1.18 A portfolio manager has maintained an actively managed portfolio with a beta of 0.2. During the last year, the risk-free rate was 5% and major equity indices performed very badly, providing returns of about −30%. The portfolio manager produced a return of −10% and claims that in the circumstances it was good. Discuss this claim.

$$\beta = 0.2, R_F = 0.05, R_M = -0.3,$$
$$then\ E(R_P) = R_F + \beta(R_M - R_F) = -0.02 = -2\%$$
$$\alpha = R_P - E(R_P) = -0.1 - (-0.02) = -0.08 = -8\%$$

5.31 The current price of a stock is $94, and three-month European call options with a strike price of $95 currently sell for $4.70. An investor who feels that the price of the stock will increase is trying to decide between buying 100 shares and buying 2,000 call options (= 20 contracts). Both strategies involve an investment of $9,400. What advice would you give? How high does the stock price have to rise for the option strategy to be more profitable?

The strategies are equally profitable if the stock price rises to a level, S, where

$$100 \times (S - 94) = 2000 \times (S - 95) - 9400 \Rightarrow S = 100$$

The option strategy is therefore more profitable if the stock price rises above $100

Chapter 7 How Traders Manage Their Risks

DELTA:

$$Delta = \frac{\Delta P}{\Delta S} = \frac{\partial P}{\partial S}$$

> ΔS is a small increase in the value of the variable and ΔP is the resulting change in the value of the portfolio.
>
> When the hedging trade is combined with the existing portfolio the resultant portfolio has a delta of zero. –> delta neutral

Forward contracts are linear products; options are not.

Linear Products: hedges protect against large changes as well as small ones in the value of the underlying asset. | The hedge, once it has been set up, never needs to be changed. –> the hedge and forget property

Nonlinear Products: making a nonlinear portfolio delta neutral only protects against small movements in the price of the underlying asset. | not in a hedge-and-forget situation. The hedge needs to be changed frequently –> dynamic hedging

To preserve delta neutrality, the hedge has to be adjusted periodically. –> rebalancing

delta neutrality only provides protection against small changes in the price of the underlying asset

GAMMA: measures the extent to which large changes cause problems. Gamma measures curvature

$$\Gamma = \frac{\partial^2 P}{\partial S^2}$$

Gamma is greatest for options where the stock price is close to the strike price K

Suppose the a delta-neutral portfolio has a gamma equal to Γ, and a traded option has a gamma equal to Γ_T. If the number of traded options added to the portfolio is w_T, the gamma of the portfolio is

$$w_T\Gamma_T + \Gamma$$

Including the traded option is likely to change the delta of the portfolio, so the position in the underlying asset then has to be changed to maintain delta neutrality. The portfolio is gamma neutral only for a short period of time. As time passes, gamma neutrality can be maintained only if the position in the traded option is adjusted so that it is always equal to $-\Gamma/\Gamma_T$.

VEGA: Spot positions and forwards do not depend on the volatility of underlying asset prices, but options and more complicated derivatives do. The vega of a long position in an option is positive.

$$V = \frac{\partial P}{\partial \sigma}$$

If V is the vega of the portfolio and V_T is the vage of a traded option. If the number of traded options added to the portfolio is w_T, the vega of the portfolio is

$$w_T V_T + V$$

Unfortunately, a portfolio that is gamma neutral will not, in general, be vega neutral, and vice versa. If a hedger requires a portfolio to be both gamma and vega neutral, at least two traded derivatives dependent on the underlying asset must usually be used.

The volatilities of short-dated options tend to be more variable than the volatilities of long-dated options.

Example 7.1:
Consider a portfolio dependent on the price of a single asset that is delta neutral, with a gamma of −5,000 and a vega of −8,000.

	Delta	Gamma	Vega
Portfolio	0	-5,000	-8,000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

To make the portfolio gamma and vega neutral, both Option 1 and Option 2 can be used. If w_1 and w_2 are the quantities of Option 1 and Option 2 that are added to the portfolio, we require that

$$\begin{gathered}-5000 + 0.5w_1 + 0.8w_2 = 0\\ -8000 + 2.0w_1 + 1.2w_2 = 0\\ \Rightarrow w_1 = 400, w_2 = 6000\end{gathered}$$

The portfolio can therefore be made gamma and vega neutral by including 400 of Option 1 and 6,000 of Option 2. The delta of the portfolio after the addition of the positions in the two traded options is $400 \times 0.6 + 6,000 \times 0.5 = 3,240$. Hence, 3,240 units of the underlying asset would have to be sold to maintain delta neutrality.

7.17 A financial institution has the following portfolio of over-the-counter options on sterling:

Type	Position	Delta of Option	Gamma of Option	Vega of Option
Call	-1,000	0.50	2.2	1.8
Call	-500	0.80	0.6	0.2
Put	-2,000	-0.40	1.3	0.7
Call	-500	0.7	1.8	1.4

$$\begin{gathered}Delta = -1000 \times 0.50 - 500 \times 0.80 - 2000 \times (-0.40) - 500 \times 0.7 = -450\\ \Gamma = -1000 \times 2.2 - 500 \times 0.6 - 2000 \times 1.3 - 500 \times 1.8 = -6000\\ V = -1000 \times 1.8 - 500 \times 0.2 - 2000 \times 0.7 - 500 \times 1.4 = -4000\end{gathered}$$

A traded option is available with a delta of 0.6, a gamma of 1.5, and a vega of 0.8.

(a) What position in the traded option and in sterling would make the portfolio both gamma neutral and delta neutral?

A long position in $-\Gamma/\Gamma_T = 6000/1.5 = 4000$ traded options will give a gamma-neutral portfolio. The delta of the whole portfolio after the addition of traded options is then: $-450 + 4000 \times 0.6 = 1950$.
Hence, in addition to the 4000 traded options, a short position in £1,950 is necessary so that the portfolio is both gamma and delta neutral.

(b) What position in the traded option and in sterling would make the portfolio both vega neutral and delta neutral?

A long position in $-V/V_T = 4000/0.8 = 5000$ traded options will give a vega-neutral portfolio. The delta of the whole portfolio after the addition of traded options is then: $-450 + 5000 \times 0.6 = 2550$.
Hence, in addition to the 5000 traded options, a short position in £2,550 is necessary so that the portfolio is both vega and delta neutral.

7.18 Consider again the situation in Problem 7.17. Suppose that a second traded option with a delta of 0.1, a gamma of 0.5, and a vega of 0.6 is available. How could the portfolio be made delta, gamma, and vega neutral?

Let w_1 be the position in the first traded option and w_2 be the position in the second traded option:

$$
\begin{aligned}
1.5w_1 + 0.5w_2 - 6000 = 0 \\
0.8w_1 + 0.6w_2 - 4000 = 0 \\
\Rightarrow w_1 = 3200, w_2 = 2400
\end{aligned}
$$

The whole portfolio then has a delta of $3200 \times 0.6 + 2400 \times 0.1 - 450 = 1710$. Therefore, the portfolio can be made delta, gamma and vega neutral by taking a long position in 3200 of the first traded option, a long position in 2400 of the second traded option and a short position in £1,710.

Chapter 8 Interest Rate Risk

Suppose y is a bond's yield and B is its market price.

$$D = -\frac{1}{B}\frac{\Delta B}{\Delta y}$$

where Δy is a small change in the bond's yield and ΔB is the corresponding change in its price.

