H17問4

(1) 帰納法.

$$d(y_n, x_n) \le \sum_{i=0}^{n-1} \frac{\varepsilon_{n-i}}{2^i} \quad —(*) \quad \cancel{(n \ge 0)} \ (n \ge 1)$$

① $\cancel{n=0}$ $n=1$

$\cancel{d(y_0, x_0) = 0}$ $\cancel{\sum_{i=}}$

$$\sum_{i=0}^{0} \frac{\varepsilon_{1-i}}{2^i} = \frac{\varepsilon_1}{2^0} = \varepsilon_1$$

$d(y_1, x_1)$

$d(y_1, f(y_0)) \le \varepsilon_1$

$x_1 = f(x_0)$

$$d(y_1, \overset{\shortparallel}{f(y_0)}) \le \varepsilon_1$$

$$d(y_1, \underset{\shortparallel}{f(x_0)}) \le \varepsilon_1$$

$$d(y_1, x_1) \le \varepsilon_1$$

② $n=K$のとき、(*)を仮定.

$$d(y_K, x_K) \le \sum_{i=0}^{k-1} \frac{\varepsilon_{k-i}}{2^i}$$

$$d(y_{k+1}, x_{k+1}) = d(y_{k+1}, f(x_k)) \le \underbrace{d(y_{k+1}, f(y_k))}_{\le \varepsilon_{k+1}} + \underbrace{d(f(y_k), f(x_k))}_{\le \frac{1}{2} d(y_k, x_k)}$$

$$\le \varepsilon_{k+1} + \frac{1}{2} d(x_k, y_k)$$

$$\le \varepsilon_{k+1} + \frac{1}{2} \sum_{i=0}^{k-1} \frac{\varepsilon_{k-i}}{2^i} = \frac{1}{2} \left\{ 2\varepsilon_{k+1} + \frac{\varepsilon_k}{2^0} + \frac{\varepsilon_{k-1}}{2^1} + \frac{\varepsilon_{k-2}}{2^2} + \cdots + \frac{\varepsilon_1}{2^{k-1}} \right\}$$

$$= \underbrace{\varepsilon_{k+1}}_{= \frac{\varepsilon_{k+1}}{2^0}} + \frac{\varepsilon_k}{2^1} + \frac{\varepsilon_{k-1}}{2^2} + \cdots + \frac{\varepsilon_1}{2^k} = \sum_{i=0}^{(k+1)-1} \frac{\varepsilon_{(k+1)-i}}{2^i} \quad //$$

$= \cancel{2\varepsilon}$

(2) $\varepsilon_n \to 0 \iff \forall \varepsilon > 0, \exists N, n \ge N \Rightarrow \varepsilon_n \le \varepsilon$.

$$d(y_n, x_n) \le \sum_{i=0}^{n-1} \frac{\varepsilon_{n-i}}{2^i} = \frac{\varepsilon_n}{2^0} + \frac{\varepsilon_{n-1}}{2^1} + \cdots + \frac{\varepsilon_{n-(N-1)}}{2^{N-1}} + \frac{\varepsilon_{n-N}}{2^N} + \cdots + \frac{\varepsilon_1}{2^{n-1}}$$

$$\cancel{= \sum_{i=0}^{N-1} \frac{\varepsilon_{n-i}}{2^i} + \sum_{i=N}^{n} \frac{\varepsilon_{n-i}}{2^i} \le \sum_{i=0}^{N-1} \frac{\varepsilon_{n-i}}{2^i} + \sum_{i=N}^{n} \frac{1}{2^i} \quad \varepsilon_n, \varepsilon_{n-1} \cdots \varepsilon_1}$$

$$= \frac{\varepsilon_1}{2^{n-1}} + \frac{\varepsilon_2}{2^{n-2}} + \cdots + \frac{\varepsilon_{N-1}}{2^{n-(N-1)}} + \frac{\varepsilon_N}{2^{n-N}} + \cdots + \frac{\varepsilon_n}{2^0}$$

$$\cancel{= \sum_{i=1}^{N-1} \frac{\varepsilon_i}{2^{n-i}} + \sum_{i=N}^{n} \frac{\varepsilon_i}{2^{n-i}}} \le \sum_{i=1}^{N-1} \frac{\varepsilon_i}{2^{n-i}} + \sum_{i=N}^{n} \frac{1}{2^{n-i}}$$

$$\cancel{\sum_{i=1}^{n} \frac{1}{2^{n-i}} - \sum_{i=1}^{N-1} \frac{1}{2^{n-i}}}$$

