Chapter 5

Linear Design Models

Equations of Motion

$$\dot{p}_n = (\cos\theta\cos\psi)u + (\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)v + (\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)w$$

$$\dot{p}_e = (\cos\theta\sin\psi)u + (\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi)v + (\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)w$$

$$\dot{h} = u\sin\theta - v\sin\phi\cos\theta - w\cos\phi\cos\theta$$

$$\dot{u} = rv - qw - g\sin\theta + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_X(\alpha) + C_{X_q}(\alpha)\frac{cq}{2V_a} + C_{X_{\delta_e}}(\alpha)\delta_e\right] + \frac{\rho S_{\mathrm{prop}} C_{\mathrm{prop}}}{2\mathrm{m}}\left[(k_{\mathrm{motor}}\delta_t)^2 - V_a^2\right]$$

$$\dot{v} = pw - ru + g\cos\theta\sin\phi + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right]$$

$$\dot{w} = qu - pv + g\cos\theta\cos\phi + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_Z(\alpha) + C_{Z_q}(\alpha)\frac{cq}{2V_a} + C_{Z_{\delta_e}}(\alpha)\delta_e\right]$$

$$\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta$$

$$\dot{\theta} = q\cos\phi - r\sin\phi$$

$$\dot{\psi} = q\sin\phi\sec\theta + r\cos\phi\sec\theta$$

$$\dot{p} = \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{p_0} + C_{p_\beta}\beta + C_{p_p}\frac{bp}{2V_a} + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right]$$

$$\dot{q} = \Gamma_5 pr - \Gamma_6(p^2 - r^2) + \frac{\rho V_a^2 Sc}{2J_y}\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{cq}{2V_a} + C_{m_{\delta_e}}\delta_e\right]$$

$$\dot{r} = \Gamma_7 pq - \Gamma_1 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{r_0} + C_{r_\beta}\beta + C_{r_p}\frac{bp}{2V_a} + C_{r_r}\frac{br}{2V_a} + C_{r_{\delta_a}}\delta_a + C_{r_{\delta_r}}\delta_r\right]$$

Lift and Drag Models

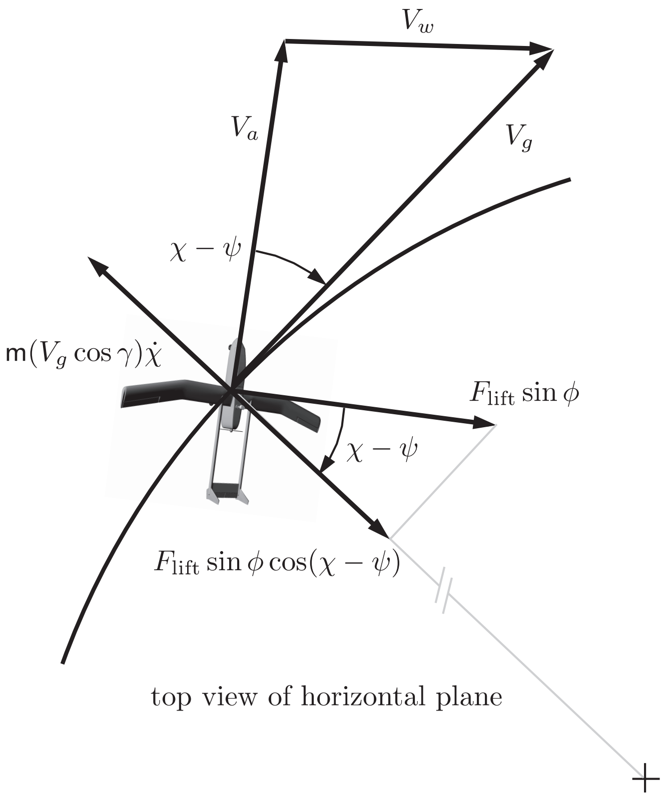
Nonlinear model with stall effects

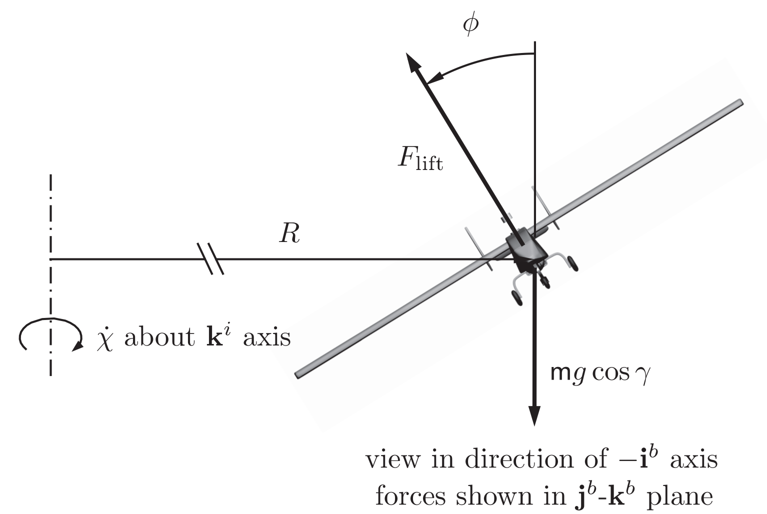
$$
\begin{aligned}
C_L(\alpha) &= (1-\sigma(\alpha))\left[C_{L_0} + C_{L_\alpha}\alpha\right] + \sigma(\alpha)\left[2\,\mathrm{sign}(\alpha)\sin^2\alpha\cos\alpha\right] \\
\sigma(\alpha) &= \frac{1 + e^{-M(\alpha-\alpha_0)} + e^{M(\alpha+\alpha_0)}}{\left(1 + e^{-M(\alpha-\alpha_0)}\right)\left(1 + e^{M(\alpha+\alpha_0)}\right)} \\
C_D(\alpha) &= C_{D_p} + \frac{(C_{L_0} + C_{L_\alpha}\alpha)^2}{\pi e AR}
\end{aligned}
$$

Linear Model

$$
\begin{aligned}
C_L(\alpha) &= C_{L_0} + C_{L_\alpha}\alpha \\
C_D(\alpha) &= C_{D_0} + C_{D_\alpha}\alpha
\end{aligned}
$$

Coordinated Turn





$$
\begin{aligned}
F_{\text{lift}} \sin\phi \cos(\chi - \psi) &= \mathsf{m}\frac{v^2}{R} \\
&= \mathsf{m} v \omega \\
&= \mathsf{m}(V_g \cos\gamma)\dot{\chi}
\end{aligned}
$$

$$
F_{\text{lift}} \cos\phi = \mathsf{m} g \cos\gamma
$$

Coordinated Turn

Dividing the two expressions gives

$$\dot{\chi} = \frac{g}{V_g} \tan\phi \cos(\chi - \psi)$$

which is the coordinated turn condition in wind.

In the absence of wind we have $V_a = V_g$ and $\psi = \chi$ which gives

$$\dot{\psi} = \frac{g}{V_a} \tan\phi$$

which is the expression commonly seen in the literature.

