Advanced Data Structure
Project VI

Strip Packing

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Abstract

The strip packing problem is a classic 2-dimensional geometric minimization problem. And in this project, we would implement some classic strip packing algorithms to solve the problem. We would test each algorithm with different dataset and examine its approximation is whether right to the theoretic analysis.

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1 Introduction

The strip packing problem is a 2-dimensional geometric minimization problem. Given a set of axis-aligned rectangles and a strip of bounded width and infinite height, determine an overlapping-free packing of the rectangles into the strip, minimizing its height. And we need to run those algorithm in different situations to test their approximation.

2 Algorithms

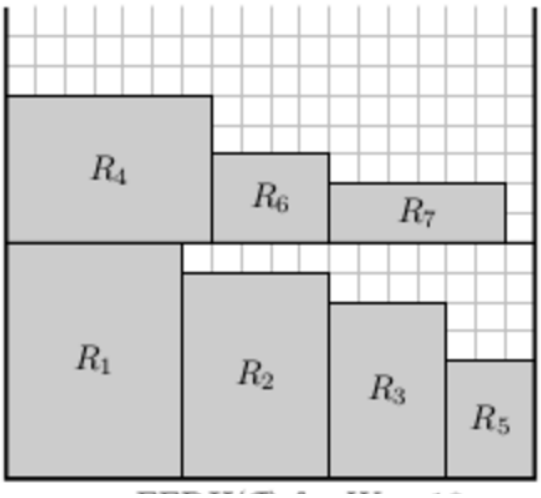
In this section, we use the below example to show how those different algorithms run:

i	1	2	3	4	5	6	7
$w(i)$	6	5	4	7	3	4	6
$h(i)$	8	7	6	5	4	3	2

And for convenience, we suppose that the width of the strip is 1 and all items' width less than 1(In implementation, the width is all integers, the proof equals to divide all that width with the total width of the strip.).

2.1 Next Fit Decreasing Height

Let $\mathcal{L}$ be the given items $R_1,\ R_2\ \ldots R_n$. The NFDH is a quite natural algorithm to store those items. Firstly, we sort all the items in a decreasing order and then put them one by one from the bottom left to the right. And once the next item exceed the right boundary, we put it on a new level and repeat that step until all items are placed. For the given example, the packing would be like the follwing picture:



Graph 1: $NFDH(\mathcal{L})$

2.1.1 Approximation

It is easy to find that the utility of space in this algorithm is quite low since we never use the level that under the top, so is its approximation. In fact, we have the theory that:

$$NFDH(\mathcal{L}) \leq 2 \times OPTIMAL(\mathcal{L}) + C$$

And the approximation constant 2 is tight.

Now we are going to prove that lemma. We denote B_i as the i-th level and t as the number of levels, or say each first item in the i-th level has the height of B_i, which we call it as H_i. And we define A_i as the total area of the i-th level while x_i and y_i be the first item's width in B_i and total width of B_i. Since we always new a level when last level cannot afford the new item then we have:

$$y_i + x_{i+1} > 1 \quad for \quad 1 < i < t \tag{1}$$

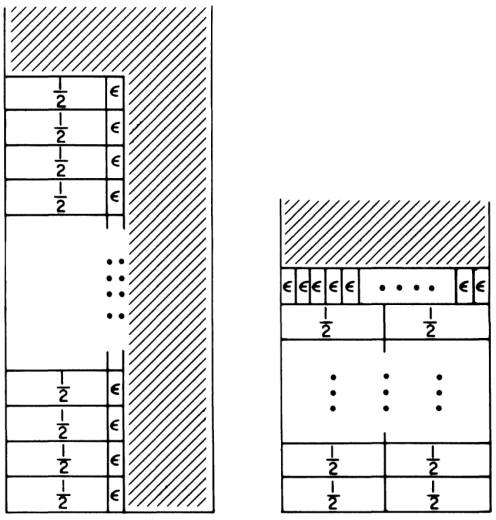
And we know all items in B_i has a height greater than H_{i+1}, so:

$$A_i + A_{i+1} \geq H_{i+1}(y_i + x_{i+1}) > H_{i+1} \tag{2}$$

Now we can get the approximation:

$$\begin{aligned} NFDH(\mathcal{L}) = \sum_{i=1}^{t} H_i &\leq H_1 + \sum_{i=1}^{t-1} A_i + \sum_{i=2}^{t} A_i \\ &\leq H_1 + 2A \\ &\leq 1 + 2\ OPTIMAL(\mathcal{L}) \end{aligned} \tag{3}$$

