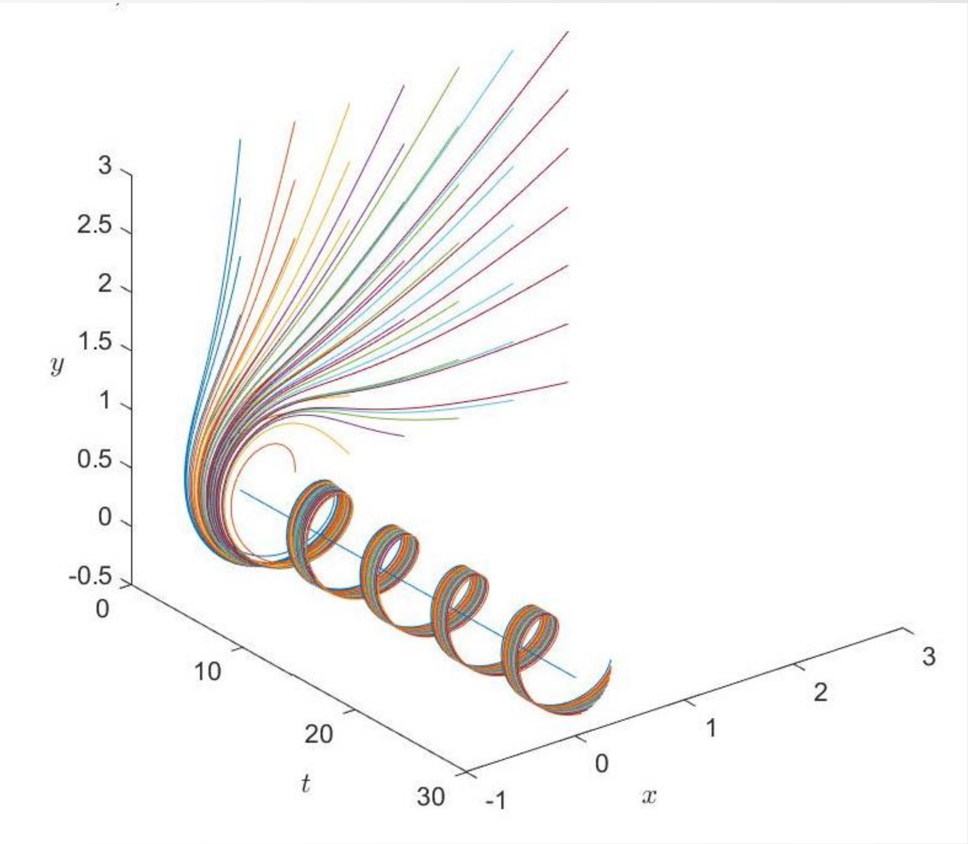
WEEK 7 DISCUSSION

ORDINARY DIFFERENTIAL EQUATIONS WITH INITIAL VALUES



Jiahe Chai
ENG 180 – Engineering Analysis
Dept. of Mechanical and Aerospace Engineering
UC Davis

REFERENCE

- Kreyszig, Chap 21.1, 21.3
- Hafez and Tavernetti, 5.3

NOMENCLATURE

- A single data point at discrete level:

$$y_{i,k}^{(n)} = y(x_i, z_k, t_n)$$

  - n: time step on axis t
  - i: location index on axis x
  - k: location index on axis z
- Adjacent points
  - in t:

$$y_{i,k}^{(n+1)} = y(x_i, z_k, t_n + \Delta t)$$

  - in x:

$$y_{i+1,k}^{(n)} = y(x_i + \Delta x, z_k, t_n)$$

  - in z:

$$y_{i,k+1}^{(n)} = y(x_i, z_k + \Delta z, t_n)$$

- Function Handle: e.g., $f(y) = y + t$

$$f\left(y^{(n+1)}, t^{(n+1)}\right) = y^{(n+1)} + t^{(n+1)}$$

PROGRAMMING TIPS

- For first order equation: $y' = f(y)$, Solve for $y^{(n+1)}$
- Write $\vec{f}(\vec{y}, t)$ as a local function handle
  - f_func = @(y, t) [-y(1)-y(2)-t; y(1)]
- Write scheme as a user-defined function
  - function y_np1 = FE(dt, yn, tn, f_func)
    - k1 = dt * f_func(yn, tn)
    - y_np1 = yn + k1
- Call the scheme function in every time step
  - yn = [v(n); x(n)]
  - y_np1 = FE(dt, yn, tn, f_func)
  - v(n+1) = y_np1(1)
  - x(n+1) = y_np1(2)

HIGHER ORDER EQUATIONS

- For example:

$$x'' + x' + x + t = 0$$

- Approach with System of first order equations

$$v' = -v - x - t$$

$$x' = v$$

- In general vector form

$$\vec{y} = \begin{bmatrix} v \\ x \end{bmatrix}$$

$$\vec{f}(\vec{y}, t) = \begin{bmatrix} -v - x - t \\ v \end{bmatrix}$$

$$\vec{y}' = \vec{f}(\vec{y}, t)$$

TIME ADVANCING SCHEMES

- Forward Euler

$$\frac{1}{\Delta t}\left(y^{(n+1)} - y^{(n)}\right) = f\left(y^{(n)}, t^{(n)}\right)$$

- Backward Euler (Implicit)

$$\frac{1}{\Delta t}\left(y^{(n+1)} - y^{(n)}\right) = f\left(y^{(n+1)}, t^{(n+1)}\right)$$

- Leap Frog

$$\frac{1}{2\Delta t}\left(y^{(n+1)} - y^{(n-1)}\right) = f\left(y^{(n)}, t^{(n)}\right)$$

- Crank Nicolson (Implicit)

$$\frac{1}{\Delta t}\left(y^{(n+1)} - y^{(n)}\right) = \frac{1}{2}\left[f\left(y^{(n+1)}, t^{(n+1)}\right) + f\left(y^{(n)}, t^{(n)}\right)\right]$$

TIME ADVANCING SCHEMES RUNGE-KUTTA (1/2)

- 2nd order Runge Kutta

$$
\begin{aligned}
k_1 &= \Delta t \cdot f\left(y^{(n)}, t^{(n)}\right) \\
k_2 &= \Delta t \cdot f\left(y^{(n)} + k_1, t^{(n+1)}\right) \\
y^{(n+1)} &= y^{(n)} + \frac{1}{2}(k_1 + k_2)
\end{aligned}
$$

- 3rd order Runge Kutta

$$
\begin{aligned}
k_1 &= \Delta t \cdot f\left(y^{(n)}, t^{(n)}\right) \\
k_2 &= \Delta t \cdot f\left(y^{(n)} + \frac{k_1}{2}, t^{(n+\frac{1}{2})}\right) \\
k_3 &= \Delta t \cdot f\left(y^{(n)} - k_1 + 2k_2, t^{(n+1)}\right) \\
y^{(n+1)} &= y^{(n)} + \frac{1}{6}(k_1 + 4k_2 + k_3)
\end{aligned}
$$

TIME ADVANCING SCHEMES RUNGE-KUTTA (2/2)

- 4th order Runge Kutta

$$k_1 = \Delta t \cdot f\left(y^{(n)}, t^{(n)}\right)$$
$$k_2 = \Delta t \cdot f\left(y^{(n)} + \frac{k_1}{2}, t^{(n+\frac{1}{2})}\right)$$
$$k_3 = \Delta t \cdot f\left(y^{(n)} + \frac{k_2}{2}, t^{(n+\frac{1}{2})}\right)$$
$$k_4 = \Delta t \cdot f\left(y^{(n)} + k_3, t^{(n+1)}\right)$$
$$y^{(n+1)} = y^{(n)} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

- 3/8 4th order Runge Kutta

$$k_1 = \Delta t \cdot f\left(y^{(n)}, t^{(n)}\right)$$
$$k_2 = \Delta t \cdot f\left(y^{(n)} + \frac{k_1}{3}, t^{(n+\frac{1}{3})}\right)$$
$$k_3 = \Delta t \cdot f\left(y^{(n)} - \frac{k_1}{3} + k_2, t^{(n+\frac{2}{3})}\right)$$
$$k_4 = \Delta t \cdot f\left(y^{(n)} + k_1 - k_2 + k_3, t^{(n+1)}\right)$$
$$y^{(n+1)} = y^{(n)} + \frac{1}{8}(k_1 + 3k_2 + 3k_3 + k_4)$$

TIME ADVANCING SCHEMES
COLLOCATION METHOD - QUADRATIC IMPLICIT

- Governing Equation:

$$y' = f(y, t)$$

- Quadratic Implicit:

$$y(t) = y^{(n)} + b\left(t - t^{(n)}\right) + c\left(t - t^{(n)}\right)^2$$
$$y'(t) = b + 2c\left(t - t^{(n)}\right)$$

  - At $t = t^{(n)}$:

$$y'^{(n)} = f\left(y^{(n)}, t^{(n)}\right) = b$$

  - At $t = t^{(n+1)}$:

$$y'^{(n+1)} = f\left(y^{(n+1)}, t^{(n+1)}\right) = b + 2c\,\Delta t$$

  - Find Coefficients:

$$b = f\left(y^{(n)}, t^{(n)}\right)$$
$$c = \frac{1}{2\Delta t}\left[f\left(y^{(n+1)}, t^{(n+1)}\right) - f\left(y^{(n)}, t^{(n)}\right)\right]$$