The turning radius is given by

$$R = \frac{V_g \cos\gamma}{\dot{\chi}} = \frac{V_g^2 \cos\gamma}{g \tan\phi \cos(\chi - \psi)}.$$

For level flight in the absence of wind we have the standard formula

$$R = \frac{V_a^2}{g \tan\phi}.$$

Trim Conditions

Given the nonlinear systems

$$\dot{x} = f(x, u).$$

The equilibria (x^*, u^*) are defined by

$$f(x^*, u^*) = 0$$

An aircraft in equilibrium is in a trim condition. In general, trim condition may include states that are not constant. Therefore, trim conditions are given by

$$\dot{x}^* = f(x^*, u^*)$$

Calculating Trim

Objective is to compute trim states and inputs when aircraft simultaneously satisfies three conditions:

- Traveling at constant speed V_a^*,
- Climbing at constant flight path angle γ^*,
- In constant orbit of radius R^*.

V_a^*, γ^*, and R^*, are inputs to the trim calculations.

$$\begin{aligned} \text{States:} \qquad & x \stackrel{\triangle}{=} (p_n, p_e, p_d, u, v, w, \phi, \theta, \psi, p, q, r)^\top \\ \text{Inputs:} \qquad & u \stackrel{\triangle}{=} (\delta_e, \delta_t, \delta_a, \delta_r)^\top \end{aligned}$$

Calculating Trim

For constant-climb orbit:

$\rightarrow$ speed of aircraft not changing $\rightarrow \dot{u}^* = \dot{v}^* = \dot{w}^* = 0$

$\rightarrow$ roll and pitch angles constant $\rightarrow \dot{\phi}^* = \dot{\theta}^* = \dot{p}^* = \dot{q}^* = 0$

Turn rate constant and given by

$$\dot{\psi}^* = \frac{V_a^*}{R^*} \cos\gamma^* \quad \rightarrow \quad \dot{r}^* = 0$$

Climb rate constant, and given by

$$\dot{h}^* = V_a^* \sin\gamma^*$$

Given parameters V_a^*, γ^*, and R^*, can specify $\dot{x}^*$ as

$$\dot{x}^* = (\dot{p}_n^* \;\; \dot{p}_e^* \;\; \dot{h}^* \;\; \dot{u}^* \;\; \dot{v}^* \;\; \dot{w}^* \;\; \dot{\phi}^* \;\; \dot{\theta}^* \;\; \dot{\psi}^* \;\; \dot{p}^* \;\; \dot{q}^* \;\; \dot{r}^* \;\;)^\top$$

...continued...

Calculating Trim

Given parameters V_a^*, γ^*, and R^*, can specify $\dot{x}^*$ as

$$\dot{x}^* = \begin{pmatrix} \dot{p}_n^* \\ \dot{p}_e^* \\ \dot{h}^* \\ \dot{u}^* \\ \dot{v}^* \\ \dot{w}^* \\ \dot{\phi}^* \\ \dot{\theta}^* \\ \dot{\psi}^* \\ \dot{p}^* \\ \dot{q}^* \\ \dot{r}^* \end{pmatrix} = \begin{pmatrix} [\text{don't care}] \\ [\text{don't care}] \\ V_a^* \sin\gamma^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{V_a^*}{R^*}\cos\gamma^* \\ 0 \\ 0 \\ 0 \end{pmatrix} = f(x^*, u^*)$$

Problem of finding x^* and u^* such that $\dot{x}^* = f(x^*, u^*)$, reduces to solving nonlinear algebraic systems of equations.

Can use Matlab trim command – see Appendix F

Lateral Transfer Functions - Roll

Objective: Find simple transfer function relationship between ϕ and δ_a

Start with: $\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta$

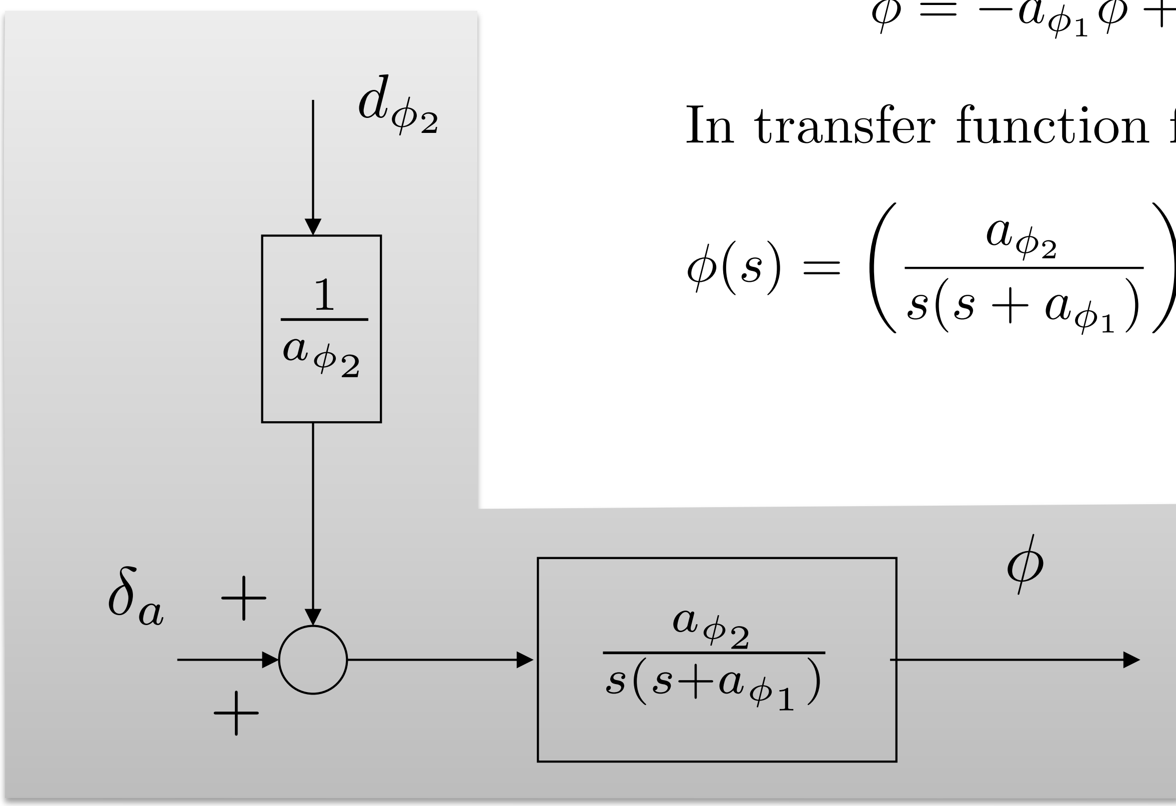
Assuming that θ is small and defining the disturbance as $d_{\phi_1} \stackrel{\triangle}{=} q\sin\phi\tan\theta + r\cos\phi\tan\theta$ gives

$$\dot{\phi} = p + d_{\phi_1}$$

Differentiating and substituting for $\dot{\phi}$ gives

$$\begin{aligned}
\ddot{\phi} &= \dot{p} + \dot{d}_{\phi_1} \\
&= \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{p_0} + C_{p_\beta}\beta + C_{p_p}\frac{bp}{2V_a} + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right] + \dot{d}_{\phi_1} \\
&= \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{p_0} + C_{p_\beta}\beta + C_{p_p}\frac{b}{2V_a}(\dot{\phi} - d_{\phi_1}) + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right] + \dot{d}_{\phi_1} \\
&= \left(\frac{1}{2}\rho V_a^2 Sb C_{p_p}\frac{b}{2V_a}\right)\dot{\phi} + \left(\frac{1}{2}\rho V_a^2 Sb C_{p_{\delta_a}}\right)\delta_a \\
&\quad + \left\{\Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{p_0} + C_{p_\beta}\beta - C_{p_p}\frac{b}{2V_a}(d_{\phi_1}) + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_r}}\delta_r\right] + \dot{d}_{\phi_1}\right\} \\
&= -a_{\phi_1}\dot{\phi} + a_{\phi_2}\delta_a + d_{\phi_2}
\end{aligned}$$

