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QUASI-TRANSIENT VEHICLE DYNAMIC MODELS

A mathematical explanation of the Milliken-Moment Method (MMMs) and GGVs and a discussion of their use in vehicle design.

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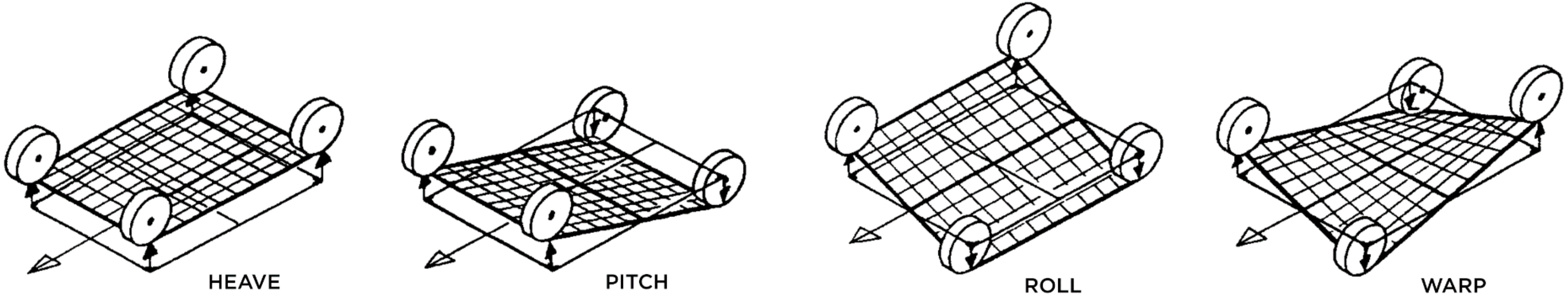
Outline

1. Preliminaries
    a) Dimensionality and Degrees of Freedom (DoFs)
    b) Linear and Nonlinear Systems
    c) Slices, Roots, Level Sets
    d) Forms of Dynamical Systems
2. Quasi-Transient Model Reduction
3. 10-DoF Chassis Dynamics
4. LASs, MMDs, GGVs
5. 14-DoF Chassis Dynamics
6. Designing with Quasi-Transient Models

Dimensionality and DoFs

- Let the ***dimension*** is the count of an object's ***degrees of freedoms (DoFs)***
  - A ***degree of freedom*** is an <u>independent</u> property that characterizes an object
  - There is often flexibility in choosing the DoFs, i.e., a "choice of basis"



- We'll be interested in the following "objects":
  - Parameters: $\boldsymbol{m} \in \mathcal{M}, \quad d_{\boldsymbol{m}} = \dim(\mathcal{M}) \sim 10 - 100$
  - States: $\boldsymbol{u} \in \mathcal{U}, \quad d_{\boldsymbol{u}} = \dim(\mathcal{U}) \sim 3 - 20$
  - Controls: $\boldsymbol{z} \in \mathcal{Z}, \quad d_{\boldsymbol{z}} = \dim(\mathcal{Z}) \sim 2 - 10$
  - Quantities of Interest (QoIs): $\boldsymbol{q} \in \mathcal{Q}, \quad d_{\boldsymbol{q}} = \dim(\mathcal{Q}) \sim 1 - 25$

Note: $\mathcal{M}, \mathcal{U}, \mathcal{Z}, \mathcal{Q} := \mathbb{R}^{d_{(\cdot)}}$

Linear and Nonlinear Systems

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b} \quad \mathbf{A} \in \mathbb{R}^{m \times n}, \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{b} \in \mathbb{R}^m$$

- Linear systems are ubiquitous in modeling and simulation
  - Linear maps are ***isomorphic*** to matrices*
- Linear systems are commonly considered in ***residual form***

$$\boldsymbol{r}(\boldsymbol{x}) := \boldsymbol{b} - \mathbf{A}\boldsymbol{x} = \mathbf{0}$$

- Nonlinear systems involve more general maps than matrices

$$\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0} \quad \boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^m, \boldsymbol{x} \in \mathbb{R}^n$$