Duration measures the sensitivity of percentage changes in the bond's price to changes in its yield.

Consider a bond that provides cash flows $c_1, c_2, \cdots, c_n$ at times $t_1, t_2, \cdots, t_n$. (The cash flows consist of the coupon and principal payments on the bond.) The bond yield, y, is defined as the discount rate that equates the bond's theoretical price to its market price. We denote the present value of the cash flow c_i, discounted from time t_i to today at rate y, by v_i so that the price of the bond is:

$$
\begin{aligned}
B = \sum_{i=1}^{n} v_i = \sum_{i=1}^{n} c_i e^{-yt_i} \\
\Rightarrow D = \sum_{i=1}^{n} t_i \frac{c_i e^{-yt_i}}{B} = \sum_{i=1}^{n} t_i \frac{v_i}{B}
\end{aligned}
$$

Duration is a weighted average of the times when payments are made, with the weight applied to time t_i being equal to the proportion of the bond's total present value provided by the cash flow at time t_i. Duration is a measure of how long the bondholder has to wait for cash flows.

Duration relates proportional changes in a bond's price to its yield.

basis points: A basis point is 0.01% per annum.

Consider a three-year 10% coupon bond with a face value of $100. Suppose that the yield on the bond is 12% per annum with continuous compounding.

Time(years)	Cash Flow($)	Present Value	Weight	Time × Weight
0.5	5	4.709	0.050	0.025
1.0	5	4.435	0.047	0.047
1.5	5	4.176	0.044	0.066
2.0	5	3.933	0.042	0.083
2.5	5	3.704	0.039	0.098
3.0	105	73.256	0.778	2.333
Total	130	94.213	1.000	2.653

Example 8.1

For the bond in Table, the bond price, B, is 94.213 and the duration, D, is 2.653 so that equation (9.1) gives:

$$\Delta B = -94.213 \times 2.653 \Delta y = -249.95 \Delta y$$

When the yield on the bond increases by 10 basis points (= 0.1%), $\Delta y = +0.001$. The duration relationship predicts that $\Delta B = -249.95 \times 0.001 = -0.250$ so that the bond price goes down to $94.213 - 0.250 = 93.963$.

When the bond yield increase by 10 basis points to 12.1%, the bond price is:

$$5e^{-0.121\times0.5} + 5e^{-0.121\times1.0} + 5e^{-0.121\times1.5} + 5e^{-0.121\times2.0} + 5e^{-0.121\times2.5} + 105e^{-0.121\times3.0} = 93.963$$

which is (to three decimal places) the same as that predicted by the duration relationship.

Example 8.3

Modified Duration: More generally, when y is expressed with a compounding frequency of m times per year, it must be divided by $1 + y/m$.

$$D_{modified} = \frac{D}{1 + y/m}$$

The bond in Table has a price of 94.213 and a duration of 2.653. The yield, expressed with semiannual compounding is 12.3673%. The (modified) duration appropriate for calculating sensitivity to the yield when it is expressed with semiannual compounding is:

$$D_{modified} = \frac{D}{1 + y/m} = \frac{2.653}{1 + 0.123673/2} = 2.4985$$
$$\Rightarrow \Delta B = -BD_{modified} \Delta y = -94.213 \times 2.4985 \Delta y = -235.39 \Delta y$$

When the yield (semiannually compounded) increases by 10 basis points (= 0.1%), $\Delta y = +0.001$. The duration relationship predicts that we expect ΔB to be $-235.39 \times 0.001 = -0.235$ so that the bond price goes down to $94.213 - 0.235 = 93.978$.

When the bond yield (semiannually compounded) increases by 10 basis points to 12.4673% (or to 12.0941% with continuous compounding), an exact calculation similar to that in the previous example shows that the bond price becomes 93.978. This shows that the modified duration is accurate for small yield changes.

Dollar Duration: $D_\$ = BD \Rightarrow \Delta B = -D_\$ \Delta y, D_\$ = -\frac{dB}{dy}$.

dollar duration relates actual changes in the bond's price to its yield

8.16

Portfolio A consists of a one-year zero-coupon bond with a face value of $2,000 and a 10-year zero-coupon bond with a face value of $6,000. Portfolio B consists of a 5.95-year zero-coupon bond with a face value of $5,000. The current yield on all bonds is 10% per annum (continuously compounded)

(a) Show that both portfolios have the same duration.

The duration of Portfolio A is:

$$B_A = 2000e^{-0.1\times1} + 6000e^{-0.1\times10} = 4016.95$$
$$D_A = \frac{1\times 2000e^{-0.1\times1} + 10\times 6000e^{-0.1\times10}}{B_A} = 5.95$$

As Portfolio B consists of a 5.95-year zero-coupon bond with a face value of $5,000, the duration of portfolio B is 5.95 as well, i.e. the two portfolios do have the same duration.

(b) Show that the percentage changes in the values of the two portfolios for a 0.1% per annum increase in yields are the same.

The value of Portfolio A is $B_A = 4016.95$. When yields increase by 10 basis points, its value becomes: $B'_A = 2000e^{-0.101\times1} + 6000e^{-0.101\times10} = 3993.18$

The percentage change in the value of Portfolio A is: $\frac{B'_A - B_A}{B_A} \times 100\% = -0.59\%$

The value of Portfolio B is $B_B = 5000e^{-0.1\times5.95} = 2757.81$. When yields increase by 10 basis points, its value becomes $B'_B = 5000e^{-0.101\times5.95} = 2741.45$.

The percentage change in the value of Portfolio B is: $\frac{B'_B - B_B}{B_B} \times 100\% = -0.59\%$

The percentage changes in the values of the two portfolios for a 0.1% per annum increase in yields are the same.

(c) What are the percentage changes in the values of the two portfolios for a 5% per annum increase in yields?

When yields increase by 5%, the values of the Portfolio A becomes: $B''_A = 2000e^{-0.15\times1} + 6000e^{-0.15\times10} = 3060.20$, and the value of Portfolio B becomes $B''_B = 5000e^{-0.15\times5.95} = 2048.15$.

The percentage reductions in the values of the two portfolios are:
Portfolio A: $(B''_A - B_A)/B_A \times 100\% = -23.82\%$.
Portfolio B: $(B''_B - B_B)/B_B \times 100\% = -25.73\%$

> The duration relationship measures exposure to small changes in yields.

Convexity:

$$C = \frac{1}{B}\frac{d^2B}{dy^2} = \frac{\sum_{i=1}^{n} c_i t_i^2 e^{-yt_i}}{B}$$

where y is the bond's yield measured with continuous compounding. This is the weighted average of the square of the time to the receipt of cash flows.

A second order approximation to the change in the bond price is:

$$\Delta B = \frac{dB}{dy}\Delta y + \frac{1}{2}\frac{d^2B}{dy^2}\Delta y^2$$
$$\Rightarrow \frac{\Delta B}{B} = -D\Delta y + \frac{1}{2}C(\Delta y)^2$$

Dollar Convexity: $C_\$ = CB \Rightarrow C_\$ = \frac{d^2B}{dy^2}$

8.17 What are the convexities of the portfolios in Problem 8.16? To what extent does (a) duration and (b) convexity explain the difference between the percentage changes calculated in part (c) of Problem 8.16?

$$C_A = \frac{1^2\times 2000e^{-0.1\times1} + 10^2\times 6000e^{-0.1\times10}}{B_A} = 55.40$$
$$C_B = 5.95^2 = 35.4025$$

Because the two portfolios do have the same duration, i.e. $D_A = D_B$, The percentage changes in the two portfolios predicted by the duration measure $\frac{\Delta B}{B} = -D\Delta y$, is the same and equal to -0.2975.