  - At $t = t^{(n+1)}$:

$$y^{(n+1)} = y^{(n)} + b\,\Delta t + c\,\Delta t^2$$

  - Solve for $y^{(n+1)}$ implicitly

TIME ADVANCING SCHEMES
COLLOCATION METHOD – 2ND ORDER QUADRATIC COLLOCATION

- Quadratic Explicit (Predictor-Corrector)
  - Predictor: y^* at t^{n+1} by Forward Euler

$$\frac{1}{\Delta t}(y^* - y^n) = f(y^n, t^n)$$

$$y^* = y^n + \Delta t\, f(y^n, t^n)$$

  - Corrector: y^{n+1} by Quadratic Spline with $f(y^*, t^{n+1})$

$$b = f(y^n, t^n)$$

$$c = \frac{1}{2\Delta t}[f(y^*, t^{n+1}) - f(y^n, t^n)]$$

  - At $t = t^{n+1}$:

$$y^{n+1} = y^n + b\,\Delta t + c\,\Delta t^2$$

  - Solve for y^{n+1} explicitly

$$y^{n+1} = y^n + \frac{1}{2}\Delta t[f(y^*, t^{n+1}) + f(y^n, t^n)]$$

- Note: Equivalent to RK2

TIME ADVANCING SCHEMES
COLLOCATION METHOD - 3RD ORDER CUBIC COLLOCATION

- 3rd Order Cubic Collocation
- **First** Function Evaluation: $f(y^n, t^n)$
  - Algebraic Evaluation 1: $y^{n+1/2}$ at $t^{n+1/2}$ by Forward Euler

$$y^{n+1/2} = y^n + \frac{1}{2}\Delta t\, f(y^n, t^n)$$

- **Second** Function Evaluation: $f\left(y^{n+1/2}, t^{n+1/2}\right)$
  - Algebraic Evaluation 3: y^* at t^{n+1} by Cubic Spline fitted with $y^n, f(y^n, t^n), y^{n+1/2}, f(y^{n+1/2}, t^{n+1/2})$

$$c = \frac{2}{\Delta t^2}\left[6y^{n+1/2} - 6y^n - 2\Delta t\, f(y^n, t^n) - \Delta t\, f\left(y^{n+1/2}, t^{n+1/2}\right)\right]$$

$$d = \frac{4}{\Delta t^3}\left[-4y^{n+1/2} + 4y^n + \Delta t\, f(y^n, t^n) + \Delta t\, f\left(y^{n+1/2}, t^{n+1/2}\right)\right]$$

$$y^* = y^n + f(y^n, t^n)(\Delta t) + c(\Delta t)^2 + d(\Delta t)^3$$

- **Third** Function Evaluation: $f\left(y^*, t^{n+1}\right)$
  - Algebraic Evaluation 4: y^{n+1} by Cubic Spline fitted with $y^n, f(y^n, t^n), f\left(y^{n+1/2}, t^{n+1/2}\right), f(y^*, t^{n+1})$

$$y^{n+1} = y^n + \frac{1}{6}\Delta t\left[f(y^n, t^n) + 4f\left(y^{n+1/2}, t^{n+1/2}\right) + f\left(y^*, t^{n+1}\right)\right]$$

- Note: Equivalent to RK3

TIME ADVANCING SCHEMES
COLLOCATION METHOD – 4TH ORDER CUBIC COLLOCATION (1/2)

- 4th Order Cubic Collocation (Option 1)
- **First** Function Evaluation: $f(y^n, t^n)$
  - Algebraic Evaluation 1: $y^{n+1/2}$ at $t^{n+1/2}$ by Forward Euler

$$y^{n+1/2} = y^n + \frac{1}{2}\Delta t\, f(y^n, t^n)$$

- **Second** Function Evaluation: $f\left(y^{n+1/2}, t^{n+1/2}\right)$
  - Algebraic Evaluation 2: y^* at $t^{n+1/2}$ by Quadratic Spline fitted with $y^n, f(y^n, t^n), f(y^{n+1/2}, t^{n+1/2})$

$$c = \frac{1}{\Delta t}\left[f\left(y^{n+1/2}, t^{n+1/2}\right) - f(y^n, t^n)\right]$$

$$y^* = y^n + f(y^n, t^n)\left(\frac{1}{2}\Delta t\right) + c\left(\frac{1}{2}\Delta t\right)^2$$

- **Third** Function Evaluation: $f\left(y^*, t^{n+1/2}\right)$
  - Algebraic Evaluation 3: y^{**} at t^{n+1} by Cubic Spline fitted with $y^n, f(y^n, t^n), y^*, f(y^*, t^{n+1/2})$

$$c = \frac{2}{\Delta t^2}\left[6y^* - 6y^n - 2\Delta t\, f(y^n, t^n) - \Delta t\, f\left(y^*, t^{n+1}\right)\right]$$

$$d = \frac{4}{\Delta t^3}\left[-4y^* + 4y^n + \Delta t\, f(y^n, t^n) + \Delta t\, f\left(y^*, t^{n+1}\right)\right]$$

$$y^{**} = y^n + f(y^n, t^n)(\Delta t) + c(\Delta t)^2 + d(\Delta t)^3$$

- **Fourth** Function Evaluation: $f\left(y^{**}, t^{n+1}\right)$
  - Algebraic Evaluation 4: y^{n+1} by Cubic Spline fitted with $y^n, f(y^n, t^n), f\left(y^*, t^{n+1/2}\right), f(y^{**}, t^{n+1})$

$$y^{n+1} = y^n + \frac{1}{6}\Delta t\left[f(y^n, t^n) + 4f\left(y^*, t^{n+1/2}\right) + f\left(y^{**}, t^{n+1}\right)\right]$$

- Note: Equivalent to RK4

TIME ADVANCING SCHEMES
COLLOCATION METHOD – 4TH ORDER CUBIC COLLOCATION (2/2)

- 4th Order Cubic Collocation (Option 2)
- **First** Function Evaluation: $f(y^n, t^n)$
  - Algebraic Evaluation 1: y^* at t^{n+1} by Forward Euler

$$y^* = y^n + \Delta t\, f(y^n, t^n)$$

- **Second** Function Evaluation: $f\left(y^*, t^{n+1}\right)$
  - Algebraic Evaluation 2: y^{**} at t^{n+1} by Quadratic Spline fitted with $y^n, f(y^n, t^n), f(y^{n+1/2}, t^{n+1/2})$

$$c = \frac{1}{2\Delta t}\left[f\left(y^*, t^{n+1}\right) - f(y^n, t^n)\right]$$

$$y^{**} = y^n + f(y^n, t^n)(\Delta t) + c(\Delta t)^2$$

- **Third** Function Evaluation: $f\left(y^{**}, t^{n+1}\right)$
  - Algebraic Evaluation 3: $y^{n+1/2}$ at $t^{n+1/2}$ by Quadratic Spline fitted with $y^n, f(y^n, t^n), f(y^{**}, t^{n+1})$

$$c = \frac{1}{2\Delta t}\left[f\left(y^{**}, t^{n+1}\right) - f(y^n, t^n)\right]$$

$$y^{n+1/2} = y^n + f(y^n, t^n)\left(\frac{1}{2}\Delta t\right) + c\left(\frac{1}{2}\Delta t\right)^2$$

- **Fourth** Function Evaluation: $f\left(y^{n+1/2}, t^{n+1/2}\right)$
  - Algebraic Evaluation 4: y^{n+1} by Cubic Spline fitted with $y^n, f(y^n, t^n), f\left(y^{n+1/2}, t^{n+1/2}\right), f(y^{**}, t^{n+1})$

$$y^{n+1} = y^n + \frac{1}{6}\Delta t\left[f(y^n, t^n) + 4f\left(y^{n+1/2}, t^{n+1/2}\right) + f\left(y^{**}, t^{n+1}\right)\right]$$

