H16問8

(1)

$P(X_{(1)} \le x) = P(\min\{X_1,\cdots,X_n\} \le x)$

$= 1 - P(\min\{X_1,\cdots,X_n\} > x)$

$= 1 - P(X_1 > x, X_2 > x, \cdots, X_n > x)$

$= 1 - P(X_1 > x)\cdots P(X_n > x)$

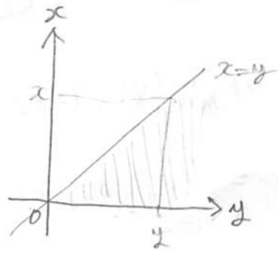
$= 1 - (e^{-x})^n = 1 - e^{-nx}$

一方、$P(\frac{X_n}{n} \le x) = P(X_n \le nx) = 1 - e^{-nx}$ $(x \ge 0)$

(2)

$Y_n \triangleq \frac{X_n}{n}$, $Y_{n-1} = \frac{X_{n-1}}{n-1}$ $f_{Y_n}(x) = \frac{\partial}{\partial x}P(Y_n \le x) = ne^{-nx}$ $(x \ge 0)$

$\cancel{P(Y_n + Y_{n-1} \le x)}$ $f_{Y_n+Y_{n-1}}(x)$ $= \int_{-\infty}^{\infty} f_{Y_n}(x-y) f_{Y_{n-1}}(y)\,dy$



$x - y \ge 0$
$y \ge 0$

$= \int_0^x ne^{-n(x-y)} \cdot (n-1)e^{-(n-1)y}\,dy$

$= n(n-1)\int_0^x \exp(-nx + \cancel{ny} - \cancel{ny} + y)\,dy$

$= n(n-1)\int_0^x e^{-nx+y}\,dy$

$= n(n-1)\underline{e^{-nx}}\underbrace{\int_0^x e^y\,dy}_{e^x - e^0} = n(n-1)e^{-nx}(e^x - 1)$ // $(x \ge 0)$

・$P(Y_n + Y_{n-1} \le x) = \int_0^x f_{Y_n+Y_{n-1}}(x)\,dx = n(n-1)\int_0^x (e^x - 1)e^{-nx}\,dx$ $\cancel{= n(n-1)\cdot[e^x - x]_0^x}$

$\cancel{= n(n-1)(e^x - x - 1)}$ ~~//~~ $\cancel{(x \ge 0)}$

$\cancel{P(Y_n + Y_{n-1} \le x) = 0}$ $(x < 0)$

$= n(n-1)\int_0^x (e^{-(n-1)x} - e^{-nx})\,dx$

$= n(n-1)\left[-\frac{1}{n-1}e^{-(n-1)x} + \frac{1}{n}e^{-nx}\right]_0^x$ $\frac{-n+n-1}{n(n-1)}$

$= n(n-1)\left\{-\frac{1}{n-1}e^{-(n-1)x} + \frac{1}{n}e^{-nx} - \left(-\frac{1}{n-1} + \frac{1}{n}\right)\right\}$

$= -ne^{-(n-1)x} + (n-1)e^{-nx} + 1$ // $(x \ge 0)$

・$P(Y_n + Y_{n-1} \le x) = 0$ $(x < 0)$

■

(3) $X_{(1)} = X_k$ $(k \in \{1, \cdots, n\})$とすると、 $X_{(2)} = \min\{X_1, \cdots, X_{k-1}, X_{k+1}, \cdots X_n\} = \min\{X_i ; i = 1, 2, \cdots, k-1, k+1, \cdots, n\}$

$$P\{X_{(2)} > x, X_{(1)} \le y\} = P\{\min\{X_i ; i \ne k\} > x, X_{(1)} \le y\}$$

$$= P(X_1 > x, \cdots, X_{k-1} > x, X_{k+1} > x, \cdots, X_n > x, X_k \le y) \quad \text{—} (*)$$

$$= \sum_{k=1}^{n} P(X_1 > x, \cdots, X_{k-1} > x, X_{k+1} > x, \cdots, X_n > x, X_k \le y)$$ (∵) kの任意性と、kの各場合が排反.