However, the convexity measure predicts that the percentage change in Portfolio A will be $\frac{\Delta B_A}{B_A} = -D_A\Delta y + \frac{1}{2}C_A(\Delta y)^2 = -0.228$ and that for Portfolio B will be $\frac{\Delta B_B}{B_B} = -D_B\Delta y + \frac{1}{2}C_B(\Delta y)^2 = -0.253$

Duration does not explain the difference between the percentage changes. Convexity explains part of the difference. 5% is such a big shift in the yield curve that even the use of the convexity relationship does not give accurate results. Better results would be obtained if a measure involving the third partial derivative with respect to a parallel shift, as well as the first and second, was considered.

Chapter 9 Value at Risk

Example 9.1
Suppose that the gain from a portfolio during six months is normally distributed with a mean of \$2million and a standard deviation of \$10million. From the properties of the normal distribution, the one-percentile point of this distribution is $2 - 2.326 \times 10$ or –\$21.3 million. The VaR for the portfolio with a time horizon of six months and confidence level of 99% is therefore \$21.3 million.

> VaR 99% 2.326

Example 9.2
Suppose that for a one-year project all outcomes between a loss of \$50 million and a gain of \$50 million are considered equally likely. In this case, the loss from the project has a uniform distribution extending from –\$50 million to +\$50 million. There is a 1% chance that there will be a loss greater than \$49 million. The VaR with a one-year time horizon and a 99% confidence level is therefore \$49 million.

Example 9.3
A one-year project has a 98% chance of leading to a gain of \$2 million, a 1.5% chance of leading to a loss of \$4 million and a 0.5% chance of leading to a loss of \$10 million. The cumulative loss distribution is shown in Figure 12.3. The point on this cumulative distribution that corresponds to a cumulative probability of 99% is \$4 million. It follows that VaR with a confidence level of 99% and a one-year time horizon is \$4 million.

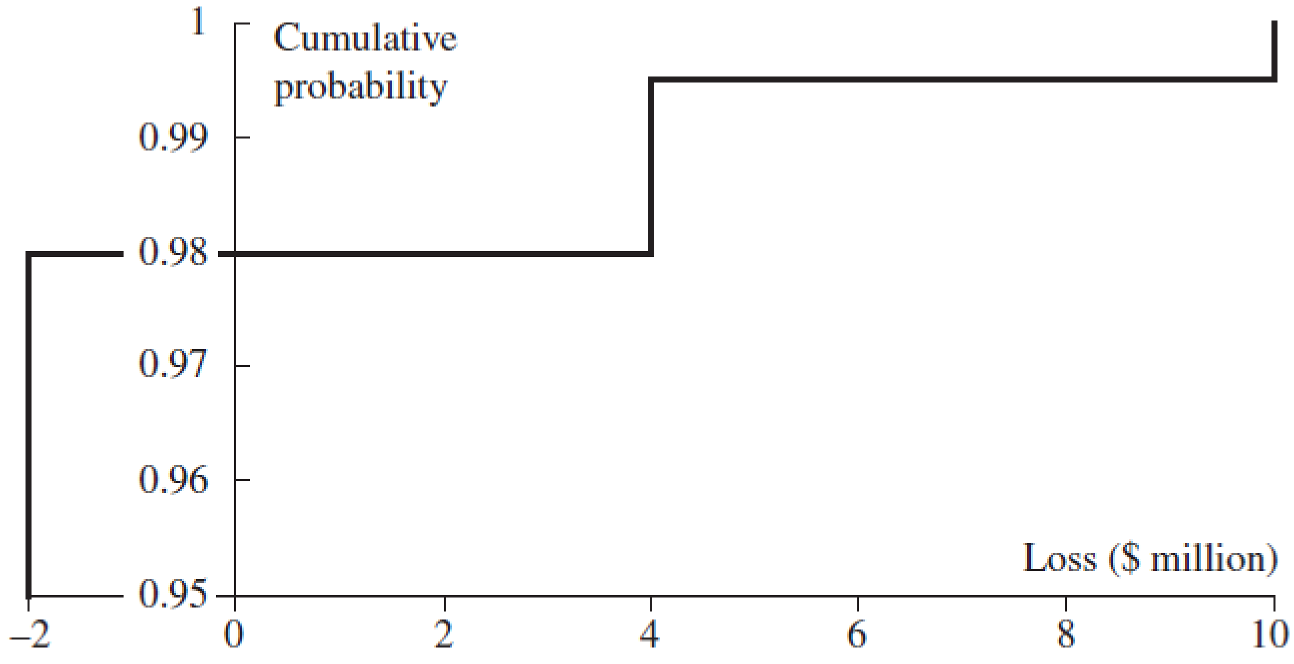

FIGURE 12.3 Cumulative Loss Distribution for Examples 12.3 and 12.4

Example 9.4
Consider again the situation in Example 9.3. Suppose that we are interested in calculating a VaR using a confidence level of 99.5%. In this case, Figure 12.3 shows that all losses between \$4 and \$10 million have a probability of 99.5% of not being exceeded. Equivalently, there is a probability of 0.5% of any specified loss level between \$4 and \$10 million being exceeded. VaR is therefore not uniquely defined. <u>One reasonable convention in this type of situation is to set VaR equal to the midpoint of the range of possible VaR values.</u> This means that, in this case, VaR would equal \$7 million.

Expected Shortfall: **ES**, conditional VaR.
It is the expected loss during time T conditional on the loss being greater than the Xth percentile of the loss distribution.

---

1. Monotonicity: If a portfolio produces a worse result than another portfolio for every state of the world, its risk measure should be greater.
2. Translation Invariance: If an amount of cash K is added to a portfolio, its risk measure should go down by K.
3. Homogeneity: Changing the size of a portfolio by a factor λ while keeping the relative amounts of different items in the portfolio the same, should result in the risk measure being multiplied by λ.
4. Subadditivity: The risk measure for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged.

Risk measures satisfying all four conditions given above are referred to as coherent.
VaR is not coherent. ES measure is always coherent.

Example 9.5
Suppose each of two independent projects has a probability of 0.02 of a loss of $10 million and a probability of 0.98 of a loss of $1 million during a one-year period. The one-year, 97.5% VaR for each project is $1 million. When the projects are put in the same portfolio, there is a 0.02 × 0.02 = 0.0004 probability of a loss of $20 million, a 2 × 0.02 × 0.98 = 0.0392 probability of a loss of $11 million, and a 0.98 × 0.98 = 0.9604 probability of a loss of $2 million. The one-year 97.5% VaR for the portfolio is $11 million. The total of the VaRs of the projects considered separately is $2 million. The VaR of the portfolio is therefore greater than the sum of the VaRs of the projects by $9 million. This violates the subadditivity condition.

Example 9.6

A bank has two $10 million one-year loans. The probabilities of default are as indicated in the following table.