Roll Dynamics



Therefore, we have the second order differential equation

$$\ddot{\phi} = -a_{\phi_1}\dot{\phi} + a_{\phi_2}\delta_a + d_{\phi_2}.$$

In transfer function form we have

$$\phi(s) = \left(\frac{a_{\phi_2}}{s(s+a_{\phi_1})}\right)\left(\delta_a(s) + \frac{1}{a_{\phi_2}}d_{\phi_2}(s)\right).$$

Lateral Transfer Functions - Course

To derive a transfer function from roll ϕ to course χ, start with the coordinated turn condition

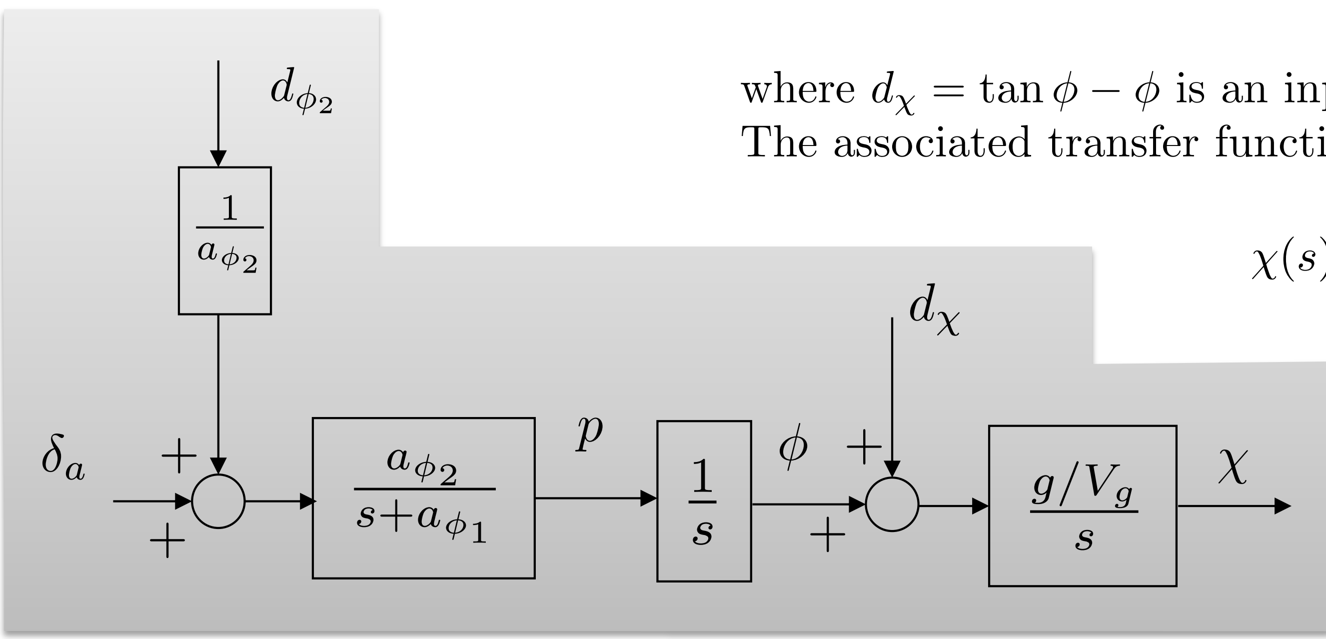
$$\dot{\chi} = \frac{g}{V_a} \tan \phi$$

add and subtract $\frac{g}{V_a}\phi$ to get

$$\begin{aligned} \dot{\chi} &= \frac{g}{V_a}\phi + \frac{g}{V_a}(\tan \phi - \phi) \\ &= \frac{g}{V_a}\phi + \frac{g}{V_a} d_\chi \end{aligned}$$

where $d_\chi = \tan \phi - \phi$ is an input disturbance. The associated transfer function is

$$\chi(s) = \frac{g/V_a}{s}\left(\phi(s) + d_\chi(s)\right)$$



Lateral Transfer Functions - Sideslip

To derive the transfer function from rudder to sideslip, note that in zero wind

$$v = V_a \sin\beta.$$

Differentiating and assuming constant airspeed gives

$$\dot{v} = (V_a \cos\beta)\dot{\beta}.$$

Substituting for $\dot{v}$ from the dynamic gives

$$\begin{aligned}(V_a \cos\beta)\dot{\beta} &= pw - ru + g\cos\theta\sin\phi \\ &\quad + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right].\end{aligned}$$

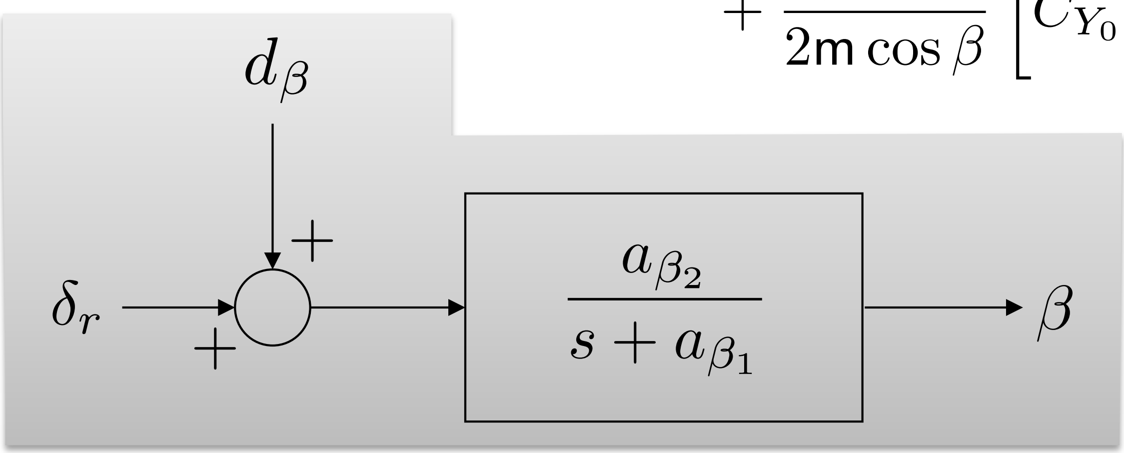
Solving for $\dot{\beta}$ and grouping terms gives:

Sideslip Dynamics

$$\dot{\beta} = -a_{\beta_1}\beta + a_{\beta_2}\delta_r + d_\beta$$

where

$$\begin{aligned} a_{\beta_1} &= -\frac{\rho V_a S}{2\mathrm{m}\cos\beta} C_{Y_\beta} \\ a_{\beta_2} &= \frac{\rho V_a S}{2\mathrm{m}\cos\beta} C_{Y_{\delta_r}} \\ d_\beta &= \frac{1}{V_a\cos\beta}(pw - ru + g\cos\theta\sin\phi) \\ &\quad + \frac{\rho V_a S}{2\mathrm{m}\cos\beta}\left[C_{Y_0} + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta_a}}\delta_a\right] \end{aligned}$$



