Exercise 1.

1.

$$\begin{aligned} v \wedge w &= (e_1 + 2e_2) \wedge (e_2 + 2e_3) \\ &= e_1 \wedge e_2 + 2e_1 \wedge e_3 + 2e_2 \wedge e_2 + 4e_2 \wedge e_3 \\ &= e_1 \wedge e_2 + 2e_1 \wedge e_3 + 4e_2 \wedge e_3. \end{aligned}$$

2.

$$w \wedge v = -(v \wedge w) = -e_1 \wedge e_2 - 2e_1 \wedge e_3 - 4e_2 \wedge e_3.$$

3.

$$v \wedge v = 0.$$

Exercise 2.

$$\begin{aligned} \alpha_0 \wedge \alpha_1 \wedge \alpha_2 &= (e_1 + e_2) \wedge (e_1 + 2e_2) \wedge (e_1 + 4e_2) \\ &= (e_1 \wedge e_2) \wedge (e_1 + 4e_2) \\ &= 0. \end{aligned}$$

The result is zero because 3-vectors can not exist in $\mathbb{R}^2$.

Exercise 3.

$$\begin{aligned} u \wedge v &= (e_1 + e_2 + e_3) \wedge (e_1 - e_2 + e_3) \\ &= -e_1 \wedge e_2 + e_1 \wedge e_3 - e_1 \wedge e_2 + e_2 \wedge e_3 - e_1 \wedge e_3 + e_2 \wedge e_3 \\ &= -2e_1 \wedge e_2 + 2e_2 \wedge e_3. \end{aligned}$$

$$\begin{aligned} u \times v &= (e_1 + e_2 + e_3) \times (e_1 - e_2 + e_3) \\ &= -2e_1 \times e_2 + 2e_2 \times e_3 \\ &= 2e_1 - 2e_3. \end{aligned}$$

The relation between $u \wedge v$ and $u \times v$ is given by

$$u \times v = *(u \wedge v).$$

Exercise 4.

1.

$$\begin{aligned} u \wedge v + v \wedge w &= u \wedge v - w \wedge v \\ &= (u - w) \wedge v \\ &= (-2e_1 - e_3) \wedge (e_1 - e_2 + 2e_3) \\ &= 2e_1 \wedge e_2 - 4e_1 \wedge e_3 + e_1 \wedge e_3 - e_2 \wedge e_3 \\ &= 2e_1 \wedge e_2 - 3e_1 \wedge e_3 - e_2 \wedge e_3. \end{aligned}$$

2.

$$\begin{aligned} (u \wedge v) \wedge w &= (e_1 + e_2 - e_3) \wedge (e_1 - e_2 + 2e_3) \wedge (3e_1 + e_2) \\ &= (-2e_1 \wedge e_2 + 3e_1 \wedge e_3 + e_2 \wedge e_3) \wedge (3e_1 + e_2) \\ &= 3e_1 \wedge e_3 \wedge e_2 + 3e_2 \wedge e_3 \wedge e_1 \\ &= 0. \end{aligned}$$

Exercise 5. (Hodge star in different dimensions.)

1. $$*e_1 = e_2.$$
2. $$*e_1 = e_2 \wedge e_3.$$
3. Since the Hodge star maps k-vectors to $(n-k)$-vectors, the dimension of the result changes with the dimension of the underlying space.

Exercise 6.

1. $$\begin{aligned} *\alpha &= *(e_1 + e_2 + e_3) = e_2 \wedge e_3 - e_1 \wedge e_3 + e_1 \wedge e_2 \\ *\beta &= *(e_1 - e_2 + 2e_3) = e_2 \wedge e_3 + e_1 \wedge e_3 + 2e_1 \wedge e_2. \end{aligned}$$
2. $$\begin{aligned} *(\alpha \wedge \beta) &= *(-2e_1 \wedge e_2 + e_1 \wedge e_3 + 3e_2 \wedge e_3) \\ &= -2e_3 - e_2 + 3e_1. \end{aligned}$$
3. $$\begin{aligned} (*\alpha) \wedge (*\beta) &= (e_2 \wedge e_3 - e_1 \wedge e_3 + e_1 \wedge e_2) \wedge (e_2 \wedge e_3 + e_1 \wedge e_3 + 2e_1 \wedge e_2) \\ &= 0. \end{aligned}$$
4. For (b), $\alpha \wedge \beta$ is a 2-vector, so $*(\alpha \wedge \beta)$ is a 1-vector. While for (c), since $*\alpha$ and $*\beta$ are both 2-vectors, so the wedge between them is a 4-vector and thus zero in $\mathbb{R}^3$.

