EXERCISE 2.1 Euler's Polyhedral Formula—Simplicial

Proof. First, get a trigulation of given simplicial disk by adding the diagonals, these diagonals are not crossed by each others. Note that adding a diagonal into polygonal disk will create a new face and a new edge, it dose not change the value of $V - E + F$.
Then, consider to remove the boundary triangles one by one.At each step we remove either one edge and one face, or two edges, a face, and a vertex. In both cases $V - E + F$ remains unchanged. Eventually we reach a graph formed by a single triangle at which point

$$V - E + F = 3 - 3 + 1 = 1.$$

For the case of simplicial sphere(polyhedron), we can remove one face and deform the rest into simplicial disk by pulling the edges of the missing face away from each other. For simplicial disk, it has $V - E + F = 1$ according to the above proof, so for the polyhedron, we have $V - E + F = 2$.

□

EXERCISE 2.2 Platonic Solids

Proof. The key is Euler's polyhedral formula that $V - E + F = 2$, and the fact that $pF = 2E = qV$, where p stands for the number of edges of each face and q for the number of edges meeting at each vertex. Combining these equations we obtain that

$$\frac{2E}{q} - E + \frac{2E}{p} = 2,$$

which induces that

$$\frac{1}{q} + \frac{1}{p} = \frac{1}{2} + \frac{1}{E}.$$

Since E is strictly positive we must have

$$\frac{1}{q} + \frac{1}{p} > \frac{1}{2}.$$

Using the fact that p and q must both be at least 3, we can easily see that there are only five possibilities for $\{p, q\}$:

$$\{3,3\}, \{4,3\}, \{3,4\}, \{5,3\}, \{3,5\}.$$

□

EXERCISE 2.3 Regular Valence

Proof. For simplicial surface, each face contains three vertices and three edges, each vertex is shared by six faces, and each edge is shared by two faces.So we have $6V = 2E = 3F$. Applying the Euler-Poincaré formula yields

$$V - E + F = \frac{1}{3}E - E + \frac{2}{3}E = 0,$$

which means the genus $g = 1$ and the simplicial surface is a torus. □

EXERCISE 2.4 Minimum Irregular Valence

Proof. Denote the number of irregular valence by n, and the valences of these n vertices are $p_1, p_2, \cdots, p_n$. So the number of faces F and the number of vertices V satisfy the following relation

$$6 * (V - n) + \sum_{i=1}^{n} p_i = 3F.$$

Apply the Euler-Poincaré formula, and we obtain

$$\begin{aligned} V - E + F &= 2 - 2g \\ \implies V - \frac{3}{2}F + F &= 2 - 2g \\ \implies V - \frac{1}{2}F &= 2 - 2g \\ \implies V - \frac{6*(V-n)+\sum_{i=1}^{n} p_i}{6} &= 2 - 2g \\ \implies n - \frac{\sum_{i=1}^{n} p_i}{6} &= 2 - 2g \\ \implies n &= 2 - 2g + \frac{\sum_{i=1}^{n} p_i}{6}. \end{aligned}$$

Since $p_i \geq 3$ holds for each $i = 1, \cdots, n$, we have $\frac{\sum_{i=1}^{n} p_i}{6} \geq \frac{n}{2}$ and thus

$$\begin{aligned} n &\geq 2 - 2g + \frac{n}{2} \\ \implies n &\geq 4 - 4g. \end{aligned}$$

Therefore, when $g = 0$, we have $n \geq 4$, i.e. $m(K) = 4$.
When $g = 1$, it can be obtained from **EXERCISE 2.3** that $m(K) = 0$.
When $g \geq 2$, there exists

$$n - \frac{\sum_{i=1}^{n} p_i}{6} \leq -2.$$

In this case, $m(K) = 1$ is just valid to satisfy the inequality. □

EXERCISE 2.5 Mean Valence (Triangle Mesh)

Proof. Each face contains three edges and each edge is shared by two faces, so it has $E : F = 3 : 2$. The relation between V and F can be written as

$$F = \frac{\sum_{i=1}^{V} p_i}{3},$$

where p_i is the valence of $i-th$ vertex.
By Euler-Poincaré formula, we have

$$\begin{aligned} V - E + F &= 2 - 2g \\ \implies V - \frac{F}{2} &= 2 - 2g \\ \implies V - \frac{\sum_{i=1}^{V} p_i}{6} &= 2 - 2g \\ \implies \frac{\sum_{i=1}^{V} p_i}{V} &= 6 - \frac{12-2g}{V}. \end{aligned}$$

So as $V \to \infty$, the mean valence $\frac{\sum_{i=1}^{V} p_i}{V} \to 6$, which implies $F = 2V$. Therefore, we have $V : E : F = 1 : 3 : 2$. □

EXERCISE 2.6 Mean Valence (Quad Mesh)

Proof. Similar to the previous exercise, we can easily obtain that $\sum_{i=1}^{V} p_i = 2E = 4Q$. Therefore, as the number of vertices approaches infinity, the limit of the ratio $V : E : Q = 1 : 2 : 1$. □

EXERCISE 2.7 Mean Valence (Tetrahedral)

Proof. TODO □

EXERCISE 2.8 Star, Closure, and Link

- St($\mathcal{S}$)
  $\{e\}, \{k\}, \{o\}$,
  $\{a, e\}, \{b, e\}, \{f, e\}, \{k, e\}, \{j, e\}, \{d, e\}, \{j, k\}, \{p, k\}, \{o, k\}, \{f, k\}, \{o, f\}, \{o, l\}, \{o, q\}, \{o, p\}$,
  $\{a, e, b\}, \{b, e, f\}, \{f, e, k\}, \{k, e, j\}, \{j, e, d\}, \{d, e, a\}, \{j, k, p\}, \{p, k, o\}, \{o, k, f\}, \{f, o, l\}, \{l, o, q\}, \{q, o, p\}, \{j, n, p\}$.

