SEVERAL COMPLEX VARIABLES

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REFERENCES: Dror Varolin, Riemann surfaces by way of complex analytic geometry. Graduate studies in Mathematics 125. ISBN 978-0-8218-5369-6

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On L^p estimates for dbar. Several Complex Variables and Complex Geometry, Part 3. 129-163. Proc. Symp. Pure Math. 52, part 3. 1991. ISBN 08218 14915/08218 14885 Check Berits office for preprints

Bo Berndtsson
L^2 methods for the dbar equation
Kass University Press-Johanneberg-Masthugget-Sisjon, 1995

Abstract:

A function $f : \mathbb{R}(x) \to \mathbb{R}(y)$ is real-analytic if it can be expanded in a power series, $y = f(x) = \sum_n a_n x^n$. A function $g : \mathbb{C}(z) \to \mathbb{C}(w)$ is complex-analytic if it can be expanded in a power-series $w = g(z) = \sum_n c_n z^n$. Complex-analytic (also called holomorphic) functions can be characterized as solutions to the homogeneous Cauchy-Riemann equation $\frac{\partial g}{\partial \overline{z}} = 0$. In complex analysis the inhomogeneous Cauchy-Riemann equation, $\frac{\partial g}{\partial \overline{z}} = u(z)$ is an extremely important tool. It's main use is to produce holomorphic functions with powerful properties. In this course we will explain the remarkable classical theory developed by Lars Hörmander to handle this equation. We will focus on complex dimension one. This will make the proofs very simple and understandable, but will show all the ideas needed in the general higher dimensional case. The basic text is the book, Several complex variables by Hörmander and notes from a course I gave in Beijing which added extra details to Hörmanders book (which is a little brief at times).

The course will have three parts. The first is a **BASIC** course about holomorphic funtions and subharmonic functions. The second part concerns **HILBERT SPACES** about Hilbert spaces of holomorphic functions and the third part is **HÖRMANDERS THEOREM**.

1. **BASICS 1/5-** HILBERT-$\partial/\partial\overline{z}$

Hörmander Chapter 1.1/1.2

1.1 Preliminaries:

Here holomorphic (=analytic) functions are introduced. These are complex valued $\mathcal{C}^1$ functions u on domains Ω in $\mathbb{C}$ which satisfy the Cauchy-Riemann equation

$$\frac{\partial u}{\partial \overline{z}} = 0$$

The set of all such functions is denoted by $A(\Omega)$.

Here

$$\begin{aligned} \frac{\partial u}{\partial z} &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) \\ \frac{\partial u}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) \end{aligned}$$

Generally,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial \overline{z}}d\overline{z}.$$

For analytic functions, $du = \frac{\partial u}{\partial z}dz$, i.e. du and dz are parallel. For analytic functions we write $\frac{\partial u}{\partial z} = u'$.

Examples of analytic functions are all polynomials $P(z) = \sum_i a_i z^i$ and the exponential function $e^z = e^x(\cos y + i \sin y)$. We have that products, compositions and inverses of analytic functions are analytic.

1.2 Cauchy's integral formula and its applications.

Let ω be a bounded open set in $\mathbb{C}$ with boundary consisting of finitely many $\mathcal{C}^1$ Jordan curves. For ease of reference, we list the results using the same numbering as in Hormander. The proofs can be read easily in Hormanders book. But we will discuss them in the problem sessions.

Cauchy integral formula for general functions:

Theorem 1.2.1 Let $u \in \mathcal{C}^1(\overline{\omega})$. Then for $\zeta \in \Omega$,

$$u(\zeta) = \frac{1}{2\pi i}\left(\oint_{\partial\omega} \frac{u(z)}{z-\zeta}dz + \int\int_{\omega} \frac{\frac{\partial u}{\partial \overline{z}}}{z-\zeta}dz \wedge d\overline{z}\right)$$

Corollary 1.2.3 Every $u \in A(\Omega)$ is in $\mathcal{C}^\infty$. Also $u' \in A(\Omega)$ if $u \in A(\Omega)$.

Theorem 1.2.4 For every compact set $K \subset \Omega$ and every open neighborhood $\omega \subset \Omega$ of K there are constants $C_j, j = 0, 1, \dots$ such that

$$(1.2.4) \sup_{z \in K} |u^{(j)}(z)| \leq C_j \|u\|_{L^1(\omega)}$$

for all $u \in A(\Omega)$, where $u^{(j)} = \frac{\partial^j u}{\partial z^j}$.

2. BASICS 2/5-HILBERT-$\partial/\partial\overline{z}$

Hörmander Chapter 1.2 continued. We will discuss proofs in the problem session.

Corollary 1.2.5 If $u_n \in A(\Omega)$ and $u_n \to u$ when $n \to \infty$, uniformly on compact subsets of Ω, it follows that $u \in A(\Omega)$.

Proof. Pick a point $\zeta \in \Omega$ and choose a disc $\overline{\Delta}(w,r) \subset \Omega$ with $\zeta \in \Delta(w,r)$. For each n, we have by theorem 1.2.1. that

$$u_n(\zeta) = \frac{1}{2\pi i}\int_{|z-w|=r} \frac{u_n(z)}{z-\zeta}dz.$$

Using the uniform convergence we then get u is continuous and

$$u(\zeta) = \frac{1}{2\pi i}\int_{|z-w|=r} \frac{u(z)}{z-\zeta}dz.$$

From this formula we see that u is $\mathcal{C}^1$ and analytic on $\Delta(w,r)$.

□

Corollary 1.2.6 If $u_n \in A(\Omega)$ and the sequence $|u_n|$ is uniformly bounded on every compact subset of Ω, there is a subsequence u_{n_j} converging uniformly on every compact subset of Ω to a limit $u \in A(\Omega)$.

Corollary 1.2.7 The sum of a power series $\sum_0^\infty a_n z^n$ is analytic in the interior of the circle of convergence.

Theorem 1.2.8 If u is analytic in $\Omega = \{z; |z| < r\}$, we have

$$u(z) = \sum_0^\infty u^{(n)}(0) z^n/n!$$

with uniform convergence on every compact subset of Ω.

Uniqueness of analytic continuation:

Corollary 1.2.9 If $u \in A(\Omega)$ and there is some point $z \in \Omega$ where $u^{(k)}(0) = 0$ for all $k \geq 0$, it follows that $u = 0$ in Ω if Ω is connected.

Corollary 1.2.10 If u is analytic in the disc $\Omega = \{z; |z| < r\}$ and if u is not identically 0, one can write u in one and only one way in the form

$$u(z) = z^n v(z)$$

where $n \geq 0$ is an integer and $v \in A(\Omega)$, $v(0) \neq 0$ (which means that $1/v$ is also analytic in a neighborhood of 0.

Theorem 1.2.11 If u is analytic in $\{z; |z - z_0| < r\} = \Omega$ and if $|u(z)| \leq |u(z_0)|$ when $z \in \Omega$, then u is constant in Ω.

Maximum Principle:

Corollary 1.2.12 Let Ω be bounded and let $u \in \mathcal{C}(\overline{\Omega})$ be analytic in Ω. Then the maximum of $|u|$ in $\overline{\Omega}$ is attained on the boundary.

Hörmander Chapter 1.6.

1.6 Subharmonic Functions.

Our main topic is holomorphic functions. But they are very rigid and hard to construct. The subharmonic functions are intimately connected to analytic functions. Moreover they are very flexible and easy to construct. The key point of the course is that Hormanders theorem provides a way to use subharmonic functions to construct holomorphic functions.

Definition: A $\mathcal{C}^2$ function is said to be harmonic if $\Delta h = 4\frac{\partial^2 h}{\partial z \partial \overline{z}} = 0$. This is equivalent to the equation $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$. If $h = \mathrm{Re}(f)$, f analytic, then h is harmonic:

$$
\begin{aligned}
\frac{\partial^2 h}{\partial z \partial \overline{z}} &= \frac{\partial^2 (\frac{f+\overline{f}}{2})}{\partial z \partial \overline{z}} \\
&= \frac{1}{2}\frac{\partial^2 f}{\partial z \partial \overline{z}} + \frac{1}{2}\frac{\partial^2 \overline{f}}{\partial z \partial \overline{z}} \\
&= \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \overline{z}}\right) + \frac{1}{2}\frac{\partial}{\partial \overline{z}}\left(\frac{\partial \overline{f}}{\partial z}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial \overline{z}}\overline{\left(\frac{\partial f}{\partial \overline{z}}\right)} \\
&= 0
\end{aligned}
$$

Conversely, suppose that u is $\mathcal{C}^2$ and harmonic on a disc D. Then for each Jordan curve $\gamma \in D$ bounding a domain U, we get by Stokes theorem that $\oint_\gamma (-u_y dx + u_x dy) = \int_U u_{xx} + u_{yy} = 0$. Hence the function $v(q) = \int_{z_0}^q -u_y dx + u_x dy$ defines a function v on D. This function satisfies $v_x = -u_y$ and $v_y = u_x$ so $u + iv$ is analytic. So on a disc, we see that h is harmonic if and only if h is the real part of an analytic function.

Definition 1.6.1 A function u defined in an open set $\Omega \subset \mathbb{C}$ with values in $[-\infty, \infty >$ is called subharmonic if
a) u is upper semicontinuous, that is $\{z \in \Omega; u(z) < c\}$ is open for every real number c.
b) For every compact $K \subset \Omega$ and every continuous function h on K which is harmonic in the interior of K and such that $u \leq h$ on ∂K, we have $u \leq h$ on K.

Note: The function $u \equiv -\infty$ is called subharmonic in this text.
Note: A function is u upper semicontinuous if and only if there exists a

sequence of continuous functions u_j such that $u_j \searrow u$.

Note: Harmonic functions are subharmonic: Let u be harmonic on Ω and let K be a compact subset of Ω. Let h be a continuous function on K which is harmonic on the interior of K. Also suppose that $u \leq h$ on ∂K. Suppose there exists a point $p \in$ the interior of K such that $u(p) > h(p)$. Then if $\epsilon > 0$ is small enough we have that the function $v = u - h + \epsilon |z|^2$ satisfies $v(p) > \sup_{\partial K} v$. We can choose p to be a point where v takes on a maximum value. Note however that $v_{xx} + v_{yy} = 4\epsilon > 0$. This contradicts that p is a max point.

Theorem 1.6.2 If u is subharmonic and $0 < c \in \mathbb{R}$, it follows that cu is subharmonic. If $u_\alpha, \alpha \in A$, is a family of subharmonic functions, then $u = \sup_\alpha u_\alpha$ is subharmonic if $u < \infty$ and u is upper semicontinuous, which is always the case if A is finite. If $u_1, u_2, \dots$ is a decreasing sequence of subharmonic functions, then $u = \lim_{j \to \infty} u_j$ is subharmonic.

The proof will be discussed in the exercise sessions. It can also be read in Hörmanders book.

3. BASICS 3/5-HILBERT-$\partial/\partial\overline{z}$

Hörmander Chapter 1.6

Theorem 1.6.3 Let u be defined on an open set $\Omega \subset \mathbb{C}$, with values in $[-\infty, \infty >$ and assume that u is upper semicontinuous. Then each of the following conditions are necessary and sufficient for u to be subharmonic:
(i) If D is a closed disc in Ω and f is an analytic polynomial such that $u \leq \mathrm{Re}(f)$ on ∂D, then it follows that $u \leq \mathrm{Re}(f)$ on D.
(ii) If $\Omega_\delta = \{z \in \Omega; d(z, \Omega^c) > \delta\}$, we have

$$(1.6.1)\quad u(z)2\pi \int d\mu(r) \leq \int_0^{2\pi} \int u(z + re^{i\theta}) d\theta d\mu(r), z \in \Omega_\delta$$

for every positive finite measure $d\mu$ on the interval $[0, \delta]$.
(iii) For every $\delta > 0$ and every $z \in \Omega_\delta$ there exists some positive finite measure $d\mu$ with support in $[0, \delta]$ such that $d\mu$ has some mass outside the origin and (1.6.1) is valid.