  - Iterative nonlinear solvers often involve a sequence of linear systems
- The ***Jacobian*** takes the argument to the map's derivative at the argument:

$$\nabla \boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^{m \times n}$$

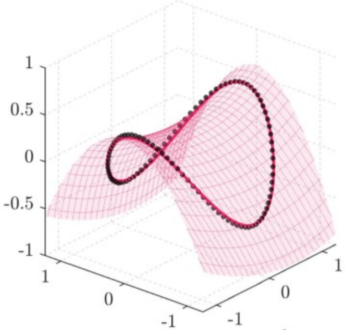
  - First order derivative (Jacobian) is a matrix, higher order derivatives are tensors

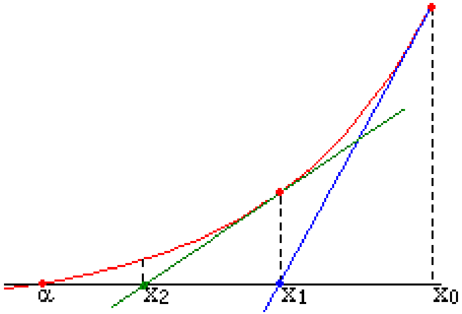
Slices, Roots, Level Sets

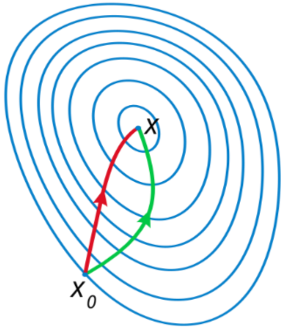
- A ***slice*** of a map, $\boldsymbol{f}$, restricts some components of the argument, $\boldsymbol{x}$
  - e.g. $g(a, b)$ may be sliced: $g(a, 0)$, $g(2.7, b)$, or $g(a, h(a))$
- The ***roots*** of a map, $\boldsymbol{f}$, are the arguments, $\boldsymbol{x}$, such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$
- ***Newton's method*** is a ***fixed-point iteration*** for finding nonlinear roots:

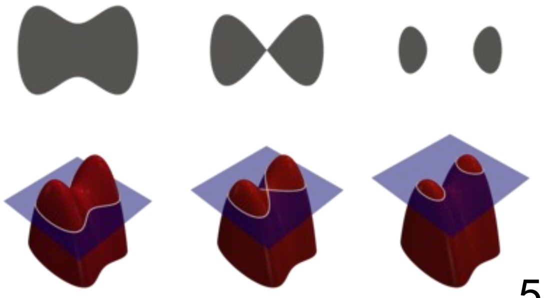
$$\underbrace{\nabla \boldsymbol{f}(\boldsymbol{x}_k)}_{\mathbb{R}^{m\times n}} \boldsymbol{\delta}_k = -\boldsymbol{f}(\boldsymbol{x}_k), \qquad \boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{\delta}_k$$

- ***Level sets*** are the arguments, $\boldsymbol{x}$, such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}$
  - Determining the level set of $\boldsymbol{y}$ is equivalent to the roots of $\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{y} = \boldsymbol{0}$









Forms of Dynamical Systems

Explicit Algebraic:	$\boldsymbol{q} = \boldsymbol{h}_{\boldsymbol{m}}(t, \boldsymbol{u}, \boldsymbol{z})$
Implicit Algebraic:	$\boldsymbol{g}_{\boldsymbol{m}}(t, \boldsymbol{u}, \boldsymbol{z} \mid \boldsymbol{q}) = \boldsymbol{0}$
Explicit ODEs:	$\dot{\boldsymbol{u}} = \boldsymbol{f}_{\boldsymbol{m}}(t, \boldsymbol{u}, \boldsymbol{z})$
Implicit ODEs:	$\boldsymbol{f}_{\boldsymbol{m}}(t, \boldsymbol{u}, \boldsymbol{z} \mid \dot{\boldsymbol{u}}) = \boldsymbol{0}$
Semi-Structured DAEs:	$\begin{cases} \dot{\boldsymbol{u}}_{\mathrm{d}} = \boldsymbol{f}_{\boldsymbol{m}}(t, \boldsymbol{u}_{\mathrm{d}}, \boldsymbol{u}_{\mathrm{a}}, \boldsymbol{z}) \\ \boldsymbol{0} = \boldsymbol{g}_{\boldsymbol{m}}(t, \boldsymbol{u}_{\mathrm{d}}, \boldsymbol{u}_{\mathrm{a}}, \boldsymbol{z}) \end{cases}$
Unstructured DAEs:	$\boldsymbol{F}_{\boldsymbol{m}}(t, \boldsymbol{u}_{\mathrm{d}}, \boldsymbol{z} \mid \boldsymbol{u}_{\mathrm{a}}, \dot{\boldsymbol{u}}_{\mathrm{d}}) = \boldsymbol{0}$