Outcome	**Probability**
Neither loan defaults	97.50%
Loan 1 defaults; Loan 2 does not default	1.25%
Loan 2 defaults; Loan 1 does not default	1.25%
Both loans default	0.00%

If a default occurs, all losses between 0% and 100% of the principal are equally likely. If the loan does not default, a profit of $0.2 million is made.

Consider first Loan 1. This has a 1.25% chance of defaulting. When a default occurs the loss experienced is evenly distributed between zero and $10 million. This means that there is a 1.25% chance that a loss greater than zero will be incurred; there is a 0.625% chance that a loss greater than $5 million is incurred; there is no chance of a loss greater than $10 million. The loss level that has a probability of 1% of being exceeded is $2 million. (Conditional on a loss being made, there is an 80% or 0.8 chance that the loss will be greater than $2 million. Because the probability of a loss
is 1.25% or 0.0125, the unconditional probability of a loss greater than $2 million
is 0.8 × 0.0125 = 0.01 or 1%.) The one-year 99% VaR is therefore $2 million. The same applies to Loan 2.

Consider next a portfolio of the two loans. There is a 2.5% probability that a default will occur. As before, the loss experienced on a defaulting loan is evenly distributed between zero and $10 million. The VaR in this case turns out to be $5.8 million. This is because there is a 2.5% (0.025) chance of one of the loans defaulting and conditional on this event is a 40% (0.4) chance that the loss on the loan that defaults is greater than $6 million. The unconditional probability of a loss from a default being greater than $6 million is therefore 0.4 × 0.025 = 0.01 or 1%. In the event that one loan defaults, a profit of $0.2 million is made on the other loan, showing that the one-year 99% VaR is $5.8 million.

The total VaR of the loans considered separately is $2+2=4$ million. The total VaR after they have been combined in the portfolio is $1.8 million greater at $5.8 million. This shows that the subadditivity condition is violated. (This is in spite of the fact that there are clearly very attractive diversification benefits from combining the loans into a single portfolio—particularly because they cannot default together.)

Example 9.7

Consider again the situation in Example 9.5. The VaR for one of the projects considered on its own is $1 million. To calculate the ES for a 97.5% confidence level we note that, of the 2.5% tail of the loss distribution, 2% corresponds to a $10 million loss and 0.5% to a $1 million loss. (Note that the other 97.5% of the distribution also corresponds to a loss of $1 million.)

Conditional that we are in the 2.5% tail of the loss distribution, there is therefore an 80% probability of a loss of $10 million and a 20% probability of a loss of $1 million. The expected loss is $0.8\times10+0.2\times1$ or $8.2 million.

When the two projects are combined, of the 2.5% tail of the loss distribution, 0.04% corresponds to a loss of $20 million and 2.46% corresponds to a loss of $11 million. Conditional that we are in the 2.5% tail of the loss distribution, the expected loss is therefore $(0.04/2.5)\times20+(2.46/2.5)\times11$ or $11.144 million. This is the ES.

Because $8.2+8.2>11.144$, the ES measure does satisfy the subadditivity condition for this example.

Example 9.8

Consider again the situation in Example 9.6. We showed that the VaR for a single loan is $2 million. The ES from a single loan when the time horizon is one year and the confidence level is 99% is therefore the expected loss on the loan conditional on a loss greater than $2 million. Given that losses are uniformly distributed between zero and $10 million, the expected loss conditional on a loss greater than $2 million is halfway between $2 million and $10 million, or $6 million.

The VaR for a portfolio consisting of the two loans was calculated in Example 9.6 as $5.8 million. The ES from the portfolio is therefore the expected loss on the portfolio conditional on the loss being greater than $5.8 million. When one loan defaults, the other (by assumption) does not and outcomes are uniformly distributed between a gain of $0.2 million and a loss of $9.8 million. The expected loss, given that we are in the part of the distribution between $5.8 million and $9.8 million, is $7.8 million. This is therefore the ES of the portfolio. Because $7.8 million is less than 2 × $6 million, the ES measure does satisfy the subadditivity condition.

9.12

Suppose that each of two investments has a 4% chance of a loss of $10 million, a 2% chance of a loss of $1 million, and a 94% chance of a profit of $1 million. They are independent of each other.

(a) What is the VaR for one of the investments when the confidence level is 95%?

A loss of $1 million extends from the 94% point of the loss distribution to the 96% point. The 95% VaR is therefore $1 million.

(b) What is the expected shortfall when the confidence level is 95%?

$$ES=1\%/5\%\times1+4\%/5\%\times10=8.2$$

(c) What is the VaR for a portfolio consisting of the two investments when the confidence level is 95%?

Outcome(loss)	**Probability**
20	$0.04\times0.04=0.0016$
11	$2\times0.04\times0.02=0.0016$
9	$2\times0.04\times0.94=0.0752$
2	$0.02\times0.02=0.0004$
0	$2\times0.02\times0.94=0.0376$
-2	$0.94\times0.94=0.8836$

Therefore, the 95% VaR is $9 million.

(d) What is the expected shortfall for a portfolio consisting of the two investments when the confidence level is 95%?

$$0.0016/0.05 \times 20 + 0.0016/0.05 \times 11 + (0.05 - 0.0016 - 0.0016)/0.05 \times 9 = 9.416$$

(e) Show that, in this example, VaR does not satisfy the subadditivity condition whereas expected shortfall does.

VaR does not satisfy the subadditivity condition because $9 > 1 + 1$; expected shortfall does because $9.416 < 8.2 + 8.2$

$$\sigma^2_{\sum_{i=1}^{T} \Delta P_i} = \sigma^2[T + 2(T-1)\rho + 2(T-2)\rho^2 + \cdots + 2\rho^{T-1}]$$

9.13

Suppose that daily changes for a portfolio have first-order correlation with correlation parameter 0.12. The 10-day VaR, calculated by multiplying the one-day VaR by $\sqrt{10}$, is \$2 million. What is a better estimate of the VaR that takes account of autocorrelation?

$$2 \times \frac{\sqrt{10 + 2 \times 9 \times 0.12 + 2 \times 8 \times 0.12^2 + \cdots + 2 \times 1 \times 0.12^9}}{\sqrt{10}} = 2 \times \sqrt{10.417}/\sqrt{10} = 2.229$$

Chapter 10 Volatility

EWMA:

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$$

GARCH(1,1):

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2, \gamma + \alpha + \beta = 1$$
$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2, \omega = \gamma V_L, \alpha + \beta < 1$$

10.19

Suppose that the price of an asset at close of trading yesterday was \$300 and its volatility was estimated as 1.3% per day. The price at the close of trading today is \$298. Update the volatility estimate using

$$u_{n-1} = \frac{S_{n-1} - S_{n-2}}{S_{n-2}} = -0.00667, \sigma_{n-1} = 0.013$$

(a) The EWMA model with $\lambda = 0.94$

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2 = 0.94\sigma_{n-1}^2 + 0.06u_{n-1}^2 = 0.00016153$$
$$\Rightarrow \sigma_n = 0.01271$$

(b) The GARCH(1,1) model with $\omega = 0.000002, \alpha = 0.04, \beta = 0.94.$

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2 = 0.000002 + 0.04u_{n-1}^2 + 0.94\sigma_{n-1}^2 = 0.00016264$$
$$\Rightarrow \sigma_n = 0.01275$$