$\le \varepsilon$ としたい

$$\sum \frac{\varepsilon}{2^i} = \varepsilon \text{ かい.}$$

$\forall \varepsilon > 0, \exists N_i \in \mathbb{N}, n \geq N_i \Rightarrow \varepsilon_n \leq \dfrac{\varepsilon_{n-i}}{2^i}$

$\cancel{\frac{N}{2}}^{2^0}$ $\frac{n}{2}$

$$d(y_n, x_n) \leq \sum_{i=0}^{n-1} \frac{\varepsilon_{n-i}}{2^i}$$

$$= \frac{\varepsilon_n}{2^0} + \frac{\varepsilon_{n-1}}{2^1} + \cdots + \frac{\varepsilon_N}{2^{n-N}} + \frac{\varepsilon_{N-1}}{2^{n-(N-1)}} + \cdots + \frac{\varepsilon_1}{2^{n-1}} + \frac{\varepsilon_0}{2^n}$$

$$\cancel{\leq \frac{\varepsilon}{2^0} + \frac{\varepsilon}{2^1} + \cdots + \frac{\varepsilon}{2^{n-N}}}$$

$$= \frac{2^n \varepsilon_n + \varepsilon_0}{2^n} + \frac{2^{n-1}\varepsilon_{n-1} + \varepsilon_1 2^1}{2^n} + \frac{\varepsilon_{n-2}2^{n-2} + \varepsilon_2 2^2}{2^n} + \cdots + \frac{\cancel{\varepsilon_N 2^{n-N} + \varepsilon_{N-1} 2^{n-(N-1)}}}{2^n}$$

$$= \left(\frac{\varepsilon_n}{2^0} + \frac{\varepsilon_0}{2^n}\right) + \left(\frac{\varepsilon_{n-1}}{2^1} + \frac{\varepsilon_1}{2^{n-1}}\right) + \cdots + \cancel{\left(\frac{\varepsilon_N}{2^{n-N}} + \frac{\varepsilon_{N-1}}{2^{n-(N-1)}}\right)}$$

$n = 2m$として.

$$\left(\frac{\varepsilon_{n-m}}{2^m} + \frac{\varepsilon_{n-(m-1)}}{2^{m-1}}\right)$$

$$\leq 2\sqrt{\frac{\varepsilon_n}{2^0} \cdot \frac{\varepsilon_0}{2^n}} + 2\sqrt{\frac{\varepsilon_{n-1}}{2^1} \cdot \frac{\varepsilon_1}{2^{n-1}}} + \cdots + 2\sqrt{\frac{\varepsilon_{n-m}}{2^m} \cdot \frac{\varepsilon_{n-(m-1)}}{2^{n-1}}}$$

$a + b \leq 2\sqrt{ab}$

$$= 2\sqrt{\frac{\varepsilon_0}{2^0} \cdot \frac{\varepsilon_n}{2^n}} + 2\sqrt{\frac{\varepsilon_1}{2^1} \cdot \frac{\varepsilon_{n-1}}{2^{n-1}}} + \cdots 2\sqrt{\frac{\varepsilon_m}{2^m} \cdot \frac{\varepsilon_m}{2^m}}$$

$\varepsilon_n \to 0$

$\eta_n = \left(\frac{\varepsilon_n}{2}\right) \to 0$

$$= 2\sqrt{\eta_0 \cdot \eta_n} + 2\sqrt{\eta_1 \cdot \eta_{n-1}} + \cdots + 2\sqrt{\eta_m \cdot \eta_m}$$

$n \gg 1$で $\leq \eta_0 \cdot \eta_n + \eta_1 \cdot \eta_{n-1} + \cdots + \eta_m \cdot \eta_m$

$\eta_n < \varepsilon \quad (n \geq N)$

η_{n-i}

?

$1 \geq \frac{n}{R} \geq 0$

$0 \geq \frac{n}{R} - \geq 1$

$x^n \geq \frac{n}{R} - 1$

$\frac{n}{R} \geq x^n - 1 \quad \cdot n$