問5 H29

(1) ~~1回目の試行 X_1 (表=1, 下=0)~~ 表の出るまでの回数 X

母関数

$$P(X=n) = (1-p)^{n-1}p \quad (n=1,2,\cdots)$$

$$E(X) = \sum_{i=1}^{\infty} i(1-p)^{i-1}p = p \cdot \frac{1}{p^2} = \frac{1}{p}$$

$$V(X) = \frac{1}{p^2} ?$$

$$E(X^2) = \sum_{i=1}^{\infty} i^2(1-p)^{i-1}p = ?$$

$$\sum x^n = \frac{1}{1-x}$$

$$\sum nx^{n-1} = \frac{1}{(1-x)^2}$$

$$\sum n(n-1)x^{n-2} = +3 \cdot \frac{1}{(1-x)^3}$$

$$\sum n^2x^{n-2} - \sum nx^{n-2} = \frac{1}{x(1-x)^2}$$

まじめ

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}$$

$$\sum_{n=2} n^2x^{n-2} - \sum_{n=2} nx^{n-2}$$

$$x^{-1}\sum_{n=2} n^2x^{n-1} - x^{-1}\sum_{n=2} nx^{n-1}$$

$$x^{-1}\sum_{n=1} n^2x^{n-1} - x^{-1}x^0 - x^{-1}\underbrace{\sum_{n=1} nx^{n-1}}_{\frac{1}{(1-x)^2}} + x^{-1}\cdot x^0$$

$$x^{-1}\sum_{n=1} n^2x^{n-1} - \cancel{x^{-1}} - x^{-1}\frac{1}{(1-x)^2} + \cancel{x^{-1}}$$

$$x^{-1}\sum_{n=1} n^2x^{n-1} = \frac{1}{x(1-x)^2} + \frac{2}{(1-x)^3}$$

~~$E(X^2) = p \cdot \frac{1}{p^2} + \frac{2(1-p)}{p^3}$~~

$$\sum_{n=1} n^2x^{n-1} = \frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3}$$

$$E(X^2) = p\left\{\frac{1}{p^2} + \frac{2(1-p)}{p^3}\right\} = \frac{1}{p} + \frac{2(1-p)}{p^2} \quad \left(= \frac{2-p}{p^2}\right)$$

$$\therefore V(X) = E(X^2) - E(X)^2 = \frac{1}{p} + \frac{2-2p-1}{p^2} = \frac{p+1-2p}{p^2} = \frac{1-p}{p^2}$$ //

(2) (i)

$$P(S_{k+1}-S_k=n)=\left(\frac{k}{N}\right)^{n-1}\left(\underset{\substack{\shortparallel\\ p}}{\frac{N-k}{N}}\right)^1 \quad (n=1,\cdots,\infty)$$

より

$$E(S_{k+1}-S_k)=\frac{1}{p}=\frac{N}{N-k}$$

$$V(S_{k+1}-S_k)=\frac{1-p}{p^2}=\frac{k}{N}\cdot\left(\frac{N}{N-k}\right)^2=\frac{Nk}{(N-k)^2} \quad //$$

(ii)

$$E(S_N)=E\Big(\underbrace{\sum_{k=1}^{N-1}(S_{k+1}-S_k)}_{(\cancel{S_2}-S_1)+(\cancel{S_3}-\cancel{S_2})+\cdots+(S_N-\cancel{S_{N-1}})}+S_1\Big)=E(S_1)+\sum_{k=1}^{N-1}E(S_{k+1}-S_k)$$

$=$ or S_0 も含められる. $E\left(\sum_{k=0}^{N-1}(S_{k+1}-S_k)\right)=\sum_{k=0}^{N-1}\frac{N}{N-k}$

$$=\frac{N}{N}+\frac{N}{N-1}+\frac{N}{N-2}+\cdots+\frac{N}{2}+\frac{N}{1}=N\sum_{i=1}^{N}\frac{1}{i} \quad //$$

$$V(S_N)=V\Big(\sum_{k=0}^{N-1}\underbrace{(S_{k+1}-S_k)}_{Y_k}\Big)$$

$$=\sum_{k=0}^{N-1}V(\underbrace{S_{k+1}-S_k}_{Y_k})+\underbrace{\sum_{i\neq j}\mathrm{Cov}(Y_i,Y_j)}_{2\sum_{i>j}\mathrm{Cov}(Y_i,Y_j)}.$$

・ $\underset{i\neq j}{\mathrm{Cov}(Y_i,Y_j)}=E(Y_iY_j)-E(Y_i)E(Y_j)=\sum_{i,j}$

・ $S_{k+1}-S_k$ は互いに独立…

$$\Rightarrow V(S_N)=\sum_{k=0}^{N-1}\underbrace{V(S_{k+1}-S_k)}_{\frac{Nk}{(N-k)^2}}=\left(\sum_{k=1}^{N-1}\frac{k}{(N-k)^2}\right)N=N\sum_{l=1}^{N-1}\frac{N-l}{l^2} \quad //$$

$l=(N-k)$