問9 H19

(1)

$$E(\hat{Y}_0) = E\left(\sum_{i=1}^{n} w_i Y_i\right) = \sum w_i E(Y_i) = \sum w_i E(\beta_0 + \beta_1 x_i + \varepsilon_i)$$

$$= \beta_0 \sum_{i=1}^{n} w_i + \beta_1 \sum_{i=1}^{n} w_i x_i + \underbrace{\sum w_i \cdot 0}_{=0} \quad //$$

$$V(\hat{Y}_0) = \sum_{i=1}^{n} V(w_i Y_i) + \underbrace{\sum_{i \neq j} \mathrm{Cov}(w_i Y_i, w_j Y_j)}_{=0 \ \text{iid.}} = \sum w_i^2 \cdot V(\beta_0 + \beta_1 x_i + \varepsilon_i)$$

$$= \sum w_i^2 \cdot \sigma^2 = \sigma^2 \sum_{i=1}^{n} w_i^2 \quad //$$

(2)

$$E(Y_0) = \beta_0 + \beta_1 x_0 = E(\hat{Y}_0) \quad (\forall \beta_0, \beta_1)$$

$$\Leftrightarrow \begin{cases} (\beta_0) & 1 = \sum_{i=1}^{n} w_i \\ (\beta_1) & x_0 = \sum_{i=1}^{n} w_i x_i \end{cases}$$

$$V(\hat{Y}_0) = \sigma^2 \sum_{i=1}^{n} w_i^2 \ \cancel{\triangleq f(w) とおく}.$$

(ラグランジュ)

$$f(w) \triangleq \sigma^2 \sum_{i=1}^{n} w_i^2 - \lambda\left(1 - \sum w_i\right) - \eta\left(x_0 - \sum w_i x_i\right)$$

$$\frac{\partial f}{\partial w_i} = 2\sigma^2 w_i + \lambda + \eta x_i = 0 \ \rightarrow \ 2\sigma^2 w_i^2 + \lambda w_i + \eta x_i w_i = 0$$

$$\downarrow$$

$$2\sigma^2 + \lambda n = 0 \quad \lambda = -\frac{2\sigma^2}{n} \qquad \underbrace{2\sigma^2 \sum w_i^2}_{= V(\hat{Y}_0)} + \lambda + x_0 \eta = 0$$

$$2V(\hat{Y}_0) - \frac{2\sigma^2}{n} + x_0 \eta = 0$$

$$2\sigma^2 w_i x_i + \lambda x_i + \eta x_i^2 = 0$$

$$2\sigma^2 x_0 + \eta \sum x_i^2 = 0, \quad \eta = -\frac{2\sigma^2 x_0}{\sum x_i^2}$$

$$\therefore w_i = -\frac{\lambda + \eta x_i}{2\sigma^2} = -\frac{\left(-\frac{2\sigma^2}{n}\right)}{2\sigma^2} - \frac{1}{\cancel{2\sigma^2}} \cdot x_i \cdot \left(-\frac{\cancel{2\sigma^2} x_0}{\sum x_i^2}\right)$$

$$= \frac{1}{n} + \frac{x_0 x_i}{\sum_{j=1}^{n} x_j^2} \quad (\forall i = 1, \cdots, n) \quad //$$

(3)

$$\hat{Y}_0 = \sum_{i=1}^{n} w_i Y_i$$

$x_1 = \frac{1960-1980}{10} = -2$

$x_2 = -1$

$x_3 = 0$

$x_4 = 1$

$x_5 = 2$

$x_0 = 12.$

・ $\sum x_i^2 = 4+1+1+4 = 10.$

・ $n = 5$

・ $W_1 = \frac{1}{5} + \frac{12 \cdot (-2)}{10} = \frac{1}{5} - \frac{12}{5} = -\frac{11}{5}$

$w_2 = \frac{1}{5} + \frac{12 \cdot (-1)}{10} = \frac{1}{5} - \frac{6}{5} = -1$

$w_3 = \frac{1}{5}.$

$w_4 = \frac{1}{5} + \frac{6}{5} \cdot x_4 = \frac{7}{5}$

$w_5 = \frac{1}{5} + \frac{6}{5} \cdot 2 = \frac{13}{5}.$

・ ~~$y_0 =$~~ ~~$y_1 =$~~

$\therefore \hat{y}_0 = -\frac{11}{5} \cdot \underset{(15+0.4)}{15.4} - 1 \cdot 15.2 + \frac{1}{5} \cdot 15.4 + \frac{7}{5} \cdot 17 + \frac{13}{5} \cdot 16.9$

$= \frac{1}{5}\{-11 \cdot 15.4 - 5 \cdot 15.2 + 15.4 + 7 \cdot 17 + 13 \cdot 16.9\}$

$= \frac{1}{5}\{-11(16-0.6) - 5(16-0.8) + (16-0.6) + 7(16+1) + 13 \cdot (16+0.9)\}$

$= \frac{1}{5}\{16\{\cancel{-11} - \cancel{5} + 1 + \overset{2}{\cancel{7}} + \overset{2}{\cancel{13}}\} + (\overset{6}{\cancel{6.6}} + \overset{16}{\cancel{4}} - 0.6 + 7 + \overset{18.7}{11.7})\}$

$= \frac{1}{5}\{80 + 28.7\} = \frac{1}{5} \cdot 108.7 = 21.74$ //

21.7.