Project 7: Ordinary Differential Equations with Initial Value Problem

Problem 1: Dynamics and Oscillation

Part a: Falling Body

$$m\dot{v} = c_1 - c_2 v$$
$$\dot{x} = v$$
$$v(0) = 1, \qquad x(0) = 0$$

Where $c_1 = mg$, $c_2 = m$, with $m = 3, g = 2$

Use each of the methods below, with $\Delta t = 0.01$ and $t \in [0, 10]$.

i. Forward Difference, Backward Difference, Central Difference, Trapezoidal Rule
ii. Second, third, fourth Runge-Kutta Methods
iii. Collocation Method with piecewise quadratic and cubic polynomials

Part b: Newton's Cannon/ Orbital Mechanics and Conics Sections

$$\ddot{x} + \omega^2 x = 0, \qquad \ddot{y} + \omega^2 y = 0, \qquad \omega^2 = \frac{\mu}{(x^2 + y^2)^{3/2}}$$
$$x(0) = 1, \qquad \dot{x}(0) = 0, \qquad y(0) = 0, \qquad \dot{y}(0) = 1$$

For $\mu = (0.35, 0.55, 0.75, 1)$, calculate and plot H and M:

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - (x^2 + y^2)^{-\frac{1}{2}}, \qquad M = x\dot{y} - y\dot{x}$$

With $\Delta t = 0.01$ and $t \in [0, 50]$. Plot the results (i.e. plot $x(t)$ vs $y(t)$).

Part c: Nonlinear Mass-Spring Damping System (Pick one between part i and ii)

i. Duffing's Equation

$$\ddot{x} + \delta\dot{x} - \beta x + \alpha x^3 = f\cos(\omega t)$$
$$x(0) = 0, \qquad \dot{x}(0) = 1$$

Use $\delta = 0.24$, $\omega = \beta = \alpha = 1$, $f = 0.22$

ii. Van del Pol Equation:

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$$
$$x(0) = 0, \qquad \dot{x}(0) = 1$$

Use $\epsilon = 8$.

For both equations, use $\Delta t = 0.01$ and $t \in [0, 50]$. Plot t vs x, t vs $\dot{x}$, and x vs. $\dot{x}$.

Problem 2: Rocket Propulsion and Chemical Reactions

Part a: Solid Propellant Engine Explosion

$$\tau\dot{P} = k_b P^n - k_t P$$
$$P(0) = 1.5, \qquad \tau = 1$$

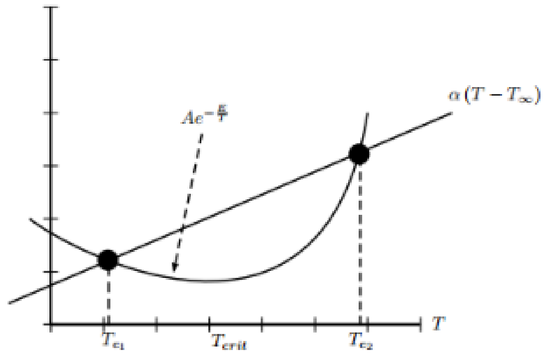
i. $n = 0.7$, $k_b = 1.5$, $k_t = 1.1$
ii. $n = 0.7$, $k_b = 1.1$, $k_t = 1.5$
iii. $n = 1.8$, $k_b = 1.1$, $k_t = 1.1$

With $\Delta t = 0.01$ and $t \in [0, 15]$

Part b: Thermal Explosion

$$\dot{T} = -\alpha(T - T_\infty) + Ae^{-E/T}$$
$$T_\infty = 0.15$$

Find a value for A to produce two equilibrium points. Do this by setting $\dot{T} = 0$. Expect something like this:



Choose an initial condition near each of the equilibrium points (e.g. $1.1T_{e1}, 1.1T_{e2}$, etc.) with $\alpha = E = 1$.

Part c: Heat Radiation

$$\dot{T} = -\beta(T^4 - T_\infty^4)$$
$$T_\infty = 0.1$$

With $\Delta t = 0.02, t \in [0, 15], \beta = 2.5$, and $T_0 = 1$.

Problem 3: Bifurcations and Chaos

Part a: Landau Equation

$$\dot{u} = au - bu^3$$
$$u(0) = 1$$

With $\Delta t = 0.02, a = 2.2$, and $b = 0.08$. Compute for $t \in [0, 10]$.

