H13

問1.（基礎問題）

$$-t\left(x^2-\frac{c}{t}x\right)=-t\left(x-\frac{c}{2t}\right)^2+t\cdot\frac{c^2}{4t^2}$$

(1) $$F(t)=\int_{-\infty}^{\infty}e^{-tx^2}\cdot e^{cx}dx=\int_{-\infty}^{\infty}e^{-tx^2+cx}dx=e^{\frac{c^2}{4t}}\int_{-\infty}^{\infty}e^{-t(x-\frac{c}{2t})^2)}dx$$

$x-\frac{c}{2t}=y$ とおくと、 $dx=dy$,

$$=e^{\frac{c^2}{4t}}\int_{-\infty}^{\infty}e^{-ty^2}dy.$$

$\sqrt{t}y=z$ とおくと、 tはfixedなので、 $\sqrt{t}dy=dz$,

$$=e^{\frac{c^2}{4t}}\int_{-\infty}^{\infty}e^{z^2}\cdot\frac{1}{\sqrt{t}}dz=\boxed{\frac{1}{\sqrt{t}}\cdot e^{\frac{c^2}{4t}}\cdot\sqrt{\pi}}\ //$$

(2)

$f(x)=x^2$のとき、

× $F(t)=\int_{-\infty}^{\infty}e^{-tx^2}\cdot x^2dx=$ ~~($x^2=y$ とおく. $2x\,dx=dy$) $=\int_0^{\infty}e^{-ty}$~~

$\left(\frac{x^3}{3}\right.$

× $$F(t)=\int e^{-tx^2}\cdot\left(\frac{x^3}{3}\right)'dx=\left[e^{-tx^2}\cdot\frac{x^3}{3}\right]_{-\infty}^{\infty}+\int 2tx\cdot e^{-tx^2}\cdot\frac{x^3}{3}dx.$$

$$\frac{2}{3}t\,e^{-tx^2}x^4dx.$$

$-2tx$

$$F(t)=\int(e^{-tx^2})'\cdot\underbrace{\frac{1}{-2tx}\cdot x^2}_{-\frac{x}{2t}}dx=\underbrace{\left[e^{-tx^2}\cdot\left(-\frac{x}{2t}\right)\right]_{-\infty}^{\infty}}_{e^{-\infty}=0\ 0}-\int_{-\infty}^{\infty}e^{-tx^2}\cdot\left(-\frac{1}{2t}\right)dx.$$

$$=\frac{1}{2t}\underbrace{\int_{-\infty}^{\infty}e^{-tx^2}dx}_{\frac{1}{\sqrt{t}}\sqrt{\pi}}=\boxed{\frac{\sqrt{\pi}}{2t\sqrt{t}}}\ //$$

(3) $$F(t)=\underbrace{\int_{-\infty}^{\infty}e^{-tx^2}x^n dx}_{I_n}=\int_{-\infty}^{\infty}(e^{-tx^2})'\cdot\underbrace{\frac{x^n}{-2tx}}_{-\frac{1}{2t}x^{n-1}}dx=\underbrace{\left[e^{-tx^2}\cdot-\frac{x^{n-1}}{2t}\right]_{-\infty}^{\infty}}_{0}+\int_{-\infty}^{\infty}e^{-tx^2}\cdot\frac{(n-1)}{2t}x^{n-2}dx$$

$$=\frac{(n-1)}{2t}I_{n-2}$$

$$I_4=\frac{3}{2t}I_2=\frac{3}{2t}\cdot\frac{1}{2t}\cdot I_0$$

$$I_0=\int_{-\infty}^{\infty}e^{-tx^2}dx=\sqrt{\frac{\pi}{t}},\qquad F(t)=I_n=\begin{cases}n=2m\Rightarrow 4\ \dfrac{\overset{(n-1)}{n!!}}{(2t)^m}\cdot\sqrt{\dfrac{\pi}{t}}\\ n=2m+1\Rightarrow\dfrac{\overset{(n-1)}{n!!}}{(2t)^m}\cdot\underset{0}{I_1}=0\end{cases}$$

$$I_5=\frac{4}{2t}I_3=\frac{4}{2t}\cdot\frac{2}{2t}\cdot I_1$$ △

$$I_1=\int_{-\infty}^{\infty}e^{-tx^2}\cdot x\,dx=\int(e^{-tx^2})'\cdot\left(-\frac{1}{2t}\right)dx$$

$$\frac{x}{-2tx}\quad -\frac{1}{2t}$$

$$=\underbrace{\left[e^{-tx^2}\cdot\left(-\frac{1}{2t}\right)\right]_{-\infty}^{\infty}}_{0}+\int_{-\infty}^{\infty}\underbrace{\left(-\frac{1}{2t}\right)'}_{0}dx=0$$ （奇関数）

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(3) $F(t) = \frac{(n-1)}{2t} I_{n-2}$

$F(t) = I_n$

$$I_n(t) = \frac{(n-1)}{2t} I_{n-2}$$

$$= \frac{(n-1)}{(2t)^2}(n-3) I_{n-4}$$

$$= \frac{1}{(2t)^3}(n-1)(n-3)(n-5) I_{n-6} \cdots$$

$$= \begin{cases} \frac{1}{(2t)^{\frac{n}{2}}}(n-1)!! \; I_1 & (n: \text{~~even~~ odd}) \\ \frac{1}{(2t)^{\frac{n}{2}-1}} \;'' \; I_2 & (n: \text{~~odd~~ even}) \end{cases}$$

$$I_1 = \int_{-\infty}^{\infty} e^{-tx^2} x \, dx = 0 \quad (奇)$$

$$I_2 = \int_{-\infty}^{\infty} e^{-tx^2} x^2 \, dx \underset{\underset{(2)}{\uparrow}}{=} \frac{\sqrt{\pi}}{2t\sqrt{t}}$$

$$\frac{\sqrt{\pi}}{2t\sqrt{t}} \cdot \frac{1}{2t}$$

$$\therefore I_n(t) = \begin{cases} 0, & \text{if } n: \text{odd.} \\ \frac{1}{(2t)^{\frac{n}{2}\cancel{-1}}}(n-1)!! \cdot \sqrt{\frac{\pi}{t}} & \text{if : even},, \end{cases}$$

$$I_0(t) = \sqrt{\frac{\pi}{t}}.$$

$$\therefore I_n(t) = \begin{cases} 0 & \text{if } n: \text{odd} \\ \frac{(n-1)!!}{(2t)^{\frac{n}{2}}} \sqrt{\frac{\pi}{t}} & \text{if : } n: \text{even}. \end{cases}$$

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