《计算机辅助几何设计》第四次作业

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思考题 1

1.

- a.

证明. 由于$C_n^i = C_{n-1}^i + C_{n-1}^{i-1}$，因此

$$M_{i,n}(t) = C_n^i t^i = (C_{n-1}^i + C_{n-1}^{i-1})t^i = M_{i,n-1}(t) + tM_{i-1,n-1}(t).$$

QED

- b.

Algorithm 1 类 de Casteljau算法

Input: $c_i, i = 0, 1, \ldots, n, \quad t \in [0, 1]$

Output: c_0^n

for $i = 0, 1, \ldots, n$ **do**

  $c_i^0 = c_i$

end for

for $k = 1, 2, \ldots, n$ **do**

  for $i = 0, 1, \ldots, n-k$ **do**

    $c_i^k = c_i^{k-1} + tc_{i+1}^{k-1}$

  end for

end for

return c_0^n

2.

证明. 令

$$B'_{i,n}(t) = C_n^i (1-t)^{n-i-1} t^{i-1} (i - nt) = 0,$$

可得$t = \dfrac{i}{n}$，因此$B_{i,n}(t)$在$t = \dfrac{i}{n}$处达到极值，并且该极值是最大值。 QED

3.

证明．设随机变量ξ服从二项分布，即

$$\xi \sim B(n,t).$$

则其分布列为

$$P(\xi = i) = B_{i,n}(t), \quad i = 0,1,\ldots,n.$$

于是，ξ的期望为

$$E\xi = \sum_{i=0}^{n} iB_{i,n}(t).$$

根据二项分布对参数n的可加性，以及两点分布η的期望为t，可知

$$E\xi = nE\eta = nt.$$

因此，我们有

$$\sum_{i=0}^{n} iB_{i,n}(t) = nt.$$

QED

思考题 2

1.

证明．

$$\begin{aligned}
\int_0^1 \|P'(t)\|dt &= n\int_0^1 \|\sum_{i=0}^{n-1} B_{i,n-1}(t)(P_{i+1}-P_i)\|dt \\
&\le n\int_0^1 \sum_{i=0}^{n-1} \|B_{i,n-1}(t)(P_{i+1}-P_i)\|dt \\
&= n\sum_{i=0}^{n-1} \|P_{i+1}-P_i\| \int_0^1 B_{i,n-1}(t)dt \\
&= \sum_{i=0}^{n-1} \|P_{i+1}-P_i\|.
\end{aligned}$$

QED

2.

证明．

- 由升阶公式可知，$Q_i = \frac{i}{n+1}P_{i-1} + (1-\frac{i}{n+1})P_i, i = 0,1,\ldots,n+1$，其中$P_{-1} = P_{n+1} = 0$。于是，$\forall i = 1,\ldots,n-1$，

$$\begin{aligned}
\|Q_{i+1}-Q_i\| &= \|\frac{i+1}{n+1}P_i + (1-\frac{i+1}{n+1})P_{i+1} - \frac{i}{n+1}P_{i-1} - (1-\frac{i}{n+1})P_i\| \\
&= \|\frac{i}{n+1}(P_i - P_{i-1}) + (1-\frac{i+1}{n+1})(P_{i+1}-P_i)\| \\
&\le \frac{i}{n+1}\|P_i - P_{i-1}\| + (1-\frac{i+1}{n+1})\|P_{i+1}-P_i\| \\
&\le \frac{i}{n+1}s_{\max}(P) + (1-\frac{i+1}{n+1})s_{\max}(P) \\
&= \frac{n}{n+1}s_{\max}(P).
\end{aligned}$$

当$i=0$时，

$$\|Q_1-Q_0\|=\|\frac{1}{n+1}P_0+(1-\frac{1}{n+1})P_1-P_0\|=\|\frac{n}{n+1}(P_1-P_0)\|\le\frac{n}{n+1}s_{\max}(P).$$

类似地，当$i=n$时，$\|Q_{n+1}-Q_n\|\le\frac{n}{n+1}s_{\max}(P)$.

综上，$s_{\max}(Q)\le\frac{n}{n+1}s_{\max}(P)$。

- 由a的结论可知，升阶r次后控制多边形最长边

$$s_{\max}(P_r)\le(\frac{n}{n+1})^r s_{\max}(P)\to 0,\quad r\to\infty.$$

QED

思考题 3

1.

