H29 問3.

(1) $\|\cdot\| : C[0,1] \to \mathbb{R}$

・ $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$

・ $\|ax\| = |a|\|x\|$

・ $\|x+y\| \leq \|x\| + \|y\|$

(2) $C[0,1] = X$, ~~P~~ ~~∋~~ $P \subset C[0,1]$,

$^\forall p, q \in P$,

$$p = \sum_{i=1}^{n} a_i x^i \quad , \quad q = \sum_{j=1}^{m} b_j x^j \quad (a_n \neq 0, b_m \neq 0)$$

$$\times\ \alpha p + \beta q = \sum_{i=1}^{\min\{n,m\}} (\underset{\alpha}{\overset{\wedge}{}}a_i + \underset{\beta}{\overset{\wedge}{}}b_i) x^i + \sum_{i=\min\{n,m\}+1}^{\max\{n,m\}} ??$$

$n \geq m$ としても一般性失わない。

$$\alpha p + \beta q = \sum_{i=1}^{m} (\alpha a_i + \beta b_i) x_i + \sum_{i=m+1}^{n} \alpha a_i \cancel{x_i} \in P.$$

(3)

$$u_n(x) = 1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \cdots + \left(\frac{x}{2}\right)^{n-1}$$

Xは完備なノルム空間であるから、u_nがコーシー列であることを示せばよい。

$$\|u_n - u_m\| = \left\| \left\{1 + \left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 + \cdots + \left(\frac{x}{2}\right)^{n-1}\right\} - \left\{1 + \left(\frac{x}{2}\right)^1 + \cdots + \left(\frac{x}{2}\right)^{m-1}\right\} \right\|$$

$(n > m)$

$$= \left\| \sum_{i=m}^{n-1} \left(\frac{x}{2}\right)^i \right\| = \max_{0 \leq x \leq 1} \left| \sum_{i=m}^{n-1} \left(\frac{x}{2}\right)^i \right| \leq \sum_{i=m}^{n-1} \max_{0 \leq x \leq 1} \left| \left(\frac{x}{2}\right)^i \right|$$

$$\cancel{= \sum_{i=m}^{n-1} \frac{\max_{0 \leq x \leq 1} |x|^i}{2^i}} \quad \leq \sum_{i=m}^{n-1} \left(\frac{1}{2}\right)^i = \sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^i - \sum_{i=0}^{m-1} \left(\frac{1}{2}\right)^i$$

$$\cancel{\leq \sum_{+\infty}^{\infty} \text{''} - \sum_{+\infty}^{\infty} \text{''} = 0 , \quad \therefore \|u_n - u_m\|}$$

$$\xrightarrow[n > m \to \infty]{} 2 - 2 = 0 \quad //$$

(4) (3)より. $^\exists u_\infty \in X$, $\lim_{n \to \infty} u_n = u_\infty$,

$$\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^{n} = \frac{\frac{x}{2}}{\cancel{2}\, 1 - \frac{x}{2}} = \frac{x}{2-x} = \frac{-(2-x)+2}{2-x} = -1 + \frac{2}{2-x}$$

~~$\frac{2}{2+x}$は~~ ~~連続は~~ $\notin P$,