The proof can be found in Hörmanders book. We will also discuss the proof in the exercise sessions.

Corollary 1.6.4 If u_1, u_2 are subharmonic, then $u_1 + u_2$ is subharmonic.

Proof. We use (1.6.1). □

Corollary 1.6.5 A function u defined in an open set $\Omega \subset \mathbb{C}$ is subharmonic if every point in Ω has a neighborhood on which u is subharmonic.

We use property (iii) of Theorem 1.6.3.

Corollary 1.6.6 If $f \in A(\Omega)$, then $\log|f|$ is subharmonic.

Proof. Use property (i) in Theorem 1.6.3. So suppose that $\log|f| \leq \mathrm{Re}(g)$ on the boundary of a disc, where g is a holomorphic polynomial. Then $|f| \leq e^{\mathrm{Re}(g)} = |e^g|$ on the boundary of the disc. Hence $|fe^{-g}| \leq 1$ on the boundary of the disc. Hence also on the inside. Therefore $\log|f| \leq \mathrm{Re}(g)$ on the whole disc. □

Definition A function $\phi(x) : \mathbb{R}(x) \to \mathbb{R}$ is convex if for every $a < b$, and every $t \in < 0, 1 >$, $\phi(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$, i.e. the graph lies under the chord.

Observation: An immediate consequence is that outside (a, b) the graph lies above the straight line continuing the chord.

Theorem 1.6.7 Let ϕ be a convex increasing function on $\mathbb{R}$ and set $\phi(-\infty) = \lim_{x \to -\infty} \phi(x)$. Then $\phi(u)$ is subharmonic if u is subharmonic.

Lemma 1.6.7a Suppose that $\phi(x)$ is convex and suppose that $x_0 \in \mathbb{R}$. Then there exists a constant $k \in \mathbb{R}$ so that $\phi(x) \geq \phi(x_0) + k(x - x_0)$ for all x. Also ϕ is continuous.

Proof of the Lemma. Take any sequence $a_n < x_0 < b_n$ where both converge to x_0. Let k be any limit for the slopes of the chords from a_n to b_n.

To prove continuity, suppose that $x_n \searrow x_0$. Fix $a < x_0 < b$. Considering the chord from a to x_n shows that $\liminf \phi(x_n) \geq \phi(x_0)$. Considering the chord from x_0 to b shows that the $\limsup \phi(x_n) \leq \phi(x_0)$. A similar argument applies for $x_n \nearrow x_0$.

Proof of theorem 1.6.7:
Let $x_0 \in \mathbb{R}$ and let k be as in Lemma 1.6.7a. Let $x = u(z + re^{i\theta})$. Then

$$\phi(u(z + re^{i\theta})) \geq \phi(x_0) + k(u(z + re^{i\theta}) - x_0).$$

Hence

$$(*)\ \frac{1}{2\pi}\int_0^{2\pi} \phi(u(z + re^{i\theta}))d\theta \geq \phi(x_0) + k(\frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta - x_0).$$

We want to show that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi(u(z + re^{i\theta}))d\theta \geq \phi(u(z)).$$

If $\frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta = -\infty$, this is clear. So assume the integral is finite.

Let $x_0 = \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta$. Then the right side of (*) reduces to $\phi(x_0) = \phi(\frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta)$. Hence we get from (*) that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi(u(z + re^{i\theta}))d\theta \geq \phi(\frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta).$$

Since $\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta \geq u(z)$ and since ϕ is increasing, $\phi(\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta) \geq \phi(u(z))$.

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} \phi(u(z + re^{i\theta}))d\theta \geq \phi(u(z)).$$

It follows from Theorem 1.6.3 (iii) that $\phi(u)$ is subharmonic.

Corollary 1.6.7b If u is subharmonic, then e^u is subharmonic. If f is analytic, $|f|$ is subharmonic.

The first part follows from Theorem 1.6.7 since e^x is a convex increasing function. Since $\log|f|$ is subharmonic by Corollary 1.6.6, it follows that $|f| = e^{\log|f|}$ also is subharmonic.

Corollary 1.6.8 Let u_1, u_2 be nonnegative and assume that $\log u_j$ is subharmonic in Ω. Then $\log(u_1 + u_2)$ is subharmonic.

The proof is in Hörmanders book. We will discuss this in the exercise session.

Theorem 1.6.9 Let u be subharmonic in the open set Ω and not identically $-\infty$ in any connected component of Ω. Then u is integrable on all compact subsets of Ω (we write $u \in L^1_{loc}(\Omega)$), which implies that $u > -\infty$ almost everywhere.

Proof. Suppose that $u(z) > -\infty$. Pick a closed disc D centered at z contained in Ω. If we let $\mu = rdr$ we get from (1.6.1) that $u(z)A \leq \int_D udA$. Note that u is bounded above on D. Hence it follows that u is in L^1 on D. It follows that $u > -\infty$ a.e. on D. Hence we can repeat the argument for points z near the boundary of D. It follows that the set U of points $z \in \Omega$ where u is integrable in some neighborhood, is open and closed. By hypotheses U is nonempty. Hence $U = \Omega$. $\square$

4. BASICS 4/5-HILBERT-$\partial/\partial\overline{z}$

Hörmander Chapter 1.6.

Theorem 1.6.10 If u is subharmonic in Ω and not $-\infty$ identically in any component of Ω, then we have that

$$(1.6.3) \quad \int u \Delta v d\lambda \geq 0$$

for all $v \in \mathcal{C}_0^2(\Omega)$ with $v \geq 0$. Here λ denotes Lebesgue measure.

Proof. Let $0 < r < d(\text{supp}(v), \Omega^c)$. Then by 1.6.1 we have for every $z \in \text{supp}(v)$ that

$$2\pi u(z) \leq \int_0^{2\pi} u(z + re^{i\theta}) d\theta.$$

Since $v \geq 0$, we get for every $z \in \text{supp}(v)$ that

$$2\pi u(z)v(z) \leq \int_0^{2\pi} u(z + re^{i\theta})v(z) d\theta.$$

We integrate with respect to λ.

$$\begin{aligned}
\int\int (2\pi u(z)v(z))d\lambda &\leq \int\int (\int_0^{2\pi} u(z+re^{i\theta})v(z)d\theta)d\lambda \\
&= \int_0^{2\pi} (\int\int u(z+re^{i\theta})v(z)d\lambda)d\theta \\
&= \int_0^{2\pi} (\int\int u(z)v(z-re^{i\theta})d\lambda)d\theta \\
&= \int\int u(z)(\int_0^{2\pi} v(z-re^{i\theta})d\theta)d\lambda
\end{aligned}$$

We can also rewrite the left side:

$$\int\int (2\pi u(z)v(z))d\lambda = \int\int u(z)(\int_0^{2\pi} v(z)d\theta)d\lambda$$

Hence we see that

$$\int\int u(z)(\int_0^{2\pi} (v(z-re^{i\theta}) - v(z))d\theta)d\lambda \geq 0.$$

We Taylor expand the intergrand $v(z - re^{i\theta}) - v(z)$.

$$\begin{aligned} v(z-re^{i\theta})-v(z) &= -v_x(z)r\cos\theta - v_y(z)r\sin\theta + \frac{1}{2}v_{xx}r^2\cos^2\theta \\ &+ \frac{1}{2}v_{yy}r^2\sin^2\theta + v_{xy}r^2\cos\theta\sin\theta + o(r^2) \end{aligned}$$

We hence get an expression

$$\begin{aligned} -\int\int u(z)(\int_0^{2\pi} v_x(z)r\cos\theta)d\theta)d\lambda &- \int\int u(z)(\int_0^{2\pi} v_y(z)r\sin\theta)d\theta)d\lambda \\ &+ \int\int u(z)(\int_0^{2\pi} \frac{1}{2}v_{xx}r^2\cos^2\theta)d\theta)d\lambda \\ &+ \int\int u(z)(\int_0^{2\pi} \frac{1}{2}v_{yy}r^2\sin^2\theta)d\theta)d\lambda \\ &+ \int\int u(z)(\int_0^{2\pi} v_{xy}r^2\cos\theta\sin\theta)d\theta)d\lambda \\ &+ \int\int u(z)(\int_0^{2\pi} o(r^2))d\theta)d\lambda \\ &\geq 0. \end{aligned}$$

Hence after carrying out the inner integrals we see that

$$\int\int u(z)(\frac{1}{2}v_{xx}\pi r^2 + \frac{1}{2}v_{yy}\pi r^2 + o(r^2))d\lambda \geq 0.$$

If we divide by $\frac{\pi r^2}{2}$ and let $r \to 0$, we see that

$$\int\int u(z)\Delta v(z) \geq 0.$$

$\square$

Corollary 1.6.10a If u is a $\mathcal{C}^2$ subharmonic function, then $\Delta u \geq 0$.

Proof. Let $v \geq 0$ be a compactly supported $\mathcal{C}^2$ function in the domain of u. By (1.6.3), we have that $\int u\Delta v \geq 0$. Integrating by parts twice, we see that $\int v\Delta u \geq 0$. Since this is valid for all compactly supported nonnegative $\mathcal{C}^2$ functions v, it follows that $\Delta u \geq 0$.

5. BASICS 5/5-HILBERT-$\partial/\partial\overline{z}$

Hörmander Chapter 1.6.

Theorem 1.6.11 Let $u \in L^1_{loc}(\Omega)$ and assume that (1.6.3) holds. Then there is one and only one subharmonic function U in Ω which is equal to u almost everywhere. If ϕ is an integrable non-negative function of $|z|$ with compact support and $\int \phi = 1$, we have for every $z \in \Omega$

$$(1.6.4) \quad U(z) = \lim_{\delta \to 0} \int u(z - \delta z')\phi(z')d\lambda(z').$$

We will divide the proof into lemmas.

Lemma 1.6.11a Assume $u \in L^1_{loc}(\Omega)$. Let $\psi = \psi(|z|)$ be a nonnegative $\mathcal{C}^\infty$ function with compact support in the unit disc and $\int \psi = 1$. Then the function

$$u_\delta(z) := \int u(z - \delta z')\psi(z')d\lambda(z') = \frac{1}{\delta^2}\int u(w)\psi(\frac{z-w}{\delta})d\lambda(w)$$

is $\mathcal{C}^\infty$ in Ω_δ. If $V \subset\subset \Omega$, we have that $\|u_\delta\|_{L^1(V_\delta)} \leq \|u\|_{L^1(V)}$. Moreover $u_\delta \to u \in L^1$ on compact subsets.

Proof. The last equality is obtained by the change of variable, $w = z - \delta z'$. The fact that $u \in \mathcal{C}^\infty$ follows from differentiation under the integral sign in the last integral.

The inequality in the L^1 norm follows by integration with respect to z first.
To show the last statement, write $u = u_1 + u_2$ where u_1 is continuous and u_2 has small L^1 norm. The convergence for $(u_1)_\delta$ is obvious. And the L^1 norm of $(u_2)_\delta$ is as small as we wish. □

Lemma 1.6.11b Suppose U is subharmonic. Then (1.6.4) holds when we set $u = U$ in the integral on the right side. In particular it follows that if two subharmonic functions are equal almost everywhere, they are identical.

Proof. It follows by (1.6.1) that for small δ,

$$U(z) \leq \int U(z - \delta z')\phi(z')d\lambda(z').$$

By upper semicontinuity of U it follows that the upper limit of the right side when $\delta \to 0$ is at $\leq U(z)$. Hence (1.6.4) holds with $u = U$. □

Lemma 1.6.11c Assume that $u \in \mathcal{C}^2(\Omega)$ and that $\Delta u \geq 0$. Then u is subharmonic. Moreover $u_\delta \searrow u$. (Here u_δ is as in the right side of 1.6.4.)