Chapter 11 Correlations and Copulas

EWMA:

$$cov_n = \lambda cov_{n-1} + (1-\lambda)x_{n-1}y_{n-1}$$

GARCH(1,1):

$$cov_n = \omega + \alpha x_{n-1}y_{n-1} + \beta cov_{n-1}$$

11.16 Suppose that the price of Asset X at close of trading yesterday was $300 and its volatility was estimated as 1.3% per day. The price of X at the close of trading today is $298. Suppose further that the price of Asset Y at the close of trading yesterday was $8, its volatility was estimated as 1.5% per day, and its correlation with X was estimated as 0.8. The price of Y at the close of trading today is unchanged at $8. Update the volatility of X and Y and the correlation between X and Y using

$$x_{n-1} = \frac{X_{n-1} - X_{n-2}}{X_{n-2}} = -0.00667, \sigma_{x,n-1} = 0.013,$$
$$y_{n-1} = \frac{Y_{n-1} - Y_{n-2}}{Y_{n-2}} = 0, \sigma_{y,n-1} = 0.015,$$
$$\rho_{n-1} = 0.8 \Rightarrow cov_{n-1} = \rho_{n-1} \times \sigma_{x,n-1} \times \sigma_{y,n-1} = 0.000156$$

(a) The EWMA model with $\lambda = 0.94$

$$\sigma^2_{x,n} = \lambda\sigma^2_{x,n-1} + (1-\lambda)x^2_{n-1} = 0.00016153 \Rightarrow \sigma_{x,n} = 0.01271$$
$$\sigma^2_{y,n} = \lambda\sigma^2_{y,n-1} + (1-\lambda)y^2_{n-1} = 0.0002115 \Rightarrow \sigma_{y,n} = 0.01454$$
$$cov_n = \lambda cov_{n-1} + (1-\lambda)x_{n-1}y_{n-1} = 0.00014664$$
$$\Rightarrow \rho_n = \frac{cov_n}{\sigma_{x,n}\sigma_{y,n}} = 0.7934$$

(b) The GARCH(1,1) model with $\omega = 0.000002, \alpha = 0.04, \beta = 0.94.$
In practice, is the ω parameter likely to be the same for X and Y?

$$\sigma^2_{x,n} = \omega + \alpha x^2_{n-1} + \beta\sigma^2_{x,n-1} = 0.0001624 \Rightarrow \sigma_{x,n} = 0.01275$$
$$\sigma^2_{y,n} = \omega + \alpha y^2_{n-1} + \beta\sigma^2_{y,n-1} = 0.0002135 \Rightarrow \sigma_{y,n} = 0.01461$$
$$cov_n = \omega + \alpha x_{n-1}y_{n-1} + \beta cov_{n-1} = 0.00014864$$
$$\Rightarrow \rho_n = \frac{cov_n}{\sigma_{x,n}\sigma_{y,n}} = 0.07977$$

For a given α and β, the ω parameter defines the long run average value of a variance or a covariance. There is no reason why we should expect the long run average daily variance for X
and Y should be the same. There is also no reason why we should expect the long run average covariance between X and Y to be the same as the long run average variance of X or the long run average variance of Y. In practice, therefore, we are likely to want to allow ω in a GARCH(1,1) model to vary from market variable to market variable.

Chapter 12 Basel I, Basel II, and Solvency II

The Cooke Ratio: risk-weighted assets, risk-weighted amount.

Credit risk exposures can be divided into three categories:

1. Those arising from on-balance sheet assets (excluding derivatives)

   Risk Weights for On-Balance-Sheet Items

Risk Weight(%)	Asset Category
0	Cash, gold bullion, claims on OECD governments such as Treasury bonds or insured residential mortgages
20	Claims on OECD banks and OECD public sector entities such as securities issued by U.S. government agencies or claims on municipalities
50	Uninsured residential mortgage loans
100	All other claims such as corporate bonds and less-developed country debt, claims on non-OECD banks

$$\sum_{i=1}^{N} w_i L_i$$

where L_i is the principal amount of the ith item and w_i is its risk weight.

Example 12.1

The assets of a bank consist of $100 million of corporate loans, $10 million of OECD government bonds, and $50 million of residential mortgages. The total of the riskweighted assets is $1.0\times100 + 0.0\times10 + 0.5\times50 = 125$ or $125 million.

2. Those arising for off-balance sheet items (excluding derivatives)
3. Those arising from over-the-counter derivatives

Add-On Factors as a Percent of Principal for Derivatives(%)

Remaining Maturity(yr)	Interest Rate	Exchange Rate and Gold	Equity	Precious Metals Except Gold	Other Commodities
< 1	0.0	1.0	6.0	7.0	10.0
1 to 5	0.5	5.0	8.0	7.0	12.0
> 5	1.5	7.5	10.0	8.0	15.0

$$\text{credit equivalent amount} = \max(V, 0) + aL$$

where V is the current value of the derivative to the bank, a is an add-on factor, and L is the principal amount. $\max(V, 0)$ is the current exposure. The add-on amount, aL, is an allowance for the possibility of the exposure increasing in the future.

Example 12.2:

A bank has entered into a $100 million interest rate swap with a remaining life of four years. The current value of the swap is $2.0 million. In this case, the add-on amount is 0.5% of the principal so that the **credit equivalent amount** is $2.0 million plus $0.5 million or $2.5 million.

The credit equivalent amount arising from either the second or third category of exposures is multiplied by the risk weight for the counterparty in order to calculate the risk-weighted assets. The risk weights are similar to those in Table 12.1 except that the risk weight for a corporation is 0.5 rather than 1.0.

Example 12.3

Consider again the bank in Example 12.2. If the interest rate swap is with a corporation, the risk-weighted assets are 2.5 × 0.5 or $1.25 million. If it is with an OECD bank, the risk-weighted assets are 2.5 × 0.2 or $0.5 million.

Putting all this together, the total risk-weighted assets for a bank with N onbalance-sheet items and M off-balance-sheet items is:

$$\text{credit risk RWA} = \sum_{i=1}^{N} w_i L_i + \sum_{j=1}^{M} w_j^{\star} C_j$$

Here, L_i is the principal of the ith on-balance-sheet asset and w_i is the risk weight for the asset; C_j is the credit equivalent amount for the jth derivative or other off-balance sheet item and $w_j^{\star}$ is the risk weight of the counterparty for this jth item.

Tier 1 Capital. equity and noncumulative perpetual preferred stock.3 (Goodwill is subtracted from equity.)

Tier 2 Capital. (Supplementary Capital). cumulative perpetual preferred stock, certain types of 99-year debenture issues, and subordinated debt (i.e., debt subordinated to depositors) with an original life of more than five years.

