《计算机辅助几何设计》第四次作业

姓名：殷文良　　学号：12435063

2024 年 10 月 29 日

思考题 1

1.

- a.

  证明. 由于$C_n^i = C_{n-1}^i + C_{n-1}^{i-1}$，因此

  $$M_{i,n}(t) = C_n^i t^i = (C_{n-1}^i + C_{n-1}^{i-1})t^i = M_{i,n-1}(t) + tM_{i-1,n-1}(t).$$

  QED

- b.

Algorithm 1 类 de Casteljau算法

Input: $c_i, i = 0, 1, \dots, n, \quad t \in [0, 1]$

Output: c_0^n

```
for i = 0, 1, ..., n do
    c_i^0 = c_i
end for
for k = 1, 2, ..., n do
    for i = 0, 1, ..., n - k do
        c_i^k = c_i^{k-1} + t c_{i+1}^{k-1}
    end for
end for
return c_0^n
```

2.

证明. 令

$$B'_{i,n}(t) = C_n^i (1-t)^{n-i-1} t^{i-1} (i - nt) = 0,$$

可得$t = \dfrac{i}{n}$，因此$B_{i,n}(t)$在$t = \dfrac{i}{n}$处达到极值，并且该极值是最大值。　QED

3.

证明. 设随机变量ξ服从二项分布，即

$$\xi \sim B(n,t).$$

则其分布列为

$$P(\xi = i) = B_{i,n}(t), \quad i = 0, 1, \ldots, n.$$

于是，ξ的期望为

$$E\xi = \sum_{i=0}^{n} i B_{i,n}(t).$$

根据二项分布对参数n的可加性，以及两点分布η的期望为t，可知

$$E\xi = nE\eta = nt.$$

因此，我们有

$$\sum_{i=0}^{n} i B_{i,n}(t) = nt.$$

QED

思考题 2

1.

证明.

$$\begin{aligned}
\int_0^1 \|P'(t)\| dt &= n \int_0^1 \|\sum_{i=0}^{n-1} B_{i,n-1}(t)(P_{i+1} - P_i)\| dt \\
&\leq n \int_0^1 \sum_{i=0}^{n-1} \|B_{i,n-1}(t)(P_{i+1} - P_i)\| dt \\
&= n \sum_{i=0}^{n-1} \|P_{i+1} - P_i\| \int_0^1 B_{i,n-1}(t) dt \\
&= \sum_{i=0}^{n-1} \|P_{i+1} - P_i\|.
\end{aligned}$$

QED

2.

证明. • 由升阶公式可知，$Q_i = \frac{i}{n+1} P_{i-1} + (1 - \frac{i}{n+1}) P_i, i = 0, 1, \ldots, n+1$，其中$P_{-1} = P_{n+1} = 0$。于是，$\forall i = 1, \ldots, n-1$，

$$\begin{aligned}
\|Q_{i+1} - Q_i\| &= \|\frac{i+1}{n+1} P_i + (1 - \frac{i+1}{n+1}) P_{i+1} - \frac{i}{n+1} P_{i-1} - (1 - \frac{i}{n+1}) P_i\| \\
&= \|\frac{i}{n+1}(P_i - P_{i-1}) + (1 - \frac{i+1}{n+1})(P_{i+1} - P_i)\| \\
&\leq \frac{i}{n+1} \|P_i - P_{i-1}\| + (1 - \frac{i+1}{n+1}) \|P_{i+1} - P_i\| \\
&\leq \frac{i}{n+1} s_{\max}(P) + (1 - \frac{i+1}{n+1}) s_{\max}(P) \\
&= \frac{n}{n+1} s_{\max}(P).
\end{aligned}$$

当$i=0$时，

$\|Q_1-Q_0\|=\|\frac{1}{n+1}P_0+(1-\frac{1}{n+1})P_1-P_0\|=\|\frac{n}{n+1}(P_1-P_0)\|\leq\frac{n}{n+1}s_{\max}(P)$.

类似地，当$i=n$时，$\|Q_{n+1}-Q_n\|\leq\frac{n}{n+1}s_{\max}(P)$.

综上，$s_{\max}(Q)\leq\frac{n}{n+1}s_{\max}(P)$。

- 由a的结论可知，升阶r次后控制多边形最长边

$$s_{\max}(P_r)\leq(\frac{n}{n+1})^r s_{\max}(P)\to 0,\quad r\to\infty.$$

QED

思考题 3

1.