Exercise 7. (Applying the Hodge star twice.)

1. if $n = 2$, let $w = ae_1 + be_2$, then we have
$$*(*w) = *(ae_2 - be_1) = -ae_1 - be_2 = -w.$$
In $\mathbb{R}^2$, the Hodge star means a quarter-rotation in the counter-clockwise direction, thus applying it twice results in a half-rotation, eqivalent to taking the negative.
2. if $n = 3$, let $w = ae_1 + be_2 + ce_3$, then we have
$$*(*w) = *(ae_2 \wedge e_3 + be_3 \wedge e_1 + ce_1 \wedge e_2) = ae_1 + be_2 + ce_3 = w.$$
3. For any $n \geq 2$, let $w = \sum_{i=1}^n a_i e_i$, then we have
$$\begin{aligned} *(*w) &= *(*(\sum_{i=1}^n a_i e_i)) \\ &= \sum_{i=1}^n a_i **e_i \\ &= \sum_{i=1}^n a_i (-1)^{n-1} e_i \\ &= (-1)^{n+1} \sum_{i=1}^n a_i e_i \\ &= (-1)^{n+1} w. \end{aligned}$$
4. $$*(*w) = (-1)^{k(n-k)} w.$$

Exercise 8. (Putting it all together.)

1.

$$\begin{aligned}\alpha \wedge (\beta + *\gamma) &= 2e_3 \wedge (e_1 - e_2 + e_1)\\ &= 2e_3 \wedge (2e_1 - e_2)\\ &= -4e_1 \wedge e_3 + 2e_2 \wedge e_3.\end{aligned}$$

2.

$$\begin{aligned}*(\gamma \wedge *(\alpha \wedge \beta)) &= *(e_2 \wedge e_3 \wedge *(-2e_1 \wedge e_3 + 2e_2 \wedge e_3))\\ &= *(e_2 \wedge e_3 \wedge (2e_2 + 2e_1))\\ &= 2.\end{aligned}$$

Exercise 9.

1.

$$\alpha = 2zdx + 3x^2dy + 5\cos(y)dz.$$

2. Obviously, at the point p, we have

$$\begin{aligned}\alpha(p) &= 6z + 6x^2 + 5\cos(y)\\ &= 18 + 6 + 5\cos 2\\ &= 24 + 5\cos 2.\end{aligned}$$

3.

$$-\alpha = -2zdx - 3x^2dy - 5\cos(y)dz.$$

Exercise 10.

1. $\alpha(U)$ is a scalar field whose value at each position is the projection of vector $(1, 2, x)$ onto vector $(0, x, 0)$.

2.

$$\alpha(U) = 2x, \quad \alpha(V) = x^2y, \quad \beta(U) = 1 + x, \quad \beta(V) = 4.$$

3.

$$(\alpha \wedge \beta)(U, V) = \alpha(U)\beta(V) - \alpha(V)\beta(U) = 8x - x^2y(1 + x) = -x^3y - x^2y + 8x.$$

4.

$$(\alpha \wedge \beta)(V, U) = -(\alpha \wedge \beta)(U, V) = x^3y + x^2y - 8x.$$

Exercise 11.

1.

$$\begin{aligned}(*[d(e^ydx + \sin(z)dz)]) \wedge dz &= (*[d(e^ydx) + d(\sin(z)dz)]) \wedge dz\\ &= (*[e^ydy \wedge dx + \cos(z)dz \wedge dz]) \wedge dz\\ &= *(-e^ydx \wedge dy) \wedge dz\\ &= -e^ydz \wedge dz\\ &= 0.\end{aligned}$$

2.

$$\begin{aligned}d[*(d(dx \wedge z^2dy)) + *(xyzdx \wedge dz \wedge dy)] &= d[*(2zdz \wedge dx \wedge dy) + *(-xyzdx \wedge dy \wedge dz)]\\ &= d(2z - xyz)\\ &= -yzdx - xzdy + (2 - xy)dz.\end{aligned}$$

Exercise 12. Coderivative.