- Note: Equivalent to RK4

PROBLEM 1A: FALLING BODY

- Exact Analytical Solution: Solve nonhomogeneous ODE

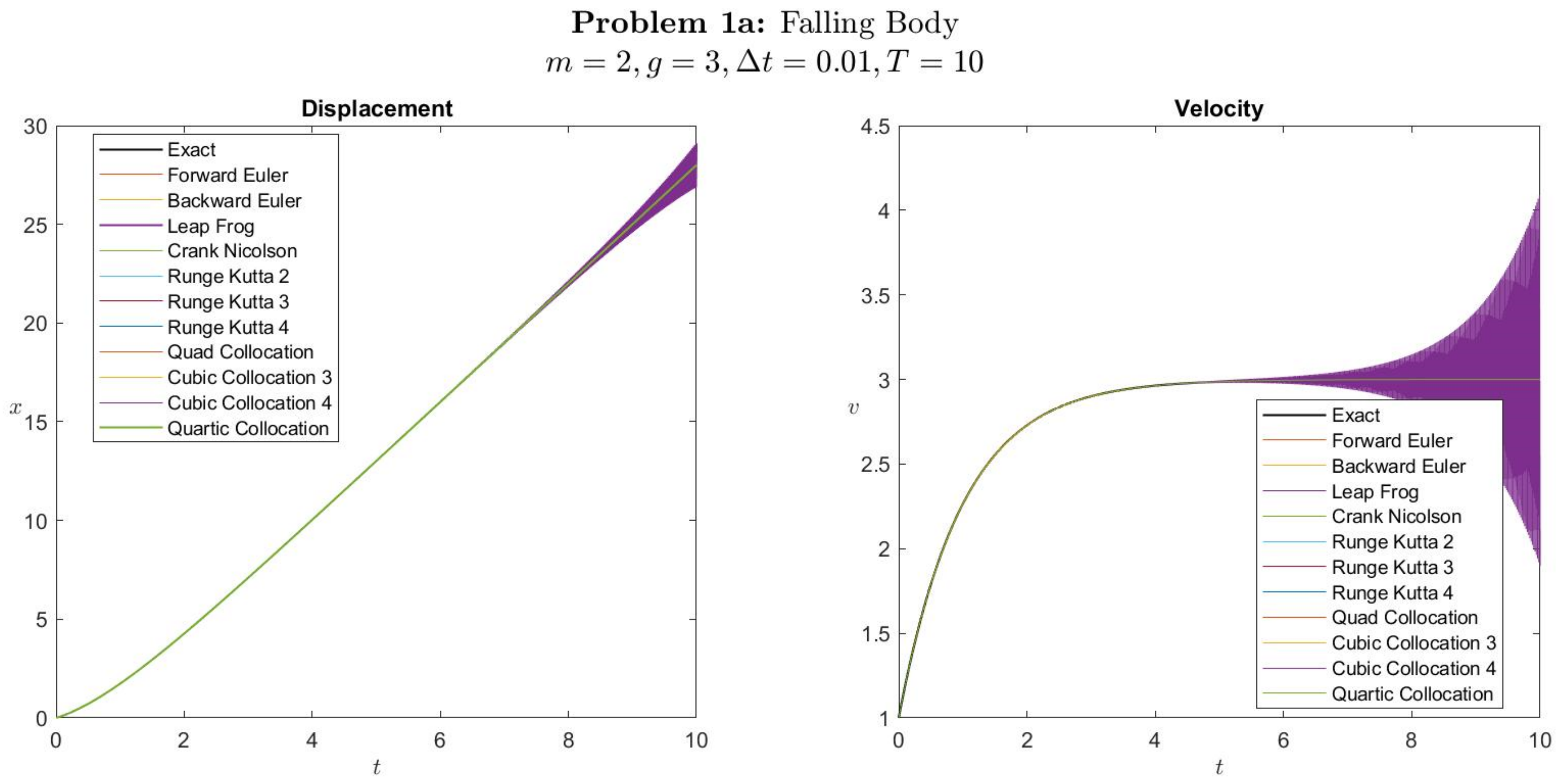
$$mx'' + c_2 x' = c_1$$

- Result:

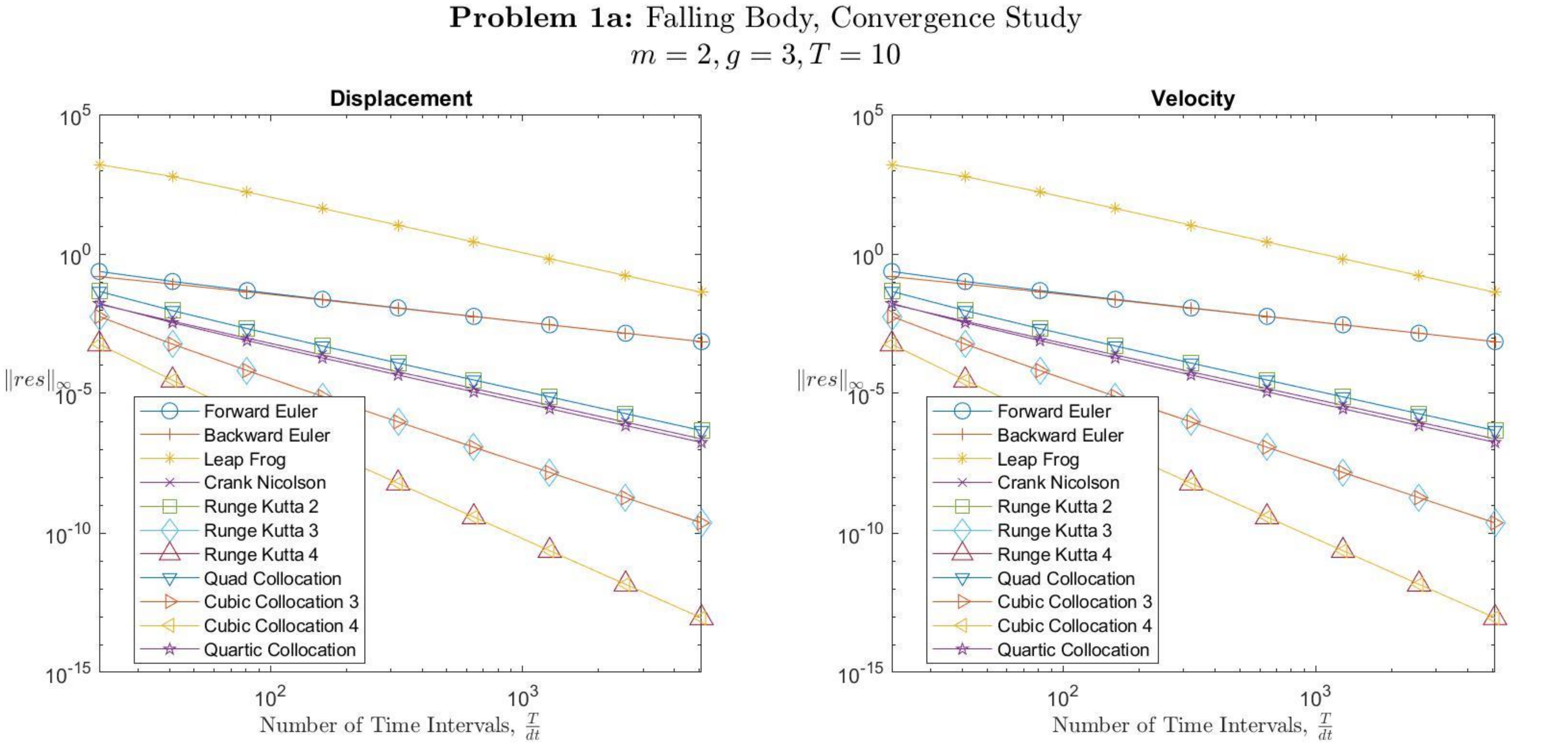
$$x = \frac{m}{c_2}\left(\frac{c_1}{c_2} - 1\right)e^{-\frac{c_2}{m}t} - \frac{m}{c_2}\left(\frac{c_1}{c_2} - 1\right) + \frac{c_1}{c_2}t$$