Proof. Fix $z_0 \in \Omega_\delta$. Let $u_{z_0}(w) = \int_0^{2\pi} u(z_0 + e^{i\theta}w)d\theta$ for $|w| < \delta$. Then u_{z_0} is $\mathcal{C}^2$ and $\Delta u_{z_0} \geq 0$. Moreover u_{z_0} only depends on $|w|$. We calculate the Laplacian of u_{z_0} at points $w = x + iy, x \geq 0, y = 0$. We can write $u_{z_0}(x, y) =$

$u_{z_0}(\sqrt{x^2+y^2},0) = g(\sqrt{x^2+y^2})$. We get that $g''(x)+g'(x)/x \geq 0$ for $x > 0$, so $xg''(x)+g'(x) \geq 0, x \geq 0$. It follows that $xg'(x)$ increases. The value at $x = 0$ is 0, so $g'(x) \geq 0$. So $g(x)$ is increasing. By Theorem 1.6.3, it follows that u is subharmonic, and we also get that u_δ decreases to u when $\delta \to 0$. □

Lemma 1.6.11d Assume $u \in L^1_{loc}$ satisfies (1.6.3). Let $\psi = \psi(|z|)$ be a nonnegative $\mathcal{C}^\infty$ function with compact support in the unit disc and $\int \psi = 1$. Then the function

$$u_\delta(z) := \int u(z-\delta z')\psi(z')d\lambda(z')$$

is $\mathcal{C}^\infty$ and subharmonic in Ω_δ. Moreover $u_\delta \to u \in L^1$ on compact subsets.

Proof. Suppose that $u \in L^1_{loc}$ and that $\int u(z)\Delta v(z) \geq 0$ for all functions that are $\mathcal{C}^2$ with compact support and with $v \geq 0$. Then it follows that also u_δ has this property. Then, by Lemma 1.6.11c it follows that u_δ is subharmonic. Also by Lemma 1.6.11a we have convergence in L^1 norm to u. □

Lemma 1.6.11e Let u and ψ be as in Lemma 1.6.11d. Then if $\delta, \epsilon > 0$,

$$\begin{array}{ccc} & \text{Let } \epsilon \searrow 0: & \\ (u_\delta)_\epsilon & \searrow & u_\delta \\ & \text{Let } \delta \searrow 0: & \\ (u_\epsilon)_\delta & \searrow & u_\epsilon \\ (u_\delta)_\epsilon & = & (u_\epsilon)_\delta \\ & \text{Let } \epsilon \searrow 0: & \\ u_\epsilon & \searrow & V \end{array}$$

for some subharmonic function V.

Proof. We first show that $(u_\delta)_\epsilon \searrow u_\delta$. By Lemma 1.6.11d we have that u_δ is $\mathcal{C}^\infty$ and subharmonic. Hence by Corollary 1.6.10.a it follows that $\Delta u_\delta \geq 0$. Hence it follows by Lemma 1.6.11c that $(u_\delta)_\epsilon \searrow u_\delta$.
The second limit holds for the same reason.

To prove the following equality, we see that

$$\begin{aligned} (u_\delta)_\epsilon(z) &= \int u_\delta(z-\epsilon z')\psi(z')d\lambda(z') \\ &= \int (\int u(z-\epsilon z'-\delta z'')\psi(z'')d\lambda(z''))\psi(z')d\lambda(z') \\ &= (u_\epsilon)_\delta(z) \end{aligned}$$

We show that $u_{\epsilon_1}(z) \geq u_{\epsilon_2}(z)$ if $\epsilon_1 > \epsilon_2$. We have shown that for each $\delta > 0$, $(u_\delta)_{\epsilon_1}(z) \geq (u_\delta)_{\epsilon_2}(z)$, hence $(u_{\epsilon_1})_\delta(z) \geq (u_{\epsilon_2})_\delta(z)$. Hence by Lemma 1.6.11c, $u_{\epsilon_1}(z) \geq u_{\epsilon_2}(z)$.

Finally, by Theorem 1.6.2, the limit of u_ϵ is subharmonic.

□

We now can prove Theorem 1.6.2. So suppose that $u \in L^1_{loc}$ and that 1.6.3 is satisfied. Then by Lemma 1.6.11d, there exist subharmonic functions u_δ which converge in L^1_{loc} to u. Moreover, by Lemma 1.6.11e, these functions u_δ monotonically decrease pointwise to a subharmonic function V. It follows that $u = V$ a.e. So u equals a subharmonic function V a.e. By Lemma 1.6.11b is follows that this V is unique and also that 1.6.4. holds.

6. BASICS-**HILBERT SPACES 1/6**-$\partial/\partial\overline{z}$

Hörmander Chapter 4.1.

We next move on to Chapter 4 in Hormander: L^2 estimates and existence theorems for the $\overline{\partial}$ operator.

We start with some functional analysis. Let G_1, G_2 denote two complex Banach spaces with norms $\|\cdot\|_1, \|\cdot\|_2$ respectively. Let E denote a complex subspace of G_1, not necessarily closed. We will consider linear maps $T : E \to G_2$. Let $G_3 = G_1 \times G_2$ denote the product space with Banach norm $\|(x,y)\|_3^2 = \|x\|_1^2 + \|y\|_2^2$. We say that T is closed if the graph of T, $G_T = \{(x, Tx); x \in E\}$ is a closed subspace of G_3.

Example 4.0.1 $G_1 = G_2 = L^2(0,1)$ and $Tx = x'$. Here the derivative is in the sense of distributions. So $Tx(\phi) = -\int x\phi'$. Here E consists of those L^2 functions x for which Tx is an L^2 function. Then T is a closed operator: If (x_n, Tx_n) converge to (x,y) then for any test function ϕ we have $\int Tx_n\phi = -\int x_n\phi'$ so taking limit one gets $\int y\phi = -\int x\phi'$. Therefore y equals x' in the sense of distributions.

Let G_i' denote the dual Banach space of G_i. So an element $y \in G_i'$ is a continuous linear function (also called functional) from G_i to $\mathbb{C}, x \to y(x)$. Moreover there is a constant C_y so that $|y(x)| \leq C_y\|x\|_i$. The smallest such constant C_y is denoted by $\|y\|_i$.

An important theorem is the Hahn-Banach Theorem.

Theorem 4.0.2, Hahn-Banach Let $L \subset G$ be a linear subspace of a Banach space. Suppose that $\phi : L \to \mathbb{C}$ is a linear function with bounded norm, i.e. $|\phi(x)| \leq C\|x\|$. Then ϕ extends to a linear function $\tilde{\phi} : G \to \mathbb{C}$ such that $\tilde{\phi}(x) = \phi(x)$ for all $x \in L$ and $|\tilde{\phi}(x)| \leq C\|x\|$ for all $x \in G$. In particular, the extension belongs to G'.

We will next add the hypothesis that the operator T is densely defined, i.e. the subspace E is dense. We will define the adjoint operator $T^* : D_{T^*} \to G_1'$ where $D_{T^*} \subset G_2'$ is a linear subspace. We say that an element y in G_2' belongs to D_{T^*} if there exists a constant C so that $|y(Tx)| \leq C\|x\|_1$ for all $x \in E$. Hence, by Hahn-Banach, $x \to y(Tx)$ extends to a continuous linear functional z on $\overline{E} = G_1$ such that $\|z\|_{G_1'} \leq C$. We set $T^*(y) = z$.

If $(y_1, y_2) \in G_1' \times G_2'$ then this defines a continuous linear functional on $G_1 \times G_2$ by $(y_1, y_2)(x_1, x_2) = y_1(x_1) - y_2(x_2)$.

Let $G_T^\perp \subset G_1' \times G_2'$ denote those (y_1, y_2) for which $(y_1, y_2)(x_1, x_2) = 0$ for all $(x_1, x_2) \in G_T$. Clearly $G_T^\perp$ is a closed subspace of $G_1' \times G_2'$.

Lemma 4.0.3 $G_T^\perp = G_{T^*}$.

Proof. Suppose first that $(y_1, y_2) \in G_T^\perp$. Then if $x \in D_T$ we have that $y_1(x) - y_2(Tx) = 0$. This implies that $|y_2(Tx)| = |y_1(x)| \leq \|y_1\|\|x\|$. Hence y_2 satisfies the requirement to be in D_{T^*}. Moreover y_1 satisfies the equirement to equal $T^*(y_2)$. Hence $(y_1, y_2) \in G_{T^*}$.

Suppose next that $(y_1, y_2) \in G_{T^*}$. Hence $y_1 = T^*(y_2)$. This implies that for any $x \in D_T$, we have that $y_1(x) = y_2(Tx)$. Hence $(y_1, y_2) \in G_T^\perp$. $\square$

Corollary 4.0.4 The graph of T^* is closed.

If G is a complex Banach space, we define G'' to be the dual of G'. There is a natural isometric embedding ϕ of G into G'': For $x \in G$, define $\phi(x)(y) = y(x)$ for $y \in G_1'$. Then $\phi(x) \in G_1''$. Also $\|\phi(x)\| \leq \|x\|$ and by the Hahn Banach theorem you have equality: Given $x \neq 0$, choose a linear function $\tilde{y}$ on $\mathbb{C}x$ by $\tilde{y}(x) = \|x\|$ and extend to G by Hahn-Banach. Call the extension y. Then $y \in G'$ and $\|y\| = 1$. Now $\phi(x)(y) = y(x) = \|x\| = \|x\|\|y\|$ so $\|\phi(x)\| \geq \|x\|$. Hence $\|\phi(x)\| = \|x\|$.

We say that $G'' = G$ if this map is surjective. The Banach space is called reflexive in this case. In this case, we also have that $G' = G'''$. Note that if G_1, G_2 are Banach spaces, then the dual of $G_1 \times G_2$ equals $G_1' \times G_2'$. Namely, if ϕ is a continuous linear function on $G_1 \times G_2$, then $\phi(x_1, x_2) = \phi(x_1, 0) + \phi(0, x_2) = y_1(x_1) + y_2(x_2)$.

Lemma 4.0.5 Let G be a reflexive Banach space, i.e. $\phi(G) = G''$, and let H be a closed subspace of G. Then $(H^\perp)^\perp = \phi(H)$.

Proof. Since ϕ is an isometry and H is closed, $\phi(H)$ is also closed.

$\phi(H) \subset H^{\perp\perp}$:

Suppose that $x \in H$ and $y \in H^\perp$. Then $y(x) = 0$ and hence $\phi(x)(y) = y(x) = 0$. Hence $\phi(x) \in H^{\perp\perp}$. Hence $\phi(H) \subset H^{\perp\perp}$.

$H^{\perp\perp} \subset \phi(H)$:

Suppose next that $z \in G'' \setminus \phi(H)$. Hence there exists an $\eta \in G'''$ so that $\eta(z) \neq 0$ while η vanishes on $\phi(H)$. (Here we use that $\phi(H)$ is closed.) By reflexivity, we have that there exists a $y \in G'$ so that $z(y) = \eta(z) \neq 0$ while $w(y) = \eta(w) = 0$ for all $w \in \phi(H)$. Hence if $x \in H$ and $w = \phi(x) \in \phi(H)$, then $y(x) = (\phi(x))(y) = w(y) = 0$. So, $y \in H^\perp$. Since $z(y) \neq 0$, it follows that z cannot be in $H^{\perp\perp}$, so $H^{\perp\perp} \subset \phi(H)$. $\square$

Corollary 4.0.6 If G_1, G_2 are reflexive and $T : G_1 \to G_2$ is a densely defined closed linear operator, then $T^{**} = T$.