Note: Implicit ODEs and DAEs differ by the singularity of the state Jacobian

$$\det[\nabla_{\boldsymbol{u}} \boldsymbol{f}_{\boldsymbol{m}}(t, \boldsymbol{u}, \boldsymbol{z} \mid \dot{\boldsymbol{u}})] \neq 0 \quad \forall t, \boldsymbol{m}, \boldsymbol{u}, \boldsymbol{z}$$

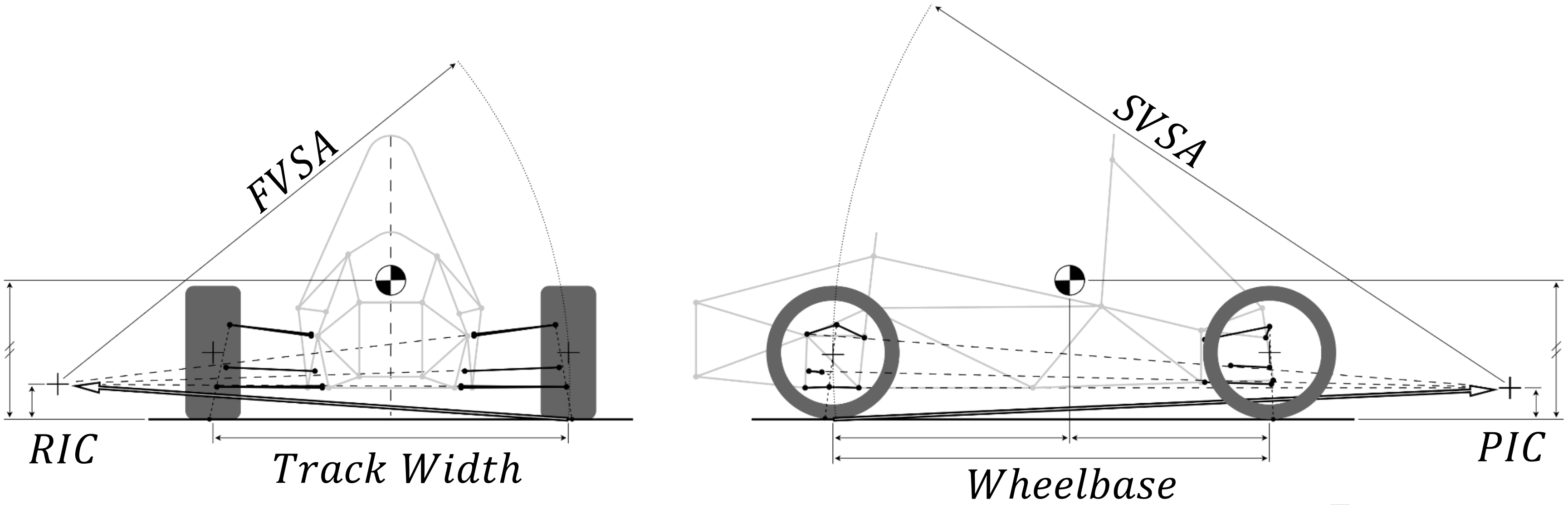
Quasi-transient Model Reduction

- Cases for model reduction:
  - Too many parameters, states, controls, QoIs to make sense of (or compute!)
  - Some quantities or modes are more impactful (relevant) than others
- ***Quasi-transient reduction*** demotes differential states to algebraic states
  - Dynamics of the reduced mode are taken to a ***steady (final time)*** point
  - Allows for the model to carry steady state information of reduced modes
    - Useful when the alternative is losing all reduced mode information!
  - Reduced model is typically small enough to precompute and remains interpretable
    - Precomputed reduced model can then be used as a ***response surface surrogate*** model