1. If α is a differential 0-form on $\mathbb{R}^n$, then $*\alpha$ is a n-form and thus $d(*\alpha)$ is a $(n+1)$-norm on $\mathbb{R}^n$. Since the $(n+1)$-norm on $\mathbb{R}^n$ must be zero, we have $\delta\alpha = 0$.

2. If α is a differential k-form on $\mathbb{R}^n$, we know that $*\alpha$ is a $(n-k)$-form and thus $d(*\alpha)$ is a $(n-k+1)$-form. Finally, by the definition of Hodge star, $\delta\alpha$ is a differential $(n-(n-k+1))$-form, i.e. a $(k-1)$-form.

3. By the definition of $\delta\alpha$, we have

$$\begin{aligned}
\delta\alpha &= *(d(*(e^y dx + (x+y)^2 dy))) \\
&= *(d(e^y dy \wedge dz + (x+y)^2 dz \wedge dx)) \\
&= *((2x+2y) dx \wedge dy \wedge dz) \\
&= 2x + 2y.
\end{aligned}$$

Exercise 13. k-form Laplacian

1. when we apply the term $d*d*$ to a 0-form ϕ: $*\phi$ is an n-form, and so $d*\phi$ must be an $(n+1)$-form. But there are no $(n+1)$-form on an n-dimensional space. So this term is often omitted when writing the scalar Laplacian.

2.

$$\begin{aligned}
\Delta\phi &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \\
&= \frac{\partial}{\partial x}(\frac{\partial\phi}{\partial x}) + \frac{\partial}{\partial y}(\frac{\partial\phi}{\partial y}) \\
&= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x+4y) \\
&= 0 + 4 = 4.
\end{aligned}$$

3. By definition, we have

$$\begin{aligned}
\Delta\phi &= *d*d\phi \\
&= *d*(xdy + ydx + 4ydy) \\
&= *d(-xdx + ydy - 4ydx) \\
&= *(-4dy \wedge dx) \\
&= 4.
\end{aligned}$$

4.

$$\begin{aligned}
\Delta\alpha &= *d*d(xdx + zdy - ydz) + d*d*(xdx + zdy - ydz) \\
&= *d*(dz \wedge dy - dy \wedge dz) + d*d(xdy \wedge dz + zdz \wedge dx - ydx \wedge dy) \\
&= *d(-2dx) + d*(dx \wedge dy \wedge dz) \\
&= 0 + d(1) \\
&= 0.
\end{aligned}$$

Exercise 14. Exactness.

Proof. For single-variable calculus, we have $d(d(x^3)) = d(3x^2dx) = d(3x^2)\cdot dx + 3x^2 d(d(x)) = 6x(dx)^2$, thus $\frac{d^2}{dx^2}f = 6x$. While for exterior derivative, we know $d(d(\alpha)) = 0$ holds, regardless of the degree of the form α. For 0-form $a(x) = x^3$, $d(d(x^3)) = d(3x^2dx) = d(3x^2) \wedge dx = 6xdx \wedge dx = 0$.
These two expressions are different because they involve different types of derivatives (ordinary derivative vs. exterior derivative) applied to different kinds of mathematical objects (functions in single-variable calculus vs. forms in exterior calculus). □

Exercise 15. Integration practice.

1. Let's first compute the edge length L and unit tangent T:

$$L = |B - A| = \sqrt{2}, \qquad T = (B - A)/L = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}).$$

   Hence, $\alpha(T) = \sqrt{2} + \frac{\sqrt{2}}{2}x$.
   An arc-length parameterization of the edge is given by

$$p(s) = A + \frac{s}{L}(B - A), \quad s \in [0, L].$$

   By plugging in all these expressions, the integral simplies to

$$\hat{\alpha}(A, B) = \int_0^L \alpha(T)_{p(s)} ds = \int_0^L (\sqrt{2} + \frac{s}{2}) ds = \frac{5}{2}.$$

2. Similarly, it is obtained that

$$\hat{\alpha}(B, A) = \int_0^L \alpha(-T)_{p(s)} ds = -\frac{5}{2}.$$

3. These two discrete 1-forms have the same value with opposite signs, i.e. they add up to zero.

Exercise 16. Exactness.