Proof. Set $G_j'' = \phi_j(G_j)$. Then

$$G_{T^{**}} = (G_{T^*})^{\perp} = G_T^{\perp\perp} = \{(\phi_1(x), \phi_2(Tx)); (x, Tx) \in G_T\}.$$

We write this imprecisely as $G_{T^{**}} = G_T$, or $T^{**} = T$. □

Lemma 4.0.7 Assume that G_1, G_2 are reflexive Banach spaces. Then the operator T^* is densely defined.

Proof. Suppose that there exists a $y_0 \in G_2' \backslash \overline{D}_{T^*}$. Then there exists a $z_0 \in G_2''$ so that $z_0(y_0) \neq 0$ while $z_0(y) = 0$ for all $y \in D_{T^*}$. So $z_0 \neq 0$. Hence for the point $(0, z_0) \in G_1'' \times G_2''$ we have that $0(T^*y) - z_0(y) = 0$ for all $y \in D_{T^*}$. It follows that $(0, z_0) \in G_{T^*}^{\perp} = G_T^{\perp\perp}$. We write $(0, z_0) = (\phi_1(0), \phi_2(x))$ for some $(0, x) \in G_1 \times G_2$. Then $(0, x) \in G_T$. But $T(0) = 0$ and x cannot be zero since $\phi_2(x) = z_0 \neq 0$, a contradiction. □

7. BASICS-HILBERT SPACES 2/6-$\partial/\partial\overline{z}$

Hörmander Chapter 4.1.

We recall the uniform boundedness principle (Banach-Steinhaus theorem).

Theorem 4.0.8 Let $\mathcal{F}$ denote a family of continuous linear functionals on a Banach space B. Suppose that for every $x \in B$ there exists a constant c_x so that $|F(x)| \leq c_x \|x\|_B$ for all $F \in \mathcal{F}$. Then there exists a constant C so that $\|F\| \leq C$ for all $F \in \mathcal{F}$.

The proof uses Baire category. For any number A the set of x where $c_x \leq A$ is closed. (Closedness follows from continuity of the functionals.) Hence for some A the set has interior.

Notation: Let $y \in G'$ and let $H \subset G$ be a linear subspace. We denote by $\|y_{|H}\|_{G'}$ the norm of the linear functional y restricted to H. So $\|y_{|H}\|_{G'}$ is the smallest c so that $|y(x)| \leq c\|x\|$ for all $x \in H$.

We next give a more general version of Theorem 4.1.1 in Hörmander.

Theorem 4.1.1' Let G_1, G_2 be reflexive Banach spaces and let $T : G_1 \to G_2$ be a densely defined closed linear operator. Let $F \subset G_2$ be a closed subspace containing the range of T, R_T. Then $F = R_T$ if and only if there exists a constant $C \geq 0$ such that

$$(4.1.1)' \quad \|y_{|F}\|_{G_2'} \leq C\|T^* y\|_{G_1'} \ \forall\, y \in D_{T^*}.$$

If any of the two equivalent conditions are satisfied, then there exists for every $z \in F$ an $x \in D_T$ with $Tx = z$ and $\|x\|_{G_1} \leq C\|Tx\|_{G_2} = C\|z\|_{G_2}$ for the same constant C.

Remark 7.1. In our application of this theorem, we will only need the case when $F = G_2$.

Proof. Suppose that $R_T = F$. We will apply the Banach-Steinhaus theorem to a family of linear functionals on the Banach space F. Namely, let $\mathcal{G}$ denote the family of $y \in D_{T^*}$ for which $\|T^* y\|_{G_1'} \leq 1$. Define for each such y a linear function $L_y \in \mathcal{F}$ on F given by $L_y(x) = y(x)$. This is a continuous linear functional defined on F. For a given $x \in F$, pick some $z \in D_T$ for which $x = Tz$. Then we have that for any $L_y \in \mathcal{F}$, that

$$|L_y(x)| = |y(x)| = |y(Tz)| = |(T^*(y))(z)| \leq \|T^*(y)\|_{G_1'}\|z\|_{G_1} \leq \|z\|_{G_1}.$$

Hence the family $\mathcal{F}$ is bounded uniformly on any given $x \in F$. Hence by the Banach-Steinhaus Theorem, there is a constant C so that $\|(L_y)_{|F}\|_{G_2'} \leq C$ for any $y \in D_{T^*}$ with $\|T^* y\|_{G_1'} \leq 1$. Then it follows that $\|(L_y)_{|F}\|_{G_2'} \leq C\|T^* y\|_{G_1'} \ \forall\, y \in D_{T^*}$. Since $(L_y)_{|F} = y(x), x \in F$ we get that $\|y_{|F}\|_{G_2'} \leq C\|T^* y\|_{G_1'}$ for all $y \in D_{T^*}$.

We next suppose that (4.1.1)' is satisfied. We will then show that $F = R(T)$ and in the process of the proof also obtain the last part of the Theorem. Fix a $z \in F$. If $y \in D_{T^*}, w = T^*(y) \in R_{T^*}$, set $\phi(w) = y(z)$. Note that if

$w = T^*(y_1) = T^*(y_2)$, then $T^*(y_1 - y_2) = 0$. Hence by (4.1.1)', $(y_1 - y_2)(z) = 0$, so $y_1(z) = y_2(z)$ and therefore $\phi(w)$ is well defined. Also, by (4.1.1)',

$$
\begin{aligned}
|\phi(w)| &= |y(z)| \\
&\leq \|y_{|F}\|_{G_2'} \| \|z\|_{G_2} \\
&\leq C\|T^*y\|_{G_1'}\|z\|_{G_2} \\
&= C\|w\|_{G_1'}\|z\|_{G_2},
\end{aligned}
$$

hence ϕ is a bounded linear functional with norm at most $C\|z\|_{G_2}$. This is then a bounded linear function on the Range of T^* in G_1' with norm $\leq C\|z\|_{G_2}$. We extend ϕ to all of G_1' using the Hahn-Banach theorem. Then $\phi \in G_1''$ with norm $\|\phi\|_{G_1''} \leq C\|z\|_{G_2}$.

Recall the definition of $*$ as it applies in this situation. We say that $\phi \in G_1''$ belongs to $D_{(T^*)^*} \subset G_1''$ if there exists a constant c so that $|\phi(T^*y)| \leq c\|y\|_{G_2'}$ for all $y \in D_{T^*}$. Since $|\phi(T^*y)| = |y(z)| \leq \|z\|_{G_2}\|y\|_{G_2'}$ and our z is fixed, it follows that $\phi \in D_{T^{**}} \subset G_1''$. By reflexivity there is an $x \in G_1$ with norm $\leq C\|z\|$ such that $u(x) = \phi(u)$ for all $u \in G_1'$. Moreover $x \in D_T$. So whenever $y \in D_{T^*}$,

$$y(z) = \phi(T^*y) = (T^*(y))(x) = y(Tx).$$

So we have shown that for any fixed $z \in F$, there is an $x \in D_T, \|x\|_{G_2} \leq C\|z\|_{G_1}$ so that $y(z) = y(Tx)$ for all $y \in D_{T^*}$. Since D_{T^*} is dense in G_2', it follows that $y(z - Tx) = 0$ for all $y \in G_2'$. By the Hahn-Banach theorem this implies that $z - Tx = 0$. $\square$

Let N_T denote the nullspace of T. Clearly N_T is contained in D_T and N_T is a closed subspace.

8. BASICS-HILBERT SPACES 3/6-$\partial/\partial\overline{z}$

Hörmander Chapter 4.1.

In the rest of the course, we will assume that all Banach spaces that we use are Hilbert spaces.
Recall a few facts about complex Hilbert spaces, H. We have an inner product $< x, y >$ for $x, y \in H$. The inner product satisfies

$$< ax, by >= a\overline{b} < x, y > .$$

The norm is $\|x\|^2 =< x, x >$. There is a natural identification between H and the dual H'. If $x \in H$, then $\lambda(x)$ defined by $\lambda(x)(y) =< y, x >$ defines a continuous linear functional on H. The map $\lambda : H \to H'$ is norm preserving and antilinear: $\lambda(cx) = \overline{c}\lambda(x)$.

We show that λ is surjective: If $g \in H', g \neq 0$, pick an x which is not in the nullspace of g, N_g with $\|x\| = 1$. Set $c = \overline{g(x)}$. We show that $g = \lambda(cx)$. Any $y \in H$ can be written uniquely as $y = ax + z, z \in N_g$. We get

$$\begin{aligned}
g(y) &= g(ax + z)\\
&= ag(x) + g(z)\\
&= a\overline{c}\\
&= \overline{c} < ax + z, x >\\
&= < y, cx >\\
&= \lambda(cx)(y)\\
&\Rightarrow\\
g &= \lambda(cx)
\end{aligned}$$

We introduce some Hilbert spaces. Let Ω be an open subset of $\mathbb{C}$. If ϕ is a continuous real function on Ω, we define

$$L^2(\Omega, \phi) = \{f : \Omega \to \mathbb{C}, \int_\Omega |f|^2 e^{-\phi} d\lambda =: \|f\|^2_{L^2(\Omega,\phi)} < \infty\}.$$

Here $d\lambda$ is Lebesgue measure and f is assumed to be measurable and locally in L^2, $f \in L^2_{loc}(\Omega)$. We know that the dual of $L^2(\Omega, \phi)$ is $L^2(\Omega, \phi)$. In particular these spaces are reflexive. We have for $f \in L^2(\Omega, \phi)$ and $g \in L^2(\Omega, \phi)$ that $g(f) = \int_\Omega f\overline{g}e^{-\phi}$. We write $g(f) =< f, g >_\phi$ and get $< f, g >_\phi= \overline{< g, f >_\phi}$. Also we have $|g(f)| \leq \|f\|_{L^2(\Omega,\phi)}\|g\|_{L^2(\Omega,\phi)}$.

Remark 8.1. Let $c \in \mathbb{C}$, $g \in L^2(\Omega, \phi)$. Then $(cg)(f) = \int f\overline{cg}e^{-\phi} = \overline{c}g(f)$. Hence one should use the complex conjugate structure on $L^2(\Omega, \phi)$ when we use it as a dual space. We won't be explicit about this, just keep it in mind.

Set $f \cdot g = f\overline{g}$ for the pointwise product.

Let $D(\Omega)$ denote the space of $\mathcal{C}^\infty$ functions on Ω with compact support in Ω. We observe that $D(\Omega)$ is dense in $L^2(\Omega, \phi)$.

Let ϕ_1, ϕ_2 be continuous functions on Ω. Consider the operator $\frac{\partial}{\partial \overline{z}}$. This gives rise to a linear densely defined closed operator

$$T : L^2(\Omega, \phi_1) \to L^2(\Omega, \phi_2).$$

An element $u \in L^2(\Omega, \phi_1)$ is in D_T if $\frac{\partial u}{\partial \overline{z}}$, defined in the sense of distributions belongs to $L^2(\Omega, \phi_2)$ and then we set $Tu = \frac{\partial u}{\partial \overline{z}}$. The operator is densely defined since it is defined on $D(\Omega)$. The closedness is as in Example 2.1. Our goal is to show that the range of T consists of all $f \in L^2(\Omega, \phi_2)$ for some choices of ϕ_j.

9. BASICS-HILBERT SPACES 4/6-$\partial/\partial\overline{z}$

Hörmander Chapter 4.1.

Lemma 4.1.3_a Let η_ν be a sequence of $\mathcal{C}^\infty$ functions with compact support in Ω. Suppose that $0 \le \eta_\nu \le 1$ and that on any given compact subset of Ω we have $\eta_\nu = 1$ for all large ν. Suppose that

$$(4.1.6)'' \ \ e^{-\phi_2}|\partial\eta_\nu/\partial\overline{z}|^2 \le e^{-\phi_1}.$$

Then for every $f \in D_T^*$ the sequence $\eta_\nu f \to f$ in G_2'. Moreover $\eta_\nu f \in D_{T^*}$ and $T^*(\eta_\nu f) \to T^*(f)$ in G_1'.