Part b: Hopf Bifurcation

$$\dot{x} = -y + (a - x^2 - y^2)x$$
$$\dot{y} = x + (a - x^2 - y^2)y$$

Using $a = 0.3$, $\Delta t = 0.02$, find solutions from various initial conditions ($7x7$ mesh):

$$0 \le x(0) \le 3$$
$$0 \le y(0) \le 3$$

For $t \in [0, 30]$. Plot $x(t)$ vs $y(t)$.

Part c: Lorenz Equation

$$\dot{x} = 13(y - x)$$
$$\dot{y} = x(31 - z) - y$$
$$\dot{z} = xy - \frac{13}{3}z$$
$$x(0) = 5, \qquad y(0) = 5, \qquad z(0) = 5$$

For $t \in [0, 30]$, $\Delta t = 0.02$. There are a few ways to plot this. (E.g., Plot the solution in 3D; Plot the solution projected into each plane (x-y plane, x-z plane, and y-z plane)).

Problem 4: Third Order Equations (Pick one between part b and c)

Part a: Jerk Equation

$$\frac{d^3x}{dt^3} + A\frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 + x = 0$$

The third order equation can be replaced by three first order equations as,

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = a, \qquad \frac{da}{dt} = -Aa + v^2 - x$$

Use $x_0 = 0.02, v_0 = 0, a_0 = 0, A = 2.022, \Delta t = 0.02$, and $t \in [0, 20]$. Plot v vs x.

Part b: Blasius Equation

$$f''' + f\,f'' = 0$$
$$f(0) = 0, \qquad f'(0) = 0, \qquad f' \to 1 \text{ as } \ t \to \infty$$

Solve this problem using shooting method by decomposing the third and second derivative to first derivatives of the form,

$$\frac{df}{dt} = u, \qquad \frac{du}{dt} = a, \qquad \frac{da}{dt} = -fa$$

Part c: Soliton Equation

$$-cf' + f\,f' + \delta^2 f''' = 0$$
$$c = 0.9$$
$$f(0) = 3, \qquad f'(0) = 0, \qquad f(L) = 3, \qquad L \gg 1.5$$

Solve the equation using the shooting method (decompose the problem in three first-order equations).

Problem 5: Spin Stability (Extra Credit)

Solve Euler equations of rigid body rotation (with free torque and $\beta = 0$)

$$\frac{d^2\omega_1}{dt^2} + \beta\frac{d\omega_1}{dt} = \alpha_1\omega_1 \qquad \omega_1(0) = 1, \ \ \frac{d\omega_1}{dt}(0) = 0$$

$$n = 4000$$

$$\frac{d^2\omega_2}{dt^2} + \beta\frac{d\omega_2}{dt} = \alpha_2\omega_2 \qquad \omega_2(0) = 0.02, \ \ \frac{d\omega_2}{dt}(0) = 0$$

$$t < 70$$

$$\frac{d^2\omega_3}{dt^2} + \beta\frac{d\omega_3}{dt} = \alpha_3\omega_3 \qquad \omega_3(0) = 0.02, \ \ \frac{d\omega_3}{dt}(0) = 0$$

where

$$\alpha_1 = \omega_3^2\frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2} + \omega_2^2\frac{(I_2 - I_3)(I_1 - I_2)}{I_1 I_3}$$

$$\alpha_2 = \omega_1^2\frac{(I_3 - I_1)(I_1 - I_2)}{I_2 I_3} + \omega_3^2\frac{(I_2 - I_3)(I_3 - I_1)}{I_2 I_1}$$

$$\alpha_3 = \omega_2^2\frac{(I_1 - I_2)(I_2 - I_3)}{I_3 I_1} + \omega_1^2\frac{(I_1 - I_2)(I_3 - I_1)}{I_3 I_2}$$

and $I_1 = 3, \ \ I_2 = 2, \ \ I_3 = 1.$

Plot the kinetic energy (E) and the squares of the angular momentum (A) given by

$$E = \frac{1}{2}(\omega_1^2 I_1 + \omega_2^2 I_2 + \omega_3^2 I_3) \qquad A = \omega_1^2 I_1^2 + \omega_2^2 I_2^2 + \omega_3^2 I_3^2$$

For $\beta = 0$, show that spinning around the axis of maximum or minimum moment of inertia is stable while for $\beta > 0$ only spinning around the axis of maximum moment of inertia is stable.