H27-6

(1) 連鎖公式より、

$$\frac{\partial v}{\partial \xi} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial \xi} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} = \frac{1}{2}(u_t + u_x)$$

$$\frac{\partial v}{\partial \xi \partial \eta} = \frac{\partial v_\xi}{\partial t}\frac{\partial t}{\partial \eta} + \frac{\partial v_\xi}{\partial x}\frac{\partial x}{\partial \eta} = \frac{1}{4}(u_{xx} + u_{tx} - u_{xt} - u_{tt}) = 0 \qquad (\because C^2\text{級}, u_{xx} = u_{tt})$$

(2)

(1)を解くと、ある関数 $h(\eta)$, $\varphi(\xi)$ を用いて、

$$V(\xi,\eta) = \underbrace{\int h(\eta)\,d\eta}_{\psi(\eta)\text{とおく。}} + \varphi(\xi) \quad \text{となる}$$

$$u(t,x) = \varphi(x+t) + \psi(x-t)$$
$$u_t(t,x) = \varphi'(x+t) - \psi'(x-t)$$

に初期条件を代入して、

$$\varphi(x) + \psi(x) = f(x)$$
$$\varphi'(x) - \psi'(x) = g(x) \longrightarrow \varphi(x) - \psi(x) = \int_0^x g(s)\,ds + C \qquad (C:\text{積分定数})$$

より、

$$\varphi(x) = \frac{1}{2}\left(f(x) + \int_0^x g(s)\,ds + C\right)$$
$$\psi(x) = \frac{1}{2}\left(f(x) - \int_0^x g(s)\,ds - C\right)$$

従って、

$$u(t,x) = \frac{1}{2}\left(f(x+t) + f(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

(3)

$$u(1,x) = \frac{1}{2}\left(f(x+1) + f(x-1)\right) + \frac{1}{2}\int_{x-1}^{x+1} g(s)\,ds$$
$$= 0 \qquad (\because |x+1|, |x-1| \geq 1)$$

(4)

$M(t) = t+1$ とすると、$|x| \geq M(t)$ となる $\forall x$ に対して、

$$|x+t|, |x-t| \geq \big||x| - |t|\big|$$
$$= |x| - |t|$$
$$\geq M(t) - t$$
$$= 1 \quad \text{より、}$$

$$u(t,x) = 0$$