問6 H27

$u_{tt} = u_{xx}$, $u(0,x) = f(x)$, $u_t(0,x) = g(x)$

(1) $\xi = x+t$, $\eta = x-t$, $V(\xi,\eta) = u(t,x)$ とおくとき、 $x = \frac{\xi+\eta}{2}$, $t = \frac{\xi-\eta}{2}$

$\frac{\partial^2 V}{\partial\eta\partial\xi} = 0$ を示す。

$$\frac{\partial V}{\partial \xi} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial \xi} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} = u_t \cdot \frac{1}{2} + u_x \cdot \frac{1}{2} = \frac{1}{2}(u_t + u_x)$$

$$\frac{\partial V}{\partial \eta} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial \eta} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} = u_t \cdot \left(-\frac{1}{2}\right) + u_x \cdot \frac{1}{2} = \frac{1}{2}(u_x - u_t)$$

$$\frac{\partial}{\partial \eta}\left(\frac{\partial V}{\partial \xi}\right) = \frac{\partial V_\xi}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial V_\xi}{\partial t}\frac{\partial t}{\partial \eta} = \frac{1}{2}(u_{tx} + u_{xx}) \cdot \frac{1}{2} + \frac{1}{2}(u_{tt} + u_{xt}) \cdot \left(-\frac{1}{2}\right)$$

$$= \frac{1}{4}(u_{xx} + u_{tx} - u_{tt} - u_{xt})$$

$$= \frac{1}{4}(u_{xx} - u_{tt}) \quad (\because C^2 級)$$

$$= 0 \quad (\because u_{xx} = u_{tt})$$

(2) $u(t,x)$ を $f(x)$, $g(x)$ で表す。

(1)より、 $\frac{\partial}{\partial \eta}\left(\frac{\partial V}{\partial \xi}\right) = 0$ を両辺 η で 積分して、 $\frac{\partial V}{\partial \xi} = C_1(\xi)$

両辺 ξ で積分すると、 $V = \underbrace{\int C_1(\xi)\,d\xi}_{= C_3(\xi)} + C_2(\eta) \rightarrow u(t,x) = V(\xi,\eta) =$

$u(0,x) = V(x,x) = f(x)$

$u_t(0,x) = \frac{\partial}{\partial t} V(x+t, x-t)$

$\cancel{\frac{\partial u}{\partial t} = \frac{\partial V}{\partial \xi}\frac{\partial \xi}{\partial t}}$

$u(t,x) = C_3(x+t) - C_2(x-t)$, $u_t(0,x) = C_3'(x) - C_2'(x) = g(x)$

$u(0,x) = C_3(x) - C_2(x) = f(x)$

$\cancel{u(0,x)} = \cancel{\int C_3(x)\,dx}$

$C_3(x) - C_2(x) = \int g(x)\,dx$

∴

∴ $\cancel{C_3(x) =}$