問6 H14

(1) $\hat{u}(\xi,t)=\int_{-\infty}^{\infty}u(x,t)e^{-i\xi x}dx$ 」

$u_x(x,0)$

形式解は、

$u(x,t)=\int_{-\infty}^{\infty}G(t,x-y)\cdot e^{-\frac{y^2}{2}}dy$

$G(t,x)=\frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right)$

両辺tでビブンすると、

$\hat{u}_t=\int_{-\infty}^{\infty}u_t e^{-i\xi x}dx=\int_{-\infty}^{\infty}\underbrace{u_{xx}}_{=(u_x)'}e^{-i\xi x}dx$

$=\Big[\underbrace{u_x e^{-i\xi x}}_{=0=u_x(-\infty,t)e^{\infty}}\Big]_{-\infty}^{\infty}+i\xi\int_{-\infty}^{\infty}\underbrace{u_x}_{=u'}e^{-i\xi x}dx=i\xi\underbrace{\Big[ue^{-i\xi x}\Big]_{-\infty}^{\infty}}_{=0}+(i\xi)^2\int_{-\infty}^{\infty}ue^{-i\xi x}dx$

0? 0?

$u(t,x)\to 0$ ($\forall t$, $x\to\pm\infty$) としてよい。u_xも。

$=\underbrace{(i\xi)^2}_{=-\xi^2}\hat{u}(\xi,t)$

$\therefore \frac{\hat{u}_t}{\hat{u}}=-\xi^2,\ \to \log\hat{u}=-\xi^2t+C_1(\xi)$ $\quad\hat{u}(\xi,t)=e^{-\xi^2t}\cdot C_2(\xi)$

Constではないから、Cは使わないべき？

$u(x,0)$: $\hat{u}(\xi,0)=\int_{-\infty}^{\infty}\underbrace{u(x,0)}_{=e^{-\frac{x^2}{2}}}e^{-i\xi x}dx=\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}-i\xi x}dx=C_2(\xi)$

$y=\frac{1}{\sqrt{2}}(x+\frac{2i\xi}{\cancel{2}})$

$=\int_{-\infty}^{\infty}e^{-\frac{1}{2}(x+\frac{2i\xi}{\cancel{2}})^2+\frac{2i^2\xi^2}{\cancel{8}}}=e^{\frac{-\xi^2\cdot 2}{\cancel{8}}}\int_{-\infty}^{\infty}e^{-y^2}\cdot dy\cdot\sqrt{2}$

$=\sqrt{\frac{2}{\pi}}e^{-\frac{2\xi^2}{\cancel{8}}}$ $\quad\therefore\hat{u}(\xi,t)=\sqrt{\frac{2}{\pi}}e^{-(t+\frac{1}{8})\xi^2}$

$\hat{u}(\xi,0)=\int_{-\infty}^{\infty}e^{-\frac{1}{2}(x+i\xi)^2-\frac{\xi^2}{2}}dx$

$-\frac{x^2}{2}-i\xi x$

$=-\frac{1}{2}(x^2+2i\xi x)$

$=-\frac{1}{2}(x+i\xi)^2+\underbrace{\frac{i^2\xi^2}{2}}_{=-\frac{\xi^2}{2}}$

$=\underbrace{\int_{-\infty}^{\infty}e^{-y^2}dy}_{=\sqrt{\pi}}\cdot\sqrt{2}\cdot e^{-\frac{\xi^2}{2}}=C_2(\xi)$

$\frac{1}{\sqrt{\pi}}$? $\frac{2}{\sqrt{\pi}}$?

$\therefore\hat{u}(\xi,t)=\sqrt{\frac{2}{\pi}}e^{-(\frac{1}{2}+t)\xi^2}$ × $=\sqrt{2\pi}\,e^{-(\frac{1}{2}+t)\xi^2}$

(2) 逆変換

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\xi,t)\, e^{i\xi x}\, d\cancel{x}\,\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sqrt{2\pi}\, e^{-(\frac{1}{2}+t)\xi^2} e^{i\xi x}\, \cancel{dx}\ d\xi$$

だったかも…

$$\cancel{= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left\{-\left(\frac{1}{2}+t\right)\xi\right.}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-(\frac{1}{2}+t)\xi^2} \underbrace{\int_{-\infty}^{\infty} e^{i\xi x}\, dx}_{=}$$

$$= \underbrace{\left[\frac{1}{i\xi} e^{i\xi x}\right]_{-\infty}^{\infty}}_{=}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(\frac{1}{2}+t)\xi^2 + ix\xi}\, d\xi$$

$$-\left(\frac{1}{2}+t\right)\left(\xi^2 - \frac{ix}{\left(\frac{1}{2}+t\right)}\xi\right)$$

b

$$\left(\xi - \frac{b}{2}\right)^2 + \left(\frac{1}{2}+t\right) \cdot -\frac{b^2}{4}$$

$$\frac{1}{4}\cdot\frac{i^2x^2}{\left(\frac{1}{2}+t\right)}$$

$$\frac{-x^2}{2+4t}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(\frac{1}{2}+t)(\xi}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left\{-\left(\frac{1}{2}+t\right)\left(\xi-\frac{b}{2}\right)^2 + \frac{-x^2}{2+4t}\right\} d\xi$$

$$= \frac{1}{\sqrt{2\pi}}\cdot e^{\frac{-x^2}{2+4t}} \underbrace{\int_{-\infty}^{\infty} e^{-y^2}\, dy}_{\sqrt{\pi}} \cdot \frac{1}{\sqrt{\frac{1}{2}+t}}$$

$$\left(y = \sqrt{\frac{1}{2}+t}\cdot\left(\xi - \frac{b}{2}\right)\right)$$

$$= \frac{1}{\sqrt{1+2t}} e^{\frac{-x^2}{2+4t}} \quad // \quad = \frac{1}{\sqrt{1+2t}} e^{-\frac{x^2}{2(1+2t)}} \quad //$$

$$y = \frac{1}{\sqrt{1+2t}}x$$

より、

$$\int_{-\infty}^{\infty} u(x,t)^2\, dx = \frac{1}{1+2t}\cdot\int_{-\infty}^{\infty} e^{-\frac{x^2}{1+2t}}\, dx = \frac{1}{1+2t}\cdot\underbrace{\int_{-\infty}^{\infty} e^{-y^2}\, dy}_{\sqrt{\pi}}\cdot\sqrt{1+2t}$$

$$= \frac{\sqrt{\pi}}{\sqrt{1+2t}} = \sqrt{\frac{\pi}{1+2t}} \quad //$$

」