問8 H27

(1)

$$E(X)=\int_0^\infty x f(x)\,dx \geq \int_a^\infty x f(x)\,dx \geq \int_a^\infty a f(x)\,dx = a\int_a^\infty f(x)\,dx = aP(X>a)$$

$$\therefore P(X>a) \leq \frac{E(X)}{a}$$

(2)

$C_n := \bigcup_{k=n}^\infty A_k$ とおくと、C_nは単調減少なので、

$$P\left(\bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k\right) = P\left(\bigcap_{n=1}^\infty C_n\right) = \lim_{n\to\infty} P(C_n) = \lim_{n\to\infty} P\left(\bigcup_{k=n}^\infty A_k\right) \leq \lim_{n\to\infty} \sum_{k=n}^\infty P(A_k)$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^\infty P(A_k) - \sum_{k=1}^{n-1} P(A_k)\right) = \sum_{k=1}^\infty P(A_k) - \sum_{k=1}^\infty P(A_k) = 0$$

一方、$0 \leq P$ より、

$$P\left(\bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k\right) = 0 \quad \blacksquare$$

(3)

$$P\left(\lim_{n\to\infty} |X_n - E(X_n)| \neq 0\right) = 0 \iff P($$

limが定義されないときまずいかも.

Var

$A_{n,\varepsilon} = \{\underbrace{|X_n - E(X_n)|}_{Y_n} > \varepsilon\}$ $\quad A_{n,\varepsilon}$

$\lim_{n\to\infty} A_{n,\varepsilon} =$

$$\cancel{P\left(\lim_{n\to\infty} |X_n - E(X_n)| = 0\right)},$$

$$\cancel{P\left(\limsup_{n\to\infty} |X_n - E(X_n)| > \varepsilon\right)} =$$

$$\sum P(A_n) = \sum P(|X_n - E(X_n)| > \varepsilon) = \sum P(|X_n - E(X_n)|^2 > \varepsilon^2) \leq \sum \frac{E(|X_n - E(X_n)|^2)}{\varepsilon^2}$$

$$= \frac{1}{\varepsilon^2} \sum V(X_n) < +\infty$$ より、

$$P\left(\bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k\right) = P\left(\limsup_{n\to\infty} A_k\right) = 0$$

$$\limsup_{n\to\infty} A_k = \{\omega \in \Omega ;\ {}^\forall n \in \mathbb{N},\ {}^\exists k \geq n,\ |X_k(\omega) - E(X_k)| > \varepsilon\}$$

$$= \{\omega \in \Omega ;\ {}^\forall n \in \mathbb{N},\ \sup_{k\geq n} |X_k(\omega) - E(X_k)| > \varepsilon\}$$

$$= \{\omega \in \Omega ;\ \inf_{n\geq 1} \sup |X_k(\omega) - E(X_k)| > \varepsilon\}$$

$$= \{\omega \in \Omega ;\ \limsup_{n\to\infty} |X_n(\omega) - E(X_k)| > \varepsilon\} = \{\limsup |X_n - E(X_n)| > \varepsilon\}$$

$\therefore {}^\forall \varepsilon > 0,\ P(\limsup |X_n - E(X_n)| > \varepsilon) = 0,\ \limsup |X_n - E(X_n)| = 0 \iff \liminf = \limsup = \lim |X_n - E(X_n)| = 0$ より.

$\Longleftrightarrow \cancel{\lim_{n\to\infty} X_n}$

$\cancel{P(\lim in}$

∴ $1 = P(\limsup |X_n - E(X_n)| \leq \varepsilon) \quad (\forall \varepsilon)$

$\Longleftrightarrow 1 = P(\limsup |X_n - E(X_n)| = 0)$

$\Longleftrightarrow 1 = P(\lim |X_n - E(X_n)| = 0) \quad (\because \liminf \leq \limsup)$ //