Proof. Without loss of generality, let's consider a single triangle σ, whose vertices are v_1, v_2, v_3 and edges are e_1, e_2, e_3, respectively. Concretely, $e_1 = (v_1, v_2), e_2 = (v_2, v_3), e_3 = (v_3, v_1)$. And suppose ϕ is a discrete 0-form at vertices. Hence, $\hat{\alpha}_i = (d_0\phi)_{e_i} = \phi(v_{i+1}) - \phi(v_i), (i \mod 3)$, $(d_1\alpha)_\sigma = \sum_{i=1}^{3} \hat{\alpha}_i$. Therefore, we have

$$d_1 \circ d_0(\phi) = \sum_{i=1}^{3} \hat{\alpha}_i = (\phi(v_2) - \phi(v_1)) + (\phi(v_3) - \phi(v_2)) + (\phi(v_1) - \phi(v_3)) = 0,$$

which implies that $d_1 \circ d_0 = 0$. □

Exercise 17. Discrete operator practice.

1. *df* is a 1-form represented by numbers on oriented edges.

2. The domain of *df* are the values of f at vertices and the range of *df* are the values on oriented edges.

3. Let's first assign an index to each edge:

$$1 - (A, B), 2 - (A, D), 3 - (D, B), 4 - (B, C), 5 - (C, D).$$

   Then by the definition of discrete exterior derivative, we get that

$$df = \begin{bmatrix} -3 \\ 1 \\ -4 \\ 1 \\ 3 \end{bmatrix}.$$

4. From the conclusion of Exercise 16, we know that $d_1 \circ d_0 = 0$, thus

$$d(df) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Exercise 18. Discrete wedge practice.

1.

$$
\begin{aligned}
f \wedge_{0,0} h(A) &= f(A)h(A) = -15, \\
f \wedge_{0,0} h(B) &= f(B)h(B) = 0, \\
f \wedge_{0,0} h(C) &= f(C)h(C) = 6, \\
f \wedge_{0,0} h(D) &= f(D)h(D) = 18.
\end{aligned}
$$

2.

$$
\begin{aligned}
(df) \wedge_{1,0} h(A,B) &= df(A,B)\frac{h(A)+h(B)}{2} = \frac{9}{2}, \\
(df) \wedge_{1,0} h(A,D) &= df(A,D)\frac{h(A)+h(D)}{2} = 0, \\
(df) \wedge_{1,0} h(D,B) &= df(D,B)\frac{h(D)+h(B)}{2} = -6, \\
(df) \wedge_{1,0} h(B,C) &= df(B,C)\frac{h(B)+h(C)}{2} = 1, \\
(df) \wedge_{1,0} h(C,D) &= df(C,D)\frac{h(C)+h(D)}{2} = \frac{15}{2}.
\end{aligned}
$$

3.

$$
\begin{aligned}
[d((df) \wedge_{1,0} h)] \wedge_{2,0} h(A,D,B) &= [d((df) \wedge_{1,0} h)](A,D,B)\frac{h(A)+h(D)+h(B)}{3} = -\frac{21}{2} \times 0 = 0, \\
[d((df) \wedge_{1,0} h)] \wedge_{2,0} h(B,C,D) &= [d((df) \wedge_{1,0} h)](B,C,D)\frac{h(B)+h(C)+h(D)}{3} = \frac{5}{2} \times \frac{5}{3} = \frac{25}{6}.
\end{aligned}
$$

4. Similar in previous exercise, we obtain that

$$
dh = \begin{bmatrix} 3 \\ 6 \\ -3 \\ 2 \\ 1 \end{bmatrix}.
$$

So we can compute that

$$
\begin{aligned}
(df) \wedge_{1,1} (dh)(A,D,B) = \frac{1}{6}[&df(A,D)dh(D,B) - df(D,B)dh(A,D) \\
&+ df(D,B)dh(B,A) - df(B,A)dh(D,B) \\
&+ df(B,A)dh(A,D) - df(A,D)dh(B,A)] = \frac{21}{2}, \\
(df) \wedge_{1,1} (dh)(B,C,D) = \frac{1}{6}[&df(B,C)dh(C,D) - df(C,D)dh(B,C) \\
&+ df(C,D)dh(D,B) - df(D,B)dh(C,D) \\
&+ df(D,B)dh(B,C) - df(B,C)dh(D,B)] = -\frac{5}{2}.
\end{aligned}
$$

Exercise 19. Commutativity of d.

1.

$$
\hat{g}(A) = 2, \quad \hat{g}(B) = 3, \quad \hat{g}(C) = 0, \quad \hat{g}(D) = 0.
$$

2.

$$
dg = d[y^2(x+2y)] = 2ydy \wedge (x+2y) + y^2 \wedge d(x+2y) = y^2dx + (2xy+6y^2)dy.
$$

3. Similarly, we have

$$
d\hat{g} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -3 \\ 0 \end{bmatrix},
$$

where the entries are ordered in correspondence to the index of edges we have assigned previously.