Longitudinal Transfer Functions - Pitch

To derive the transfer function from elevator to pitch angle:

$$\begin{aligned}\dot{\theta} &= q\cos\phi - r\sin\phi \\ &= q + q(\cos\phi - 1) - r\sin\phi \\ &\stackrel{\triangle}{=} q + d_{\theta_1}\end{aligned}$$

Differentiating and substituting $\dot{q}$ from the dynamics gives

$$\begin{aligned}\ddot{\theta} &= \Gamma_6(r^2 - p^2) + \Gamma_5 pr + \frac{\rho V_a^2 cS}{2J_y}\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{cq}{2V_a} + C_{m_{\delta_e}}\delta_e\right] + \dot{d}_{\theta_1} \\ &= \Gamma_6(r^2 - p^2) + \Gamma_5 pr + \frac{\rho V_a^2 cS}{2J_y}\left[C_{m_0} + C_{m_\alpha}(\theta - \gamma) + C_{m_q}\frac{c}{2V_a}(\dot{\theta} - d_{\theta 1}) + C_{m_{\delta_e}}\delta_e\right] + \dot{d}_{\theta_1} \\ &= \left(\frac{\rho V_a^2 cS}{2J_y}C_{m_q}\frac{c}{2V_a}\right)\dot{\theta} + \left(\frac{\rho V_a^2 cS}{2J_y}C_{m_\alpha}\right)\theta + \left(\frac{\rho V_a^2 cS}{2J_y}C_{m_{\delta_e}}\right)\delta_e + \Big\{\Gamma_6(r^2 - p^2) \\ &\qquad + \Gamma_5 pr + \frac{\rho V_a^2 cS}{J_y}\left[C_{m_0} - C_{m_\alpha}\gamma - C_{m_q}\frac{c}{2V_a}d_{\theta_1}\right] + \dot{d}_{\theta_1}\Big\} \\ &= -a_{\theta_1}\dot{\theta} - a_{\theta_2}\theta + a_{\theta_3}\delta_e + d_{\theta_2}\end{aligned}$$

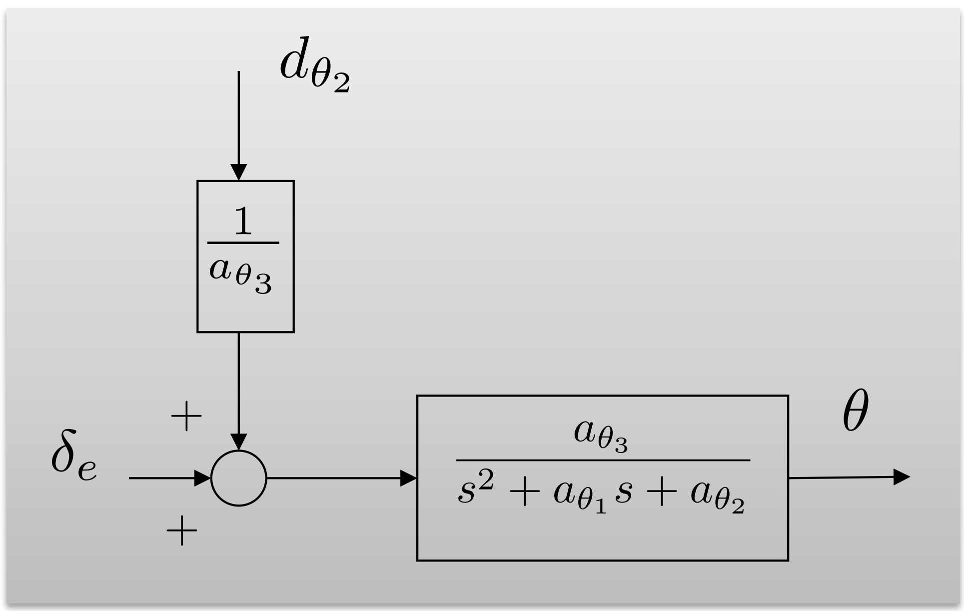
Longitudinal Transfer Functions - Pitch

The ODE model is

$$\ddot{\theta} = -a_{\theta_1}\dot{\theta} - a_{\theta_2}\theta + a_{\theta_3}\delta_e + d_{\theta_2}$$

with associated transfer function

$$\theta(s) = \left(\frac{a_{\theta_3}}{s^2 + a_{\theta_1}s + a_{\theta_2}}\right)\left(\delta_e(s) + \frac{1}{a_{\theta_3}}d_{\theta_2}(s)\right)$$



Longitudinal TF - Altitude from Pitch

To derive a transfer function from pitch to altitude (assuming constant airspeed):

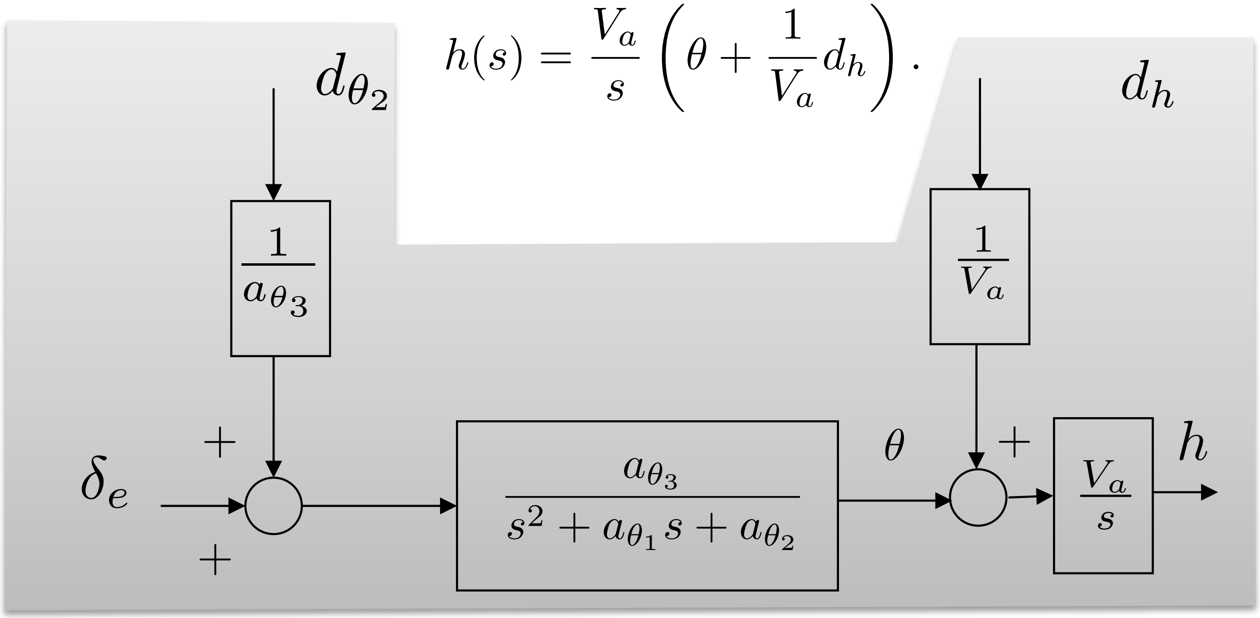
$$
\begin{aligned}
\dot{h} &= u \sin\theta - v \sin\phi \cos\theta - w \cos\phi \cos\theta \\
&= V_a \theta + (u \sin\theta - V_a \theta) - v \sin\phi \cos\theta - w \cos\phi \cos\theta \\
&= V_a \theta + d_h
\end{aligned}
$$

where

$$d_h \stackrel{\triangle}{=} (u \sin\theta - V_a \theta) - v \sin\phi \cos\theta - w \cos\phi \cos\theta.$$

The associated transfer function is

$$h(s) = \frac{V_a}{s}\left(\theta + \frac{1}{V_a} d_h\right).$$



Longitudinal Transfer Functions - Airspeed

Both pitch angle and throttle strongly affect airspeed. We wish to derive the relationship.

