H15問9

(1)

$$V(\bar{X}) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}V\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\left\{\sum_{i=1}^{n}\underbrace{V(X_i)}_{=\sigma^2} + \underbrace{\sum_{i\neq j}Cov(X_i, X_j)}_{=0}\right\} = \frac{\sigma^2}{n}$$

$$V(\bar{Y}) = \frac{\sigma^2}{n}$$

$$Cov(\bar{X}, \bar{Y}) = Cov\left(\frac{1}{n}\sum_{i=1}^{n} X_i, \frac{1}{n}\sum_{j=1}^{n} Y_j\right) = \frac{1}{n^2}\sum_{i,j} Cov(X_i, Y_j) = \frac{1}{n^2}\sum_{i=1}^{n}\underbrace{Cov(X_i, Y_i)}_{=-\frac{\sigma^2}{2}} = -\frac{\sigma^2}{2n}$$

(2)

$E(Z) = \mu$となっている.

$$V(Z) = V(c\bar{X} + (1-c)\bar{Y}) = V(c\bar{X}) + 2Cov(c\bar{X}, (1-c)\bar{Y}) + V((1-c)\bar{Y})$$

$$= c^2V(\bar{X}) + 2c(1-c)Cov(\bar{X}, \bar{Y}) + (1-c)^2V(\bar{Y})$$

$$= c^2\cdot\frac{\sigma^2}{n} + 2c(1-c)\cdot\left(-\frac{\sigma^2}{2n}\right) + (1-c)^2\cdot\frac{\sigma^2}{n}$$

$$= \frac{\sigma^2}{n}c^2 - \frac{\sigma^2}{n}c(1-c) + \frac{\sigma^2}{n}(1-c)^2$$

$$= \frac{\sigma^2}{n}\left\{c^2 - c + c^2 + 1 - 2c + c^2\right\}$$

$$= \frac{\sigma^2}{n}\left\{3c^2 - 3c + 1\right\}$$

$$= \frac{\sigma^2}{n}\cdot\left\{3\left(c - \frac{1}{2}\right)^2 + \frac{1}{4}\right\}$$

より、$c = \frac{1}{2}$のとき分散は最小で、$V(Z) = \frac{\sigma^2}{4n}$ //