- Cl($\mathcal{S}$)
  $\{e\}, \{k\}, \{o\}, \{j\}, \{p\}, \{f\}, \{n\}$,
  $\{j, k\}, \{p, k\}, \{f, o\}, \{j, p\}, \{j, n\}, \{n, p\}$,
  $\{j, k, p\}, \{j, n, p\}$.

- Lk($\mathcal{S}$)
  $\{a\}, \{b\}, \{d\}, \{l\}, \{q\}$,
  $\{a, b\}, \{a, d\}, \{q, l\}$.

EXERCISE 2.9 Boundary and Interior

- bd($\mathcal{K}'$)
  $\{a\}, \{b\}, \{f\}, \{l\}, \{m\}, \{s\}, \{q\}, \{p\}, \{j\}, \{e\}$,
  $\{a, b\}, \{b, f\}, \{f, l\}, \{l, m\}, \{m, s\}, \{s, q\}, \{q, p\}, \{p, j\}, \{j, e\}, \{e, a\}$.

- int($\mathcal{K}'$)
  $\{k\}, \{o\}$,
  $\{b, e\}, \{e, f\}, \{e, k\}, \{f, k\}, \{f, o\}, \{k, j\}, \{k, o\}, \{k, p\}, \{o, l\}, \{o, p\}, \{o, q\}, \{l, q\}, \{m, q\}$,
  $\{a, b, e\}, \{b, e, f\}, \{e, f, k\}, \{e, k, j\}, \{k, f, o\}, \{j, k, p\}, \{k, p, o\}, \{f, o, l\}, \{o, p, q\}, \{o, l, q\}, \{l, m, q\}, \{q, m, s\}$.

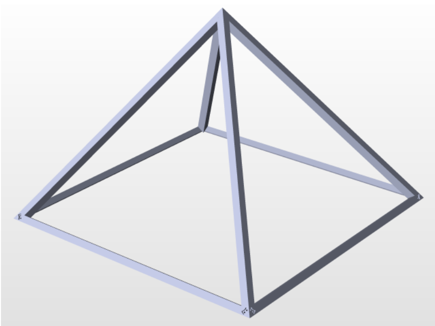
EXERCISE 2.10 Surface as Permutation

h	0	1	2	3	4	5	6	7	8	9
$\eta(h)$	4	2	1	5	0	3	7	6	9	8

h	0	1	2	3	4	5	6	7	8	9
$\rho(h)$	1	2	0	4	5	6	3	9	7	8

EXERCISE 2.11 Permutation as Surface

Proof. From the given permutation ρ, it can be easily obtained that there exist four triangles and one square, $\{0, 8, 7\}, \{1, 2, 14\}, \{3, 4, 12\}, \{5, 6, 10\}$ and $\{9, 15, 13, 11\}$, respectively. With the "twin" map η, we can assert that the combinatorial surface it describes is a square pyramid, where the permutation ρ is indeed a "next" map.



□

EXERCISE 2.12 Surface as Matrices

Proof.

$$A_0 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

□

EXERCISE 2.13 Classification of Simplicial 1-Manifolds

Proof. By definition, a simplicial 1-manifold is a pure simplicial 1-complex where the link of every vertex is a simplicial 0-sphere, that is, a pair of points.Therefore, it must contain paths of edges or closed loops of edges. □

EXERCISE 2.14 Boundary Loops

Proof. For any simplicial surface $\mathcal{K}$, it is composed of a collection of triangles with vertices and edges. Obviously, only the outer edges are proper faces of exactly one simplex of $\mathcal{K}$, whose closure connect to form a collection of closed loops. By the definition of boundary, so we know it always has to be a collection of closed loops. □

EXERCISE 2.15 Boundary Has No Boundary

Proof. Assume that $\mathcal{K}$ is a pure simplicial k-complex. Then we know that $\mathrm{bd}(\mathcal{K})$ is just a $(k-1)$-sphere, which is a $(k-1)$-complex. For those $(k-1)$-simplices in $\mathrm{bd}(\mathcal{K})$, they are not proper faces of any simplex of $\mathcal{K}$.While for those lower-degree simplices, they are shared by two or more simplices of $\mathcal{K}$. Therefore, the boundary $\mathrm{bd}(\mathcal{K})$ of a simplicial manifold has no boundary, i.e. $\mathrm{bd}(\mathrm{bd}(\mathcal{K})) = \varnothing$. □