Algorithm 2 Bézier 曲线交点求解算法

Input: $C_1(t), C_2(t)$

Output: $C_1(t)\cap C_2(t)$

```
hull1 ← ComputeConvexHull(C_1.control_points)
hull2 ← ComputeConvexHull(C_2.control_points)
if not ConvexHullsIntersect(hull1, hull2) then
    return []
end if
if ShouldSplit(C_1, C_2) then
    (C1_left, C1_right) ← SplitCurve(C_1)
    (C2_left, C2_right) ← SplitCurve(C_2)
    return BezierIntersection(C1_left, C2_left)
    + BezierIntersection(C1_left, C2_right)
    + BezierIntersection(C1_right, C2_left)
    + BezierIntersection(C1_right, C2_right)
end if
return ApproximateIntersection(C_1, C_2)
```

2.

Algorithm 3 n次多项式在$[0,1]$上的求根算法

Input: $f(x)=\sum_{i=0}^{n}a_ix^i$

Output: $\{x_i\in[0,1]|f(x_i)=0\}$

1: $f(x)=\sum_{i=0}^{n}a_ix^i=\sum_{i=0}^{n}b_iB_{i,n}(x)=p(x)$

2: **return** BezierIntersection$(p(x), y=0|_{x\in[0,1]})$

思考题 4

1.

- 2维：$p_0 = (-1, 0)', p_1 = (0, 1)', p_2 = (2, 2)', p_3 = (1, 0)', p_4 = (0, -1)'$.

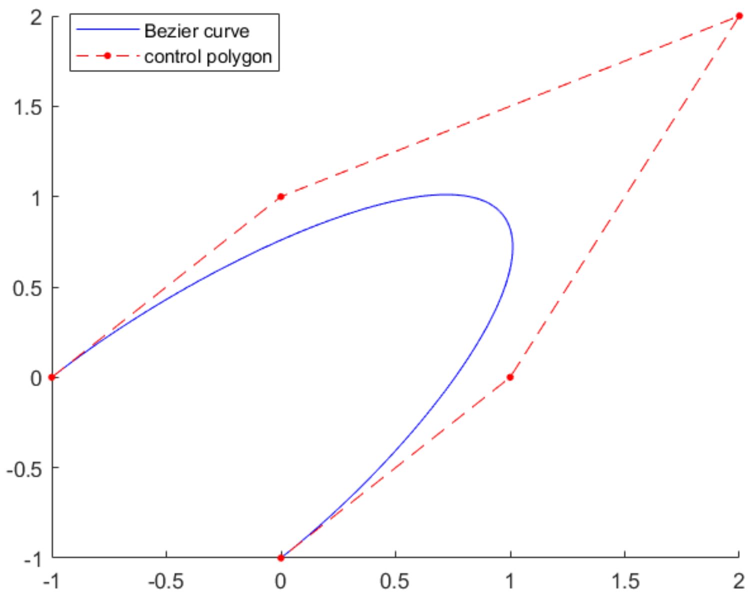

图 1: 2维Bézier曲线

- 3维：$p_0 = (-1, 0, 0)', p_1 = (0, 2, 1)', p_2 = (2, 2, 3)', p_3 = (1, 0, 2)', p_4 = (0, -1, 0)'$.