Worst case of the NFDH is we always create a new unnecessary level like the following graph:



Graph 2: Worst case

In this case, $OPTIMAL(\mathcal{L}) = \frac{1}{4}n+1$ and $NFDH(\mathcal{L}) = \frac{n}{2}$, so we can always find a big enough n to to make approximation arbitrarily close to 2, which mean **NFDH is a 2-approximation algorithm** and the constant 2 is tight.

2.1.2 Time Complexity

We use the STL function `sort()`, so the time complexity of sorting is $\mathcal{O}(logN)$ and we use `erase()` in the loop:

```
Function next_fit_decreasing_height(rectangles_list arr_rect, container_width width):
    Sort arr_rect by the second element (height) in descending order using cmp_1
    Set total_height = 0   // To store the total height used
    Set remaining_width = 0   // To track the remaining width in the current container
    While arr_rect is not empty:
        Take the first rectangle t from arr_rect
        Remove the first rectangle from arr_rect
        If remaining_width >= t.width:
            Update remaining_width by subtracting t.width
        Else:
            Update total_height by adding t.height
            Set remaining_width = width - t.width   // Start a new container with this rectangle

    Return total_height
```

So the total complexity equals to the complexity of the loop, which is $\mathcal{O}(logN)$

2.2 First Fit Decreasing Height

2.2.1 Approximation

This algorithm works similar to the NFDH algorithm. However, when placing the next item, the algorithm scans the levels from bottom to top and places the item in the first level on which it will fit. We only new a level when there's no space for the new item. It satisfies:

$$FFDH(\mathcal{L}) \leq 1.7\ OPTIMAL(\mathcal{L}) + 1 \tag{4}$$

The proof is based on the analogous proof for the First-Fit algorithm of one-dimensional bin-packing, we need to introduca a weight function:

$$W(x) = \begin{cases} \frac{6}{5}x & \text{if } 0 \leq x \leq \frac{1}{6} \\ \frac{9}{5}x - \frac{1}{10} & \text{if } \frac{1}{6} < x \leq \frac{1}{3} \\ \frac{6}{5}x + \frac{1}{10} & \text{if } \frac{1}{6} < x \leq \frac{1}{2} \\ \frac{6}{5}x + \frac{4}{10} & \text{if } \frac{1}{2} < x \leq 1 \end{cases} \tag{5}$$

We extend this function to rectangles by $W(r) = W(w(r))$, and set $A = \sum_{r \in \mathcal{L}} h(r) \cdot W(r)$.It is proved in that no collection of numbers x summing to 1 or less can have $W(x)$ summing to more than 1.7. We can apply this result to our case by cutting the optimal packing into horizontal slices. And summing all the slices we get $A \leq 1.7\ OPTIMAL(\mathcal{L})$.

Then to prove our thoery, we only need to prove that $A \geq FFDH(\mathcal{L}) - 1$. Let T_1 and T_2 the set of blockswhose first rectangle has width no more than or greater than $\frac{1}{2}$. Ant we define H_i as the total height of all blocks that satisfy T_i. Since the first item in block B_i has height H_i, and due

to our normalization assumptions, we have

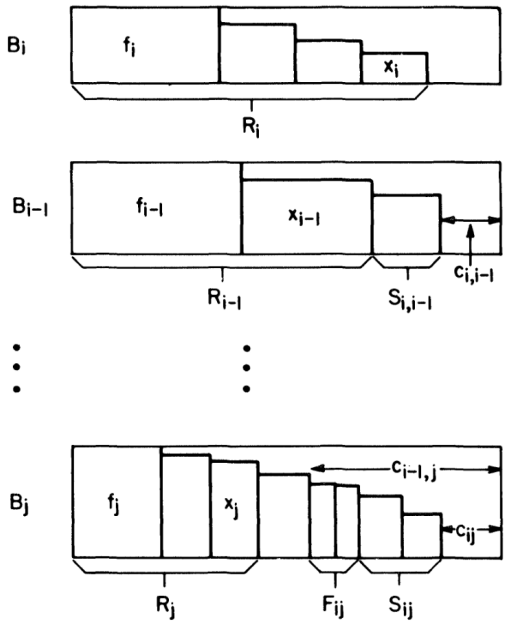
$$\sum_{B\in T_2}\sum_{r\in B} h(r)\cdot W(r) \geq H_2 \tag{6}$$