Netting: in the event of a default all transactions are considered as a single transaction. Effectively, this means that, if a company defaults on one transaction that is covered by the master agreement, it must default on all transactions covered by the master agreement.

$$\sum_{i=1}^{N} \max(V_i, 0) \rightarrow \max(\sum_{i}^{N} V_i, 0)$$

Net replacement ratio, NRR

$$NRR = \frac{\max(\sum_i^N V_i, 0)}{\sum_{i=1}^N \max(V_i, 0)}$$

The credit equivalent amount was modified to:

$$\max(\sum_{i}^{N} V_i, 0) + (0.4 + 0.6 \times NRR) \sum_{i=1}^{N} a_i L_i$$

Example 12.4:

Portfolio of Derivatives with a Particular Counterparty

Transaction	Principal, L_i	Current Value, V_i	Add-On Amount, $a_i L_i$
3-year interest rate swap	1,000	-60	5
6-year foreign exchange forward	1,000	70	75
9-month option on a stock	500	55	30

The current exposure with netting is $-60 + 70 + 55 = 65$. The current exposure without netting is $0 + 70 + 55 = 125$. The net replacement ratio is given by

$$NRR = 65/125 = 0.52$$

The total of the add-on amounts, $\sum a_i L_i$, is 5 + 75 + 30 = 110. The credit equivalent amount when netting agreements are in place is $65 + (0.4 + 0.6 \times 0.52) \times 110 = 143.32$. Without netting, the credit equivalent amount is $125 + 110 = 235$. Suppose that the counterparty is an OECD bank so that the risk weight is 0.2. This means that the risk-weighted assets with netting is 0.2 × 143.32 = 28.66. Without netting, it is 0.2 × 235 = 47.

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internal model-based approach: The value-at-risk measure used in the internal model-based approach was calculated with a 10-day time horizon and a 99% confidence level. It is the loss that has a 1%chance of being exceeded over a 10-day period.

$$\text{Market Risk Capital} = \max(VaR_{t-1}, m_c \times VaR_{avg}) + SRC$$

where m_c is a multiplicative factor, and SRC is a specific risk charge. The variable VaR_{t-1} is the previous day's value at risk and VaR_{avg} is the average value at risk over the past 60 days. The minimum value for m_c is 3.

> internal model-based approach for SRC: $m_c \geq 4$

The first term covers risks relating to movements in broad market variables such as interest rates, exchange rates, stock indices, and commodity prices. The second term, SRC, covers risks related to specific companies such as those concerned with movements in a company's stock price or changes in a company's credit spread.

$$\begin{aligned}\text{Total Capital} &= \text{Credit Risk Capital} + \text{Market Risk Capital} \\ &= 0.08 \times (\text{credit risk RWA} + \text{market risk RWA}) \\ \text{market risk RWA} &= 12.5 \times \text{Market Risk Capital}\end{aligned}$$

Basel II

The Basel II is based on three “pillars”:

1. Minimum Capital Requirements
2. Supervisory Review
3. Market Discipline

Banks hold a total capital equal to 8%of risk-weighted assets (RWA). When the capital requirement for a particular risk is calculated directly rather than in a way involving RWAs, it is multiplied by 12.5 to convert it into an RWA-equivalent. As a result it is always the case that:

$$\text{Total Capital} = \text{Credit Risk Capital} + \text{Market Risk Capital} + \text{Operational Risk Capital}$$
$$= 0.08 \times (\text{credit risk RWA} + \text{market risk RWA} + \text{operational risk RWA})$$
$$\text{market risk RWA} = 12.5 \times \text{Market Risk Capital}$$
$$\text{operational risk RWA} = 12.5 \times \text{Operational Risk Capital}$$

The Standardized Approach:

The standardized approach is similar to Basel I except for the calculation of risk weights.

TABLE 15.4 Risk Weights as a Percent of Principal for Exposures to Countries, Banks, and Corporations Under Basel II's Standardized Approach

	AAA to AA−	A+ to A−	BBB+ to BBB−	BB+ to BB−	B+ to B−	Below B−	Unrated
Country*	0	20	50	100	100	150	100
Banks**	20	50	50	100	100	150	50
Corporations	20	50	100	100	150	150	100

*Includes exposures to the country's central bank.
**National supervisors have options as outlined in the text.

The standard rule for retail lending is that a risk weight of 75% be applied. When claims are secured by a residential mortgage, the risk weight is 35%.

Example 12.5 Suppose that the assets of a bank consist of $100 million of loans to corporations rated A, $10 million of government bonds rated AAA, and $50 million of residential mortgages. Under the Basel II standardized approach, the total of the risk-weighted assets is:

$$0.5 \times 100 + 0 \times 10 + 0.35 \times 50 = 67.5$$

Adjustments for Collateral

There are two ways banks can adjust risk weights for collateral. The first is termed the simple approach and is similar to an approach used in Basel I. The second is termed the comprehensive approach. Banks have a choice as to which approach is used in the banking book, but must use the comprehensive approach to calculate capital for counterparty credit risk in the trading book.

Under the simple approach, the risk weight of the counterparty is replaced by the risk weight of the collateral for the part of the exposure covered by the collateral. (The exposure is calculated after netting.) For any exposure not covered by the collateral, the risk weight of the counterparty is used.

Under the comprehensive approach, banks adjust the size of their exposure upward to allow for possible increases in the exposure and adjust the value of the collateral downward to allow for possible decreases in the value of the collateral. A new exposure equal to the excess of the adjusted exposure over the adjusted value of the collateral is calculated and the counterparty's risk weight is applied to this exposure. Where netting arrangements apply, exposures and collateral are separately netted and the adjustments made are weighted averages.

Example 12.6

Suppose that an $80 million exposure to a particular counterparty is secured by collateral worth $70 million. The collateral consists of bonds issued by an A-rated company. The counterparty has a rating of B+. The risk weight for the counterparty is 150% and the risk weight for the collateral is 50%.

The risk-weighted assets applicable to the exposure using the simple approach is $0.5 \times 70 + 1.5 \times (80 - 70) = 50$

Consider next the comprehensive approach. Assume that the adjustment to exposure to allow for possible future increases in the exposure is +10% and the adjustment to the collateral to allow for possible future decreases in its value is –15%.
The new exposure is $80 \times (1+0.1) - 70 \times (1-0.15) = 28.5$, and a risk weight of 150% is applied to this exposure to give risk adjusted assets equal to $42.75 million.

The IRB Approach:

The capital required is the value at risk minus the expected loss.

Factor Models:

$$U_i = a_{i1}F_1 + a_{i2}F_2 + \cdots + a_{iM}F_M + \sqrt{1 - a_{i1}^2 - a_{i2}^2 - \cdots - a_{iM}^2}Z_i$$

$$cov_{ij} = \sum_{m=1}^{M} a_{im}a_{jm}$$

$F_1, F_2, \cdots, F_M, Z_i$ have standard normal distributions. The Z_i are uncorrelated with each other and with $F_1, F_2, \cdots, F_M, Z_i$

WCRD(T,X):

Define T_i as the time when company i defaults.
Make the simplifying assumption that all loans have the same cumulative probability distribution for the time to default and define PD as the probability of default by time T:

$$WCRD(T,X) = N(\frac{N^{-1}(PD) + \sqrt{\rho}N^{-1}(X)}{\sqrt{1-\rho}})$$

$$PD = Prob(T_i < T)$$

the default rate during time T that will not be exceeded with probability X%.

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the ith obligor has a one-year probability of default equal to PD_i:

$$WCDR_i = N(\frac{N^{-1}(PD_i) + \sqrt{\rho}N^{-1}(0.999)}{\sqrt{1-\rho}})$$

> the bank is 99.9% certain it will not be exceeded next year for the ith counterparty.

for a large portfolio of instruments that have the same ρ, in a one factor model the one-year 99.9% VaR is approximately:

$$\sum_i EAD_i \times LGD_i \times WCDR_i$$

where EAD_i is the exposure at default of the ith counterparty and LGD_i is the loss given default for the ith counterparty.