$$v = -\left(\frac{c_1}{c_2} - 1\right)e^{-\frac{c_2}{m}t} + \frac{c_1}{c_2}$$

PROBLEM 1A: FALLING BODY

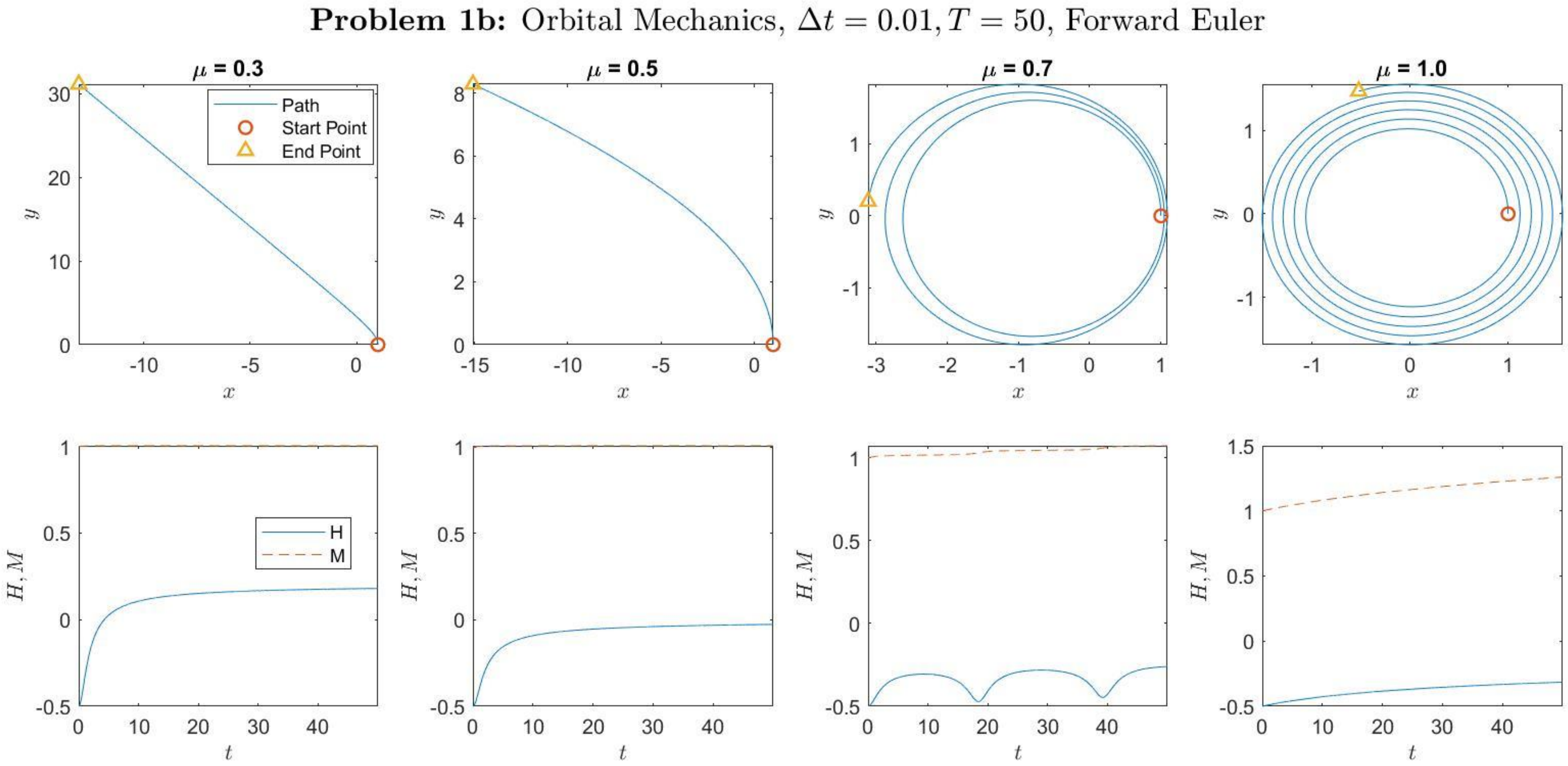


PROBLEM 1A: FALLING BODY CONVERGENCE STUDY



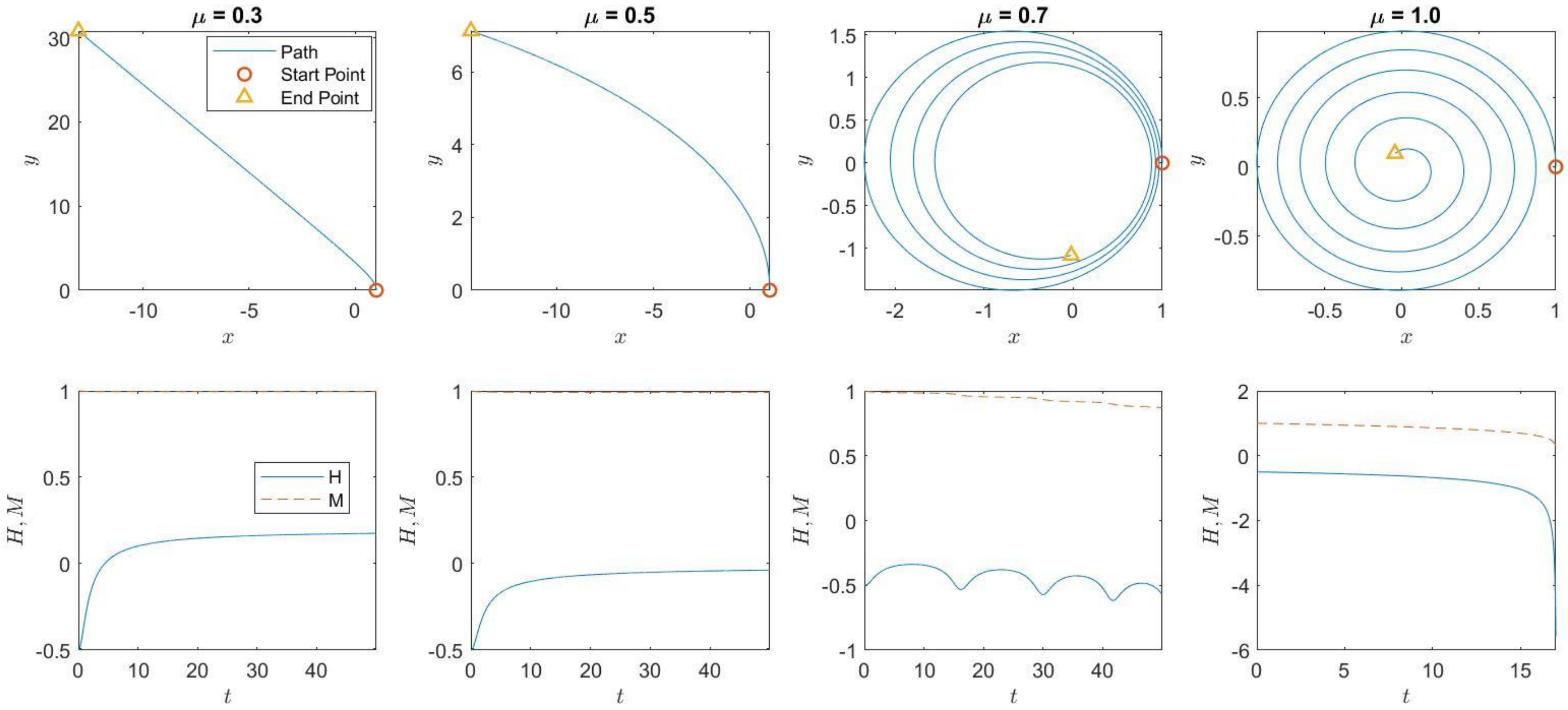
*quartic collocation is not converging properly