Now to reach the end, we only need to prove

$$\sum_{B\in T_1}\sum_{r\in B} h(r)\cdot W(r) \geq H_1 - 1 \tag{7}$$

We need to introduce a lot of definitions to help finishing that proof. Let L_1 be the rectangles in blocks of T_1. Note that the FFDH packing of L_1 would be similar to T_1. And now we denote $H_{t+1} = 0$ by convention. And we now divide rectangles into regular and fallback while f_i and g_i represents the first and last regular rectangle in B_i and R_i be the set of all regular rectangles in B_i. Moreover, we define F_{ij} be the set of all fallback rectangles in B_j before f_i but after g_{i-1} $(0 \leq i \leq t,\ 0 \leq j \leq i)$. Similarly we define S_{ij} of all falback items in Bi after f_i but before g_i. Note that F_{ij} and S_{ij} are disjoint and $F_{i,i-1} =$, so we have

$$L_1 = \cup_{i=1}^{t}[R_i \cup \cup_{j<i} F_{ij} \cup \cup_{j<i} S_{ij}] \tag{8}$$



Graph 3: An illustration figure to show the set relationship

Define c_{ij} to be the width of the empty space at the bottom right end of B_j. when the last regular item is packed in B_i. And in the picture there always holds relations:

$$\begin{aligned} w(f_i) &\geq c_{ij} + \sum_{r\in S_{ij}} w(r), \quad 1 < i \leq t \text{ and } 1 \leq j < i \\ c_{ij} &= c_{i-1,j} - \sum_{r\in S_{ij}\cup F_{ij}} w(r), \quad 2 < i \leq t \text{ and } 1 \leq j < i \end{aligned} \tag{9}$$

Define that $c_i = max_{j<i} c_{ij}$, we need to use a lemma that for every $i \in (1, t)$ exists $i' < i$ satisfies

$$\sum_{r \in R_{i-1}} W(r) + \frac{6}{5} \sum_{r \in F_{ii'} \cup S_{ii'}} w(r) \geq 1 + \frac{6}{5}(c_{i-1} - c_{ii'}) \tag{10}$$

We observe that

$$\begin{aligned} \sum_{r \in L_1} h(r) \cdot W(r) \geq & \sum_{i=1}^{t} H_{i+1} \sum_{r \in R_i} W(r) + \sum_{i=1}^{t} W(f_i)(H_i - H_{i+1}) \\ & + \sum_{i=2}^{t} H_i \sum_{r \in R_{ii'}} W(r) + \sum_{i=2}^{t} H_{i+1} \sum_{r \in S_{ii'}} W(r). \end{aligned} \tag{11}$$

Then we use the lemma to get

$$\begin{aligned} \sum_{i=2}^{t} \left[H_i \sum_{r \in R_{i-1}} W(r) + \frac{6}{5}(H_i - H_{i+1}) \right] \left(c_{ii'} + \sum_{r \in S_{ii'}} w(r) \right) + \frac{6}{5} H_i \sum_{r \in F_{ii'}} w(r) + \frac{6}{5} H_{i+1} \sum_{r \in S_{ii'}} w(r) \Bigg] \\ \sum_{i=2}^{t} \left[H_i \sum_{r \in R_{i-1}} W(r) + \frac{6}{5}(H_i - H_{i+1}) c_{ii'} + \frac{6}{5} H_i \sum_{r \in F_{ii'} \cup S_{ii'}} \right] \\ \sum_{i=2}^{t} [H_i + \frac{6}{5} H_i (c_{i-1} - c_{ii'}) + \frac{6}{5}(H_i - H_{i+1}) c_{ii'}], \end{aligned} \tag{12}$$

