H20問1

(1) $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 0 & 2-\lambda & -1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 2-\lambda & -1 \end{vmatrix}.$$

$$= (1-\lambda)\underbrace{\left\{(2-\lambda)(1-\lambda)+1\right\}}_{\lambda^2-3\lambda+3} + \underbrace{(1-(2-\lambda))}_{\lambda-1}$$

$$= \cancel{-(\lambda}\ (1-\lambda)\{\lambda^2-3\lambda+\underbrace{\cancel{3-1}}_{+2}\}$$

$$= (1-\lambda)(\lambda-2)(\lambda-1) = 0 \quad \therefore \lambda = 1, 2, 1$$

単根 $\lambda_1 = 2$

重根 $\lambda_2 = 1$

・λ_1 $(A-\lambda I)x = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{cases} -x_1-x_2+x_3 = 0 \\ -x_3 = 0. \end{cases} \begin{cases} x_3 = 0 \\ x_1 = -x_2. \end{cases}$

$c\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $(c \neq 0)$ 」

・λ_2 $(A-\lambda I)x = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{cases} x_2-x_3 = 0 \\ x_1+x_2 = 0 \end{cases} \begin{cases} x_3 = x_2 \\ x_1 = -x_2 \end{cases}$

$\begin{pmatrix} -c \\ c \\ c \end{pmatrix} = c\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ $(c \neq 0)$ 」

(2) P^{-1} $P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}$ $\quad P = (x_1\ x_2\ x_3)$

~~$Ax_3 =$~~ $AP = P \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix} = \left(P\begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix}, P\begin{pmatrix} 0 \\ \lambda_2 \\ 0 \end{pmatrix}, P\begin{pmatrix} 0 \\ 1 \\ \lambda_2 \end{pmatrix} \right)$

$\quad\ \ \| $

$(x_1 x_2 x_3)$

$(Ax_1, Ax_2, Ax_3)\ \ (\lambda_1 x_1, \lambda_2 x_2, \cancel{\lambda} x_2 + \lambda_3 x_2)$

$$\begin{cases} Ax_1 = \lambda_1 x_1 \\ Ax_2 = \lambda_2 x_2 \\ Ax_3 = x_2 + \lambda_2 x_3 \end{cases} \quad //$$

(3) $x_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},$ $\quad Ax_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + I x_3$

$(A - I)x_3 =$

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \iff \begin{cases} -y_2 + y_3 = -1 \\ y_1 + y_2 = 1 \end{cases} \iff \begin{cases} y_3 = -1 + y_2 \\ y_1 = 1 - y_2 \end{cases}$$

$$\begin{pmatrix} 1-c \\ c-1 \\ c \\ c-1 \\ 1-c \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & -1 & +1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \Bigg| \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ +1 & +1 & 0 \end{pmatrix} = P^{-1}$$

(4) $(P^{-1}AP)^n = P^{-1}A^nP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^2 & 0 & 0 \\ 0 & 1^2 & 1^2+1^2 \\ 0 & 0 & 1^2 \end{pmatrix}$

$$\begin{pmatrix} \lambda_1^2 & 0 & 0 \\ \cancel{0} & \cancel{\lambda_2^2} & \cancel{\lambda_2^2 + \lambda_2} \\ 0 & 0 & \lambda_2^2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ \cancel{0} & \cancel{\lambda_2} & \cancel{1} \\ 0 & 0 & \lambda_2 \end{pmatrix} \cancel{= \begin{pmatrix} \lambda_1 & & \\ 0, & \lambda_2^3, & \lambda_2^2 + \lambda_2^3 + 1 \end{pmatrix} \begin{pmatrix} & & 0 \\ 0 & \lambda_2 & 1 \\ & & \lambda_2 \end{pmatrix} \quad \lambda_2^3 + \lambda_2^3 + \lambda_4}$$

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 2\lambda_2 \\ 0 & 0 & \lambda_2^2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & \phi \\ 0 & 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^3 & 0 & 0 \\ 0 & \lambda_2^3 & \lambda_2^2 + 2\lambda_2^2 \\ 0 & 0 & \lambda_2^3 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1^3 & 0 & 0 \\ 0 & \lambda_2^3 & 3\lambda_2^2 \\ 0 & 0 & \lambda_2^3 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda_1^4 & 0 & 0 \\ 0 & \lambda_2^4 & \lambda_2^3 + 3\lambda_2^3 \\ 0 & 0 & \lambda_3^4 \end{pmatrix} = \underset{n}{\underline{4}}\,\lambda_2^4$$

$$\underset{I}{\underline{P^{-1}A^nP}} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & \cancel{4}n \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^n & 0 & 1 \\ 2^n & 1 & n+1 \\ -2^n & -1 & -n \end{pmatrix}\Bigg($$

$$\cancel{= \begin{pmatrix} 2^n & -2^n+1 & -2^n+1 \\ 2^n-1 & -2^n+6 & -2^n+5 \\ -2^n+1 & 2^n-5 & 2^n-4 \end{pmatrix}}$$

$$= \Bigg(2$$

$$A^n = P\begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}P^{-1} = \underline{\begin{pmatrix} 1 & -1 & +1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$= \cancel{\begin{pmatrix} 2^n \\ -2^n \end{pmatrix}} \qquad = \begin{pmatrix} 2^n & -1 & -n+1 \\ -2^n & 1 & n \\ 0 & 1 & n+1 \end{pmatrix}$$

$$2^n - 1 - n + 1 = 2^n - n$$

$$(A^n)_{11} = \sum_{i=1}^{3} b_{1i} c_{i1} = 2^n - 1 + n + 1 = 2^n$$