If there is no wind, then

$$V_a = \sqrt{u^2 + v^2 + w^2}.$$

Differentiating gives

$$\dot{V}_a = \dot{u}\cos\alpha\cos\beta + \dot{v}\sin\beta + \dot{w}\sin\alpha\cos\beta = \dot{u}\cos\alpha + \dot{w}\sin\alpha + d_{V_1}$$

where

$$d_{V_1} = -\dot{u}(1-\cos\beta)\cos\alpha - \dot{w}(1-\cos\beta)\sin\alpha + \dot{v}\sin\beta.$$

Substituting from the dynamics gives

$$\begin{aligned}
\dot{V}_a &= rV_a\cos\alpha\sin\beta - pV_a\sin\alpha\sin\beta - g\cos\alpha\sin\theta + g\sin\alpha\cos\theta\cos\phi \\
&\quad + \frac{\rho V_a^2 S}{2\mathsf{m}}\left[-C_D(\alpha) - C_{D_\alpha}\alpha - C_{D_q}\frac{cq}{2V_a} - C_{D_{\delta_e}}\delta_e\right] + \frac{1}{\mathsf{m}}T_p(\delta_t, V_a) + d_{V_1} \\
&= (rV_a\cos\alpha - pV_a\sin\alpha)\sin\beta - g\sin(\theta-\alpha) - g\sin\alpha\cos\theta(1-\cos\phi) \\
&\quad + \frac{\rho V_a^2 S}{2\mathsf{m}}\left[-C_{D_0} - C_{D_\alpha}\alpha - C_{D_q}\frac{cq}{2V_a} - C_{D_{\delta_e}}\delta_e\right] + \frac{1}{\mathsf{m}}T_p(\delta_t, V_a) + d_{V_1} \\
&= -g\sin\gamma + \frac{\rho V_a^2 S}{2\mathsf{m}}\left[-C_{D_0} - C_{D_\alpha}\alpha - C_{D_q}\frac{cq}{2V_a} - C_{D_{\delta_e}}\delta_e\right] + \frac{1}{\mathsf{m}}T_p(\delta_t, V_a) + d_{V_2}
\end{aligned}$$

Longitudinal Transfer Functions - Airspeed

Evaluate at trim: α^*, V_a^*, θ^*, δ_e^*, and let

$$\bar{V}_a = V_a - V_a^*, \qquad \bar{\theta} = \theta - \theta^*, \qquad \bar{\delta}_e = \delta_e - \delta_e^*$$

to get

$$\begin{aligned}
\dot{\bar{V}}_a &= -g\cos(\theta^* - \alpha^*)\bar{\theta} + \left\{ \frac{\rho V_a^* S}{\mathsf{m}} \left[-C_{D_0} - C_{D_\alpha}\alpha^* - C_{D_{\delta_e}}\delta_e^* \right] + \frac{1}{\mathsf{m}} \frac{\partial T_p}{\partial V_a}(\delta_t^*, V_a^*) \right\} \bar{V}_a \\
&\quad + \frac{1}{\mathsf{m}} \frac{\partial T_p}{\partial \delta_t}(\delta_t^*, V_a^*)\bar{\delta}_t + d_V \\
&= -a_{V_1}\bar{V}_a + a_{V_2}\bar{\delta}_t - a_{V_3}\bar{\theta} + d_V
\end{aligned}$$

where

$$\begin{aligned}
a_{V_1} &= \frac{\rho V_a^* S}{\mathsf{m}} \left[C_{D_0} + C_{D_\alpha}\alpha^* + C_{D_{\delta_e}}\delta_e^* \right] - \frac{1}{\mathsf{m}} \frac{\partial T_p}{\partial V_a}(\delta_t^*, V_a^*) \\
a_{V_2} &= \frac{1}{\mathsf{m}} \frac{\partial T_p}{\partial \delta_t}(\delta_t^*, V_a^*), \\
a_{V_3} &= g\cos(\theta^* - \alpha^*),
\end{aligned}$$

Longitudinal Transfer Functions - Airspeed

The associated transfer function is

$$\bar{V}_a(s) = \frac{1}{s + a_{V_1}} \left(a_{V_2} \bar{\delta}_t(s) - a_{V_3} \bar{\theta}(s) + d_V(s) \right)$$

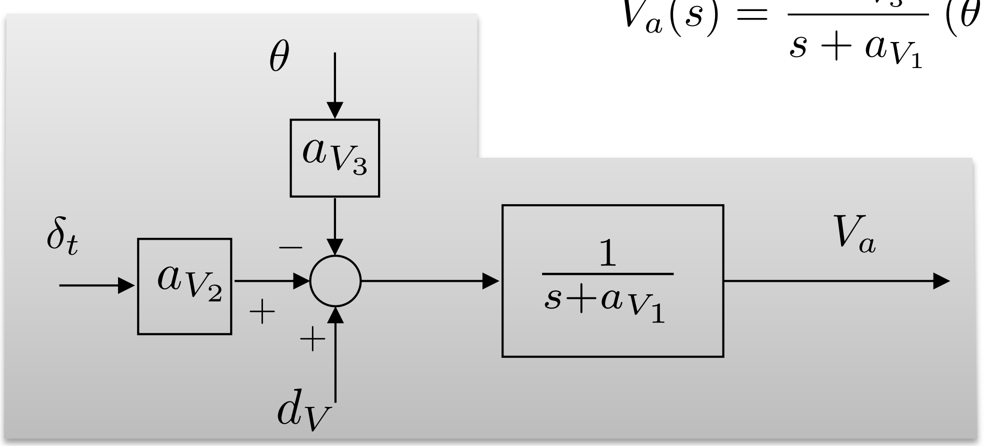
For UAVs we don't know trim values. Therefore the untrimmed variables are used and the trim values are absorbed into the disturbances.

For throttle we have

$$V_a(s) = \frac{a_{V_2}}{s + a_{V_1}} \left(\delta_t(s) + d_{V_1}(s) \right)$$

and for pitch angle we have

$$V_a(s) = \frac{-a_{V_3}}{s + a_{V_1}} \left(\theta(s) + d_{V_2}(s) \right)$$



Linear State-space Models

We will derive the state space equations about trim.
Recall that the nonlinear equations are given by $\dot{x} = f(x, u)$. The trim condition is $\dot{x}^* = f(x^*, u^*)$. Let $\bar{x} = x - x^*$ be the deviation from trim. Then

$$\begin{aligned}
\dot{\bar{x}} &= \dot{x} - \dot{x}^* \\
&= f(x, u) - f(x^*, u^*) \\
&= f(x + x^* - x^*, u + u^* - u^*) - f(x^*, u^*) \\
&= f(x^* + \bar{x}, u^* + \bar{u}) - f(x^*, u^*)
\end{aligned}$$

Using a Taylor series expansion around trim gives

$$\begin{aligned}
\dot{\bar{x}} &= f(x^*, u^*) + \frac{\partial f(x^*, u^*)}{\partial x}\bar{x} + \frac{\partial f(x^*, u^*)}{\partial u}\bar{u} + H.O.T - f(x^*, u^*) \\
&\approx \frac{\partial f(x^*, u^*)}{\partial x}\bar{x} + \frac{\partial f(x^*, u^*)}{\partial u}\bar{u} \\
&\stackrel{\triangle}{=} A\bar{x} + B\bar{u}
\end{aligned}$$