which means

$$\begin{aligned} \sum_{r \in L_1} h(r) \cdot W(r) \equiv & \sum_{i=2}^{t} \left[H_i - \frac{6}{5} H_{i+1} c_{ii'} + \frac{6}{5} H_i c_{i-1} \right] \\ & H(T_1) - H_1 - \frac{6}{5} \sum_{i=2}^{t} H_{i+1} c_{ii'} + \frac{6}{5} \sum_{i=1}^{t-1} H_{i+1} c_i \\ & H(T_1) - H_1 + \frac{6}{5} \sum_{i=2}^{t-1} H_{i+1} (c_i - c_{ii'}) H(T_1) - 1, \end{aligned} \tag{13}$$

and by our normalization assumption, here comes to the end, which means that **FFDH is a 1.7-approximation algorithm**.

2.2.2 Time Complexity

Like the NFDH, FFDH consists of sorting and putting. All details are the same as NFDH except we need to check each level when putting the item, which takes a complexity of constant. So the time complexity is also $\mathcal{O}(N^2)$

3 Testing Results

We design a data genrator that can generates N rectangles whose width are within the strip width. We choose $N = 10,\ 50,\ 100,\ 500,\ 1000,\ 5000,\ 10000$ and some different boxes' width $w = 50,\ 100,\ 500$.

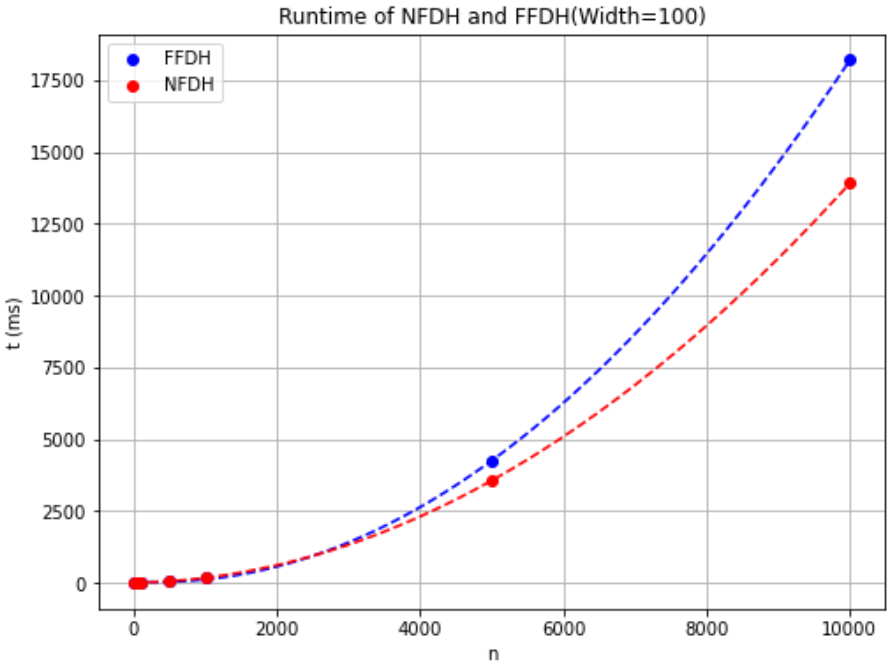
3.1 Runtime Analysis

The theoretical complexity of FFDH and NFDH are the same, but it is clear that in FFDH we do branch operations, which means a bigger constant, the testing results show the same trend.

Since the width of the strip won't influence the runtime so much, so here only show the data of a width of 100:

Numbers	10	50	100	500	1000	5000	10000
FFDH/TIME(ms)	9.5	10.5	13.6	71	190	4219	18196
NFDH/TIME(ms)	8.8	9.6	13.2	56	159	3565	13900

Figure 1: FFDH/TIME vs. NFDH/TIME (Transposed)



Graph 4: Runtime comparion(width=100)

3.2 Approximation Analysis

See the data in attached folder, here we only show the figure. To inspect our approximation analysis, we mainly see two point, whether the ascending trend of the algorithm in H is nearly linear in big N and whether it satisfies about $\frac{H(FFDH)}{H(NFDH)} \approx 0.85$ in big N.(In small N, FFDH would degenerate into almost NFDH). Here's the result for three widths:

Numbers	10	50	100	500	1000	5000	10000
FFDH/NFDH(width=50)	0.9283	0.8452	0.8468	0.8342	0.8371	0.8313	0.8326
FFDH/NFDH(width=100)	0.8920	0.8678	0.8931	0.8294	0.8231	0.8189	0.8188
FFDH/NFDH(width=500)	0.9649	0.7846	0.8035	0.8077	0.7964	0.8018	0.8005

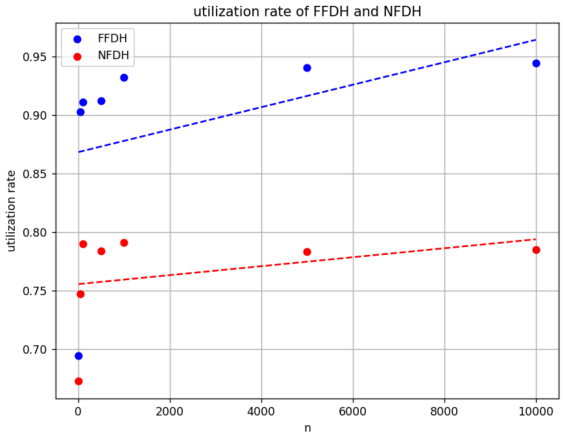
Figure 2: Height Ratio of FFDH and DFDH

And here are the detailed figures:

We can see that the trend of height is linear and the ratio is just within our expectation. In big width, the ratio becomes smaller since our generator generates typically small rectangles. So

the reuse of blocks under the top is more usual in FFDH, lowering height and the ratio.

We also draw a figure of utilization in strip of the two algorithms under $width = 50$, we can see the result with our expectation that FFDH use the space more efficiently. That's because in NFDH we never use lower levels again, causing much waste:



Graph 5: Utilization of FFDH and NFDH

4 Bonus

4.1 Introduction

We now use tetrominos instead of rectangles. A tetromino is a geometric shape composed of four squares, connected orthogonally. We now need to pack the strip with those tetreminos.

4.2 Algorithm

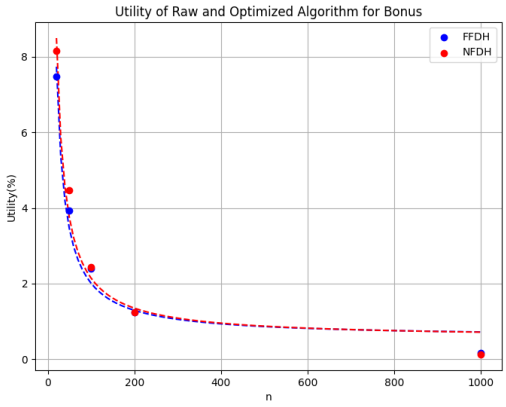
A really simple idea is to turn the tetreminos into rectangles, so we just see a tetreminos's four sides as a rectangle and then use NFDH and FFDH. Clearly it would be a worse than 2 or 1.7 approximation algorithm since the optimal become smaller. And we design a randomly-generated tetremino whose shape can be input. The random generator ensure all tetreminos have a limited size and can be different.

4.3 Optimization

We implement a simple optimization - assemble some tetreminos together to become a rectangle so that space can be less wasted. The optimized ration depends on the tetreminos structure.

4.4 Testing Results

We run $N = 10,\ 50,\ 100,\ 1000,\ 5000,\ 10000$, and in small case, the packing is visualized. and here is the utility between raw and optmized algorithm:



Graph 6: Utility between raw and optimized

We can see that our optimization makes sense and the utility becomes lower as the N becomes bigger. However, in extreme cases, optimization effects utility negatively because our optimization is a greedy policy indeed. Assembling tetreminos may get a really high rectangle, which ruins the utility.

4.5 Potential Optimization

Our algorithm is still so simple that the utility isn't high enough, we can further optimize by:

- iteratively running the assembling part to generate more rectangles
- Introduce DP to lower the waste of assembling(try to generate more rectangles have the same height or say)
- Design new packing algorithm since that view tetreminos as rectangles is not wise

5 Conclusion

In this project, we do theoretical analysis of the NFDH and FFDH algorithm to partly solve the NP problem - strip packing in a finite time. And by using a data generator to examine our program in datasets of different scales, we check that our theoretical analysis of complexity and approximation is right.