$$\dot{\boldsymbol{u}} = \boldsymbol{f}(\boldsymbol{u}), \qquad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_\mathrm{r} \\ \bar{\boldsymbol{u}} \end{bmatrix}, \qquad \dot{\bar{\boldsymbol{u}}} = \boldsymbol{0} \;\longrightarrow\; \begin{bmatrix} \dot{\boldsymbol{u}}_\mathrm{r} \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{f}\left(\begin{bmatrix} \boldsymbol{u}_\mathrm{r} \\ \bar{\boldsymbol{u}} \end{bmatrix}\right)$$

  - Consider the quasi-transient system as a map, $\boldsymbol{f}_\mathrm{r}\colon \mathcal{U}_\mathrm{r} \ni \boldsymbol{u}_\mathrm{r} \to (\bar{\boldsymbol{u}}, \dot{\boldsymbol{u}}_\mathrm{r}) \in \bar{\mathcal{U}} \times \mathcal{U}_\mathrm{r}$
    - This isn't a "bulletproof" paradigm

10-DoF Chassis Dynamics



$$\boldsymbol{p}_\mathrm{s} \coloneqq [x_\mathrm{s}, y_\mathrm{s}, z_\mathrm{s}, \phi, \theta, \psi]^T, \qquad \boldsymbol{p}_\mathrm{u} \coloneqq \left[z_{\mathrm{u},1}, z_{\mathrm{u},2}, z_{\mathrm{u},3}, z_{\mathrm{u},4}\right]^T,$$

$$\boldsymbol{p} \coloneqq \begin{bmatrix} \boldsymbol{p}_\mathrm{s} \\ \boldsymbol{p}_\mathrm{u} \end{bmatrix}, \qquad \boldsymbol{u} \coloneqq \begin{bmatrix} \boldsymbol{p} \\ \dot{\boldsymbol{p}} \end{bmatrix}, \qquad \boldsymbol{z} \coloneqq \begin{bmatrix} \delta_\mathrm{sw} \\ \eta_\mathrm{th} \end{bmatrix}$$

$$\dot{\boldsymbol{u}} = \boldsymbol{f}_{\boldsymbol{m}}(t, \boldsymbol{u}, \boldsymbol{z})$$

- Some mechanics required for $\boldsymbol{f}$: $\mathcal{M} \times \mathbb{R}^+ \times \mathbb{R}^{20} \times \mathbb{R}^2 \to \mathbb{R}^{20}$

10-DoF Quasi-transient Reduction

1) First, we reduce ***attitude*** modes (heave, pitch, roll) and ride dynamics

$$\boldsymbol{p}_r := [x_\mathrm{s}, y_\mathrm{s}, \psi]^T, \qquad \bar{\boldsymbol{p}} := \left[z_\mathrm{s}, \phi, \theta, z_{\mathrm{u},1}, z_{\mathrm{u},2}, z_{\mathrm{u},3}, z_{\mathrm{u},4}\right]^T$$

$$\boldsymbol{u}_r := \begin{bmatrix} \boldsymbol{p}_r \\ \dot{\boldsymbol{p}}_r \end{bmatrix}, \qquad \bar{\boldsymbol{u}} := \begin{bmatrix} \bar{\boldsymbol{p}} \\ \boldsymbol{0} \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} \dot{\boldsymbol{u}}_\mathrm{r} \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{f}_{\boldsymbol{m}}\left(t, \begin{bmatrix} \boldsymbol{u}_\mathrm{r} \\ \bar{\boldsymbol{u}} \end{bmatrix}, \boldsymbol{z}\right)$$

- Only retain longitudinal, lateral, and yaw dynamics
  - Reduces 10(20)-DoF model to 3(6)-DoF while retaining steady attitude mode behavior

2) “Ideal environment” assumptions eliminate dependence of $\boldsymbol{f}$ on $\boldsymbol{p}_r$

- Flat ground, constant friction surface, no wind…

$$\boldsymbol{f}_\mathrm{r}: \mathcal{M} \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^2 \ni (\boldsymbol{m}, t, \dot{\boldsymbol{p}}_r, \boldsymbol{z}) \to (\bar{\boldsymbol{p}}, \ddot{\boldsymbol{p}}_\mathrm{r}) \in \mathbb{R}^7 \times \mathbb{R}^3$$

