H31-5

(1)

$$E(g'(X)) = \int_0^\infty g'(x)f(x)\,dx = \mu\int_0^\infty g'(x)e^{-\mu x}dx = \mu\left[g(x)e^{-\mu x}\right]_0^\infty + \mu\int_0^\infty g(x)\mu e^{-\mu x}dx$$

$$= \mu\int_0^\infty g(x)f(x)\,dx = \mu E(g(X))$$

(2)

$$g'(x) = \begin{cases} \frac{1}{h} & a \le x < a+h \\ 0 & \text{o.w.} \end{cases}$$

fの原始関数を$F(x) = \int_{-\infty}^{x} f(x)dx$とする。

$$E(g'(X)) = \int_a^{a+h}\frac{1}{h}f(x)\,dx = \frac{F(a+h)-F(a)}{h} \longrightarrow f(a) \quad (h\to 0)$$

$$E(g(X)) = \int_a^{a+h}\frac{x-a}{h}f(x)dx + \int_{a+h}^\infty f(x)\,dx$$

$$= \frac{1}{h}\int_a^{a+h}xf(x)\,dx - \frac{a}{h}\int_a^{a+h}f(x)dx + \int_{a+h}^\infty f(x)dx$$

$$= \underbrace{\frac{1}{h}\left[xF(x)\right]_a^{a+h}}_{F(a+h)+a\cdot\frac{F(a+h)-F(a)}{h}} - \frac{1}{h}\int_a^{a+h}F(x)\,dx - \frac{a}{h}\int_a^{a+h}f(x)dx + \int_{a+h}^\infty f(x)dx.$$

$$\longrightarrow \cancel{F(a)} + \cancel{a\cdot f(a)} - \cancel{F(a)} - \cancel{af(a)} + 1 - F(a)$$

より、

$$\begin{cases} F'(a) = \mu(1-F(a)) \\ F(0) = 0 \end{cases} \quad (a \ge 0)$$

これを解くと、

$$F(x) = 1 - e^{-\mu x} \quad (x \ge 0)$$

$$\therefore f(x) = \mu e^{-\mu x}$$