問12 8 H15

(1) $U_1, U_2, \cdots \overset{iid}{\sim} \text{Uniform}(0,1)$

~~$X_n \triangleq \max\{U_1, U_2, \cdots\}$~~

$X_n \triangleq \max\{U_1, \cdots, U_n\}$, $F_n(x) \triangleq \Pr(X_n \le x)$

$F_n(x) = \Pr(\max(U_1, \cdots, U_n) \le x) = \Pr(U_1 \le x, U_2 \le x, \cdots, U_n \le x)$

$= \prod_{i=1}^{n} \Pr(U_i \le x) = \prod_{i=1}^{n} x = x^n \quad (x \in [0,1])$

$\therefore F_n(x) = \begin{cases} 1 & x \ge 1 \\ x^n & x \in [0,1] \\ 0 & \text{o.w. } x < 0 \end{cases}$ ⌋

(2) $Y_n \triangleq n(1 - X_n)$

$G_n(y) = \Pr(Y_n \le y) = \Pr(n(1-X_n) \le y) = \Pr(1 - \frac{y}{n} \le X_n) = 1 - \Pr(X_n < 1 - \frac{y}{n})$

$= 1 - F_n(1 - \frac{y}{n} - 0) = \begin{cases} 1 & x \ge 1 \\ 1 - (1 - \frac{y}{n})^n & \text{if } 0 \le 1 - \frac{y}{n} \le 1 \quad (0 \le y \le n) \\ 0 & \text{o.w. } x < 0 \end{cases}$ ⌋

$G(y) = \lim_{n \to \infty} G_n(y) = \begin{cases} 1 - e^{-y} & y \ge 0 \\ 0 & y < 0 \end{cases} \sim Po(1)$ ⌋

$\left(\begin{array}{l} (\because) \\ (1 + \frac{1}{n})^n \to e \text{ より } \lim_{n \to \infty} (1 + \frac{y}{n})^n = \lim_{m \to \infty} (1 + \frac{1}{m})^{my} = e^y \\ \qquad \frac{y}{n} = \frac{1}{m}, \ n = my \end{array} \right)$

(3)

$\Pr(Z_n \le z) = \Pr(-\max(V_1, \cdots, V_n) + \log n \le z)$

$= \Pr(\log n - z \le \max(V_1, \cdots, V_n))$

$= 1 - \Pr(\max(V_1, \cdots, V_n) < \log n - z)$

$= 1 - \prod_{i=1}^{n} \Pr(V_i < \log n - z)$

$= 1 - \prod_{i=1}^{n} (1 - e^{-\log n + z}) \quad (\text{if } \log n - z > 0)$

$= 1 - (1 - \frac{1}{n} e^{z})^n$

$\to 1 - e^{-e^z}$

$\Pr(V_i \le v) = \int_0^v e^{-v} dv = [-e^{-v}]$
$= -e^{-v} + 1$

$\therefore H(z) = 1 - e^{-e^z}$ // ⌋