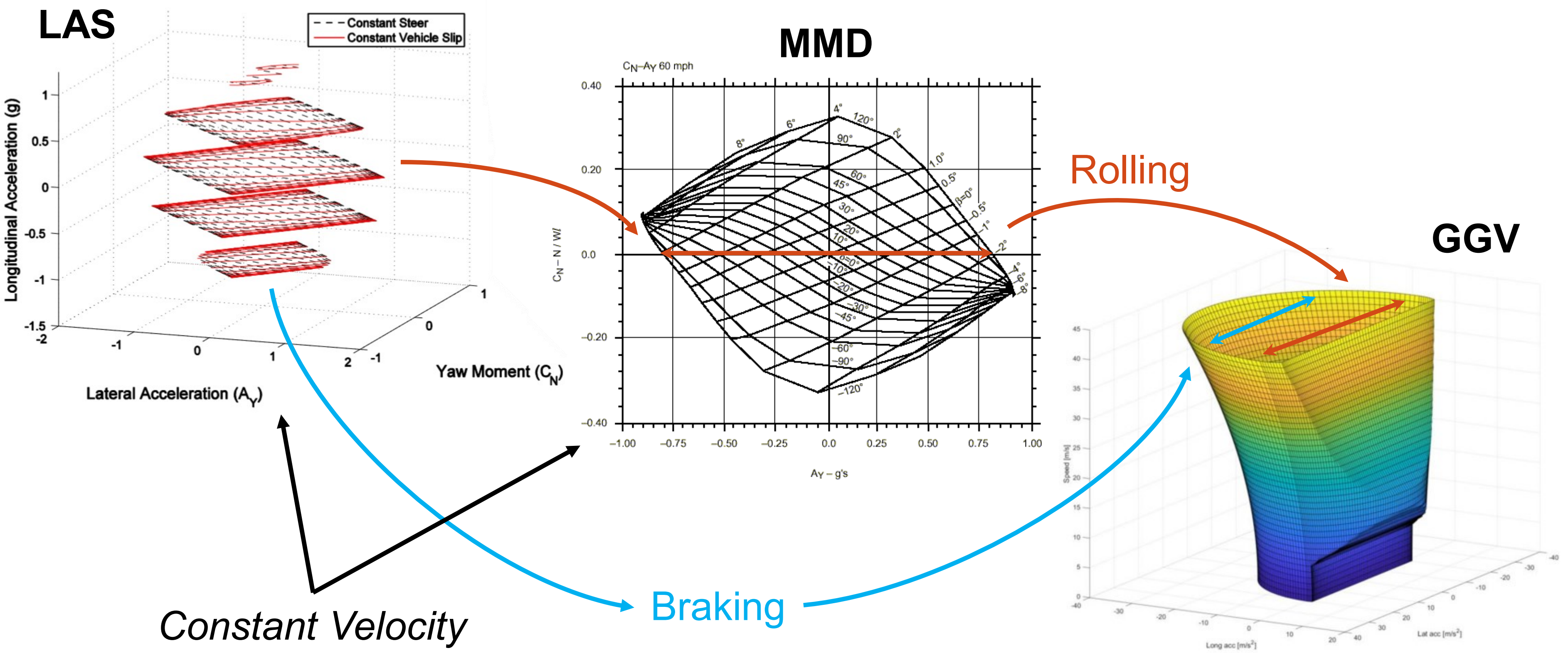
3) Planar velocities are re-expressed as tangential velocity and body slip

$$(\dot{x}_\mathrm{s}, \dot{y}_\mathrm{s}) \mapsto (v, \beta) := \left(\sqrt{\dot{x}_\mathrm{s}^2 + \dot{y}_\mathrm{s}^2}, \tan^{-1}\left(\frac{\dot{y}_\mathrm{s}}{\dot{x}_\mathrm{s}}\right)\right)$$