PROBLEM 1B: ORBITAL MECHANICS FORWARD EULER

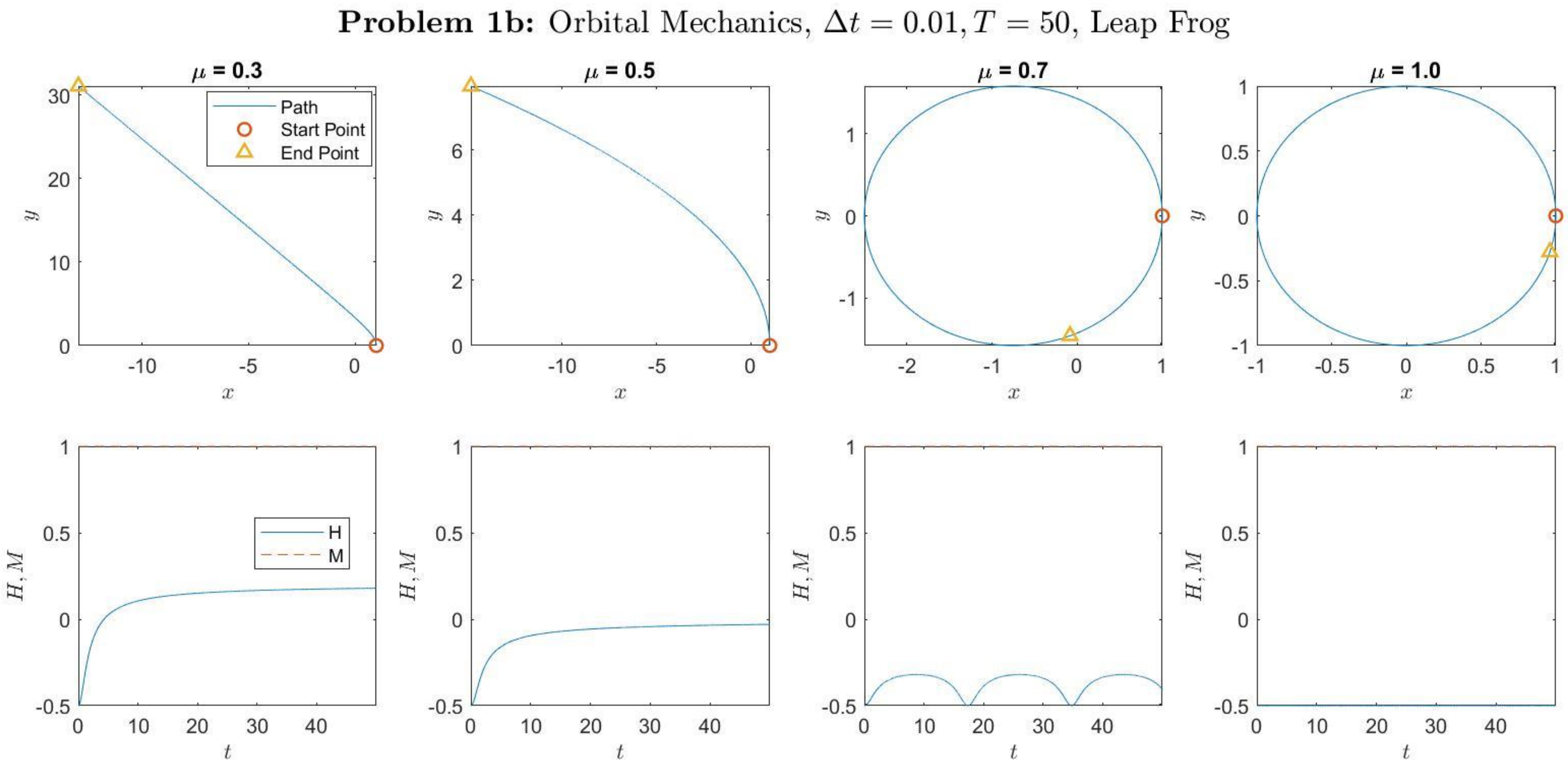


PROBLEM 1B: ORBITAL MECHANICS BACKWARD EULER

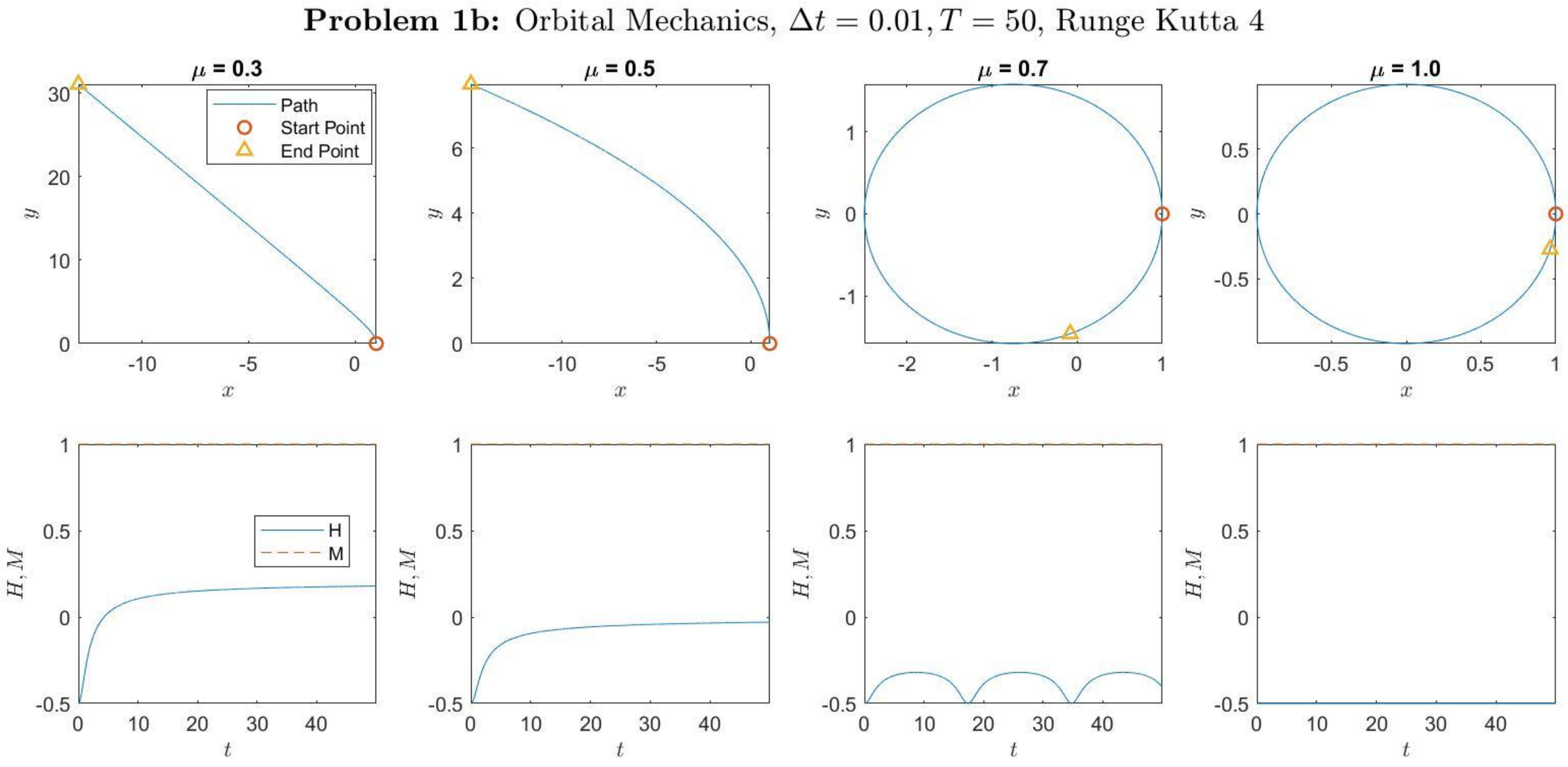
Problem 1b: Orbital Mechanics, $\Delta t = 0.01, T = 50$, Backward Euler



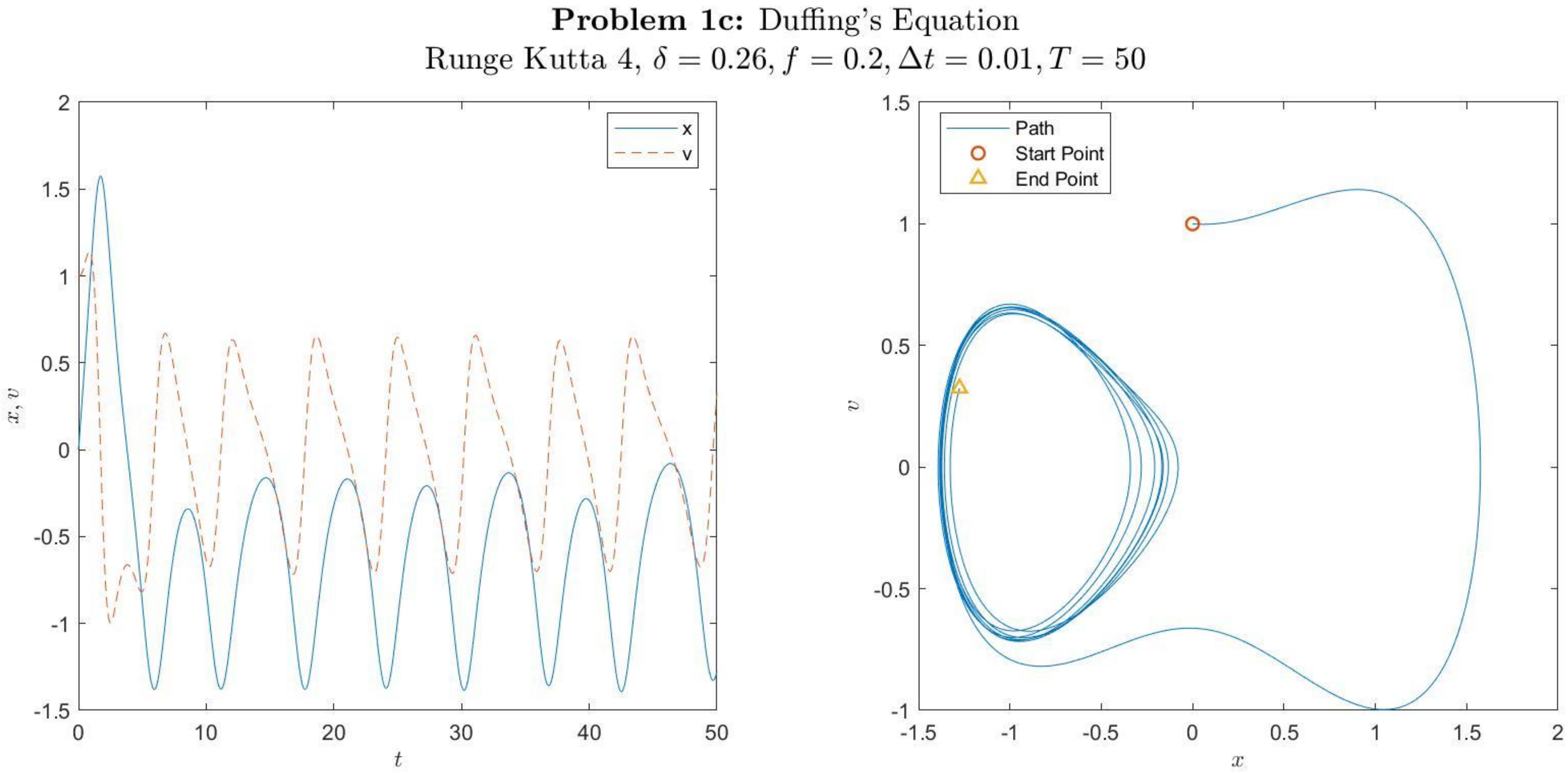
PROBLEM 1B: ORBITAL MECHANICS LEAP FROG



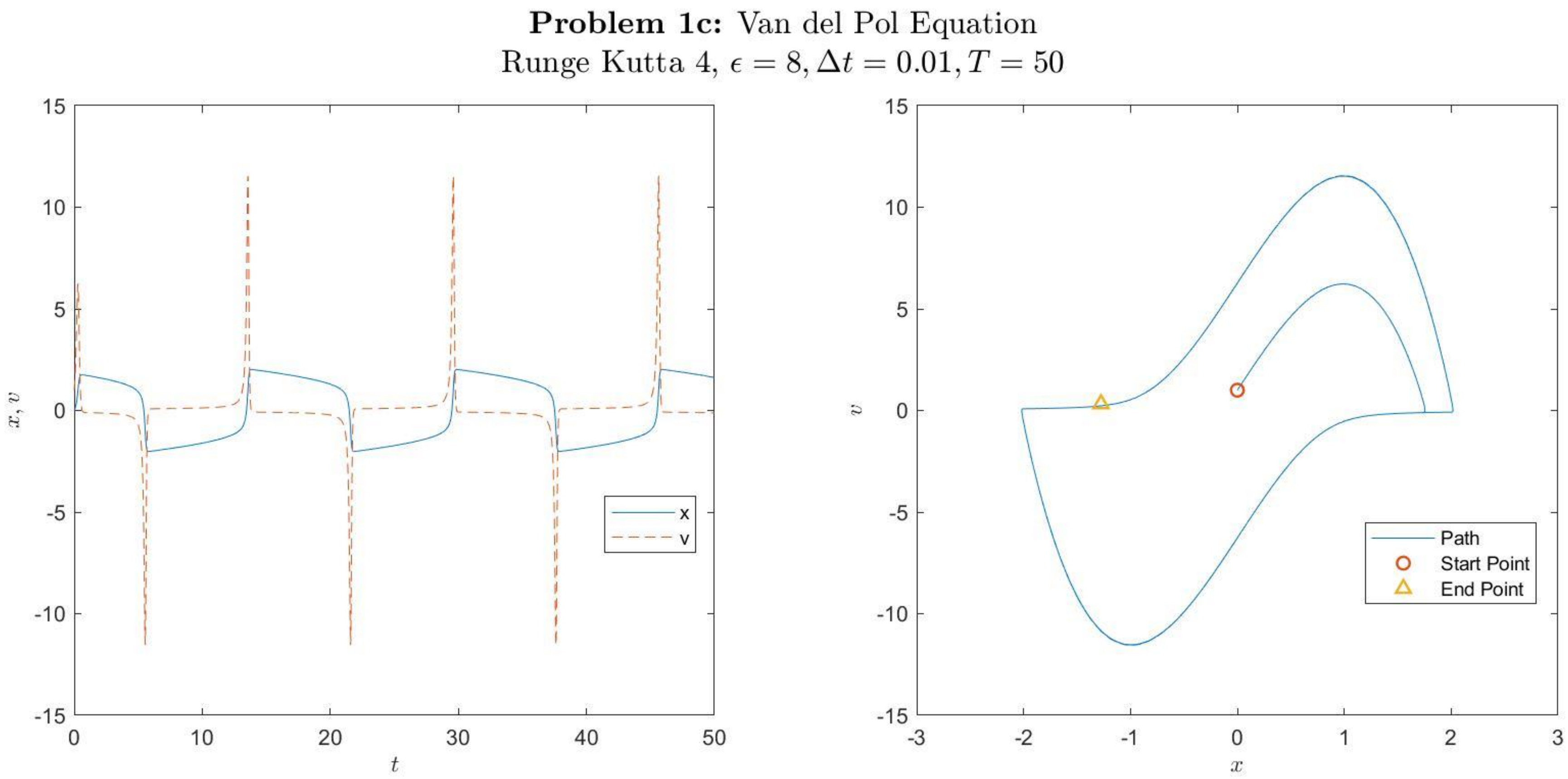
PROBLEM 1B: ORBITAL MECHANICS RUNGE KUTTA 4



