H19問1

(1)

$z_1 = (S-\alpha_n E)z$

$z_2 = (S-\alpha_{n-1}E)(S-\alpha_n E)z$

$z_n = (S-\alpha_1 E)(S-\alpha_2 E)\cdots(S-\alpha_{n-1}E)(S-\alpha_n E)z$

$(\forall z)$

・① $i=1$のとき、

$$z_i = z_1 = (S-\alpha_n E)z = \begin{pmatrix} \alpha_1-\alpha_n & & & & * \\ & \alpha_2-\alpha_n & & & \\ & & \ddots & & \\ & 0 & & \alpha_{n-1}-\alpha_n & \\ & & & & \alpha_n-\alpha_n \end{pmatrix} z$$

z_iの第 $n-i+1 = n$番目は、

$S-\alpha_n E$が上三角形かつ、

特に、第n,n成分は、0 なので、第n行は全て0

$\therefore\ z_{1,n}=0$.

② $i=k$のとき成立と仮定

z_kの $z_k = \begin{pmatrix} z_{k,1} \\ \vdots \\ z_{k,n-k+1} \\ \vdots \\ z_n \end{pmatrix}$ $\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} = 0$ となっている。

$z_k = (S-\alpha_{n-k+1}E)\cdots(S-\alpha_n E)z$

$z_{k+1} = (S-\alpha_{n-(k+1)+1}E)(S-\alpha_{n-k+1}E)\cdots(S-\alpha_n E)z$

$= (S-\alpha_{n-k}E)z_k z$

$$= \begin{pmatrix} \alpha_1-\alpha_{n-k} & & & * \\ & \ddots & & \\ & & \alpha_{n-k-1}-\alpha_{n-k} & \\ & & 0 & \\ 0 & & \alpha_{n-k+1}-\alpha_{n-k} & \\ & & & \ddots \\ & & n-k & \alpha_n-\alpha_{n-k} \end{pmatrix} \begin{pmatrix} z_{k,1} \\ \vdots \\ z_{k,n-k} \\ 0 \\ \vdots \\ 0 \end{pmatrix} z = 0$$

・ ~~$(z_{k+1})_{n-k+1} \sim n$~~ は、

(2)

$S = P^{-1}AP \rightarrow PSP^{-1} = A$

$Az = \alpha z$

$(A - \alpha_1 E) = (PSP^{-1} - \alpha_1 E)$

$z_n = (S - \alpha_1 E) \cdots (S - \alpha_n E) z = 0$

$\overset{P^{-1}AP}{}$

$\alpha_n z = Az$

$z_1 = (S - \alpha_1 E) z$

$P^{-1}AP$

A

$= (S - Az)$

$z_1 = (P^{-1}AP - Az)$

$$(P^{-1}AP - \alpha_1 P^{-1}P)(P^{-1}AP - \alpha_2 P^{-1}P) \cdots (\overset{P^{-1}AP}{S} - \alpha_n P^{-1}P) z = 0$$

$$\Rightarrow P^{-1}(A - \alpha_1 E)PP^{-1}(A - \alpha_2 E)PP^{-1}(A - \alpha_n E)Pz = 0$$

$$\cancel{P^{-1}}\underbrace{(A - \alpha_1 E) \cdots (A - \alpha_n E)}_{B}\underbrace{Pz}_{y\ (\forall y)} = 0$$

$By = 0\ (\forall y) \Rightarrow B = 0$

(3)

$f(x) = c_0 x^m + c_1 x^{m-1} + \cdots + c_{m-1} x + c_m \quad (c_0 \neq 0)$

$f(A) = c_0 A^m + c_1 A^{m-1} + \cdots + c_{m-1} A' + c_m E$: ~~$f(A) = (A^m - \alpha_1 E) - (\quad) = 0$~~

$m > n$ かつ $f(A) \neq 0$ のとき、 $f(A) = 0$ とすると

$(A - \alpha_1 E) \cdots (A - \alpha_n E) = 0$

示 $m > n$ fixed $\wedge$ $f(A) \neq 0 \Rightarrow \exists g \in P(n-1),\ f(A) = g(A)$

対 ~~$\forall g \in P(n-1),\ f(A) \neq g(A) \Rightarrow m \leq n$~~

~~$f(A) = 0$ とすると、$\exists g \in P(n)$~~ 以下, ~~$g(A) = f(A) = (A - \alpha_1 E) \cdots (A - \alpha_n E)$~~ $= 0$ がある。

$(x - \alpha_1)(x - \alpha_2) \cdots$

g $f(A) = 0$ とすると、$f(A) = 0 = (A - \alpha_1 E) \cdots (A - \alpha_n E)$ より

$\exists g(x) = (x - \alpha_1)(x - \cdots (x - \alpha_n)$: 次数が n 以下

$f(x) = g(x)\overset{\exists}{p} + \overset{\exists}{r}$ $\quad f(A) = \underset{=}{g}(A)$

$h(x) \triangleq (x-\alpha_1)\cdots(x-\alpha_n)$ $\quad \deg h = n$ $\quad \deg f = m$.

除法定理より、

$f(x) = h(x)q(x) + r(x)$, $\exists q, r \in \mathrm{Polynomial}$, $\deg q > \deg r$.

$f(A) = h(A)q(A) + r(A)$

$= r(A)$

$m > n$

最大で $m = n + \deg q$ より、 $\deg = m - n > \deg r$.

$m = n \times q + r$.