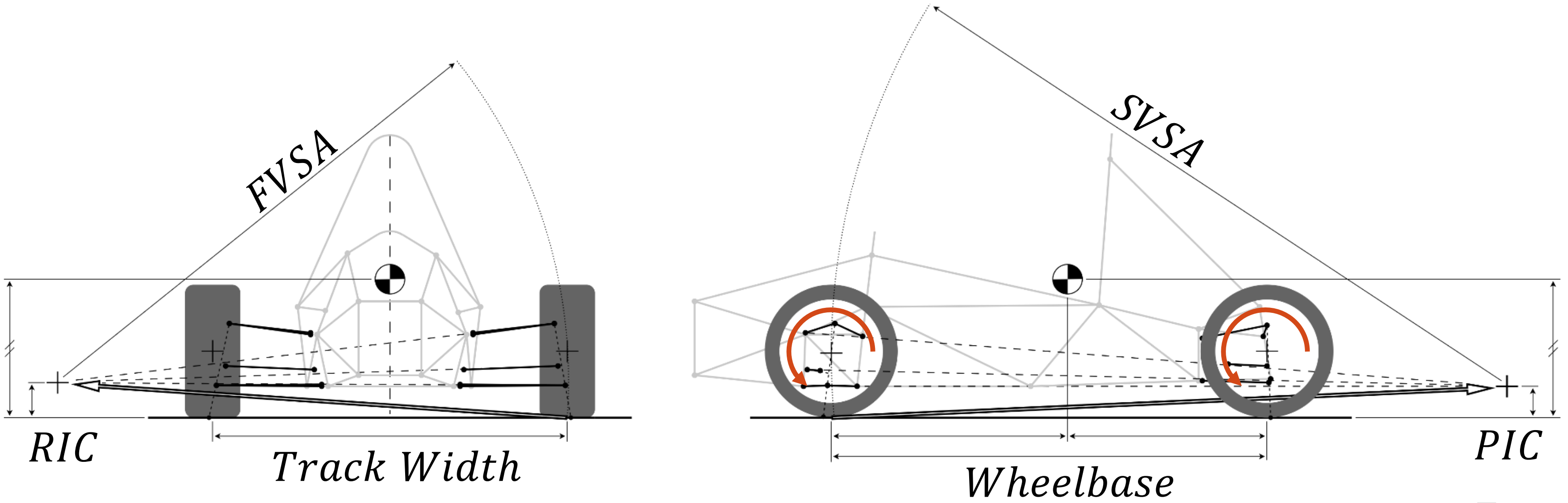
LASs, MMDs, and GGVs

- At this point, you have the ***limit acceleration surface (LAS)***
  - Refer to Patton's "*Development of Vehicle Dynamics Tools for Motorsports*"
  - Requires sweeping: $v, \beta, \dot{\psi}, \delta_{\mathrm{sw}}, \eta_{\mathrm{th}}, \ldots^{*}$
- ***Milliken Moment Diagrams (MMDs)*** present a 2D slice of the LAS
  - Fixes $v, \dot{\psi}, \eta_{\mathrm{th}}, \ldots^{*}$ and sweeps $\beta, \delta_{\mathrm{sw}}$
    - Typically, $\eta_{\mathrm{th}} = 0$ and $\dot{\psi} = 0$ for "free-rolling" "straight-line" MMDs
- ***GGVs*** further reduce by eliminating yaw dynamics
  - Unlike the attitude and ride modes, "steady yaw" loses meaning
  - Two options:
    - Disregard moment balance equations relating to yaw
    - Take slice corresponding to $\psi = 0 \quad (\Longrightarrow \dot{\psi}, \ddot{\psi} = 0)$

Diagram Relationships



14-DoF Chassis Dynamics



$$\boldsymbol{p}_\mathrm{s} \coloneqq [x_\mathrm{s}, y_\mathrm{s}, z_\mathrm{s}, \phi, \theta, \psi]^T, \qquad \boldsymbol{p}_\mathrm{u} \coloneqq [z_{\mathrm{u},1}, z_{\mathrm{u},2}, z_{\mathrm{u},3}, z_{\mathrm{u},4}, \omega_1, \omega_2, \omega_3, \omega_4]^T,$$

$$\boldsymbol{p} \coloneqq \begin{bmatrix} \boldsymbol{p}_\mathrm{s} \\ \boldsymbol{p}_\mathrm{u} \end{bmatrix}, \qquad \boldsymbol{u} \coloneqq \begin{bmatrix} \boldsymbol{p} \\ \dot{\boldsymbol{p}} \end{bmatrix}, \qquad \boldsymbol{z} \coloneqq \begin{bmatrix} \delta_\mathrm{sw} \\ \eta_\mathrm{th} \end{bmatrix}$$

$$\dot{\boldsymbol{u}} = \boldsymbol{f}_{\boldsymbol{m}}(t, \boldsymbol{u}, \boldsymbol{z})$$