Lateral State-space Equations

The lateral states and inputs are defined as

$$\dot{x}_{\text{lat}} \stackrel{\triangle}{=} (v, p, r, \phi, \psi)^\top, \qquad u_{\text{lat}} \stackrel{\triangle}{=} (\delta_a, \delta_r)^\top.$$

The nonlinear equations of motion are

$$\begin{aligned}
\dot{v} &= pw - ru + g\cos\theta\sin\phi + \frac{\rho\sqrt{u^2+v^2+w^2}S}{2\text{m}}\frac{b}{2}\left[C_{Y_p}p + C_{Y_r}r\right] \\
&\quad + \frac{\rho(u^2+v^2+w^2)S}{2\text{m}}\left[C_{Y_0} + C_{Y_\beta}\tan^{-1}\left(\frac{v}{\sqrt{u^2+w^2}}\right) + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right] \\
\dot{p} &= \Gamma_1 pq - \Gamma_2 qr + \frac{\rho\sqrt{u^2+v^2+w^2}S}{2}\frac{b^2}{2}\left[C_{p_p}p + C_{p_r}r\right] \\
&\quad + \frac{1}{2}\rho(u^2+v^2+w^2)Sb\left[C_{p_0} + C_{p_\beta}\tan^{-1}\left(\frac{v}{\sqrt{u^2+w^2}}\right) + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right] \\
\dot{r} &= \Gamma_7 pq - \Gamma_1 qr + \frac{\rho\sqrt{u^2+v^2+w^2}S}{2}\frac{b^2}{2}\left[C_{r_p}p + C_{r_r}r\right] \\
&\quad + \frac{1}{2}\rho(u^2+v^2+w^2)Sb\left[C_{r_0} + C_{r_\beta}\tan^{-1}\left(\frac{v}{\sqrt{u^2+w^2}}\right) + C_{r_{\delta_a}}\delta_a + C_{r_{\delta_r}}\delta_r\right] \\
\dot{\phi} &= p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \\
\dot{\psi} &= q\sin\phi\sec\theta + r\cos\phi\sec\theta
\end{aligned}$$

where we have used $\beta = \tan^{-1}\left(\frac{v}{\sqrt{u^2+w^2}}\right)$ and $V_a = \sqrt{u^2+v^2+w^2}$.

Jacobian Matrices

$$A_{\text{lat}} = \frac{\partial f_{\text{lat}}}{\partial x_{\text{lat}}} = \begin{pmatrix} \frac{\partial \dot{v}}{\partial v} & \frac{\partial \dot{v}}{\partial p} & \frac{\partial \dot{v}}{\partial r} & \frac{\partial \dot{v}}{\partial \phi} & \frac{\partial \dot{v}}{\partial \psi} \\ \frac{\partial \dot{p}}{\partial v} & \frac{\partial \dot{p}}{\partial p} & \frac{\partial \dot{p}}{\partial r} & \frac{\partial \dot{p}}{\partial \phi} & \frac{\partial \dot{p}}{\partial \psi} \\ \frac{\partial \dot{r}}{\partial v} & \frac{\partial \dot{r}}{\partial p} & \frac{\partial \dot{r}}{\partial r} & \frac{\partial \dot{r}}{\partial \phi} & \frac{\partial \dot{r}}{\partial \psi} \\ \frac{\partial \dot{\phi}}{\partial v} & \frac{\partial \dot{\phi}}{\partial p} & \frac{\partial \dot{\phi}}{\partial r} & \frac{\partial \dot{\phi}}{\partial \phi} & \frac{\partial \dot{\phi}}{\partial \psi} \\ \frac{\partial \dot{\psi}}{\partial v} & \frac{\partial \dot{\psi}}{\partial p} & \frac{\partial \dot{\psi}}{\partial r} & \frac{\partial \dot{\psi}}{\partial \phi} & \frac{\partial \dot{\psi}}{\partial \psi} \end{pmatrix}$$

$$B_{\text{lat}} = \frac{\partial f_{\text{lat}}}{\partial u_{\text{lat}}} = \begin{pmatrix} \frac{\partial \dot{v}}{\partial \delta_a} & \frac{\partial \dot{v}}{\partial \delta_r} \\ \frac{\partial \dot{p}}{\partial \delta_a} & \frac{\partial \dot{p}}{\partial \delta_r} \\ \frac{\partial \dot{r}}{\partial \delta_a} & \frac{\partial \dot{r}}{\partial \delta_r} \\ \frac{\partial \dot{\phi}}{\partial \delta_a} & \frac{\partial \dot{\phi}}{\partial \delta_r} \\ \frac{\partial \dot{\psi}}{\partial \delta_a} & \frac{\partial \dot{\psi}}{\partial \delta_r} \end{pmatrix}$$

Take partial derivatives and evaluate at trim to get

Lateral State-space Equations

$$\begin{pmatrix} \dot{\bar{v}} \\ \dot{\bar{p}} \\ \dot{\bar{r}} \\ \dot{\bar{\phi}} \\ \dot{\bar{\psi}} \end{pmatrix} = \begin{pmatrix} Y_v & Y_p & Y_r & g\cos\theta^*\cos\phi^* & 0 \\ L_v & L_p & L_r & 0 & 0 \\ N_v & N_p & N_r & 0 & 0 \\ 0 & 1 & \cos\phi^*\tan\theta^* & q^*\cos\phi^*\tan\theta^* - r^*\sin\phi^*\tan\theta^* & 0 \\ 0 & 0 & \cos\phi^*\sec\theta^* & p^*\cos\phi^*\sec\theta^* - r^*\sin\phi^*\sec\theta^* & 0 \end{pmatrix} \begin{pmatrix} \bar{v} \\ \bar{p} \\ \bar{r} \\ \bar{\phi} \\ \bar{\psi} \end{pmatrix} + \begin{pmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\delta}_a \\ \bar{\delta}_r \end{pmatrix}$$

