H18-4

(1)

$$\|Tu-Tv\| = \left\| \frac{1}{a}\int_0^t (u(\tau)-v(\tau))\sin(t-\tau)d\tau \right\|$$

$$= \frac{1}{a}\sup_{0\le t\le a}\left|\int_0^t (u(\tau)-v(\tau))\sin(t-\tau)d\tau\right|$$

$$\leqq \frac{1}{a}\sup_{0\le t\le a}\int_0^t \left|(u(\tau)-v(\tau))\sin(t-\tau)\right|d\tau$$

$$= \frac{1}{a}\cdot\int_0^a |u(\tau)-v(\tau)||\sin(a-\tau)|d\tau$$

$$< \frac{1}{a}\int_0^a |u(\tau)-v(\tau)|\cdot 1\, d\tau$$

$$\le \frac{1}{a}\int_0^a \|u-v\| d\tau$$

$$= \frac{1}{a}\cdot a\,\|u-v\|$$

$$= \|u-v\|$$

(2)

$$u(t) = 1 + \frac{1}{a}\int_0^t u(\tau)(\sin t\cos\tau - \cos t\sin\tau)d\tau$$

$$= 1 + \frac{1}{a}\sin t\int_0^t u(\tau)\cos\tau\, d\tau - \frac{1}{a}\cos t\int_0^t u(\tau)\sin\tau\, d\tau$$

$$u'(t) = \frac{1}{a}\cos t\int_0^t u(\tau)\cos\tau\, d\tau + \frac{1}{a}u(t)\cancel{\sin t\cos t}$$

$$+ \frac{1}{a}\sin t\int_0^t u(\tau)\sin\tau\, d\tau - \frac{1}{a}u(t)\cancel{\sin t\cos t}$$

$$u''(t) = -\frac{1}{a}\sin t\int_0^t u(\tau)\cos\tau\, d\tau + \frac{1}{a}u(t)\cos^2 t$$

$$+ \frac{1}{a}\cos t\int_0^t u(\tau)\sin\tau\, d\tau + \frac{1}{a}u(t)\sin^2 t$$

$$= -\left\{\frac{1}{a}\sin t\int_0^t u(\tau)\cos\tau\, d\tau - \frac{1}{a}\cos t\int_0^t u(\tau)\sin\tau\, d\tau\right\} + \frac{1}{a}u(t)$$

$$= -(u(t)-1) + \frac{1}{a}u(t)$$

・ $u'' + (1-\frac{1}{a})u = 1$

(同次形)

$$u'' + (1-\frac{1}{a})u = 0$$

$$t^2 + (1-\frac{1}{a}) = 0$$

$t = \pm\sqrt{1-\frac{1}{a}}$; $\xrightarrow{\text{虚数解}}$ $u(t) = C_1\cos\sqrt{1-\frac{1}{a}}\,t + C_2\sin\sqrt{1-\frac{1}{a}}\,t$ (同次形)

(非同次形)

特殊解 $u(t) = C_3$ と書け, $C_3 = \frac{a}{a-1}$ より、

$$u(t) = \frac{a}{a-1} + C_1\cos\sqrt{1-\frac{1}{a}}\,t + C_2\sin\sqrt{1-\frac{1}{a}}\,t$$

・$u(0) = 1$, $u'(0) = 0$ より、

$$C_1 = -\frac{1}{a-1}, \quad C_2 = 0$$

従って、不動点は、

$$u(t) = \frac{a}{a-1} - \frac{1}{a-1}\cos\sqrt{1-\frac{1}{a}}\,t$$