The expected loss from defaults is:

$$\sum_i EAD_i \times LGD_i \times PD_i$$

The capital required is the excess of the 99.9% worst-case loss over the expected loss:

$$\sum_i EAD_i \times LGD_i \times (WCDR_i - PD_i)$$

Corporation, Sovereign, and Bank Exposures:

The formula for the capital required for the counterparty is:

$$\text{Capital Required} = EAD \times LGD \times (WCDR - PD) \times MA$$

$$MA = \frac{1 + (M - 2.5)b}{1 - 1.5b}$$

$$b = (0.11852 - 0.05478 \ln{(PD)})^2$$

$$RWA = 12.5 \times EAD \times LGD \times (WCDR - PD) \times MA$$

> MA is the maturity adjustment. M is the maturity of the exposure.

Example 12.7

TABLE 15.6 Relationship between One-Year 99.9% WCDR and PD for Corporate, Sovereign, and Bank Exposures

PD	0.1%	0.5%	1%	1.5%	2.0%
WCDR	3.4%	9.8%	14.0%	16.9%	19.0%

Suppose that the assets of a bank consist of $100 million of loans to A-rated corporations.
The PD for the corporations is estimated as 0.1% and the LGD is 60%. The average maturity is 2.5 years for the corporate loans.

$$b = (0.11852 - 0.05478 \ln(PD))^2 = 0.247$$
$$MA = \frac{1 + (M - 2.5)b}{1 - 1.5b} = 1.59$$
$$WCDR = 3.4\%$$
$$RWA = 12.5 \times EAD \times LGD \times (WCDR - PD) \times MA = 39.3$$

Retail Exposures

There is no maturity adjustment, MA:

$$\text{Capital Required} = EAD \times LGD \times (WCDR - PD)$$
$$RWA = 12.5 \times EAD \times LGD \times (WCDR - PD)$$

TABLE 15.7 Relationship between One-Year 99.9% WCDR and PD for Retail Exposures

PD	0.1%	0.5%	1.0%	1.5%	2.0%
WCDR	2.1%	6.3%	9.1%	11.0%	12.3%

Example 12.8:

Suppose that the assets of a bank consist of $50 million of residential mortgages
where the PD is 0.005 and the LGD is 20%. In this case, $\rho = 0.15$ and

$$WCDR = N\left(\frac{N^{-1}(PD) + \sqrt{\rho}N^{-1}(0.999)}{\sqrt{1-\rho}}\right) = 0.067$$
$$RWA = 12.5 \times EAD \times LGD \times (WCDR - PD) = 7.8$$

12.21:

A bank has the following transaction with a AA-rated corporation
(a) A two-year interest rate swap with a principal of $100 million that is worth $3 million
(b) A nine-month foreign exchange forward contract with a principal of $150 million that is worth –$5 million
(c) An long position in a six-month option on gold with a principal of $50 million that is worth $7 million

What is the capital requirement under Basel I if there is no netting?

$$a: 3 + 0.005 \times 100 = 3.5$$
$$b: \max(-5, 0) + 1\% \times 150 = 1.5$$
$$c: 7 + 0.01 \times 50 = 7.5$$
$$\text{credit equivalent amount} = 3.5 + 1.5 + 7.5 = 12.5$$
$$RWA = 0.5 \times \text{credit equivalent amount} = 6.25$$
$$\text{Capital Requirement} = 0.08 \times RWA = 0.5$$

What difference does it make if the netting amendment applies?

$$\text{The current exposure with netting} = \max(3 - 5 + 7, 0) = 5$$
$$\text{The current exposure without netting} = 3 + 0 + 7 = 10$$
$$NRR = 5/10 = 0.5$$
$$\text{credit equivalent amount} = 5 + (0.4 + 0.6 \times 0.5) \sum_{i=1}^{N} a_i L_i = 6.75$$
$$RWA = 0.5 \times \text{credit equivalent amount} = 3.375$$
$$\text{Capital Requirement} = 0.08 \times RWA = 0.27$$

Therefore, the netting amendment reduces the capital requirement by $|(0.27 - 0.5)/0.5| = 0.46$

What is the capital required under Basel II when the standardized approach is used?

$$RWA = 0.2 \times 6.75 = 1.35$$
$$\text{Capital Requirement} = 0.08 \times RWA = 0.108$$

Chapter 13 Basel II.5, Basel III, and Other Post-Crisis Changes

Chapter 14 Historical Simulation and Extreme Value Theory

Accuracy of VaR: Suppose that the q-percentile of the distribution is estimated as x. The standard error of the estimate is

$$\frac{1}{f(x)}\sqrt{\frac{(1-q)q}{n}}$$

> where n is the number of observations and $f(x)$ is an estimate of the probability density function of the loss evaluated at x.

14.12: Suppose that a one-day 97.5% VaR is estimated as \$13 million from 2,000 observations. The one-day changes are approximately normal with mean zero and standard deviation \$6 million.
Estimate a 99% confidence interval for the VaR estimate.

$$x = 1.96 \times 6 = 11.76 \Rightarrow f(x) = 0.0097$$
$$\Rightarrow \frac{1}{f(x)}\sqrt{\frac{(1-q)q}{n}} = 0.358$$

A 99% confidence interval for the VaR is $13 - 2.576 \times 0.358$ to $13 + 2.576 \times 0.358$

Chapter 15 Model-Building Approach

15.16

Consider a position consisting of a \$300,000 investment in gold and a \$500,000 investment in silver. Suppose that the daily volatilities of these two assets are 1.8% and 1.2% respectively, and that the coefficient of correlation between their returns is 0.6. What is the 10-day 97.5% VaR for the portfolio? By how much does diversification reduce the VaR?

$$\sigma_{gold} = 0.018 \times 300000 = 5400, \sigma_{silver} = 0.012 \times 500000 = 6000, \rho = 0.6$$
$$\sigma_{gold+silver} = \sqrt{\sigma_{gold}^2 + \sigma_{silver}^2 + 2\rho\sigma_{gold}\sigma_{silver}} = 10200$$
$$VaR(1, 97.5\%) = 1.96 \times \sigma_{gold+silver} = 19992$$
$$VaR(10, 97.5\%) = \sqrt{10} \times 19992 = 63220$$
$$VaR_{gold} = 5400 \times 1.96 \times \sqrt{10} = 33470$$
$$VaR_{silver} = 6000 \times 1.96 \times \sqrt{10} = 37188$$

The diversification reduce the VaR by $33470 + 37188 - 63220$

Chapter 16 Estimating Default Probabilities

Z-Score:

$$Z = 1.2X_1 + 1.4X_2 + 3.3X_3 + 0.6X_4 + 0.999X_5$$

> X_1: Working capital / Total assets; X_2: Retained earning / Total asset; X_3: Earning before interest and tax / Total asset; X_4: Market value of equity / Book value of total liabilities; X_5: Sales / Total assets

If the Z-score is greater than 3.0, the company was considered unlikely to default.
If it was between 2.7 and 3.0, there was reason to be "on alert".
If it was between 1.8 and 2.7, there was a good chance of default.
If it was less than 1.8, the probability of a financial embarrassment was considered to be very high.

