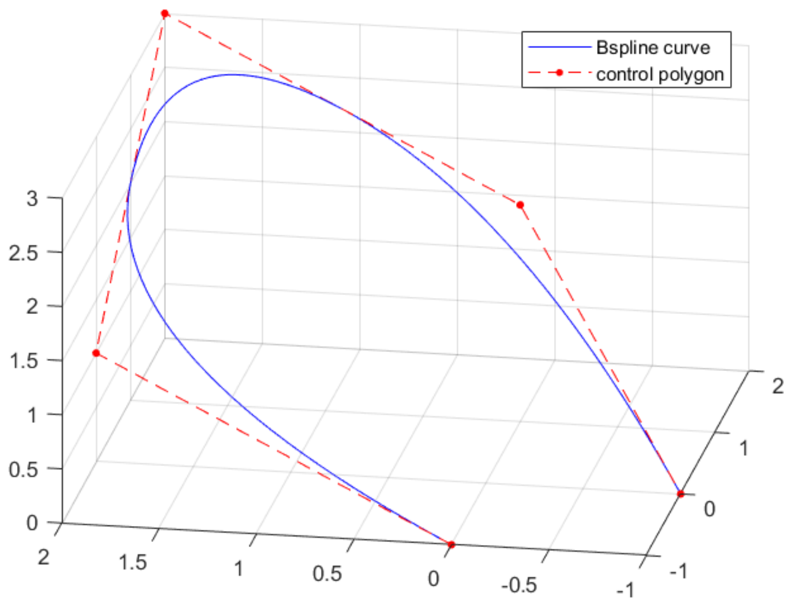

图 4: 控制顶点 $p_0 = (-1,0,0)', p_1 = (0,2,1)', p_2 = (2,2,3)', p_3 = (1,0,2)', p_4 = (0,-1,0)'$

2.

采用不同的参数化方法实现三次B-spline曲线插值，分别为均匀参数化、累积弦长参数化和切比雪夫累积弦长参数化[1]，并比较插值效果。

型值点：$p_0 = (-1,0)', p_1 = (0,1)', p_2 = (2,2)', p_3 = (1,0)', p_4 = (0,-1)'$，边界导矢：$D1 = (7.0990, 7.0990)', D2 = (-7.0990, -7.0990)'$。

[1]参考《数据插值中的参数化新方法》

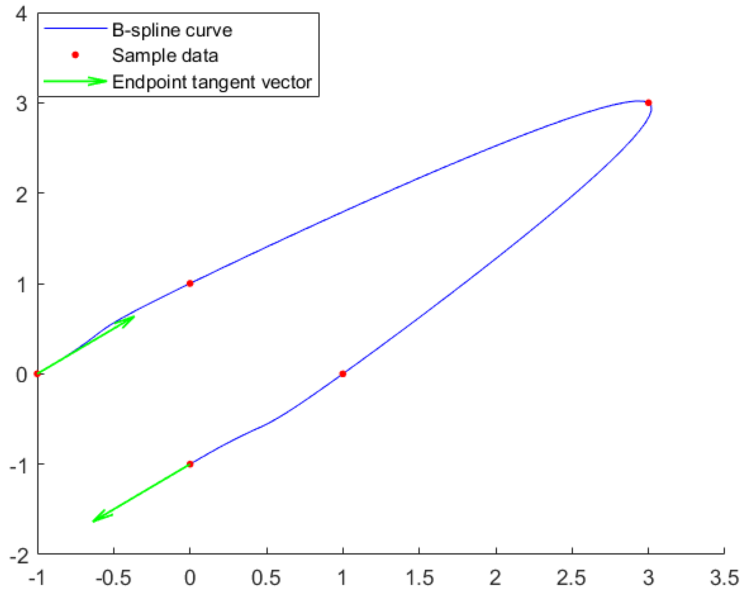

图 5: 均匀参数化

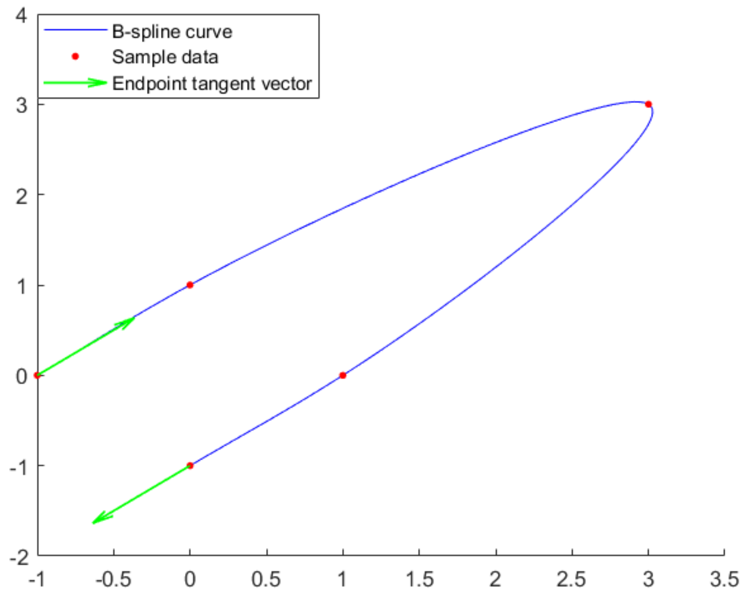

图 6: 累积弦长参数化

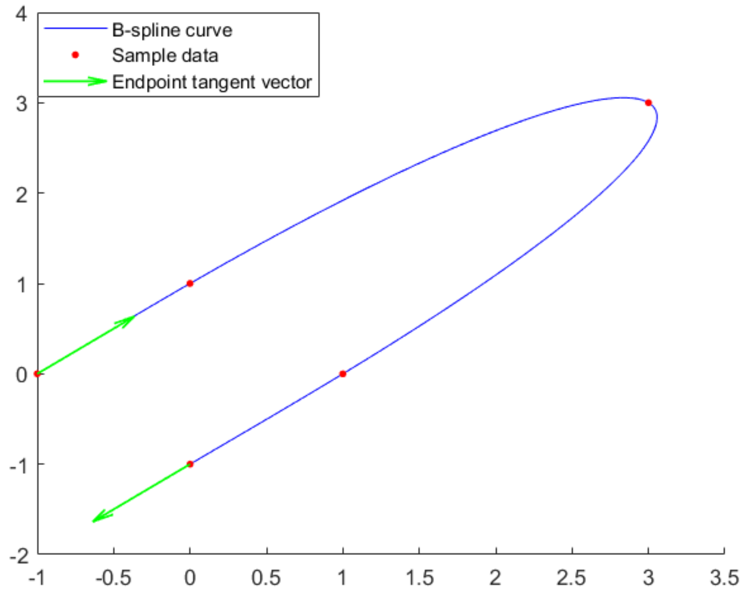

图 7: 切比雪夫累积弦长参数化