Proof. Suppose that $f \in D_{T^*}$. By the Lebesgue dominated convergence theorem, $\eta_\nu f \to f$ in G_2'. We show that $\eta f \in D_{T^*}$ if η is real valued and smooth with compact support. Let $u \in D_T$. Then $Tu \in G_2$ and

$$\begin{aligned}
(\eta f)(Tu) &= < Tu, \eta f >_{\phi_2} \\
&= \int \frac{\partial u}{\partial \overline{z}}(\overline{\eta f})e^{-\phi_2} \\
&= \int (\eta \frac{\partial u}{\partial \overline{z}})\overline{f})e^{-\phi_2} \\
&= \int (\frac{\partial(\eta u)}{\partial \overline{z}} - u\frac{\partial \eta}{\partial \overline{z}})(\overline{f})e^{-\phi_2} \\
&= \int \frac{\partial(\eta u)}{\partial \overline{z}}\overline{f}e^{-\phi_2} - \int u\frac{\partial \eta}{\partial \overline{z}}\overline{f}e^{-\phi_2} \\
&= \int \eta u\overline{T^* f}e^{-\phi_1} - \int u\frac{\partial \eta}{\partial \overline{z}}\overline{f}e^{-\phi_2} \\
&= \int u\overline{\eta T^* f}e^{-\phi_1} - \int u\frac{\partial \eta}{\partial \overline{z}}\overline{f}e^{-\phi_2}
\end{aligned}$$

So

$$\begin{aligned}
|(\eta f)(Tu)| &\le (1)\|\eta T^* f\|_{G_1'}\|u\|_{G_1} + \|f\|_{G_2'}\left(\int |u\frac{\partial\eta}{\partial\overline{z}}|^2 e^{-\phi_2}\right)^{1/2} \\
&\le (2)\|T^* f\|_{G_1'}\|u\|_{G_1} + \|f\|_{G_2'}\left(\int |u\frac{\partial\eta}{\partial\overline{z}}|^2 e^{-\phi_2}\right)^{1/2} \\
&\le (3)\|T^* f\|_{G_1'}\|u\|_{G_1} + \|f\|_{G_2'}\left(\int |u|^2(|\frac{\partial\eta}{\partial\overline{z}}|^2) e^{-\phi_2}\right)^{1/2} \\
&\le (4)\|T^* f\|_{G_1'}\|u\|_{G_1} + C_\eta\|f\|_{G_2'}\left(\int |u|^2 e^{-\phi_1}\right)^{1/2} \\
&\quad \text{where } C_\eta = 1 \text{ for } \eta = \eta_\nu
\end{aligned}$$

$$\begin{aligned} |(\eta f)(Tu)| &\leq \|T^* f\|_{G_1'}\|u\|_{G_1} + C_\eta \|f\|_{G_2'}\left(\int |u|^2 e^{-\phi_1}\right)^{1/2} \\ &= \left(\|T^* f\|_{G_1'} + C_\eta\|f\|_{G_2'}\right)\|u\|_{G_1} \end{aligned}$$

So then $\eta f \in D_{T^*}$.

It remains to show that $T^*(\eta_\nu f) \to T^* f$. It suffices to show that $T^*(\eta_\nu f) - \eta_\nu T^* f \to 0$. Let $u \in D_T$. Then

$$\begin{aligned} (T^*(\eta_\nu f) - \eta_\nu T^* f)u &= < u, T^*(\eta_\nu f) - \eta_\nu T^* f >_{\phi_1} \\ &= < Tu, \eta_\nu f >_{\phi_2} - < u, \eta_\nu T^* f >_{\phi_1} \\ &= < \eta_\nu Tu, f >_{\phi_2} - < u, \eta_\nu T^* f >_{\phi_1} \\ &= < T(\eta_\nu u) - u\frac{\partial \eta_\nu}{\partial \overline{z}}, f >_{\phi_2} - < u, \eta_\nu T^* f >_{\phi_1} \\ &= < \eta_\nu u, T^* f >_{\phi_1} - < u\frac{\partial \eta_\nu}{\partial \overline{z}}, f >_{\phi_2} - < u, \eta_\nu T^* f >_{\phi_1} \\ &= - < u\frac{\partial \eta_\nu}{\partial \overline{z}}, f >_{\phi_2} \end{aligned}$$

Hence

$$|(T^*(\eta_\nu f) - \eta_\nu T^* f)u| \leq \int |f||u\frac{\partial \eta_\nu}{\partial \overline{z}}|e^{-\phi_2}.$$

So in particular we have for any $u \in D(\Omega)$ that

$$|\int (T^*(\eta_\nu f) - \eta_\nu T^* f)\cdot u e^{-\phi_1}| \leq \int |f||\frac{\partial \eta_\nu}{\partial \overline{z}}||u|e^{-\phi_2}.$$

This implies the pointwise a.e. estimate

$$|(T^*(\eta_\nu f) - \eta_\nu T^* f)e^{-\phi_1}| \leq |f||\frac{\partial \eta_\nu}{\partial \overline{z}}|e^{-\phi_2}.$$

So

$$\begin{aligned} |(T^*(\eta_\nu f) - \eta_\nu T^* f)| &\leq |f||\frac{\partial \eta_\nu}{\partial \overline{z}}|e^{\phi_1 - \phi_2} \\ |(T^*(\eta_\nu f) - \eta_\nu T^* f)|^2 e^{-\phi_1} &\leq |f|^2|\frac{\partial \eta_\nu}{\partial \overline{z}}|^2 e^{2(\phi_1-\phi_2)}e^{-\phi_1} \\ &\leq |f|^2\left(e^{-\phi_1}e^{\phi_2}\right)e^{2(\phi_1-\phi_2)}e^{-\phi_1} \\ &= |f|^2 e^{\phi_1(-1+2-1)}e^{\phi_2(1-2)} \\ &= |f|^2 e^{-\phi_2} \end{aligned}$$

Since the functions $|(T^*(\eta_\nu f) - \eta_\nu T^* f)|$ converge pointwise to zero, it suffices by the dominated convergence theorem to show that $|f|^2 e^{-\phi_2}$ is an L^1 function. This follows since $f \in L^2(\Omega, \phi_2)$.

$\square$

10. BASICS-**HILBERT SPACES 5/6-$\partial/\partial\overline{z}$**

Hörmander Chapter 4.1.

Before we continue the discussion started in Lemma $4.1.3_a$, we will study smoothing.

We will use Minkowski's integral inequality. See Stein, Elias (1970). Singular integrals and differentiability properties of functions. Princeton University Press.

Theorem 4.1.4_0 Let $F(x,y) \geq 0$ be a measurable function on the product of two measure spaces S_1, S_2 with positive measures $d\mu_1(x), d\mu_2(y)$ respectively. Then

$$\left(\int_{S_1}\left(\int_{S_2} F(x,y)d\mu_2(y)\right)^2 d\mu_1(x)\right)^{1/2} \leq \int_{S_2}\left(\int_{S_1} F^2(x,y)d\mu_1(x)\right)^{1/2} d\mu_2(y)$$

The following is the smoothing theorem.

Lemma 4.1.4_a Let χ be a smooth function with compact support in $\mathbb{R}^N$, with $\int \chi(x)dx = 1$ and set $\chi_\epsilon(x) = \frac{1}{\epsilon^N}\chi(\frac{x}{\epsilon})$. If $g \in L^2(\mathbb{R}^N)$ then the convolution $g * \chi_\epsilon$ satisfies

$$\begin{aligned}(g * \chi_\epsilon)(x) &:= \int_{\mathbb{R}^N} g(y)\chi_\epsilon(x-y)dy \\ &= \int g(x-y)\chi_\epsilon(y)dy \\ &= \int g(x-\epsilon y)\chi(y)dy\end{aligned}$$

and is a $\mathcal{C}^\infty$ function such that $\|g * \chi_\epsilon - g\|_{L^2} \to 0$ when $\epsilon \to 0$. The support of $g * \chi_\epsilon$ has no points at distance $> \epsilon$ from the support of g if the support of χ lies in the unit ball.

Proof. The equalities for $g * \chi_\epsilon(x)$ are obvious. The first integral shows that $g * \chi_\epsilon$ is $\mathcal{C}^\infty$ since g is in L^1_{loc} and since we can differentiate under the integral sign. We apply Minkowski's integral inequality to the second integral.

$$
\begin{aligned}
\left(\int |g*\chi_\epsilon|^2 dx\right)^{1/2} &= \left(\int |\int g(x-y)\chi_\epsilon(y)dy|^2 dx\right)^{1/2} \\
&\leq \left(\int \left(\int |g(x-y)||\chi_\epsilon(y)|dy\right)^2 dx\right)^{1/2} \\
&\leq \int \left(\int \left[|g(x-y)||\chi_\epsilon(y)|\right]^2 dx\right)^{1/2} dy \\
&= \int |\chi_\epsilon(y)|\|g\|_{L^2} \\
&= C\|g\|_{L^2}, C := \int |\chi|
\end{aligned}
$$

This shows that $g*\chi_\epsilon \in L^2$ and that

$$\|g*\chi_\epsilon\|_{L^2} \leq C\|g\|_{L^2}.$$

Next pick a $\delta > 0$ and choose a continuous function h with compact support so that $\|g-h\|_{L^2} < \delta$. We then get

$$
\begin{aligned}
\|g*\chi_\epsilon - g\|_{L^2} &\leq \|g*\chi_\epsilon - h*\chi_\epsilon\|_{L^2} + \|h*\chi_\epsilon - h\|_{L^2} + \|h-g\|_{L^2} \\
&\leq \|(g-h)*\chi_\epsilon\|_{L^2} + \|h*\chi_\epsilon - h\|_{L^2} + \delta \\
&\leq (C+1)\delta + \|h*\chi_\epsilon - h\|_{L^2}
\end{aligned}
$$

We have that

$$(h*\chi_\epsilon - h)(x) = \int (h(x-\epsilon y) - h(x))\chi(y)dy.$$

Since h is continuous with compact support and χ has compact support, it follows that $h*\chi_\epsilon - h$ is supported in a ball $\|x\| \leq R$ for $\epsilon < 1$ and converges uniformly to 0 when $\epsilon \to 0$. It follows that $g*\chi_\epsilon \to g$ in L^2.

The last assertion follows from the last of the integrals in the expression for $g*\chi_\epsilon$. □

Lemma 4.1.4_b Let $f_1, \dots, f_N \in L^1_{loc}(\mathbb{R}^N)$. Also suppose that the distribution $\sum_{j=1}^N \frac{\partial f_j}{\partial x_j} \in L^1_{loc}$. Then

$$\sum_{j=1}^N \frac{\partial(f_j*\chi_\epsilon)}{\partial x_j} = \left(\sum_{j=1}^N \frac{\partial f_j}{\partial x_j}\right) * \chi_\epsilon.$$

Proof. Both sides are $\mathcal{C}^\infty$ functions. To show that they are equal, we show that for any $\phi \in \mathcal{C}_0^\infty$ that

$$\int \sum_{j=1}^N \frac{\partial (f_j * \chi_\epsilon)}{\partial x_j}(x)\phi(x)dx = \int \left(\sum_{j=1}^N \frac{\partial f_j}{\partial x_j}\right) * \chi_\epsilon(x)\phi(x)dx.$$

$$\begin{aligned}
\int \left(\sum_{j=1}^N \frac{\partial f_j}{\partial x_j}\right) * \chi_\epsilon(x)\phi(x)dx &= \int \left(\int \left(\sum_{j=1}^N \frac{\partial f_j}{\partial x_j}\right)(x-y)\chi_\epsilon(y)dy\right)\phi(x)dx \\
&= \int \chi_\epsilon(y)\left(\int \left(\sum_{j=1}^N \frac{\partial f_j}{\partial x_j}\right)(x-y)\phi(x)dx\right)dy \\
&= -\int \chi_\epsilon(y)\left(\int \sum_{j=1}^N f_j(x-y)\frac{\partial \phi}{\partial x_j}(x)dx\right)dy \\
&= -\int \left(\int \sum_{j=1}^N \frac{\partial \phi}{\partial x_j}(x) f_j(x-y)\chi_\epsilon(y)dy\right)dx \\
&= -\int \sum_{j=1}^N \frac{\partial \phi}{\partial x_j}(x)(f_j * \chi_\epsilon)(x)dx \\
&= \int \sum_{j=1}^N \frac{\partial (f_j * \chi_\epsilon)}{\partial x_j}(x)\phi(x)dx
\end{aligned}$$

□

Next we discussr T^*. We first do some preparations. In this lemma ϕ_1, ϕ_2 are continuous functions on Ω.