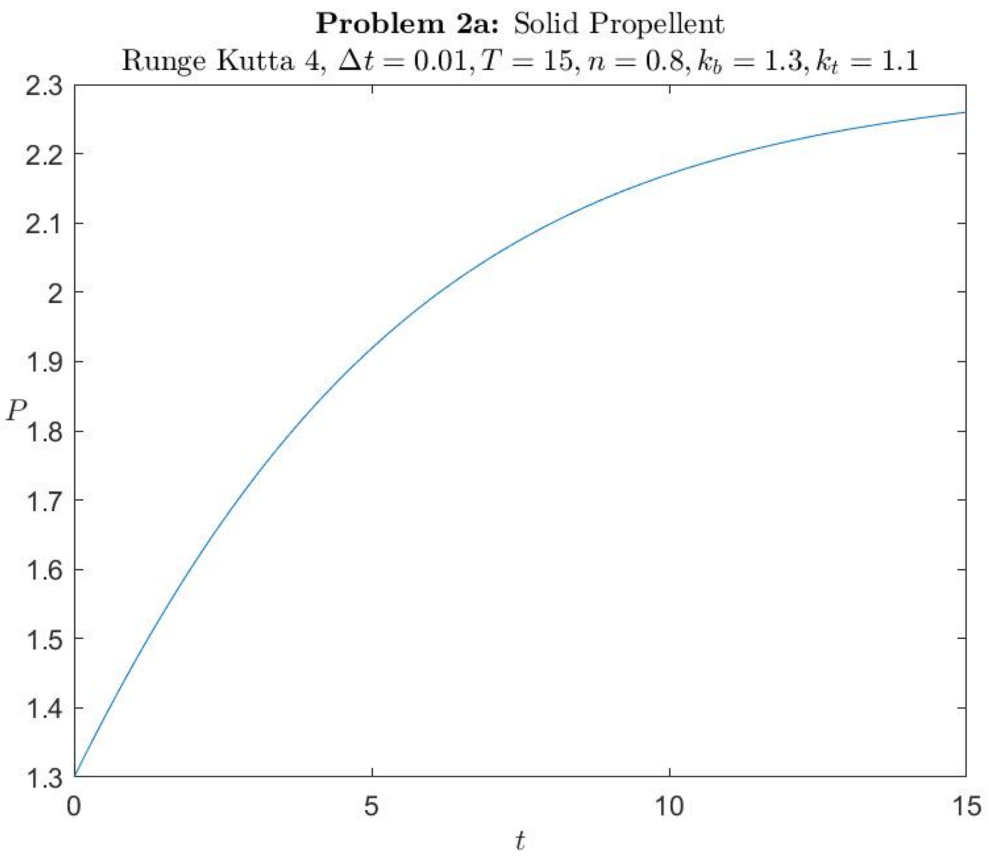
PROBLEM 1C: DUFFING'S EQUATION



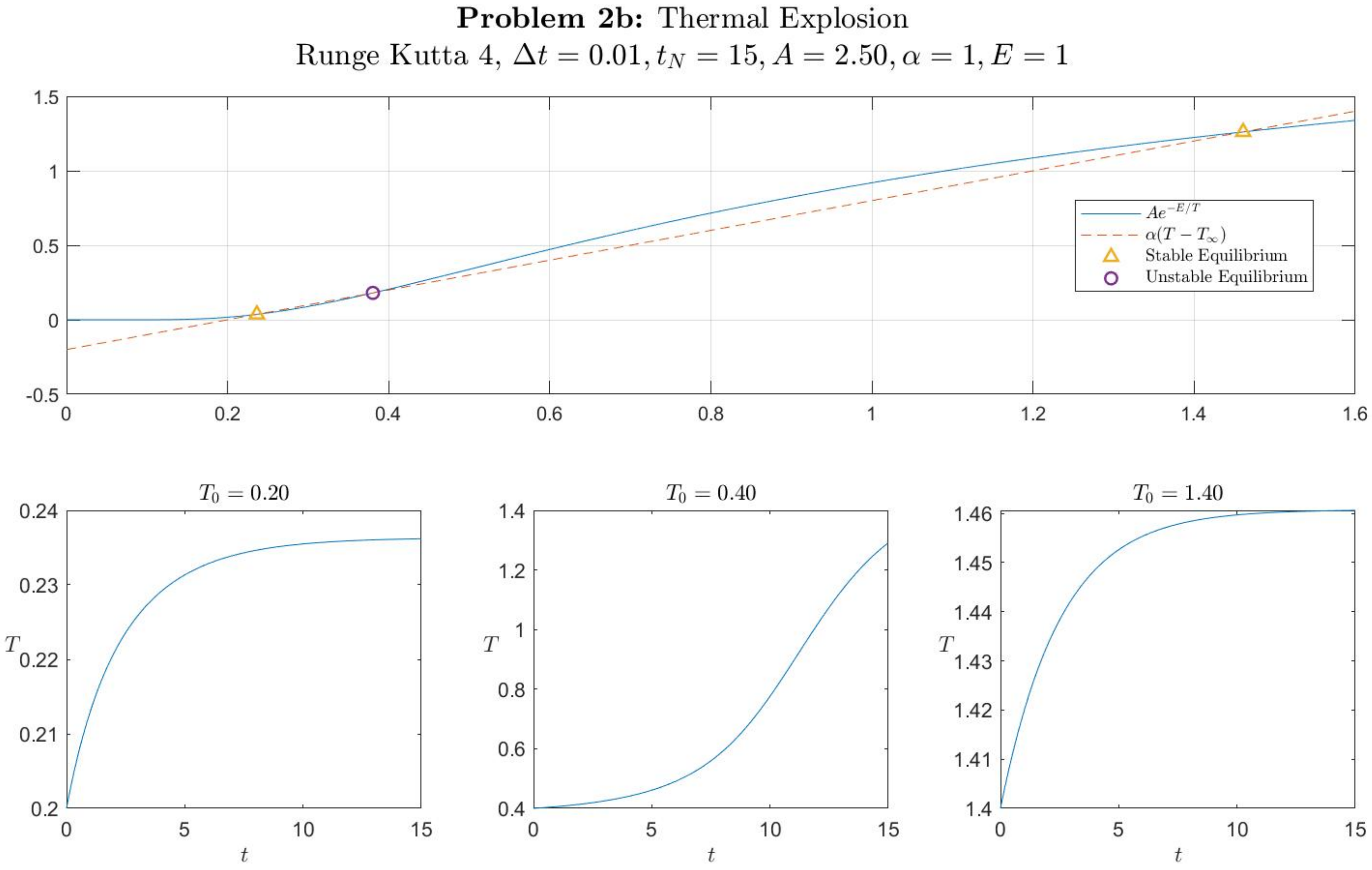
PROBLEM 1C: VAN DEL POL EQUATION



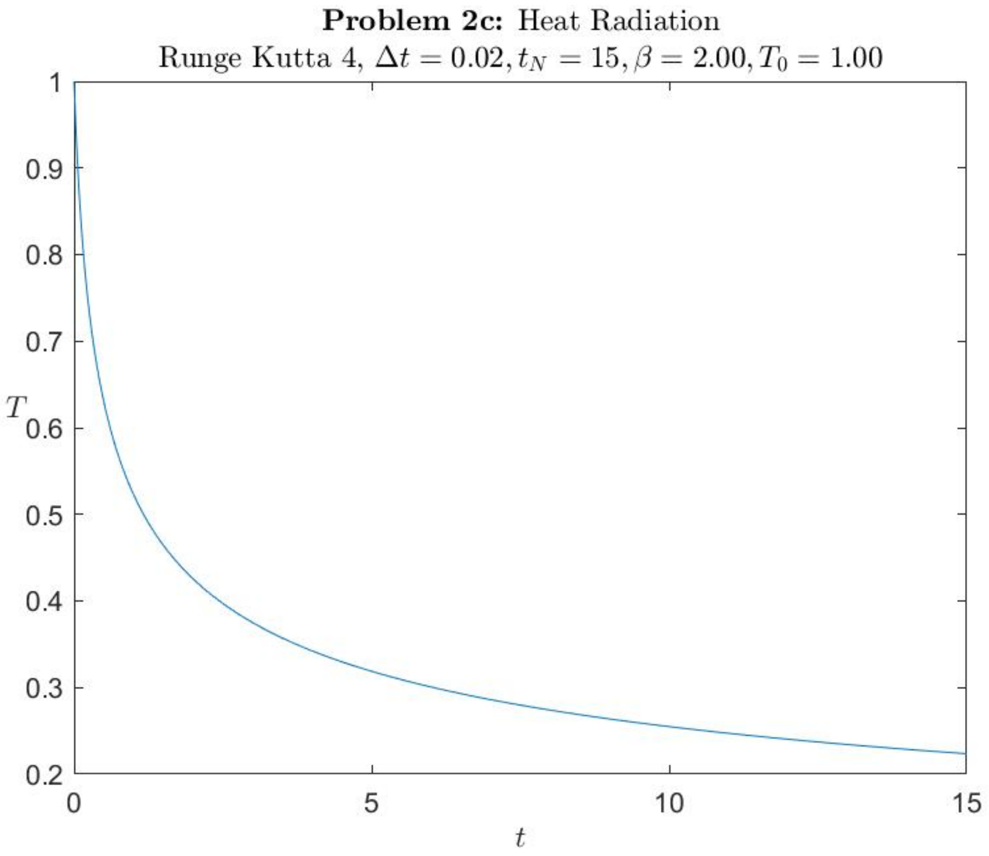
PROBLEM 2A: SOLID PROPELLENT



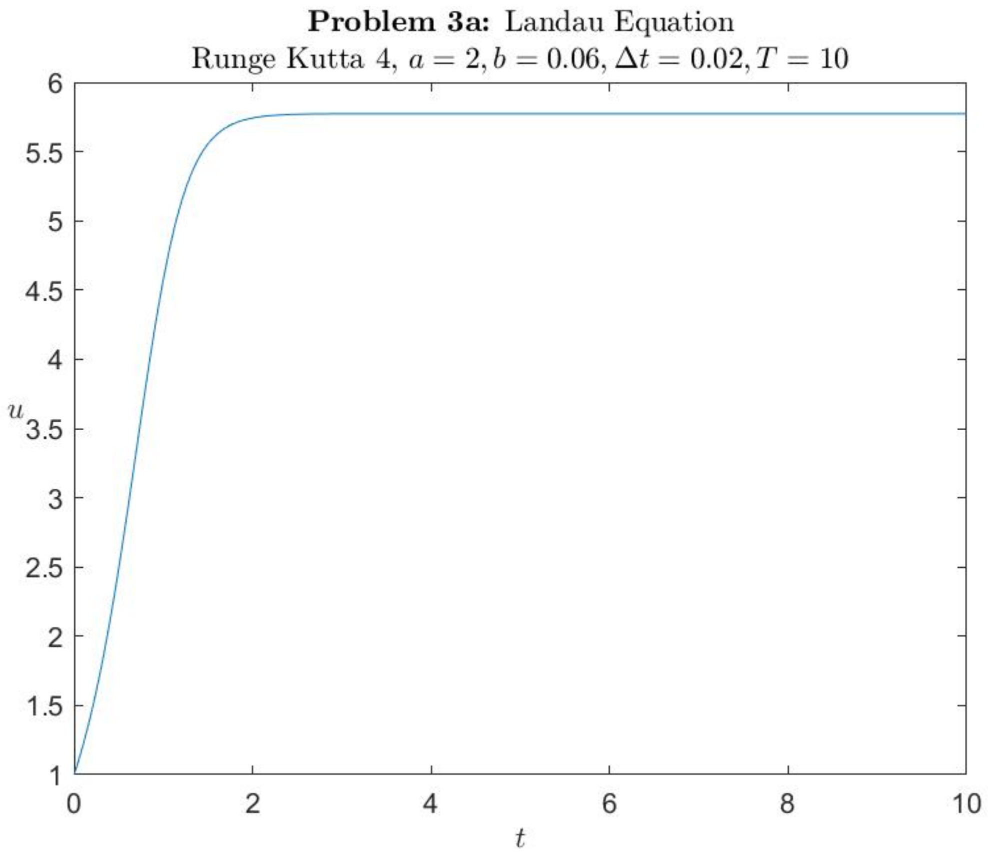
PROBLEM 2B: THERMAL EXPLOSION