Lateral	Formula
Y_v	$\frac{\rho S b v^*}{4 m V_a^*}\left[C_{Y_p} p^* + C_{Y_r} r^*\right] + \frac{\rho S v^*}{m}\left[C_{Y_0} + C_{Y_\beta}\beta^* + C_{Y_{\delta_a}}\delta_a^* + C_{Y_{\delta_r}}\delta_r^*\right] + \frac{\rho S C_{Y_\beta}}{2m}\sqrt{u^{*2}+w^{*2}}$
Y_p	$w^* + \frac{\rho V_a^* S b}{4m} C_{Y_p}$
Y_r	$-u^* + \frac{\rho V_a^* S b}{4m} C_{Y_r}$
Y_{δ_a}	$\frac{\rho V_a^{*2} S}{2m} C_{Y_{\delta_a}}$
Y_{δ_r}	$\frac{\rho V_a^{*2} S}{2m} C_{Y_{\delta_r}}$
L_v	$\frac{\rho S b^2 v^*}{4 V_a^*}\left[C_{p_p} p^* + C_{p_r} r^*\right] + \rho S b v^*\left[C_{p_0} + C_{p_\beta}\beta^* + C_{p_{\delta_a}}\delta_a^* + C_{p_{\delta_r}}\delta_r^*\right] + \frac{\rho S b C_{p_\beta}}{2}\sqrt{u^{*2}+w^{*2}}$
L_p	$\Gamma_1 q^* + \frac{\rho V_a^* S b^2}{4} C_{p_p}$
L_r	$-\Gamma_2 q^* + \frac{\rho V_a^* S b^2}{4} C_{p_r}$
L_{δ_a}	$\frac{\rho V_a^{*2} S b}{2} C_{p_{\delta_a}}$
L_{δ_r}	$\frac{\rho V_a^{*2} S b}{2} C_{p_{\delta_r}}$
N_v	$\frac{\rho S b^2 v^*}{4 V_a^*}\left[C_{r_p} p^* + C_{r_r} r^*\right] + \rho S b v^*\left[C_{r_0} + C_{r_\beta}\beta^* + C_{r_{\delta_a}}\delta_a^* + C_{r_{\delta_r}}\delta_r^*\right] + \frac{\rho S b C_{r_\beta}}{2}\sqrt{u^{*2}+w^{*2}}$
N_p	$\Gamma_7 q^* + \frac{\rho V_a^* S b^2}{4} C_{r_p}$
N_r	$-\Gamma_1 q^* + \frac{\rho V_a^* S b^2}{4} C_{r_r}$
N_{δ_a}	$\frac{\rho V_a^{*2} S b}{2} C_{r_{\delta_a}}$
N_{δ_r}	$\frac{\rho V_a^{*2} S b}{2} C_{r_{\delta_r}}$

Alternative Form - Lateral

The lateral dynamics are often expressed using β instead of v. Recall that

$$v = V_a \sin\beta.$$

Therefore

$$\bar{v} = V_a^* \cos\beta^* \bar{\beta}.$$

Differentiating gives

$$\dot{\bar{\beta}} = \frac{1}{V_a^* \cos\beta^*}\dot{\bar{v}}.$$

The associated state space equations are

$$\begin{pmatrix} \dot{\bar{\beta}} \\ \dot{\bar{p}} \\ \dot{\bar{r}} \\ \dot{\bar{\phi}} \\ \dot{\bar{\psi}} \end{pmatrix} = \begin{pmatrix} Y_v & \frac{Y_p}{V_a^* \cos\beta^*} & \frac{Y_r}{V_a^* \cos\beta^*} & \frac{g\cos\theta^*\cos\phi^*}{V_a^* \cos\beta^*} & 0 \\ L_v V_a^* \cos\beta^* & L_p & L_r & 0 & 0 \\ N_v V_a^* \cos\beta^* & N_p & N_r & 0 & 0 \\ 0 & 1 & \cos\phi^*\tan\theta^* & q^*\cos\phi^*\tan\theta^* - r^*\sin\phi^*\tan\theta^* & 0 \\ 0 & 0 & \cos\phi^*\sec\theta^* & p^*\cos\phi^*\sec\theta^* - r^*\sin\phi^*\sec\theta^* & 0 \end{pmatrix} \begin{pmatrix} \bar{\beta} \\ \bar{p} \\ \bar{r} \\ \bar{\phi} \\ \bar{\psi} \end{pmatrix} + \begin{pmatrix} \frac{Y_{\delta_a}}{V_a^* \cos\beta^*} & \frac{Y_{\delta_r}}{V_a^* \cos\beta^*} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\delta}_a \\ \bar{\delta}_r \end{pmatrix}$$

Longitudinal State-space Equations

The longitudinal states and inputs are defined as

$$\dot{x}_{\rm lon} \stackrel{\triangle}{=} (u, w, q, \theta, h)^\top, \qquad u_{\rm lon} \stackrel{\triangle}{=} (\delta_e, \delta_t)^\top.$$

The nonlinear equations of motion are

$$\begin{aligned}
\dot{u} &= -qw - g\sin\theta + \frac{\rho(u^2+w^2)S}{2\mathrm{m}}\left[C_{X_0} + C_{X_\alpha}\tan^{-1}\left(\frac{w}{u}\right) + C_{X_{\delta_e}}\delta_e\right] \\
&\qquad + \frac{\rho\sqrt{u^2+w^2}S}{4\mathrm{m}}C_{X_q}cq + \frac{\rho S_{\rm prop}}{2\mathrm{m}}C_{\rm prop}\left[(k\delta_t)^2 - (u^2+w^2)\right] \\
\dot{w} &= qu + g\cos\theta + \frac{\rho(u^2+w^2)S}{2\mathrm{m}}\left[C_{Z_0} + C_{Z_\alpha}\tan^{-1}\left(\frac{w}{u}\right) + C_{Z_{\delta_e}}\delta_e\right] \\
&\qquad + \frac{\rho\sqrt{u^2+w^2}S}{4\mathrm{m}}C_{Z_q}cq \\
\dot{q} &= \frac{1}{2J_y}\rho(u^2+w^2)cS\left[C_{m_0} + C_{m_\alpha}\tan^{-1}\left(\frac{w}{u}\right) + C_{m_{\delta_e}}\delta_e\right] + \frac{1}{4J_y}\rho\sqrt{u^2+w^2}SC_{m_q}c^2q \\
\dot{\theta} &= q \\
\dot{h} &= u\sin\theta - w\cos\theta
\end{aligned}$$

where we have used $\alpha = \tan^{-1}\left(\frac{w}{u}\right)$ and $V_a = \sqrt{u^2+w^2}$.

Jacobian Matrices

$$A_{\mathrm{lon}} = \frac{\partial f_{\mathrm{lon}}}{\partial x_{\mathrm{lon}}} = \begin{pmatrix} \frac{\partial \dot{u}}{\partial u} & \frac{\partial \dot{u}}{\partial w} & \frac{\partial \dot{u}}{\partial q} & \frac{\partial \dot{u}}{\partial \theta} & \frac{\partial \dot{u}}{\partial h} \\ \frac{\partial \dot{w}}{\partial u} & \frac{\partial \dot{w}}{\partial w} & \frac{\partial \dot{w}}{\partial q} & \frac{\partial \dot{w}}{\partial \theta} & \frac{\partial \dot{w}}{\partial h} \\ \frac{\partial \dot{q}}{\partial u} & \frac{\partial \dot{q}}{\partial w} & \frac{\partial \dot{q}}{\partial q} & \frac{\partial \dot{q}}{\partial \theta} & \frac{\partial \dot{q}}{\partial h} \\ \frac{\partial \dot{\theta}}{\partial u} & \frac{\partial \dot{\theta}}{\partial w} & \frac{\partial \dot{\theta}}{\partial q} & \frac{\partial \dot{\theta}}{\partial \theta} & \frac{\partial \dot{\theta}}{\partial h} \\ \frac{\partial \dot{h}}{\partial u} & \frac{\partial \dot{h}}{\partial w} & \frac{\partial \dot{h}}{\partial q} & \frac{\partial \dot{h}}{\partial \theta} & \frac{\partial \dot{h}}{\partial h} \end{pmatrix}$$

$$B_{\mathrm{lon}} = \frac{\partial f_{\mathrm{lon}}}{\partial u_{\mathrm{lon}}} = \begin{pmatrix} \frac{\partial \dot{u}}{\partial \delta_e} & \frac{\partial \dot{u}}{\partial \delta_t} \\ \frac{\partial \dot{w}}{\partial \delta_e} & \frac{\partial \dot{w}}{\partial \delta_t} \\ \frac{\partial \dot{q}}{\partial \delta_e} & \frac{\partial \dot{q}}{\partial \delta_t} \\ \frac{\partial \dot{\theta}}{\partial \delta_e} & \frac{\partial \dot{\theta}}{\partial \delta_t} \\ \frac{\partial \dot{h}}{\partial \delta_e} & \frac{\partial \dot{h}}{\partial \delta_t} \end{pmatrix}$$

