H18問6.

(1) $E(t) = \int_0^1 (u_t^2(x,t) + u_x^2(x,t))\,dx$,

$E(0) = \int_0^1 u_t^2(x,0) + u_x^2(x,0)\,dx =$

$E'(t) = \int_0^1 (2u_t \underbrace{u_{tt}}_{u_{xx}} + 2u_x u_{xt})\,dx$ (積分とビブンの交換)

$= \left[2u_t u_x\right]_0^1 - \{ \int_0^1 2u_{tx} u_x\,dx + \int_0^1 2u_x u_{xt}$ (部分積分)

$= 2u_t(1,t)u_x(1,t) - 2u_t(0,t)u_x(0,t)$ ($u_{tx} = u_{xt}$)

$= 0$ ($u'(0,t) = u'(1,0) = 0$)

$E(t) = c$ ($\forall t$)

従って、$E(0) = c = E(t)$ //

(2) $V \triangleq u_1 - u_2$ (u_1, u_2を古典解).

Vは(P)を満たす. ($f(x) = g(x) = 0$).

Vに対して、

$E(t) = \int_0^1 (V_t^2 + V_x^2)\,dx$

$E(0) = \int_0^1 \underbrace{V_t^2(x,0)}_{0} + \underbrace{V_x^2(x,0)}_{0\ (\because V(x,0)=0 \to V'(x,0)=0)}\,dx = 0$.

(1)より、

$E(t) = 0 \Rightarrow V_t^2 + V_x^2 = 0 \Rightarrow V_t = V_x = 0 \Rightarrow \begin{cases} V(x,t) = \varphi(x) \\ V(x,t) = \psi(t) \end{cases}$

$\because V(0,t) = \psi(t) = 0$ ($\forall t$) より、$\psi(t) = 0$ ($\forall t$),

$\because V(x,0) = \varphi(x) = 0$ より、 $\varphi \equiv 0$

$\therefore V \equiv 0$