PROBLEM 2C: HEAT RADIATION

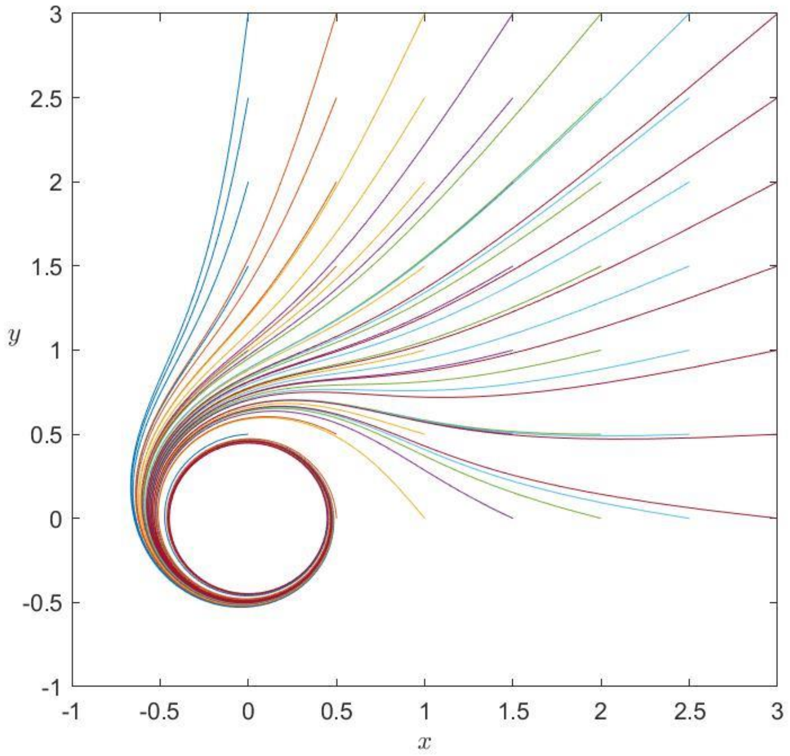


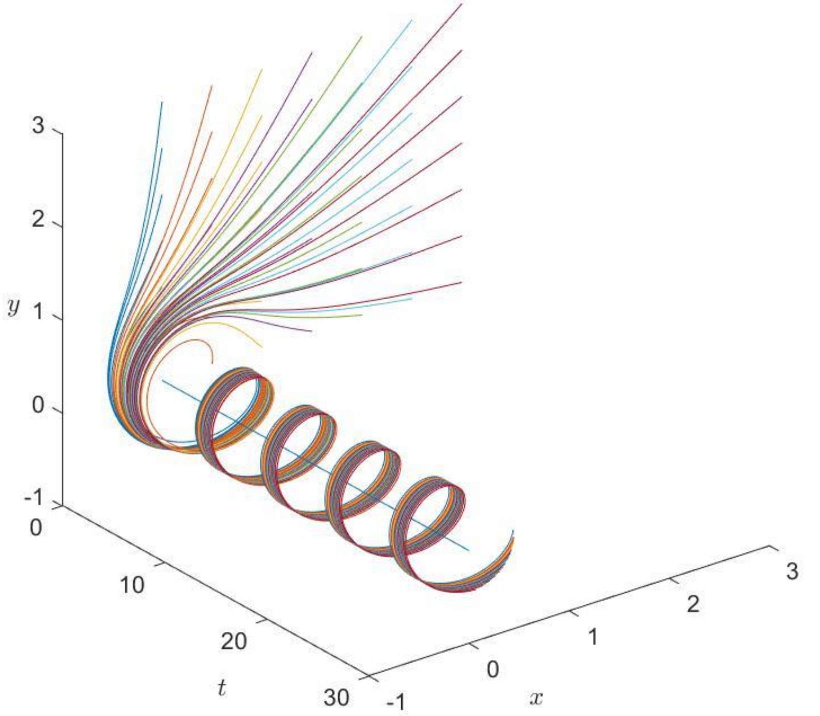
PROBLEM 3A: LANDAU EQUATION



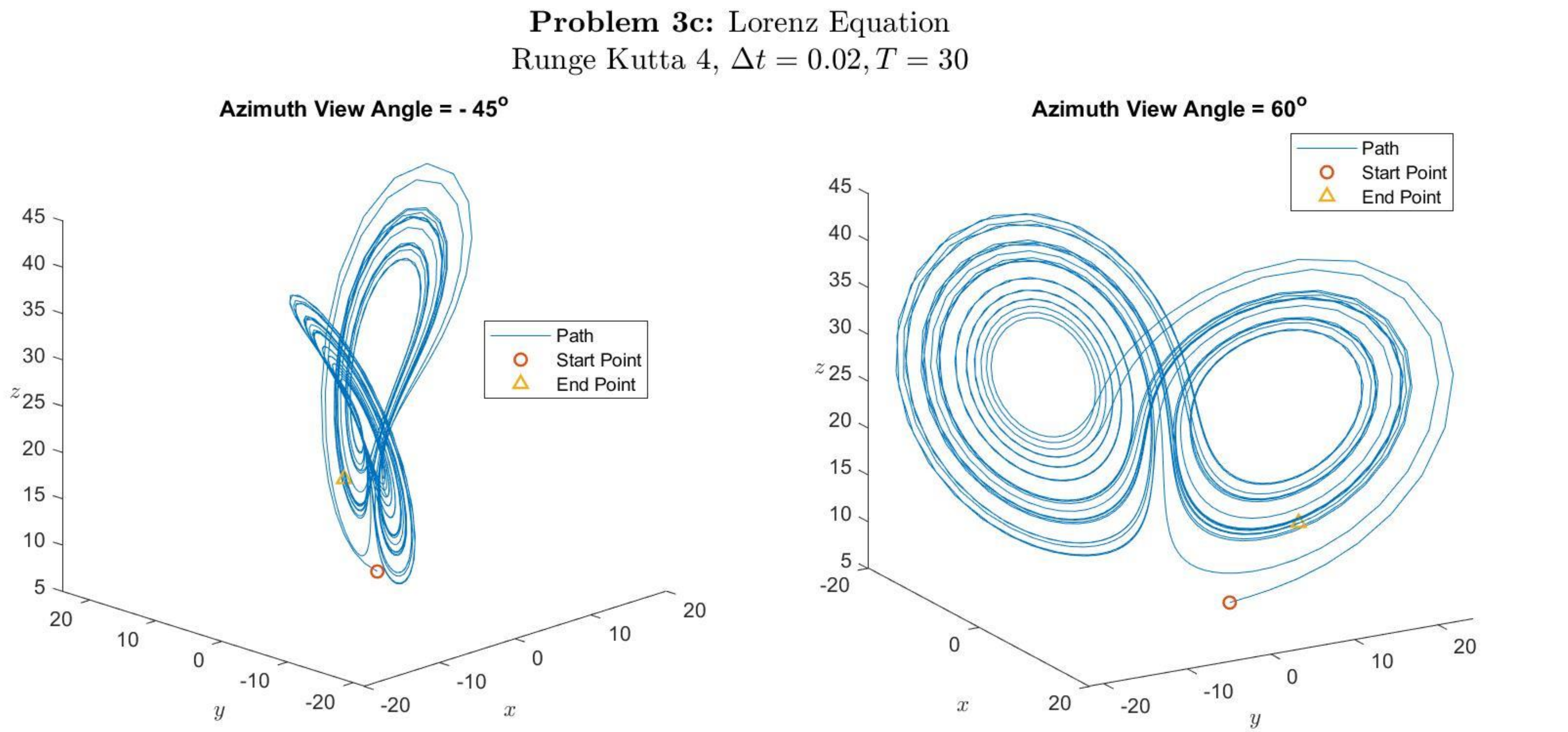
PROBLEM 3B: HOPF BIFURCATION

Problem 3b: Hopf Bifurcation
Runge Kutta 4, $a = 0.2, \Delta t = 0.02, T = 30$