Algorithm 2 Bézier 曲线交点求解算法

Input: $C_1(t), C_2(t)$

Output: $C_1(t)\cap C_2(t)$

$hull1 \leftarrow$ ComputeConvexHull(C_1.control_points)
$hull2 \leftarrow$ ComputeConvexHull(C_2.control_points)
if not ConvexHullsIntersect($hull1, hull2$) **then**
  return []
end if
if ShouldSplit(C_1, C_2) **then**
  $(C1\_left, C1\_right) \leftarrow$ SplitCurve(C_1)
  $(C2\_left, C2\_right) \leftarrow$ SplitCurve(C_2)
  return BezierIntersection($C1\_left, C2\_left$)
  + BezierIntersection($C1\_left, C2\_right$)
  + BezierIntersection($C1\_right, C2\_left$)
  + BezierIntersection($C1\_right, C2\_right$)
end if
return ApproximateIntersection(C_1, C_2)

2.

Algorithm 3 n次多项式在$[0,1]$上的求根算法

Input: $f(x)=\sum_{i=0}^{n}a_ix^i$

Output: $\{x_i\in[0,1] | f(x_i)=0\}$

$f(x)=\sum_{i=0}^{n}a_ix^i=\sum_{i=0}^{n}b_iB_{i,n}(x)=p(x)$
return BezierIntersection($p(x), y=0|_{x\in[0,1]}$)

思考题 4

1.

- 2维：$p_0 = (-1, 0)', p_1 = (0, 1)', p_2 = (2, 2)', p_3 = (1, 0)', p_4 = (0, -1)'$.

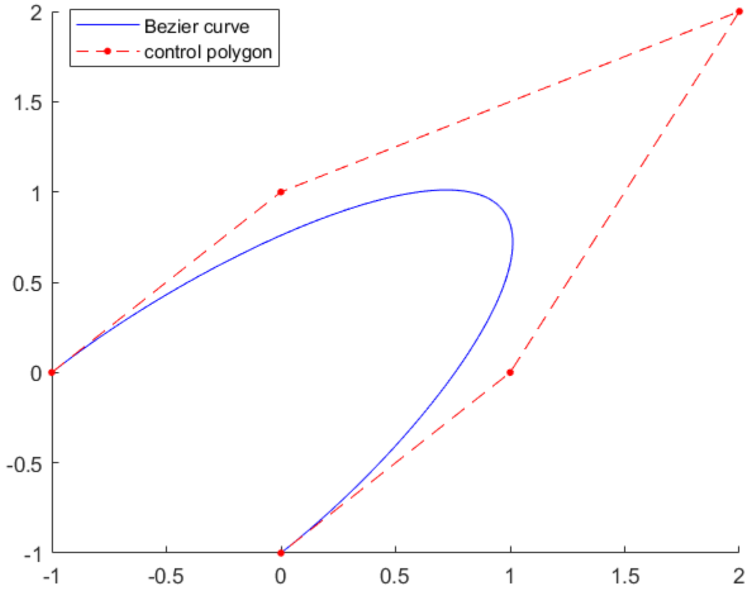

图 1: 2维Bézier曲线

- 3维：$p_0 = (-1, 0, 0)', p_1 = (0, 2, 1)'$ﬁ$p_2 = (2, 2, 3)', p_3 = (1, 0, 2)', p_4 = (0, -1, 0)'$.