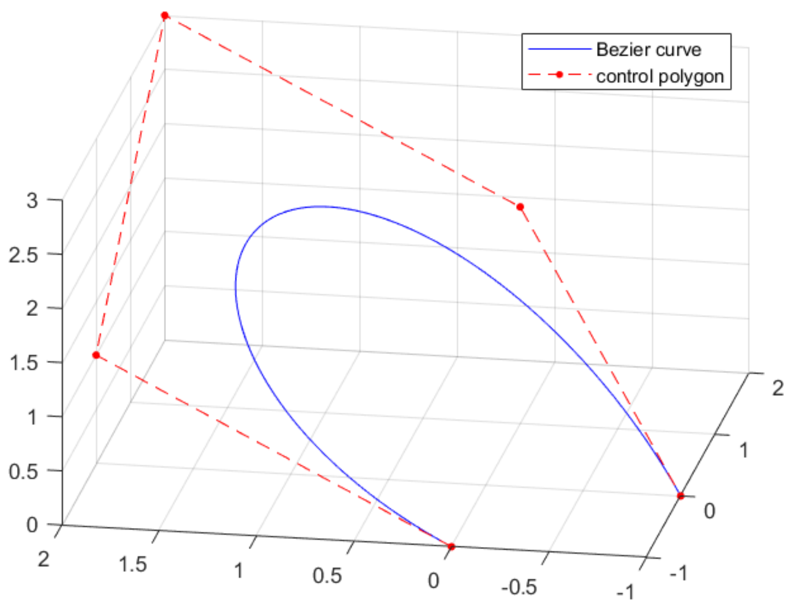

图 2: 3维Bézier曲线

2.

- 2维：$p_0 = (-1, 0)', p_1 = (0, 1)', p_2 = (3, 3)', p_3 = (1, 0)', p_4 = (0, -1)'$.

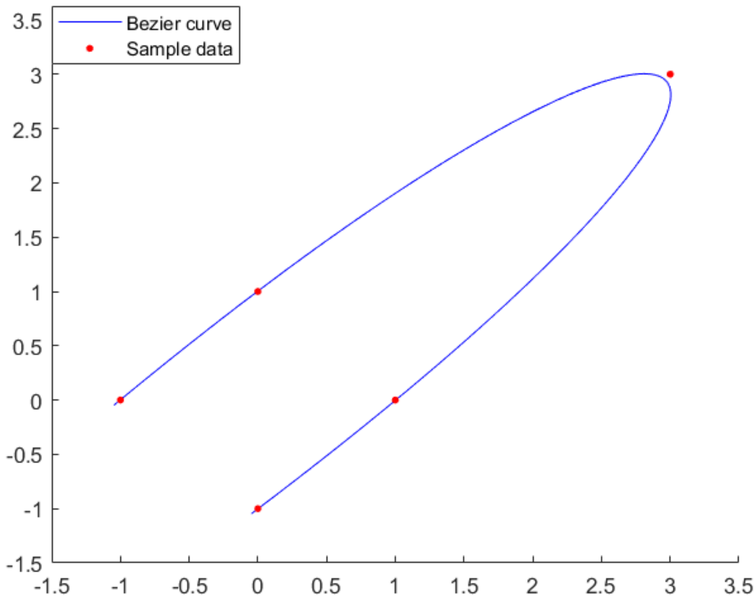

图 3: 2维Bézier曲线拟合

- 3维：$p_0 = (-1, 0, 0)', p_1 = (0, 2, 1)', p_2 = (3, 3, 3)', p_3 = (1, 0, 2)', p_4 = (0, -1, 0)'$.

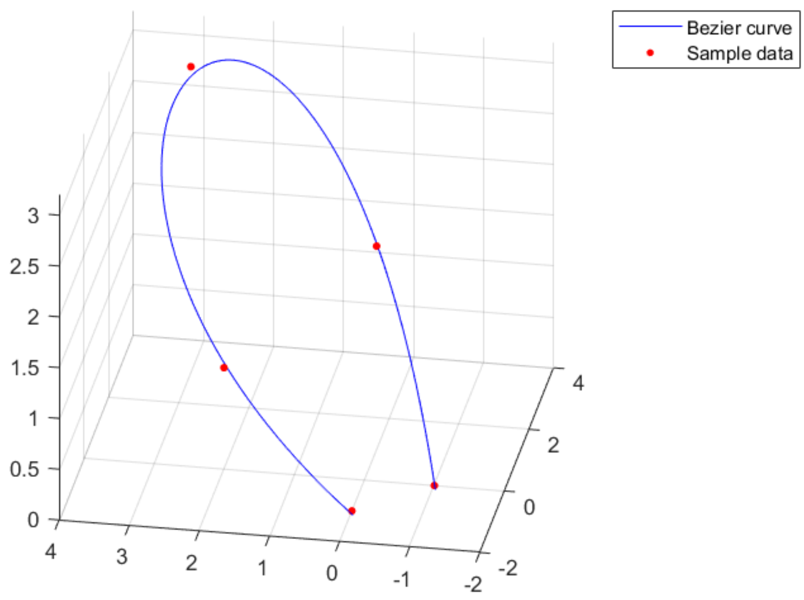

图 4: 3维Bézier曲线拟合