問9 H13

(1) $\hat{Y}$がμの不偏推定量 $\Longleftrightarrow E[\hat{Y}]=\mu$ —— (P1)

$$E[\hat{Y}] = E\left[\sum_{i=1}^{2n}\alpha_i X_i\right] \underset{\substack{\uparrow \\ E\text{の線形性}}}{=} \sum_{i=1}^{2n}\alpha_i E[X_i] \underset{\substack{\uparrow \\ \text{仮定}}}{=} \mu\sum_{i=1}^{2n}\alpha_i$$ より、

$$(P1) \Longleftrightarrow \mu\left(\sum_{i=1}^{2n}\alpha_i - 1\right) = 0 \Longleftrightarrow_{\text{and}} \begin{cases} \mu = 0 \Rightarrow (P1)\text{は常に成立} \\ \mu \neq 0 \Rightarrow \sum_{i=1}^{2n}\alpha_i = 1 \end{cases}$$

この時、

$$V[\hat{Y}] = E[\hat{Y}^2] - \underbrace{E[\hat{Y}]^2}_{\mu^2}$$

$$E[\hat{Y}^2] = E\left[\left(\sum_{i=1}^{2n}\alpha_i X_i\right)^2\right] = E\left[\sum_{i=1}^{2n}\sum_{j=1}^{2n}\alpha_i\alpha_j X_i X_j\right] \underset{\uparrow}{=} \sum_{i=1}^{2n}\sum_{j=1}^{2n}\alpha_i\alpha_j E[X_iX_j] = \mu^2\sum_{i=1}^{2n}\sum_{j=1}^{2n}\alpha_i\alpha_j$$

Eの線形性
$E[X_i]=\mu$ $(\forall i)$
~~各~~ X_iは独立

$\times$
$E[X_iX_i] \neq \mu^2$ #

$$= \mu^2\sum_{i=1}^{2n}\alpha_i\underbrace{\sum_{j=1}^{2n}\alpha_j}_{1} = \mu^2\underbrace{\sum_{i=1}^{2n}\alpha_i}_{1} = \mu^2$$

~~より V[Ŷ] = 0 ?~~

$$E[\hat{Y}^2] = \sum_{i=j}\underbrace{\cancel{\alpha_i\alpha_j}}_{\alpha_i^2}\underbrace{E[X_i^2]}_{\sigma^2+\mu^2} + \sum_{i\neq j}\alpha_i\alpha_j\underbrace{E[X_i]E[X_j]}_{\mu^2}$$

$$= \sigma^2\sum_{i=1}^{2n}\alpha_i^2 + \underbrace{\mu^2\sum_{i=1}^{2n}\alpha_i^2 + \mu^2\sum_{i\neq j}\alpha_i\alpha_j}_{\mu^2\underbrace{\sum_{i=1}^{2n}\sum_{j=1}^{2n}\alpha_i\alpha_j}_{\underbrace{\sum_{i=1}^{2n}\alpha_i}_{1}\cdot 1}} = \sigma^2\sum_{i=1}^{2n}\alpha_i^2 + \mu^2$$ より、

$$V[\hat{Y}] = E[\hat{Y}^2] - E[\hat{Y}]^2 = \sigma^2\sum_{i=1}^{2n}\alpha_i^2$$ //

☆ better

$$V[\hat{Y}] = V\left[\sum_{i=1}^{2n}\alpha_i X_i\right] \underset{\substack{\uparrow \\ V\text{の性質}}}{=} \sum_{i=1}^{2n}\alpha_i^2\underbrace{V[X_i]}_{\sigma^2} = \sigma^2\sum_{i=1}^{2n}\alpha_i^2$$ //

(2) $\operatorname{argmin} V[\hat{Y}] = \operatorname{argmin}\sum_{i=1}^{2n}\alpha_i^2$ であり、ラグランジュの未定乗数法より、

$$g(\alpha_1,\alpha_2,\cdots\alpha_{2n},\lambda) := \sum_{i=1}^{2n}\alpha_i^2 - \lambda\left(\sum_{i=1}^{2n}\alpha_i - 1\right)$$

$$\frac{\partial g}{\partial \alpha_k} = 2\alpha_k - \lambda = 0 \Longleftrightarrow \alpha_k = \frac{\lambda}{2} \quad (\forall k = 1, 2, \cdots, 2n)$$

$\frac{\partial g}{\partial \lambda} = 0$に代入して、$\sum_{i=1}^{2n}\alpha_i = 2n\cdot\frac{\lambda}{2} = n\lambda = 1 \quad \therefore \lambda = \frac{1}{n}$

$\therefore \alpha_k = \frac{1}{2n}$ $(\forall k)$ のとき、最小値をとることがわかる

このとき、

$$V[\hat{Y}] = \sigma^2\cdot 2n\cdot\frac{1}{(2n)^2} = \frac{\sigma^2}{2n}$$