Lemma 4.1.3_b Let $f \in L^2(\Omega, \phi_2)$ and suppose that $f \in D_{T^*}$. Let $T^*(f) = g \in L^2(\Omega, \phi_1)$. Then

$$(4.1.9)\quad g = -e^{\phi_1}\frac{\partial (e^{-\phi_2} f)}{\partial z}.$$

In particular, the distribution $\frac{\partial (e^{-\phi_2} f)}{\partial z} \in L^2_{loc}$.

Proof. Let $u \in D(\Omega)$. So in particular, $u \in D_T$. Then $(T^*f)(u) = f(Tu)$. Here

$$Tu = \frac{\partial u}{\partial \overline{z}}$$

We get

$$(T^*f)(u) = \int u\overline{g}e^{-\phi_1} = f(Tu) = \int \frac{\partial u}{\partial \overline{z}}\overline{f}e^{-\phi_2}.$$

Hence for all smooth functions ψ with compact support, we have that

$$\int \overline{g}e^{-\phi_1}\psi = \int \overline{f}e^{-\phi_2}\frac{\partial \psi}{\partial \overline{z}}.$$

Hence

$$\int ge^{-\phi_1}\overline{\psi} = \int fe^{-\phi_2}\frac{\partial \overline{\psi}}{\partial z}.$$

Therefore

$$ge^{-\phi_1} = -\frac{\partial(e^{-\phi_2}f)}{\partial z} \in L^2_{loc}.$$

$\square$

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Hörmander Chapter 4.1, Part F

Corollary 4.1.3_c If $\phi_2 \in \mathcal{C}^\infty(\Omega)$, then

$$g = -e^{\phi_1-\phi_2}\frac{\partial \phi_2}{\partial z} f + e^{\phi_1-\phi_2}\frac{\partial f}{\partial z}$$

where the distribution $\frac{\partial f}{\partial z}$ is in L^2_{loc}.

Proof. The distribution $\frac{\partial(fe^{-\phi_2})}{\partial z} \in L^2_{loc}$. Let ϕ be a test function.

$$\begin{aligned}
\int \left(\frac{\partial(fe^{-\phi_2})}{\partial z}\right)\phi &= -\int fe^{-\phi_2}\frac{\partial \phi}{\partial z} \\
&= -\int f\frac{\partial(e^{-\phi_2}\phi)}{\partial z} - \int fe^{-\phi_2}\frac{\partial \phi_2}{\partial z}\phi \\
&= \left(\frac{\partial f}{\partial z}\right)(e^{-\phi_2}\phi) - \left(fe^{-\phi_2}\frac{\partial \phi_2}{\partial z}\right)(\phi) \\
&= \left(e^{-\phi_2}\frac{\partial f}{\partial z}\right)(\phi) - \left(e^{-\phi_2}f\frac{\partial \phi_2}{\partial z}\right)(\phi)
\end{aligned}$$

The expression on the left is in L^2_{loc} and the second expression on the right is in L^2_{loc}. Hence the first expression on the right is in L^2_{loc}.

□

Lemma 4.1.3_d Suppose that $f \in D(\Omega), \phi_2 \in \mathcal{C}^\infty$. Then $f \in D_{T^*}$.

Proof. Let $g = -e^{\phi_1}\frac{\partial(fe^{-\phi_2})}{\partial z}$. To show that $f \in D_{T^*}$, we prove that for any $u \in D_T, f(Tu) = g(u)$. Choose such u, then $u \in G_1$, and

$$Tu = \frac{\partial u}{\partial \overline{z}}$$

Then

$$\begin{aligned} f(Tu) &= < Tu, f >_{\phi_2} \\ &= \int \overline{f} \frac{\partial u}{\partial \overline{z}} e^{-\phi_2} \\ &= - \int \overline{\frac{\partial (f e^{-\phi_2})}{\partial z}} u \\ &= - \int \left(\overline{e^{\phi_1} \frac{\partial (f e^{-\phi_2})}{\partial z}} \right) u e^{-\phi_1} \\ &= < u, g > \\ &= g(u) \end{aligned}$$

□

Lemma 4.1.3$_e$ Suppose that $\phi_2 \in \mathcal{C}^\infty$. Let $f \in D_{T^*}$ have compact support. Then $f * \chi_\epsilon \to f$ in G_2'. Moreover, $f * \chi_\epsilon \in D_{T^*}$ and $T^*(f * \chi_\epsilon) \to T^* f$ in G_1'.

Proof. Since $f * \chi_\epsilon \in D(\Omega)$ for small ϵ, we have by Lemma 4.1.3$_d$ that $f * \chi_\epsilon \in D_{T^*}$. Also by the smoothing theorem, $f * \chi_\epsilon \to f$ in G_2'. By Corollary 4.1.3$_c$ we can write $T^*(f * \chi_\epsilon) = g$ where

$$g = -e^{\phi_1 - \phi_2} \frac{\partial \phi_2}{\partial z} f * \chi_\epsilon - e^{\phi_1 - \phi_2} \frac{\partial (f * \chi_\epsilon)}{\partial z}.$$

The first term on the right converges to $-e^{\phi_1 - \phi_2} \frac{\partial \phi_2}{\partial z} f$ in L^2 by the smoothing Theorem. The second part can be written, using Lemma 4.1.4$_b$, as $-e^{\phi_1 - \phi_2} \left(\frac{\partial f}{\partial z} \right) * \chi_\epsilon$ and converges to

$$-e^{\phi_1 - \phi_2} \frac{\partial f}{\partial z}$$

in L^2. □

Theorem 4.1.3$_f$ Suppose that ϕ_1 is continuous and ϕ_2 is $\mathcal{C}^\infty$. Suppose that $\{\eta_\nu\}, 0 \leq \eta_\nu \leq 1$ is a sequence of compactly supported $\mathcal{C}^\infty$ functions such that on any compact subset of Ω all $\eta_\nu = 1$ except finitely many, as in (4.1.6)". Suppose that $f \in D_{T^*}$. Then there exist a sequence $\{f_n\} \subset D(\Omega)$ so that $f_n \in D_{T^*}$, $f_n \to f$ in G_2' and $T^* f_n \to T^* f$ in G_1'.

Proof. Let $\delta > 0$. Using Lemma 4.1.3$_a$ for T^*, we can let ν_0 be large enough that

$$\|\eta_{\nu_0} f - f\|, \|T^*(\eta_{\nu_0} f) - T^* f\| < \delta/2$$

and $\eta_{\nu_0} f \in D_{T^*}$. Then for $\epsilon > 0$ small enough, $\hat{f} = (\eta_{\nu_0} f) * \chi_\epsilon$ is in D_{T^*} and

$$\|\hat{f} - f\|, \|T^* \hat{f} - T^* f\| < \delta.$$

□

Corollary 4.1.3_g The T system in Theorem 4.1.3_f satisfies the Basic Estimate (4.1.1.') if there is a constant C so that for every $y, u \in D(\Omega)$ thought of as elements of G_2', G_2 respectively, we have that

$$|y(u)| \leq C\|T^*(y)\|_{G_1'}\|u\|_{G_2}.$$

Theorem 4.1.3_h Assume the conditions of Theorem 4.1.3_f. If for every $f \in G_2$ with $\lambda(f) \in D_{T^*}$ we have that

$$(1) \quad \|f\| \leq C\|T^* f\|$$

for a fixed constant C, independent of f, then we have that for all $y \in D_{T^*}, u \in G_2$ that

$$(2) \quad |y(u)| \leq C\|T^* y\|\|u\|).$$

We also have (2) $\Rightarrow$ (1).

We note that $N_{T^*} = R_T^{\perp}$. Namely, if $y \in D_{T^*}, T^*(y) = 0$ and $x \in D_T$, we have $(T^*(y))(x) = y(Tx) = 0$. Conversely, if $y \in R_T^{\perp}$, then $y(Tx) = 0$ for all $x \in D_T$, so $y \in N_{T^*}$. Note that we identify $R_T^{\perp} \subset H'$, the dual space with the orthogonal complement of $R_T \subset H$, i.e. the vectors in H perpendicular to the vectors in R_T. If $y \in N_{T^*}$ and $x \in \overline{R_T}$ then $y(x) = 0$. On the other hand, if y is in the subspace in H' identified with $\overline{R_T}$ and x is in the subspace of H identified with N_{T^*}, we also have $y(x) = 0$.

Proof. Assume (1). It suffices to prove (2) for all $y, u \in D(\Omega)$. We write $y = y_1 + y_2$ where $\lambda^{-1}(y_1) \in \overline{R}_T$ and $y_2 \in N_{T^*}$. Similarly, we write $u = u_1 + u_2$ where $u_1 \in \overline{R_T}$ and $\lambda(u_2) \in N_{T^*}$. Then $y_2(u_1) = 0$ and $y_1(u_2) = 0$. Note that $y_1 \in D_{T^*}$ since both y and y_2 are.

It follows that

$$\begin{aligned}
|y(u)| &= |y_1(u_1) + y_2(u_1) + y_1(u_2) + y_2(u_2)| \\
&\leq |y_1||u_1| + |y_2||u_2| \\
&\leq C\|T^* y_1\|\|u_1\| + \|y_2\|\|T^* u_2\| \\
&= C\|T^* y_1\|\|u_1\| \\
&= C\|T^* y\|\|u_1\| \\
&\leq C\|T^* y\|\|u\|
\end{aligned}$$

The reverse implication follows by applying (2) to the case $y = u \in D(\Omega)$.

□

12. BASICS-HILBERT-HÖRMANDER'S $\frac{\partial u}{\partial \overline{z}}$ THEOREM 1/4

Hörmander Chapter 4.2.

We are trying to prove the basic estimate

$$\|f\| \leq C\|T^* f\|.$$

We need a formulasfor $\|T^* f\|$.

Definition 4.2.1_a Let $\phi \in \mathcal{C}^\infty(\Omega)$ be a real valued function. If $w \in \mathcal{C}^\infty(\Omega)$ we let $\delta(w) := e^\phi \frac{\partial(we^{-\phi})}{\partial z} = \frac{\partial w}{\partial z} - w\frac{\partial \phi}{\partial z}$.

Lemma 4.2.1_b Le w_1, w_2 be $\mathcal{C}^\infty$ functions with compact support in Ω. Then $< w_1, \frac{\partial w_2}{\partial \overline{z}} >_\phi := \int w_1 \overline{\left(\frac{\partial w_2}{\partial \overline{z}}\right)} e^{-\phi} d\lambda = \int (-\delta w_1)\overline{w}_2 e^{-\phi} d\lambda =: < -\delta w_1, w_2 >_\phi$.