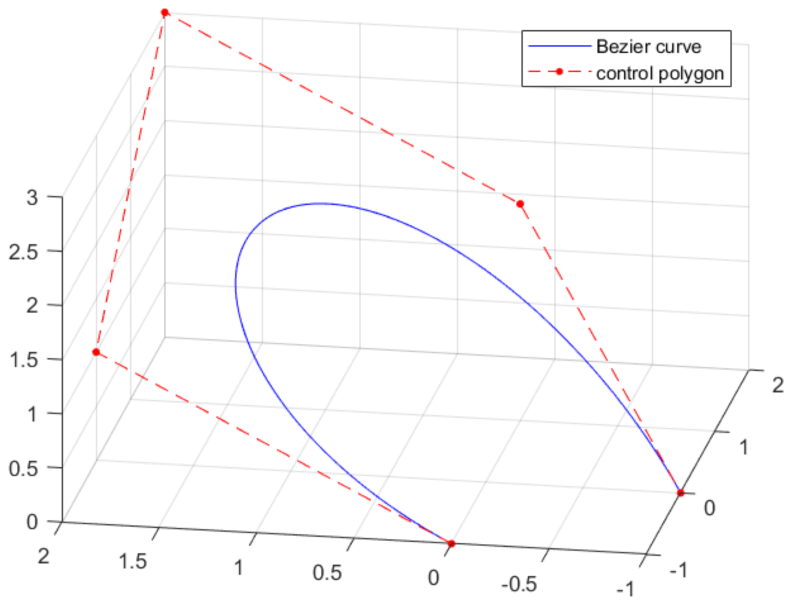

图 2: 3维Bézier曲线

2.

- 2维：$p_0 = (-1,0)', p_1 = (0,1)', p_2 = (3,3)', p_3 = (1,0)', p_4 = (0,-1)'$.

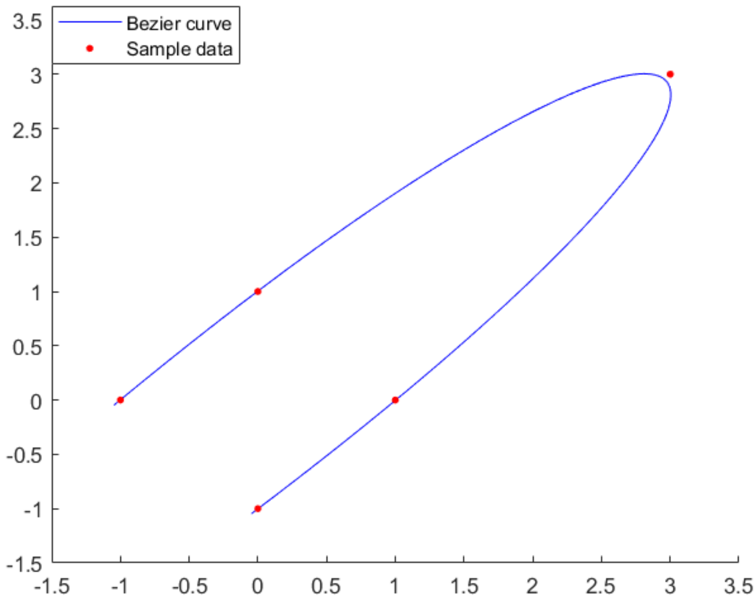

图 3: 2维Bézier曲线拟合

- 3维：$p_0 = (-1,0,0)', p_1 = (0,2,1)'fip_2 = (3,3,3)', p_3 = (1,0,2)', p_4 = (0,-1,0)'$.

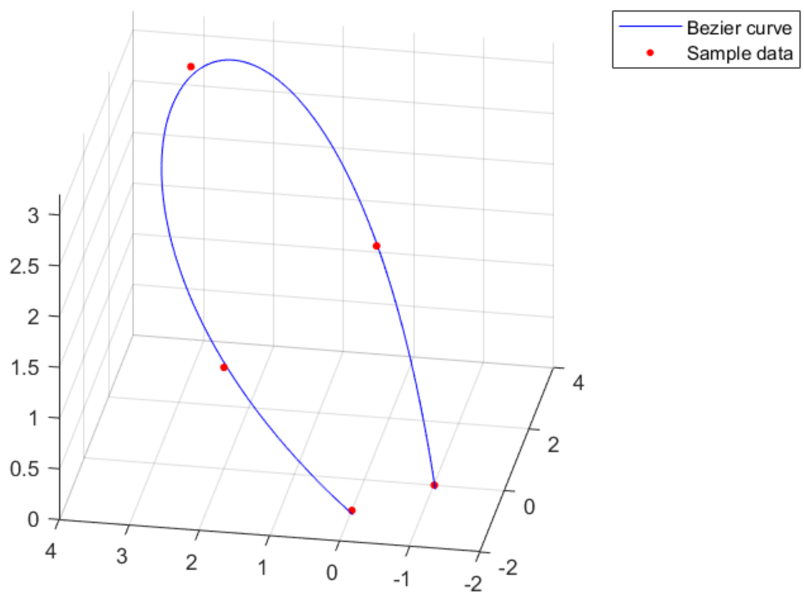

图 4: 3维Bézier曲线拟合