4.

$$\int_{(A,B)} dg(1,0) = \int_0^1 y^2 dx = 1,$$
$$\int_{(A,D)} dg(0,-1) = \int_0^1 (-2xy - 6y^2) dy = -2,$$
$$\int_{(D,B)} dg(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = \int_0^{\sqrt{2}} \frac{9\sqrt{2}}{4} s^2 ds = 3,$$
$$\int_{(B,C)} dg(0,-1) = \int_0^1 (-2xy - 6y^2) dy = -3,$$
$$\int_{(C,D)} dg(-1,0) = \int_0^1 -y^2 dx = 0.$$

5. By Stoke's theorem, we have

$$\int_{(v_i,v_j)} dg = g(v_j) - g(v_i) = \hat{g}(v_j) - \hat{g}(v_i) = d\hat{g}(v_i, v_j).$$

Exercise 20. Matrix representations.

1. Let's assume that the vertices are ordered by A, B, C, D and order of edges is the same as that in previous exercise, then

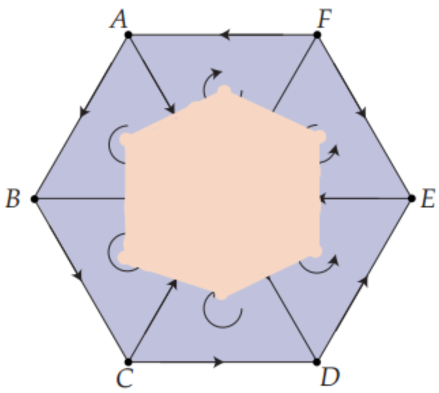
$$d_0 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

2. Assume the faces are ordered by $(A, D, B), (B, C, D)$, then

$$d_1 = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Exercise 21. Discrete Hodge star practice.

1. The dual mesh is as follows:



2. $*_0\alpha_0, *_1\alpha_1, *_2\alpha_2$ are dual 2-form, dual 1-form and dual 0-form, respectively.

3. By definition,

$$*_0\alpha_0(O^*) = |\text{Area}(O^*)|\alpha_0(O) = \frac{\sqrt{3}}{2} \times 7 = \frac{7\sqrt{3}}{2}.$$

4.

$$\begin{aligned}
*_1\alpha_1((A,O)^*) &= \frac{|\text{Length}((A,O)^*)|}{|\text{Length}((A,O))|}\alpha_1(A,O) = \frac{\sqrt{3}}{3}\times 2 = \frac{2\sqrt{3}}{3},\\
*_1\alpha_1((O,F)^*) &= \frac{|\text{Length}((O,F)^*)|}{|\text{Length}((O,F))|}\alpha_1(O,F) = \frac{\sqrt{3}}{3}\times (-2) = -\frac{2\sqrt{3}}{3},\\
*_1\alpha_1((E,O)^*) &= \frac{|\text{Length}((E,O)^*)|}{|\text{Length}((E,O))|}\alpha_1(E,O) = \frac{\sqrt{3}}{3}\times (-3) = -\sqrt{3},\\
*_1\alpha_1((O,D)^*) &= \frac{|\text{Length}((O,D)^*)|}{|\text{Length}((O,D))|}\alpha_1(O,D) = \frac{\sqrt{3}}{3}\times 1 = \frac{\sqrt{3}}{3},\\
*_1\alpha_1((C,O)^*) &= \frac{|\text{Length}((C,O)^*)|}{|\text{Length}((C,O))|}\alpha_1(C,O) = \frac{\sqrt{3}}{3}\times 3 = \sqrt{3},\\
*_1\alpha_1((O,B)^*) &= \frac{|\text{Length}((O,B)^*)|}{|\text{Length}((O,B))|}\alpha_1(O,B) = \frac{\sqrt{3}}{3}\times (-5) = -\frac{5\sqrt{3}}{3}.
\end{aligned}$$

5.

