H24問8

(1)

$$P(N=n) = P(X_1 \le a, \cdots, X_{n-1} \le a, X_n > a)$$
$$= P(X_1 \le a) \cdots P(X_{n-1} \le a) P(X_n > a)$$
$$= (1-e^{-\mu a})^{n-1} \cdot e^{-\mu a} \quad //$$

$$P(X \le x) = \int_0^x \mu e^{-\mu x} = [-e^{-\mu x}]_0^x$$
$$= -e^{-\mu x} + 1 = 1 - e^{-\mu x} \quad //$$

$$f(x | x > a) =$$

(2)

$$E(X_i | X_i > a) = \int_0^\infty x \underbrace{P(X_i \le x | X_i > a)}_{f} dx =$$

$(a < x のとき)$

$$P(X_i \le x | X_i > a) = \frac{P(X_i \le x, X_i > a)}{P(X_i > a)} = \frac{P(a < X_i \le x)}{e^{-\mu a}} = \frac{e^{-\mu a} - e^{-\mu x}}{e^{-\mu a}} = 1 - e^{-\mu(x-a)}$$

$(1-e^{-\mu x}) - (1-e^{-\mu a})$

$\downarrow$

$\mu e^{-\mu(x-a)}$

$$より E(X_i | X_i > a) = \int_a^\infty x(1-e^{-\mu(x-a)}) dx = \cancel{\int_a^\infty (x - xe^{-\mu(x-a)}) dx}$$

$x - a = y$

$$= \int_0^\infty (y+a)(1-e^{-\mu y}) dy = \int_0^\infty y(1-e$$

$$E(X_i | X_i > a) = \int_a^\infty x f(x | x > a) dx = \int_a^\infty x \mu e^{-\mu(x-a)} dx = \int_0^\infty (y+a) \mu e^{-\mu y} dy$$
$$= E(X_i) + \int_0^\infty \underbrace{a \mu e^{-\mu y}}_{\int_0^\infty f(x) = 1?} dy = \mu^{-1} + a[-e^{-\mu x}]_0^\infty = \mu^{-1} + a \quad //$$

$$E(X_i | X_i \le a) = \int_0^a x f(x | x \le a) dx = \int_0^a x \cdot \underbrace{f(x)}_{f(a)} dx = \int_0^a x \mu e^{-\mu x} = \cancel{x}$$

$(x \le a)$

$$\cancel{P(X_i \le x | X_i \le a)} =$$

$$= [-xe^{-\mu x}]_0^a + \int_0^a e^{-\mu x} dx = -ae^{-\mu a} + -\frac{1}{\mu}e^{-\mu a} + \frac{1}{\mu}$$

$$[-\frac{1}{\mu}e^{-\mu x}]$$

$$\cancel{= e^{-\mu a}(} \quad = \frac{1}{\mu} - e^{-\mu a}(a + \frac{1}{\mu}) \quad //$$

$$P(X_i \le x \mid X_i \le a) = \frac{P(X_i \le \min(x,a))}{P(X_i \le a)} = \begin{cases} x \ge a \Rightarrow 1 \\ x < a \Rightarrow \dfrac{P(X_i \le x)}{P(X_i \le a)} = \dfrac{1-e^{-\mu x}}{\cancel{e^{-\mu a}}\ \ 1-e^{-\mu a}} = \cancel{e^{\mu a} - e^{-\mu(x-a)}} \end{cases}$$

if

$$f(x \mid x \le a) = \begin{cases} 0 & x \ge a \\ \cancel{\mu e^{-\mu(x-a)}}\ \ \dfrac{\mu e^{-\mu x}}{1-e^{-\mu a}} & x < a \end{cases}$$

$$E(X_i \mid X_i \le a) = \int_0^\infty x f(x \mid x_i \le a)\, dx = \int_0^a \mu x e^{-\mu(x-a)}\, dx = \left[-x e^{-\mu(x-a)}\right]_0^a + \int_0^a e^{-\mu(x-a)}\, dx$$

$$= -a e^0 + 0 + \left[-\frac{1}{\mu} e^{-\mu(x-a)}\right]_0^a = -a$$

$$E(X_i \mid X_i \le a) = \int_0^a \frac{\mu}{1-e^{-\mu a}} x e^{-\mu x}\, dx = \cancel{\frac{\mu}{1-e^{-\mu a}} \left\{ -\frac{1}{\mu} e^{-\mu a} + \frac{1}{\mu} \right\}}$$