Proof.

$$\begin{aligned}
\int w_1 \overline{\frac{\partial w_2}{\partial \overline{z}}} e^{-\phi} d\lambda &= \int w_1 \frac{\partial \overline{w}_2}{\partial z} e^{-\phi} d\lambda \\
&= -\int \frac{\partial(w_1 e^{-\phi})}{\partial z} \overline{w}_2 d\lambda \\
&= -\int \frac{\partial w_1}{\partial z} \overline{w}_2 e^{-\phi} d\lambda + \int w_1 \frac{\partial \phi}{\partial z} \overline{w}_2 e^{-\phi} d\lambda \\
&= -\int \delta w_1 \overline{w}_2 e^{-\phi} d\lambda
\end{aligned}$$

□

We next prove a commutation relation between δ and $\frac{\partial}{\partial \overline{z}}$.

Lemma 4.2.1_c Let ψ be a smooth function. Then

$$\delta(\frac{\partial \psi}{\partial \overline{z}}) - \frac{\partial}{\partial \overline{z}}(\delta(\psi)) = \psi \frac{\partial^2 \phi}{\partial \overline{z} \partial z}.$$

Proof.

$$\begin{aligned}
\delta(\psi_{\overline{z}}) - (\delta\psi)_{\overline{z}} &= \psi_{\overline{z},z} - \psi_{\overline{z}} \frac{\partial \phi}{\partial z} - (\psi_z - \psi \frac{\partial \phi}{\partial z})_{\overline{z}} \\
&= \psi_{\overline{z},z} - \psi_{\overline{z}}\phi_z - \psi_{z,\overline{z}} + \psi_{\overline{z}}\phi_z + \psi\phi_{z,\overline{z}} \\
&= \psi\phi_{z,\overline{z}}
\end{aligned}$$

□

Lemma 4.2.1$_d$ Let f, g be $\mathcal{C}^\infty$ functions with compact support in Ω. Then

$$\int \delta f \overline{\delta g} e^{-\phi} \quad = \quad -\int \frac{\partial(\delta f)}{\partial \overline{z}} \overline{g} e^{-\phi}$$

Proof.

$$\begin{aligned}
\int \delta f \overline{\delta g} e^{-\phi} &= \overline{\int \delta g \overline{\delta f} e^{-\phi}} \\
&= -\overline{\int g \overline{\left(\frac{\partial(\delta f)}{\partial \overline{z}}\right)} e^{-\phi}} \\
&= -\int \overline{g} \frac{\partial(\delta f)}{\partial \overline{z}} e^{-\phi}
\end{aligned}$$

□

The following lemma is immediate from Lemma 4.2.1$_b$:

Lemma 4.2.1$_e$ Let f, g be $\mathcal{C}^\infty$ functions with compact support. Then

$$\int \frac{\partial f}{\partial \overline{z}} \overline{\frac{\partial g}{\partial \overline{z}}} e^{-\phi} \quad = \quad -\int \delta\left(\frac{\partial f}{\partial \overline{z}}\right) \overline{g} e^{-\phi}$$

Corollary 4.2.1$_f$ Let f, g be $\mathcal{C}^\infty$ functions with compact support. Then

$$\int \delta f \overline{\delta g} e^{-\phi} - \int \frac{\partial f}{\partial \overline{z}} \overline{\frac{\partial g}{\partial \overline{z}}} e^{-\phi} \quad = \quad \int f \overline{g} \frac{\partial^2 \phi}{\partial z \partial \overline{z}} e^{-\phi}$$

Proof. We combine Lemmas 4.2.1$_{c-e}$:

$$\begin{aligned}
\int \delta f \overline{\delta g} e^{-\phi} - \int \frac{\partial f}{\partial \overline{z}} \overline{\frac{\partial g}{\partial \overline{z}}} e^{-\phi} &= -\int \frac{\partial(\delta f)}{\partial \overline{z}} \overline{g} e^{-\phi} + \int \delta\left(\frac{\partial f}{\partial \overline{z}}\right) \overline{g} e^{-\phi} \\
&= \int f \frac{\partial^2 \phi}{\partial z \partial \overline{z}} \overline{g} e^{-\phi}
\end{aligned}$$

□

Let f be in $D(\Omega)$. Recall that

Lemma 4.2.1$_g$**.** $Tf = \frac{\partial}{\partial \overline{z}}$ and hence
$|Tf|^2 = |\frac{\partial f}{\partial \overline{z}}|^2$

13. BASICS-HILBERT-HÖRMANDER'S $\frac{\partial u}{\partial \bar{z}}$ THEOREM 2/4

Hörmander Chapter 4.2.

We will next be more specific about our L^2 spaces. First pick some smooth function ψ as in Theorem 4.1.3$_h$. We use weights $e^{-\phi_j}$ using smooth functions ϕ and ψ as follows:

$$\phi_1 = \phi - 2\psi, \phi_2 = \phi - \psi.$$

With these weights Theorem 4.1.3$_h$ applies to show that smooth compactly supported forms are dense in the graph norm of T^*.

Let $f \in D(\Omega)$. Recall from Lemma 4.1.3$_d$ that

$$\begin{aligned} T^* f &= g \\ &\text{where} \\ g &= -e^{\phi_1} \frac{\partial (e^{-\phi_2} f)}{\partial z} \end{aligned}$$

Lemma 4.2.1_h

$$e^{\psi} T^* f = -\delta f - \frac{\partial \psi}{\partial z} f$$

Proof.

$$\begin{aligned} e^{\psi} T^* f &= e^{\psi} g \\ e^{\psi} g &= -e^{\psi} e^{\phi_1} \frac{\partial (e^{-\phi_2} f)}{\partial z} \\ &= -e^{\psi + \phi_1} \frac{\partial (e^{-\phi + \psi} f)}{\partial z} \\ &= -e^{\psi + \phi_1} e^{\psi} \frac{\partial (e^{-\phi} f)}{\partial z} \\ &- \; e^{\psi + \phi_1} e^{-\phi} f e^{\psi} \frac{\partial \psi}{\partial z} \\ &= -e^{\phi} \frac{\partial (e^{-\phi} f)}{\partial z} - f \frac{\partial \psi}{\partial z} \\ &= -\delta f - f \frac{\partial \psi}{\partial z} \\ e^{\psi} T^* f &= -\delta f - f \frac{\partial \psi}{\partial z} \end{aligned}$$

□

We prove the large constant, small constant lemma:

Lemma 4.2.1$_i$. If a, b are complex numbers and $c > 0$, then

$$2|ab| \leq c|a|^2 + \frac{1}{c}|b|^2.$$

Moreover,

$$|a+b|^2 \leq (1+c)|a|^2 + (1+\frac{1}{c})|b|^2.$$

Proof. The first inequality follows from

$$\begin{aligned} c|a|^2 + \frac{1}{c}|b|^2 - 2|a||b| &= (\sqrt{c}|a| - \sqrt{1/c}|b|)^2 \\ &\geq 0 \end{aligned}$$

The second inequality follows then from

$$\begin{aligned} |a+b|^2 &= |a|^2 + a\bar{b} + \overline{a}b + |b|^2 \\ &\leq |a|^2 + 2|a||b| + |b|^2 \\ &\leq (1+c)|a|^2 + (1+\frac{1}{c})|b|^2 \end{aligned}$$

$\square$

We introduce the notation $2' = 1 + c, 2'' = 1 + \frac{1}{c}, c > 0$. Then statements involving $2', 2''$ below are valid for any choice of c.

Lemma 4.2.1$_j$.

$$\delta f \overline{\delta f} \leq 2' e^{2\psi} |T^* f|^2 + 2''|f|^2 |\frac{\partial \psi}{\partial z}|^2.$$

Proof.

$$\begin{aligned} \delta f \overline{\delta f} &= |\delta f|^2 \\ &= |e^{\psi} T^* f + f \frac{\partial \psi}{\partial z}|^2 \\ &\leq 2' |e^{\psi} T^* f|^2 + 2''|f \frac{\partial \psi}{\partial z}|^2 \\ &= 2' e^{2\psi} |T^* f|^2 + 2''|f|^2 |\frac{\partial \psi}{\partial z}|^2 \end{aligned}$$

$\square$

14. BASICS-HILBERT-HÖRMANDER'S $\frac{\partial u}{\partial \bar{z}}$ THEOREM 3/4

Lemma 4.2.1_k**.**

$$
\begin{aligned}
(\delta f \overline{\delta f} &- \frac{\partial f}{\partial \bar{z}} \overline{\frac{\partial f}{\partial \bar{z}}}) e^{-\phi} \\
&\leq 2' |T^* f|^2 e^{-\phi_1} + 2'' |f|^2 |\frac{\partial \psi}{\partial z}|^2 e^{-\phi}
\end{aligned}
$$

Proof.

$$
\begin{aligned}
(\delta f \overline{\delta f} &- \frac{\partial f}{\partial \bar{z}} \overline{\frac{\partial f}{\partial \bar{z}}}) e^{-\phi} \\
&\leq (2' e^{2\psi} |T^* f|^2 + 2'' |f|^2 |\frac{\partial \psi}{\partial z}|^2) e^{-\phi} - |\frac{\partial f}{\partial \bar{z}}|^2 e^{-\phi} \\
&\leq 2' e^{2\psi - \phi} |T^* f|^2 + 2'' |f|^2 |\frac{\partial \psi}{\partial z}|^2 e^{-\phi} \\
&= 2' |T^* f|^2 e^{-\phi_1} + 2'' |f|^2 |\frac{\partial \psi}{\partial z}|^2 e^{-\phi}
\end{aligned}
$$

□

Theorem 4.2.1_ℓ. Let Ω be a domain in $\mathbb{C}$. Let $0 \leq \eta_\nu \leq 1$ be a sequence of $\mathcal{C}^\infty$ functions with compact support such that for any given compact subset of Ω only finitely many are not identically 1. Let ψ be a $\mathcal{C}^\infty$ function with $|\frac{\partial \eta_\nu}{\partial \bar{z}}|^2 \leq e^\psi$ in Ω for all $\nu = 1, 2, \ldots$. Let $\phi \in \mathcal{C}^\infty(\Omega)$ and set $\phi_1 = \phi - 2\psi, \phi_2 = \phi - \psi$. Let T denote the $\frac{\partial}{\partial \bar{z}}$ operator from $L^2(\Omega, \phi_1)$ to $L^2(\Omega, \phi_2)$. Let $f \in D(\Omega)$. Then

$$
\int \left(f \bar{f} \frac{\partial \phi^2}{\partial z \partial \bar{z}} - 2'' |\frac{\partial \psi}{\partial z}|^2 |f|^2 \right) e^{-\phi} \leq 2' \|T^* f\|_1^2.
$$

Proof. Integrating both sides of Lemma 4.2.1_k, we get

$$
\begin{aligned}
\int (\delta f \overline{\delta f} &- \frac{\partial f}{\partial \bar{z}} \overline{\frac{\partial f}{\partial \bar{z}}}) e^{-\phi} \\
&\leq 2' \int |T^* f|^2 e^{-\phi_1} + 2'' \int |f|^2 |\frac{\partial \psi}{\partial z}|^2 e^{-\phi}
\end{aligned}
$$

We next apply Corollary 4.2.1_f. Then we get

$$
\int f \bar{f} \frac{\partial \phi^2}{\partial z \partial \bar{z}} e^{-\phi} \leq 2' \|T^* f\|_1^2 + 2'' \int |\frac{\partial \psi}{\partial z}|^2 |f|^2 e^{-\phi}
$$

□

In the following corollary, we use the case $2' = 2'' = 2$.