$$= nP(X_1 > x, \cdots, X_{k-1} > x, X_{k+1} > x, \cdots, X_n > x, X_k \le y),$$ (対称性)

$$= nP(X_2 > x, \cdots, X_n > x, X_1 \le y)$$ (∵特にk=1を用いる)

$$= nP(\min\{X_i ; i = 2, \cdots, n\} > x, X_1 \le y)$$

(4)

$$P(X_{(2)} > x) = \lim_{y \to +\infty} P(X_{(2)} > x, X_{(1)} \le y) = \lim_{y \to +\infty} nP(\min(X_2, \cdots, X_n) > x, X_1 \le y)$$

× $0 \le y \le x$

$$P(\min\{X_i ; i = 2, \cdots, n\} > x, X_1 \le y) = P(X_2 > x, \cdots, X_n > x, X_1 \le y)$$

$$= \prod_{i=2}^{n} P(X_i > x) \cdot P(X_1 \le y) = \left(\prod_{i=2}^{n} e^{-x}\right) \cdot (1 - e^{-y}) = e^{-nx}(1 - e^{-y})$$ より

$$P(X_{(2)} > x) = ne^{-nx}(1 - 0) = ne^{-nx}$$

×

$0 \le y \le x$ で $P(X_{(2)} > x, X_{(1)} \le y) > 0$ なので、

$+ P(X_{(2)} > X_{(1)} > x) = P(X_{(1)} > x) = e^{-nx}$.

$$P(X_{(2)} > x) = P(X_{(2)} > x, \underline{X_{(1)} \le x}) = nP(\min(X_2, \cdots, X_n) > x, X_1 \le x)$$

★ここが違う.

$0 \le y \le x$ だから $0 \le X_{(1)} \le x$.

$$= nP(X_1 \le x, X_2 > x, \cdots, X_n > x)$$

$$= n \cdot P(X_1 \le x) \cdot \prod_{i=2}^{n} P(X_i > x)$$

$$= n \cdot (1 - e^{-x}) \cdot (e^{-x})^{n-1}$$

$$= n(1 - e^{-x})e^{-(n-1)x}$$

$$= \cancel{ne^{-(n-1)x} - ne^{-nx}}$$

≠0?

・$P(X_{(2)} > x) = P(X_{(2)} > x, 0 \le X_{(1)} \le x) = P(X_{(2)} > x, X_{(1)} \le x) - P(X_{(2)} > x, \underline{X_{(1)} < 0})$

でも0か.

×

$= nP(\min(X_2, \cdots, X_n) > x, X_1 \le x) - nP(\min$

・$P(X_{(2)} > x) = P(X_{(2)} > x, X_{(1)} \le x) =$

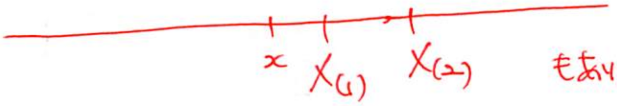
$$= (n - ne^{-x})e^{-(n-1)x} = ne^{-(n-1)x} - ne^{-x-nx+x}$$

$$= ne^{-(n-1)x} - ne^{-nx}$$

$$P(X_{(2)} \leq x) = 1 - P(X_{(2)} > x)$$

$$= 1 - ne^{-(n-1)x} + ne^{-nx}$$

じゃっかん違う → $P(X_{(2)} > x)$ は



もありえるので

∴

$$\therefore P(X_{(2)} > x) = P(X_{(2)} > x, X_{(1)} \leq x) + P(X_{(2)} > X_{(1)} > x)$$

$$= ne^{-(n-1)x} - (n-1)e^{-nx}$$

$$\therefore P(X_{(2)} \leq x) = 1 - ne^{-(n-1)x} + (n-1)e^{-nx} \quad //$$

$$= P(Y_n + Y_{n-1} \leq x) \quad //$$