$$\begin{aligned}
*_2\alpha_2((A,B,O)^*) &= \frac{1}{|\text{Area((A,B,O))}|}\alpha_2((A,B,O)) = \frac{\sqrt{3}}{4}\times 3 = \frac{3\sqrt{3}}{4},\\
*_2\alpha_2((C,B,O)^*) &= \frac{1}{|\text{Area((C,B,O))}|}\alpha_2((C,B,O)) = \frac{\sqrt{3}}{4}\times (-2) = -\frac{\sqrt{3}}{2},\\
*_2\alpha_2((D,C,O)^*) &= \frac{1}{|\text{Area((D,C,O))}|}\alpha_2((D,C,O)) = \frac{\sqrt{3}}{4}\times 1 = \frac{\sqrt{3}}{4},\\
*_2\alpha_2((D,E,O)^*) &= \frac{1}{|\text{Area((D,E,O))}|}\alpha_2((D,E,O)) = \frac{\sqrt{3}}{4}\times 0 = 0,\\
*_2\alpha_2((E,F,O)^*) &= \frac{1}{|\text{Area((E,F,O))}|}\alpha_2((E,F,O)) = \frac{\sqrt{3}}{4}\times (-1) = -\frac{\sqrt{3}}{4},\\
*_2\alpha_2((A,F,O)^*) &= \frac{1}{|\text{Area((A,F,O))}|}\alpha_2((A,F,O)) = \frac{\sqrt{3}}{4}\times (-2) = -\frac{\sqrt{3}}{2}.
\end{aligned}$$

6.

$$*_1 = \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{\sqrt{3}}{3} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} \end{bmatrix}.$$

7.

$$*_2 = \begin{bmatrix} \frac{\sqrt{3}}{4} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{\sqrt{3}}{4} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{\sqrt{3}}{4} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{\sqrt{3}}{4} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{4} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{4} \end{bmatrix}.$$

EXERCISE 4.1

Proof. The Lagrange's indentity

$$(\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2) - (\sum_{i=1}^{n} a_i b_i)^2 = \sum_{1\le i<j\le n} (a_i b_j - a_j b_i)^2$$

asserts that if a and b are vectors in $\mathbb{R}^3$ with length $|a|$ and $|b|$, then Lagrange's indentity can be written in terms of cross product and dot product, i.e.

$$|a|^2|b|^2 - (a \cdot b)^2 = |a \times b|^2.$$

Hence, we have

$$\begin{aligned} |df(u) \times df(v)| &= \sqrt{|df(u)|^2|df(v)|^2 - (df(u) \cdot df(v))^2} \\ &= \sqrt{g(u,u)g(v,v) - g(u,v)^2} \\ &= \sqrt{\det(g)}. \end{aligned}$$

□

EXERCISE 4.2

Proof. The right-hand side of Green's theorem represents adding up the tangential components of X on the boundary $\partial\Omega$, i.e.

$$\int_{\partial\Omega} t \cdot X dl = \int_{\partial\Omega} X^b.$$

Apply the Stoke's theorem, we can obtain

$$\begin{aligned} \int_{\partial\Omega} t \cdot X dl &= \int_{\Omega} d(X^b) \\ &= \int_{\Omega} d(X_1 dx^1 + X_2 dx^2 + X_3 dx^3) \\ &= \int_{\Omega} \frac{\partial X_1}{\partial x^1} dx^1 \wedge dx^1 + \frac{\partial X_1}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial X_1}{\partial x_3} dx^3 \wedge dx^1 + \frac{\partial X_2}{\partial x^1} dx^1 \wedge dx^2 + \\ &\quad \frac{\partial X_2}{\partial x^2} dx^2 \wedge dx^2 + \frac{\partial X_2}{\partial x_3} dx^3 \wedge dx^2 + \frac{\partial X_3}{\partial x^1} dx^1 \wedge dx^3 + \frac{\partial X_3}{\partial x^2} dx^2 \wedge dx^3 + \frac{\partial X_3}{\partial x_3} dx^3 \wedge dx^3 \\ &= \int_{\Omega} (\frac{\partial X_3}{\partial x^2} - \frac{\partial X_2}{\partial x^3}) dx^2 \wedge dx^3 + (\frac{\partial X_1}{\partial x^3} - \frac{\partial X_3}{\partial x^1}) dx^3 \wedge dx^1 + (\frac{\partial X_2}{\partial x^1} - \frac{\partial X_1}{\partial x^2}) dx^1 \wedge dx^2 \\ &= \int_{\Omega} \nabla \times X dA, \end{aligned}$$

which shows that Stokes' theorem also implies Green's theorem. □