Example 16.1

Consider a company for which working capital is 170,000, total assets are 670,000, earnings before interest and taxes is 60,000, sales are 2,200,000, the market value of equity is 380,000, total liabilities is 240,000, and retained earnings is 300,000. In this case, $X_1 = 0.254$, $X_2 = 0.448$, $X_3 = 0.0896$, $X_4 = 1.583$, and $X_5 = 3.284$. The Z-score is

$$Z = 1.2X_1 + 1.4X_2 + 3.3X_3 + 0.6X_4 + 0.999X_5 = 5.46$$

The Z-score indicates that the company is not in danger of defaulting in the near future.

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hazard rate / default intensity: $\lambda(t)$ –> $\lambda(t)\Delta t$ is the probability of default between time t and $t + \Delta t$ conditional on no default between time zero and time t. $V(t)$ is the cumulative probability of the company surviving to time t.

$$\frac{V(t) - V(t+\Delta t)}{V(t)} = \lambda(t)\Delta t \Rightarrow V(t) = e^{-\int_0^t \lambda(\tau)d\tau}$$

Define $Q(t)$ as the probability of default of default by time t,

$$Q(t) = 1 - V(t) = 1 - e^{-\int_0^t \lambda(\tau)d\tau} = 1 - e^{-\bar{\lambda}(t)t}$$

where $\bar{\lambda}(t)$ is the average hazard rate between time zero and time t.

Example 16.2

Suppose that the hazard rate is a constant 1.5% per year. The probability of a default by the end of the first year is $1 - e^{-0.015\times1} = 0.0149$. The probability of a default by the end of the second year is $1 - e^{-0.015\times2} = 0.0296$. The probability of a default by the end of the third, fourth, and fifth years are similarly 0.0440, 0.0582, and 0.0723. The unconditional probability of a default during the fourth year is $0.0582 - 0.0440 = 0.0142$. The probability of default in the fourth year, conditional on no earlier default is $0.0142/(1 - 0.0440) = 0.0149$.

Estimating default probabilities from credit spreads

Approximate Calculation:

$$\bar{\lambda} = \frac{s(T)}{1-R}$$

> $s(T)$ is the credit spread for a maturity of T. R is the recovery rate. $\bar{\lambda}$ is the average hazard rate between time zero and time t.

EXAMPLE 16.3
Suppose that the CDS spreads for 3-, 5-, and 10-year instruments are 50, 60, and 100 basis points and the expected recovery rate is 60%. The average hazard rate over three years is approximately $0.005/(1-0.6) = 0.0125$. The average hazard rate over five years is approximately $0.006/(1-0.6) = 0.015$. The average hazard rate over 10 years is approximately $0.01/(1-0.6) = 0.025$. From this we can estimate that the average hazard rate between year 3 and year 5 is $(5\times0.015 - 3\times0.0125)/2 = 0.01875$. The average hazard rate between year 5 and year 10 is $(10\times0.025 - 5\times0.015)/5 = 0.035$.

A More Exact Calculation:

Using Equity Prices to Estimate:

Suppose, for simplicity, that a firm has one zero-coupon bond outstanding and that the bond matures at time T. Define

> V_0: Value of company's assets today.
> V_T: Value of company's assets at time T.
> E_0: Value of company's equity today.
> E_T: Value of company's equity at time T.
> D: Amount of debt interest and principal due to be repaid at time T.
> σ_V: Volatility of assets (assumed constant).
> σ_E: Instantaneous volatility of equity.

If $V_T < D$, it is (at least in theory) rational for the company to default on the debt at time T. The value of the equity is then zero. If $V_T > D$, the company should make the debt repayment at time T and the value of the equity at this time is $V_T - D$.
The value of the firm's equity at time T:

$$E_T = \max(V_T - D, 0)$$

This shows that the equity of a company is a call option on the value of the assets of the company with a strike price equal to the repayment required on the debt.

Black-Scholes-Merton:

$$E_0 = V_0 N(d_1) - De^{-rT} N(d_2)$$
$$d_1 = \frac{\ln(V_0/D) + (r + \sigma_V^2/2)T}{\sigma_V \sqrt{T}}$$
$$d_2 = d_1 - \sigma_V \sqrt{T}$$

> N is the cumulative normal distribution function

$$\sigma_E E_0 = \frac{\partial E}{\partial V} \sigma_V V_0 = N(d_1) \sigma_V V_0$$

> The company defaults when the option is not exercised. The probability is $N(-d_2)$

Example 16.4

The value of a company's equity is \$3 million and the volatility of the equity is 80%. The debt that will have to be paid in one year is \$10 million. The risk-free rate is 5% per annum. In this case, $E_0 = 3$, $\sigma_E = 0.80$, $r = 0.05$, $T = 1$, and $D = 10$.

Solving equations yields $V_0 = 12.40$ and $\sigma_V = 0.2123$.

The parameter, d2, is 1.1408 so that the probability of default is $N(-d2) = 0.127$ or 12.7%. The market value of the debt is $V_0 - E_0$ or 9.40. The present value of the promised payment on the debt is $10e^{-0.05\times 1} = 9.51$. The expected loss on the debt is therefore $(9.51 - 9.40)/9.51$ or about 1.2% of its no-default value. The expected loss is the probability of default times one minus the recovery rate. The recovery rate (as a percentage of the no-default value) is therefore $1 - 1.2/12.7$ or about 91%.

Distance to Default:

$$\frac{\ln(V_0/D) + (r - \sigma_V^2/2)T}{\sigma_V \sqrt{T}}$$

16.24

A company has one- and two-year bonds outstanding, each providing a coupon of 8% per year payable annually. The yields on the bonds (expressed with continuous compounding are 6.0% and 6.6%, respectively. Risk-free rates are 4.5% for all maturities. The recovery rate is 35%. Defaults can take place half way through each year. Estimate the risk-neutral default rate each year.

For bond 1, the market price is: $108e^{-0.06\times 1} = 101.71$, and the default-free price is: $108e^{-0.045\times 1} = 103.25$. The expected loss from default is $103.25 - 101.71 = 1.54$.

Suppose that the probability of default between time zero and time 1 is Q_1. The default-free value of the bond 1 at time 0.5 is: $108e^{-0.045\times0.5}=105.60$. The loss given default is $105.60-100\times0.35=70.60$, therefore the present value of the expected loss is: $70.60e^{0.045\times0.5}Q_1=69.03Q_1$

Therefore, $69.03Q_1=1.54\Rightarrow Q_1=0.0223$

For bond 2, the market price is: $8e^{-0.066\times1}+108e^{-0.066\times2}=102.13$ and its default-free value is: $8e^{-0.045\times1}+108e^{-0.045\times2}=106.35$. The expected loss from default is $106.35-102.13=4.22$.

Suppose that the probability of default between time 1 and time 2 is Q_2.
The default-free value of bond 2 at time 0.5 is $8e^{-0.045\times0.5}+108e^{-0.045\times1.5}=108.77$. The loss given default is $108.77-35=73.77$. The present value of expected loss is: $73.77e^{-0.045\times0.5}Q_1=72.13Q_1$.
The default-free value of bond 2 at time 1.5 is $108e^{-0.045\times0.5}=105.60$, and the loss given default is $105.60-35=70.60$. The present value of the expected loss is: $70.60e^{-0.045\times1.5}Q_2=65.99Q_2$.

$72.13Q_1+65.99Q_2=4.22\Rightarrow Q_2=0.0396$

The risk-neutral default rates in year 1 and year 2 are therefore 2.23% and 3.96%