Corollary 4.2.1$_m$. Assume the conditions of Theorem 4.2.1$_\ell$. Suppose in addition the condition that

$$c(z) := \frac{\partial^2 \phi}{\partial z \partial \overline{z}} \geq 2(|\frac{\partial \psi}{\partial z}|^2 + e^{\psi}).$$

Then we have that for every $f \in D_{T^*} \cap D_S$ that

$$\|f\|^2_{\phi_2} \leq \|T^* f\|^2_{\phi_1}.$$

Proof. We get for any $f \in D(\Omega)$ that

$$\begin{aligned}
\|f\|^2_{\phi_2} &= \int |f|^2 e^{-\phi_2} \\
&= \int e^{\psi} |f|^2 e^{-\phi} \\
&\leq \frac{1}{2} \int f \overline{f} \frac{\partial \phi^2}{\partial z \partial \overline{z}} e^{-\phi} - \int (|\frac{\partial \psi}{\partial z}|^2) |f|^2 e^{-\phi} \\
&= \frac{1}{2} \left(\int f \overline{f} \frac{\partial \phi^2}{\partial z \partial \overline{z}} e^{-\phi} - 2 \int (|\frac{\partial \psi}{\partial z}|^2) |f|^2 e^{-\phi} \right) \\
&\leq \frac{1}{2} (2 \|T^* f\|^2) \\
&= \|T^* f\|^2
\end{aligned}$$

The corollary follows now for all $f \in D_{T^*}$ by density of $D(\Omega)$ in the graph norm. $\square$

15. BASICS-HILBERT-HÖRMANDER'S $\frac{\partial u}{\partial \bar{z}}$ THEOREM 4/4

Hormander Chapter 4.2.

Theorem 4.2.1$_n$. Assume the conditions in Theorem 4.2.1ℓ and Corollary 4.2.1$_m$. Then if $f \in L^2(\Omega, \phi_2)$, then there exists a $g \in L^2(\Omega, \phi_1)$ such that $\frac{\partial g}{\partial \bar{z}} = f$ and

$$\|g\|_{\phi_1} \leq \|f\|_{\phi_2}.$$

Proof. By Theorem 4.1.3$_j$, we have, since $\|f\|^2 \leq \|T^* f\|^2$ for all $f \in D_{T^*}$ that for any $y \in D_{T^*}$ and any $u \in G_2$ we get:

$$|y(u)| \leq \|T^* y\| \|u\|.$$

Hence we have the norm of y as a linear functional on G_2 that $\|y_{|N_S}\| \leq \|T^* y\|$ for all $y \in D_{T^*}$. We set $F = G_2$. This is certainly a closed subspace containing R_T. It follows then from Theorem 4.1.1$_b$ and Theorem 4.1.1' with constant $C = 1$ that $R_T = G_2$ and that for every $u \in G_2$ there is a $v \in D_T$ with $Tv = u$ and $\|v\| \leq \|u\|$. □

Lemma 4.2.1$_o$. Let $\{a_j, b_j\}_{j=2,3,\dots}$ be given strictly positive constants. Then there is a smooth, positive increasing, convex function $\lambda(x)$ for $x \geq 0$ such that $\lambda'(x) > a_j$ on $[j, j+1], j \geq 2$ and $\lambda(x) > b_j$ on $[j, j+1], j \geq 2$.

Proof. To find such a λ, pick a sufficiently large smooth function $\sigma(x) >> 1, x \geq 0$ and define $\nu(x) = \int_0^x \sigma(t)dt$, $\lambda(x) = \int_0^x \nu(t)dt$. □

Lemma 4.2.1$_p$. Suppose that $\Omega \subset \mathbb{C}$ is an open set and assume that ψ is as in Theorem 4.2.1$_\ell$. Also suppose that $f \in L^2_{loc}(\Omega)$. Then there exists a smooth strongly subharmonic function ϕ on Ω so that

$$\frac{\partial^2 \phi}{\partial z \partial \bar{z}} \geq 2(|\frac{\partial \psi}{\partial z}|^2 + e^{\psi})$$

and $f \in L^2(\Omega, \phi_2), \phi_2 = \phi - \psi$.

The proof uses Theorem 2.6.11. We will discuss the proof of this theorem for the case of open sets Ω in $\mathbb{C}$ in the exercises.

Proof. Using Theorem 2.6.11, we can find a smooth strongly subharmonic function ρ on Ω such that $\{\rho < c\} \subset\subset \Omega$ for any $c \in \mathbb{R}$. We can assume that $\min \rho = 2$ by adding a constant. Let $m(z)$ be the smooth function on Ω given by $m(z) = \frac{\partial^2 \rho}{\partial z \partial \bar{z}}$.

For $j = 2, 3, \dots$, let $L_j = \{z \in \Omega; j \leq \rho(z) \leq j+1\}$. Then each L_j is compact and $\Omega = \cup L_j$. Define

$$a_j = \sup_{L_j} \frac{2(|\frac{\partial \psi}{\partial z}|^2 + e^{\psi})}{m(z)}, j \geq 2.$$

Since L_j is compact, $f \in L^2(L_j)$. Pick $b_j > 0$ so that $\int_{L_j} |f|^2 e^{\psi - b_j} < \frac{1}{2^j}, j \geq 2$. Let λ be as in the Lemma 4.2.1$_o$. We define $\phi = \lambda \circ \rho$ on Ω. Then on L_j,

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial z \partial \overline{z}} &\geq \lambda'(\rho(z)) \frac{\partial^2 \rho}{\partial z \partial \overline{z}} \\
&\geq a_j \frac{\partial^2 \rho}{\partial z \partial \overline{z}} \\
&= \sup_{L_j} \frac{2(|\frac{\partial \psi}{\partial z}|^2 + e^{\psi})}{m(z)} \frac{\partial^2 \rho}{\partial z \partial \overline{z}} \\
&\geq \frac{2(|\frac{\partial \psi}{\partial z}|^2 + e^{\psi})}{m(z)} m(z) \\
&= 2(|\frac{\partial \psi}{\partial z}|^2 + e^{\psi})
\end{aligned}$$

Also

$$\begin{aligned}
\int_\Omega |f|^2 e^{\psi - \phi} &= \sum_{j=2}^{\infty} \int_{L_j} |f|^2 e^{\psi - \lambda(\rho)} \\
&\leq \sum_{j=2}^{\infty} \int_{L_j} |f|^2 e^{\psi - b_j} \\
&< \sum_{j=2}^{\infty} \frac{1}{2^j} \\
&< \infty
\end{aligned}$$

□

Corollary 4.2.1$_q$. If $f \in L^2_{loc}(\Omega)$, Ω open set in $\mathbb{C}$. Then there exists a $g \in L^2_{loc}(\Omega)$ so that $\frac{\partial g}{\partial \overline{z}} = f$ in Ω.

Proof. let $f \in L^2_{loc}(\Omega)$. We pick ϕ as in Lemma 4.2.1$_p$. Then $f \in L^2(\Omega, \phi_2)$. Using Theorem 4.2.1$_n$, we find $g \in L^2(\Omega, \phi_1)$ with $\frac{\partial g}{\partial \overline{z}} = f$. This $g \in L^2_{loc}(\Omega)$. □

In applications of these results it is useful to know that functions which satisfies Cauchy-Riemann in the sense of distributions are in fact holomorphic:

Lemma 4.2.1$_r$: Suppose that $f \in L^2(\Omega)$ and $\frac{\partial f}{\partial \overline{z}} = 0$ (in the sense of distributions. Then there is a holomorphic function u on Ω so that $u = f$ a.e.

Proof. Suppose that $f \in L^2(B(0, \delta))$, the ball of radius δ in $\mathbb{C}^n$ centered at 0. Assume that $\frac{\partial f}{\partial \overline{z}} = 0$ in the sense of distributions. We apply the smoothing theorem. Then if $0 < \epsilon < \frac{\delta}{2}$, $f * \chi_\epsilon$ is $\mathcal{C}^\infty$ in $B(0, \delta/2)$ and $\|f * \chi_\epsilon - f\|_{L^2(B(0,\delta/2))} \to 0$ when $\epsilon \to 0$.

By Lemma $4.1.4_b$, $\frac{\partial(f*\chi_\epsilon)}{\partial \bar{z}} = (\frac{\partial f}{\partial \bar{z}}) * \chi_\epsilon = 0 * \chi_\epsilon = 0$. Hence each $f * \chi_\epsilon$ is holomorphic. We can choose $\epsilon_j \searrow 0$ so that $\|f * \chi_{\epsilon_{j+1}} - f * \chi_{\epsilon_j}\|_{L^2(B(0,\delta/2))} < \frac{1}{2^j}$.

Let $u_j := f * \chi_{\epsilon_j}$. We get

$$\|u_{j+1} - u_j\|_{L^1(B(0,\delta/2))} \leq C\|u_{j+1} - u_j\|_{L^2(B(0,\delta/2))} \leq \frac{C}{2^j}.$$

By Theorem 2.2.3 we then get pointwise estimates $|u_{j+1}(z) - u_j(z)| \leq \frac{C'}{2^j}$ on $B(0, \delta/4)$. Hence u_j converges uniformly to a function u on $B(0, \delta/4)$. By Corollary 2.2.5, u is holomorphic. But necessarily $u = f$ a.e.

□

16. Problem Session 1

Recall that a function $f(z) = u(x,y) + iv(x,y)$ is analytic if $f \in \mathcal{C}^1$ and $\frac{\partial f}{\partial \overline{z}} = 0$.
Problem 1.1 Show that the equation $\frac{\partial f}{\partial \overline{z}} = 0$ is equivalent to the classical Cauchy Riemann equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

Problem 1.2 Show that the function defined by $f(z) = e^{-(z^{-4})}$ for $z \neq 0$ and $f(0) = 0$ satisfies the Cauchy Riemann equations at every point. Is $f \in \mathcal{C}^1$?

Probem 1.3 Show that

$$\overline{\left(\frac{\partial f}{\partial z}\right)} = \frac{\partial \overline{f}}{\partial \overline{z}}$$

and

$$\overline{\left(\frac{\partial f}{\partial \overline{z}}\right)} = \frac{\partial \overline{f}}{\partial z}$$

Problem 1.4 Let Ω be an open set in $\mathbb{C}$. For K compact in Ω show that $\hat{K}_\Omega$ is compact and that the distance of $\hat{K}_\Omega$ to the boundary of Ω is the same as the distance of K to the boundary.

Problem 1.5 Let $K \subset \Omega \subset \mathbb{C}$ be a compact subset. Let U be a connected component of $\mathbb{C} \setminus K$. Show that $U \subset \hat{K}_\Omega$ if and only if U is a bounded set and $U \subset \Omega$.

17. Problem Session 2

Problem 2.1 Discuss the proofs of Theorem 1.2.1, Theorem 1.2.2, Corollary 1.2.3 and Theorem 1.2.4 in the book of Hörmander.

18. Problem Session 3

Problem 3.1 Show that the function $\sup_{0<\epsilon<1} \epsilon \log|z|$ fails to be subharmonic.

Problem 3.2 Suppose that u is a $\mathcal{C}^2$ subharmonic function on $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be an analytic function defined on an open set Ω in $\mathbb{C}$. Show that

$u \circ f$ is subharmonic on Ω.

Problem 3.3 Let u a subharmonic function in $\{|z| < 2\}$. Suppose that $u(z) = 0$ for all $z, 1 < |z| < 2$. Show that $u \equiv 0$.

19. Problem Session 4

Problem 4.1 Discuss the proof of Theorem 1.6.2

Problem 4.2 Discuss the proof of Theorem 1.6.3

20. Problem Session 5

Problem 5.1 Discuss the proof of Corollary 1.6.8

Problem 5.2 Prove theorem 2.6.11 in Hörmander: If Ω is an open subset of $\mathbb{C}$, then there exists a $\mathcal{C}^\infty$ strongly subharmonic function $\rho : \Omega \to \mathbb{R}$ so that for any $c \in \mathbb{R}$, the set $\{\rho < c\} \subset\subset \Omega$.

21. Problem Session 6

Problem 6.1 Let T be the operator $\frac{\partial}{\partial \overline{z}} : L^2(\Omega) \to L^2(\Omega)$. Show that T is a closed and densely defined operator.

Problem 6.2 Work out T^* when T is a linear operator from $\mathbb{C}^2$ to $\mathbb{C}^3$.

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