$$\cancel{\left[-\frac{1}{\mu} e^{-\mu x}\right]}$$

$$= \frac{1}{1-e^{-\mu a}} \cdot \left\{ \underbrace{\left[-x e^{-\mu x}\right]_0^a}_{-a e^{-\mu a}} + \underbrace{\int_0^a e^{-\mu x}\, dx}_{\left[-\frac{1}{\mu} e^{-\mu x}\right]} \right\}$$

$$-\frac{1}{\mu} e^{-\mu a} + \frac{1}{\mu} \}$$

$$= \cancel{\frac{e^{-\mu a}\left(-a - \frac{1}{\mu}\right) + \frac{1}{\mu}}{1-e^{-\mu a}}}$$

$$= \frac{1}{1-e^{-\mu a}} \left\{ -a e^{-\mu a} + \frac{1}{\mu} \left\{ 1 - e^{-\mu a} \right\} \right)$$

$$= \frac{1}{\mu} - \frac{a e^{-\mu a}}{1-e^{-\mu a}}$$

$$= \frac{1}{\mu} - \frac{a}{e^{\mu a} - 1}$$

//

(3)

$$E\left(\sum_{i=1}^{N} X_i\right) = E\left(E_{X_i}\left(\sum_{i=1}^{n} X_i \middle| n=N\right)\right)$$

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$$= E_N\left(E\left(\sum_{i=1}^{n} X_i \middle| X_1 \le a, X_2 \le a, \cdots, X_{n-1} \le a, X_n > a\right)\right)$$

$$= E_N\left(\sum_{i=1}^{n} E(X_i | X_1 \le a, \cdots, X_{n-1} \le a, X_n > a)\right)$$

$$= E_N\left(\sum_{i=1}^{n-1} \underbrace{E(X_i | X_i \le a)}_{\cancel{(\mu^{-1}+a)}} + E(X_n | X_n > a)\right)$$

$$= E_N\Big($$

$$= E_N\left(\sum_{i=1}^{n-1}\left(\frac{1}{\mu} - \frac{a}{e^{\mu a}-1}\right) + \mu^{-1} + a \middle| N=n\right)$$

$$= E_N\left((N-1)\left(\quad " \quad \right) + \mu^{-1} + a\right)$$

$$= \cancel{\mu^{-1}} + a - \cancel{\frac{1}{\mu}} + \frac{a}{e^{-\mu a}-1} + \underbrace{E_N(N)}_{\parallel}\left(\frac{1}{\mu} - \frac{a}{e^{\mu a}-1}\right)$$

$$e^{-\mu a}\sum_{n=0}^{\infty} n(1-e^{-\mu a})^{n-1} = e^{-\mu a} \cdot \frac{1}{(1-1+e^{-\mu a})^2} = \frac{1}{e^{-\mu a}}$$

$$e^{-\mu a} \cdot \left(\sum_{n=0}^{\infty}(1-e^{-\mu a})^n\right)'$$

3^n 3^{2n}

2^n BBX

$$= \cancel{\mu^{-1}+a} \quad a + \frac{a}{e^{-\mu a}-1} + \frac{1}{\mu e^{-\mu a}} - \frac{a}{1-e^{-\mu a}}$$

(3)

$$E\left(\sum_{i=1}^{N} X_i\right) \cancel{= E\left(\sum_{i=1}^{n} X_i \middle| N=n\right)} = \cancel{E\left(E\left(\sum_{i=1}^{n} X_i\right) \middle| N=n\right)}$$

$$= E\left(E\left(\sum_{i=1}^{N} X_i \middle| N\right)\right)$$

$$= E\left(E\left(X_1 + X_2 + \cdots + X_N \middle| N\right)\right)$$

$$= \sum_{n=0}^{\infty} E(X_1 + \cdots + X_n \mid N=n)\, P(N=n)$$

$$= \sum_{n=0}^{\infty} \left(E(X_1 \mid N=n) + \cdots + E(X_n \mid N=n)\right) P(N=n)$$

$$= \sum_{n=0}^{\infty} \left(E(X_1) + \cdots + E(X_n)\right) P(N=n) \quad (\because N と X_i は独立)$$

$$= \sum_{n=0}^{\infty} n E(X_1) P(N=n) = E(X_1) E(N)$$

$$= \mu^{-1} \cdot \sum_{n=0}^{\infty} n (1 - e^{-\mu a})^{n-1} \cdot e^{-\mu a} = \mu^{-1} \cdot e^{-\mu a} \cdot \frac{1}{(1 - 1 + e^{-\mu a})^2}$$

$$= \frac{e^{+\mu a}}{\cancel{}\ \mu}$$

$$= \frac{e^{\mu a}}{\mu}$$