- Some mechanics required for $\boldsymbol{f}$: $\mathcal{M} \times \mathbb{R}^+ \times \mathbb{R}^{28} \times \mathbb{R}^2 \to \mathbb{R}^{28}$

14-DoF Quasi-transient Reduction

$$\boldsymbol{p}_r \coloneqq [x_\mathrm{s}, y_\mathrm{s}, \psi, \omega_1, \omega_2, \omega_3, \omega_4]^T, \qquad \bar{\boldsymbol{p}} \coloneqq \left[z_\mathrm{s}, \phi, \theta, z_{\mathrm{u},1}, z_{\mathrm{u},2}, z_{\mathrm{u},3}, z_{\mathrm{u},4}\right]^T$$

$$\boldsymbol{u}_r \coloneqq \begin{bmatrix} \boldsymbol{p}_r \\ \dot{\boldsymbol{p}}_r \end{bmatrix}, \qquad \bar{\boldsymbol{u}} \coloneqq \begin{bmatrix} \bar{\boldsymbol{p}} \\ \boldsymbol{0} \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} \dot{\boldsymbol{u}}_\mathrm{r} \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{f}_{\boldsymbol{m}}\left(t, \begin{bmatrix} \boldsymbol{u}_\mathrm{r} \\ \bar{\boldsymbol{u}} \end{bmatrix}, \boldsymbol{z}\right)$$

- Retain wheel slip alongside longitudinal, lateral, and yaw dynamics
  - Reduces 14(28)-DoF model to 7(14)-DoF
- Wheel’s rotational symmetry provides similar effect as “ideal environment”

$$\boldsymbol{f}_\mathrm{r} \colon \mathcal{M} \times \mathbb{R}^+ \times \mathbb{R}^7 \times \mathbb{R}^2 \ni (\boldsymbol{m}, t, \dot{\boldsymbol{p}}_r, \boldsymbol{z}) \to (\bar{\boldsymbol{p}}, \ddot{\boldsymbol{p}}_\mathrm{r}) \in \mathbb{R}^7 \times \mathbb{R}^7$$

- Recommend re-expressing the wheel rates as slip ratio:

$$\dot{\boldsymbol{p}}_{\mathrm{u},j} \coloneqq \left[\dot{x}_\mathrm{s} - y_{\mathrm{u},j}\dot{\psi},\ \dot{y}_\mathrm{s} + x_{\mathrm{u},j}\dot{\psi},\ 0\right]^T, \qquad \dot{\omega}_j \mapsto \kappa_j \coloneqq \frac{\dot{\omega}_j r_\mathrm{eff}}{\left|\dot{\boldsymbol{p}}_{\mathrm{u},j}\right| \cos(\alpha_j)} - 1$$

- Alternatively, one may try to define the level sets for various accelerations

Quasi-Transient Model Design

- Reduced models enable more design space exploration
  – Recall, the parameter space in this setting is typically the largest
- Quasi-Transient reduction makes sense if there is a steady final time value
  ✓ Body attitude (heave, pitch, roll) modes, and unsprung ride modes, …
  ✗ Yaw, wheel rotation, …
- Quasi-Transient modes cannot inform decisions that concern transients
  – e.g. LAS (MMM, GGV) models cannot be used to design kinematic instant centers
    - Transient behavior of kinematic vs sprung weight transfer is important!
- ***Multifidelity*** model hierarchies assist in progressive ***model-based design***

Low Fidelity Models	⟷	High Level Design
Reduce ↑ ↓ *Progress*	*Inform*	↓ *Constrain*
High Fidelity Models	⟷	Low Level Design

THANK YOU FOR YOUR ATTENTION!

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LHR