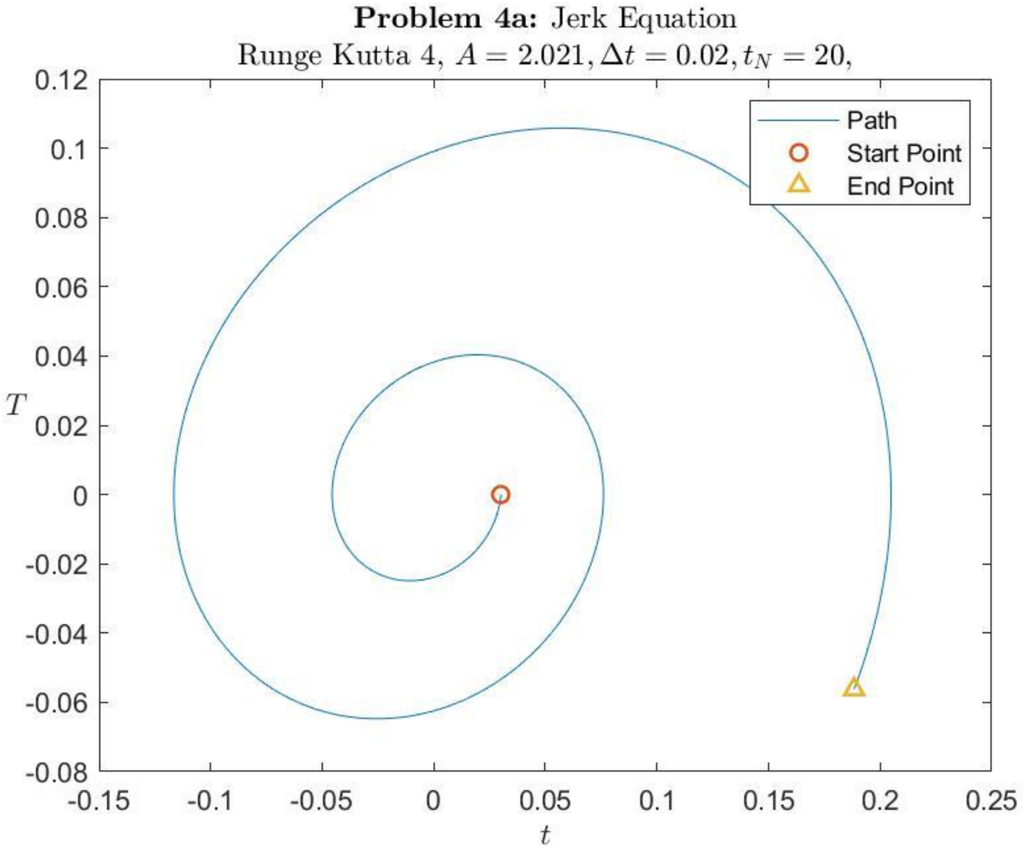




PROBLEM 3C: LORENZ EQUATION

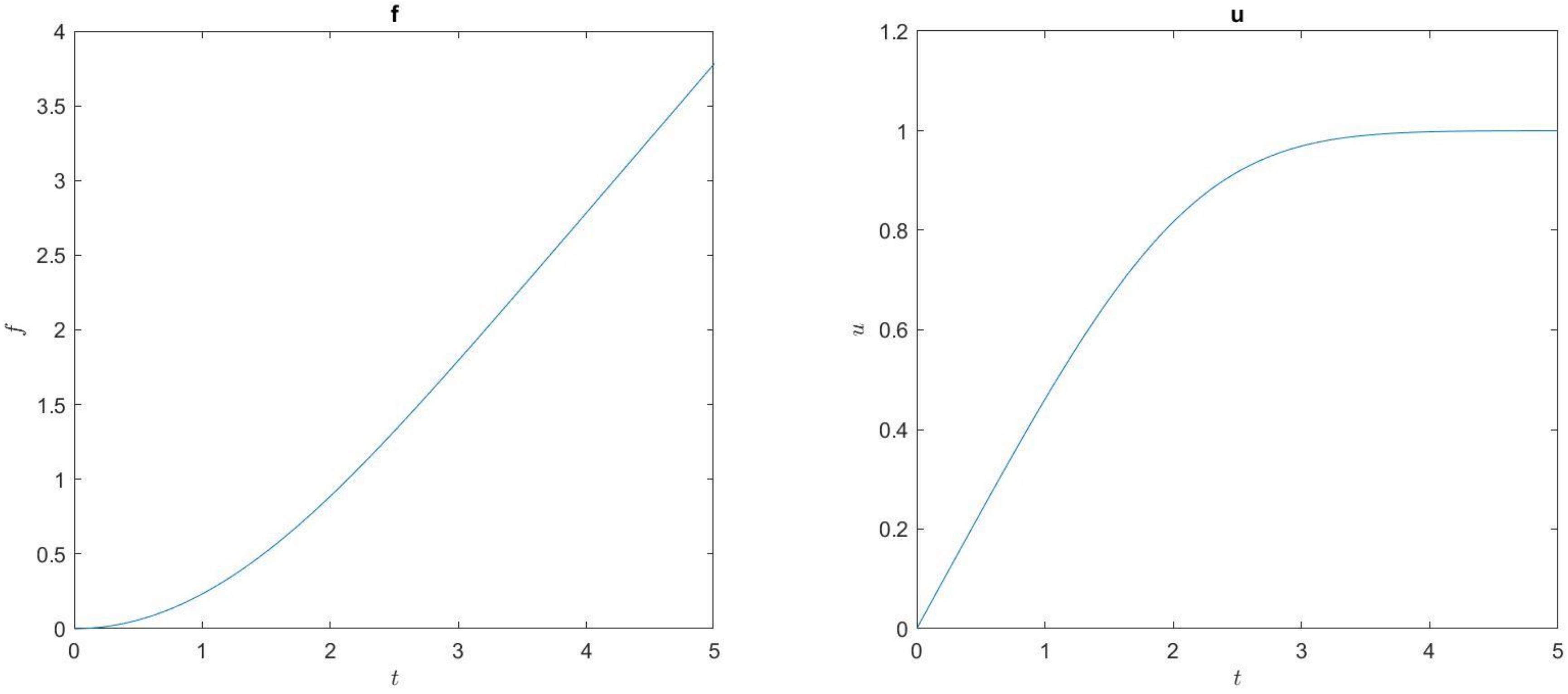


PROBLEM 4A: JERK EQUATION



PROBLEM 4B: BLASIUS EQUATION

Problem 4b: Blasius Equation
Runge Kutta 4, $\Delta y = 0.01, y^N = 5,$



PROBLEM 4C: SOLITON EQUATION