H17問6

区間が $[0,\pi]$ になっている.

$a(x) \triangleq x(\pi - x)$

$S_a(t,x) \triangleq \sum_{k=1}^{\infty} a_k e^{-\lambda_k t} \varphi_k(x)$ $\quad (\lambda_k \triangleq (k\pi)^2,\ a_k \triangleq (a, \varphi_k))$

簡単に導出できるように.

$\varphi_n(x) \triangleq \frac{\sqrt{2}}{\sqrt{\pi}} \sin(n\pi x)$, $\{\varphi_n\}$: ONS $\quad \varphi_n(x) \triangleq \sqrt{2}\sin(nx)$

$(\varphi_n, \varphi_n) = \frac{2}{\pi}\int_0^{\pi} \sin^2(n\pi x)dx = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1$

$a_k = (a, \varphi_k) = \int_0^{\pi} x(\pi - x)\sqrt{\frac{2}{\pi}}\sin(n\pi x)dx = \sqrt{\frac{2}{\pi}}\int_0^{\pi}(\pi x - x^2)\sin(n\pi x)dx$

$= \sqrt{\frac{2}{\pi}}\underbrace{\left[(\pi x - x^2)\frac{-\cos(n\pi x)}{n\pi}\right]_0^{\pi}}_{0-0} - \sqrt{\frac{2}{\pi}}\int_0^{\pi}(\pi - 2x)\cdot\frac{(-\cos(n\pi x))}{n\pi}dx$ $\quad \left(\frac{-\cos n\pi x}{n\pi}\right)'$

$= +\sqrt{\frac{2}{\pi}}\int_0^{\pi}\frac{1}{n}\cos(n\pi x) - \frac{2}{n\pi}x\cos(n\pi x)dx$

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$a_k = (a, \varphi_k) = \sqrt{2}\int_0^{\pi} x(\pi - x)\sin(nx)dx = \sqrt{2}\int_0^{\pi}(\pi x - x^2)\sin(nx)dx = \sqrt{2}\underbrace{\left[(\pi x - x^2)\cdot -\frac{\cos nx}{n}\right]_0^{\pi}}_{0} + \int\frac{\sqrt{2}}{n}(\pi - 2x)\cos(nx)$

$= \frac{\pi}{dx} = \frac{\sqrt{2}}{n}\int \pi\cos(nx) - 2x\cos(nx)dx$

$= \underbrace{\frac{\sqrt{2}\pi}{n}\left[\frac{\sin(nx)}{n}\right]_0^{\pi}}_{0} - \frac{2\sqrt{2}}{n}\left\{\underbrace{\left[x\cdot\frac{\sin(nx)}{n}\right]_0^{\pi}}_{0} - \int_0^{\pi}\frac{\sin(nx)}{n}dx\right\}$

$= \frac{2\sqrt{2}}{n^2}\underbrace{\left[-\frac{\cos(nx)}{n}\right]_0^{\pi}}_{-\frac{(-1)^n}{n} - \left(-\frac{1}{n}\right)} = \frac{2\sqrt{2}}{n^3}(1 - (-1)^n)$

$\therefore u(t,x) = S_a(t,x) = \sum_{k=1}^{\infty}\frac{2\sqrt{2}}{k^3}(1-(-1)^k)\cdot e^{-(k\pi)^2 t}\cdot\sqrt{2}\sin(kx)$

$= \sum_{k=1}^{\infty}\frac{8}{(2k-1)^3}e^{-\{(2k-1)\pi\}^2 t}\cdot\sin(2k-1)x$ // (端点条件も満たす)

1乗 $\lambda_n t$.

$\lim_{t\to\infty} e^t u(t,x) = 0$?

$= e^t S_a(t,x) = \sum a_k \underbrace{e^{(1-\lambda_k)t}}_{?} \varphi_k(x)$