Take partial derivatives and evaluate at trim to get

Longitudinal State-space Equations

$$\begin{pmatrix}\dot{\bar{u}}\\ \dot{\bar{w}}\\ \dot{\bar{q}}\\ \dot{\bar{\theta}}\\ \dot{\bar{h}}\end{pmatrix} = \begin{pmatrix} X_u & X_w & X_q & -g\cos\theta^* & 0\\ Z_u & Z_w & Z_q & -g\sin\theta^* & 0\\ M_u & M_w & M_q & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ \sin\theta^* & -\cos\theta^* & 0 & u^*\cos\theta^* + w^*\sin\theta^* & 0\end{pmatrix}\begin{pmatrix}\bar{u}\\ \bar{w}\\ \bar{q}\\ \bar{\theta}\\ \bar{h}\end{pmatrix} + \begin{pmatrix} X_{\delta_e} & X_{\delta_t}\\ Z_{\delta_e} & 0\\ M_{\delta_e} & 0\\ 0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\bar{\delta}_e\\ \bar{\delta}_t\end{pmatrix}$$

Longitudinal	Formula
X_u	$\frac{u^*\rho S}{\mathsf{m}}\left[C_{X_0} + C_{X_\alpha}\alpha^* + C_{X_{\delta_e}}\delta_e^*\right] - \frac{\rho S w^* C_{X_\alpha}}{2\mathsf{m}} + \frac{\rho S c C_{X_q} u^* q^*}{4mV_a^*} - \frac{\rho S_{\text{prop}} C_{\text{prop}} u^*}{\mathsf{m}}$
X_w	$-q^* + \frac{w^*\rho S}{\mathsf{m}}\left[C_{X_0} + C_{X_\alpha}\alpha^* + C_{X_{\delta_e}}\delta_e^*\right] + \frac{\rho S c C_{X_q} w^* q^*}{4mV_a^*} + \frac{\rho S C_{X_\alpha} u^*}{2\mathsf{m}} - \frac{\rho S_{\text{prop}} C_{\text{prop}} w^*}{\mathsf{m}}$
X_q	$-w^* + \frac{\rho V_a^* S C_{X_q} c}{4\mathsf{m}}$
X_{δ_e}	$\frac{\rho V_a^{*2} S C_{X_{\delta_e}}}{2\mathsf{m}}$
X_{δ_t}	$\frac{\rho S_{prop} C_{prop} k^2 \delta_t^*}{\mathsf{m}}$
Z_u	$q^* + \frac{u^*\rho S}{\mathsf{m}}\left[C_{Z_0} + C_{Z_\alpha}\alpha^* + C_{Z_{\delta_e}}\delta_e^*\right] - \frac{\rho S C_{Z_\alpha} w^*}{2\mathsf{m}} + \frac{u^*\rho S C_{Z_q} c q^*}{4mV_a^*}$
Z_w	$\frac{w^*\rho S}{\mathsf{m}}\left[C_{Z_0} + C_{Z_\alpha}\alpha^* + C_{Z_{\delta_e}}\delta_e^*\right] + \frac{\rho S C_{Z_\alpha} u^*}{2\mathsf{m}} + \frac{\rho w^* S c C_{Z_q} q^*}{4mV_a^*}$
Z_q	$u^* + \frac{\rho V_a^* S C_{Z_q} c}{4\mathsf{m}}$
Z_{δ_e}	$\frac{\rho V_a^{*2} S C_{Z_{\delta_e}}}{2\mathsf{m}}$
M_u	$\frac{u^*\rho S c}{J_y}\left[C_{m_0} + C_{m_\alpha}\alpha^* + C_{m_{\delta_e}}\delta_e^*\right] - \frac{\rho S c C_{m_\alpha} w^*}{2J_y} + \frac{\rho S c^2 C_{m_q} q^* u^*}{4J_y V_a^*}$
M_w	$\frac{w^*\rho S c}{J_y}\left[C_{m_0} + C_{m_\alpha}\alpha^* + C_{m_{\delta_e}}\delta_e^*\right] + \frac{\rho S c C_{m_\alpha} u^*}{2J_y} + \frac{\rho S c^2 C_{m_q} q^* w^*}{4J_y V_a^*}$
M_q	$\frac{\rho V_a^* S c^2 C_{m_q}}{4J_y}$
M_{δ_e}	$\frac{\rho V_a^{*2} S c C_{m_{\delta_e}}}{2J_y}$

Alternative Form – Longitudinal

The longitudinal dynamics are often expressed using α instead of w. Recall that

$$w = V_a \sin\alpha \cos\beta.$$

At trim, when $\beta = 0$ we have

$$\bar{w} = V_a^* \cos\alpha^* \bar{\alpha}.$$

Differentiating gives

$$\dot{\bar{\alpha}} = \frac{1}{V_a^* \cos\alpha^*} \dot{\bar{w}}.$$

The associated state space equations are

$$\begin{pmatrix} \dot{\bar{u}} \\ \dot{\bar{\alpha}} \\ \dot{\bar{q}} \\ \dot{\bar{\theta}} \\ \dot{\bar{h}} \end{pmatrix} = \begin{pmatrix} X_u & X_w V_a^* \cos\alpha^* & X_q & -g\cos\theta^* & 0 \\ \frac{Z_u}{V_a^* \cos\alpha^*} & Z_w & \frac{Z_q}{V_a^* \cos\alpha^*} & \frac{-g\sin\theta^*}{V_a^* \cos\alpha^*} & 0 \\ M_u & M_w V_a^* \cos\alpha^* & M_q & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sin\theta^* & -V_a^* \cos\theta^* \cos\alpha^* & 0 & u^*\cos\theta^* + w^*\sin\theta^* & 0 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{\alpha} \\ \bar{q} \\ \bar{\theta} \\ \bar{h} \end{pmatrix} + \begin{pmatrix} X_{\delta_e} & X_{\delta_t} \\ \frac{Z_{\delta_e}}{V_a^* \cos\alpha^*} & 0 \\ M_{\delta_e} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\delta}_e \\ \bar{\delta}_t \end{pmatrix}$$

Reduced Order Modes

<u>Longitudinal Modes</u>

- Short Period Mode
  - Short period mode is the fast mode seen in pitch rate q and pitch angle theta.
- Phugoid Mode
  - Phugoid mode is slow mode seen in theta and u.
- Induced by impulse on elevator

<u>Lateral Modes</u>

- Roll Mode
  - First order fast mode between aileron and pitch rate.
- Spiral-divergence Mode
  - First order (really) slow mode between aileron and yaw/course
- Dutch-roll Mode
  - Second order mode: coupling between roll, yaw, and side slip. Like a duck wagging its tail.
- Induced by doublet on aileron or rudder

Simulation Project

- Find trim states and inputs. For the aerosonde model use $V_a = 17$ m/s, and $R = 150$ m.
- Find the transfer functions derived in this chapter.
- Find the state space models derived in this chapter.
- Compute the eigenvalues of A_{lon} and A_{lat} and associate them with the phugoid mode, the short period mode, the roll mode, the dutch roll mode, and the spiral mode. Show these modes in the simulator.