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# Motion Planning with Model Predictive Control


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主要研究方向为 Motion Planning 以及
Nonlinear Model Predictive Control

# 目录

# Introduction: Model Predictive Control

- Model
  - System model
  - Problem model
- Prediction
  - State space
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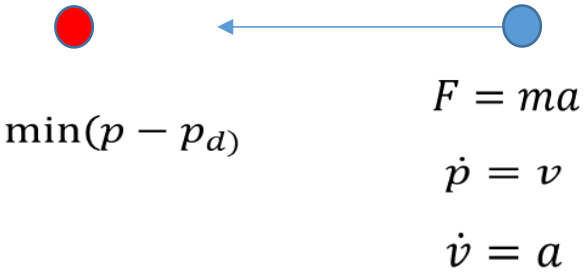
  - Parameter space
- Control
  - The process of choosing the best policy



# Introduction: Model

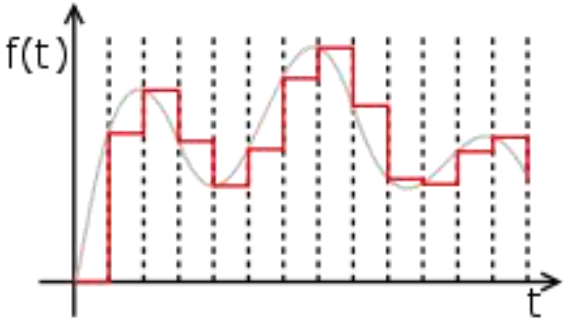
$$\min C_F\left(x(t_f)\right) + \int_{t=t_0}^{t_f} C_R(x,u)\,dt$$

$$\dot{x} = f(x,u)$$
$$g(x,u) < 0$$
$$h(x,u) = 0$$
$$x \notin Obstacle$$

# Introduction: Parameter space

$$\min_{u} C_F\left(x(t_f)\right) + \int_{t=t_0}^{t_f} C_R(x,u)\,dt$$

- Zero order hold (direct discretization)
- Polynomial
- B-spline
- Numerical mapping
  - Jerk limited trajectory
  - Neural network based method




$$u(t) = at^3 + bt^2 + ct + d$$

# Introduction: Optimization

Searching:

- Graph search
- Random sampling based search

Convex optimization:

- Quadratic programming

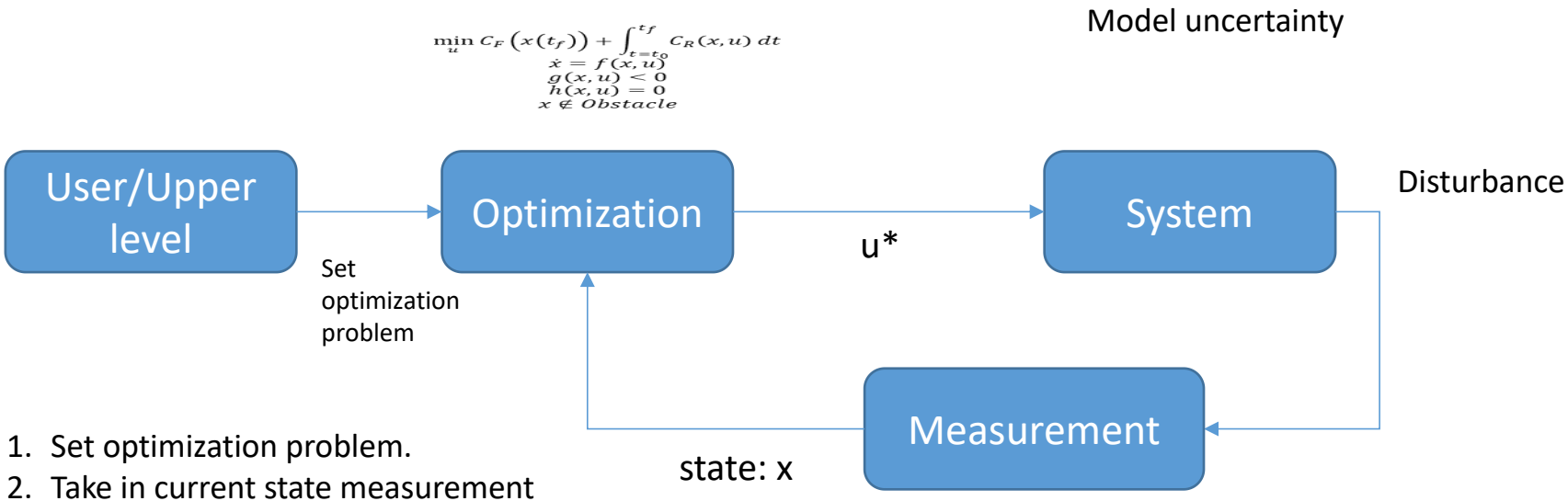
Nonconvex optimization:

- Sequential quadratic programming
- Particle swarm optimization

*Nonconvex, nonlinear, discontinuous*

# Introduction: Control

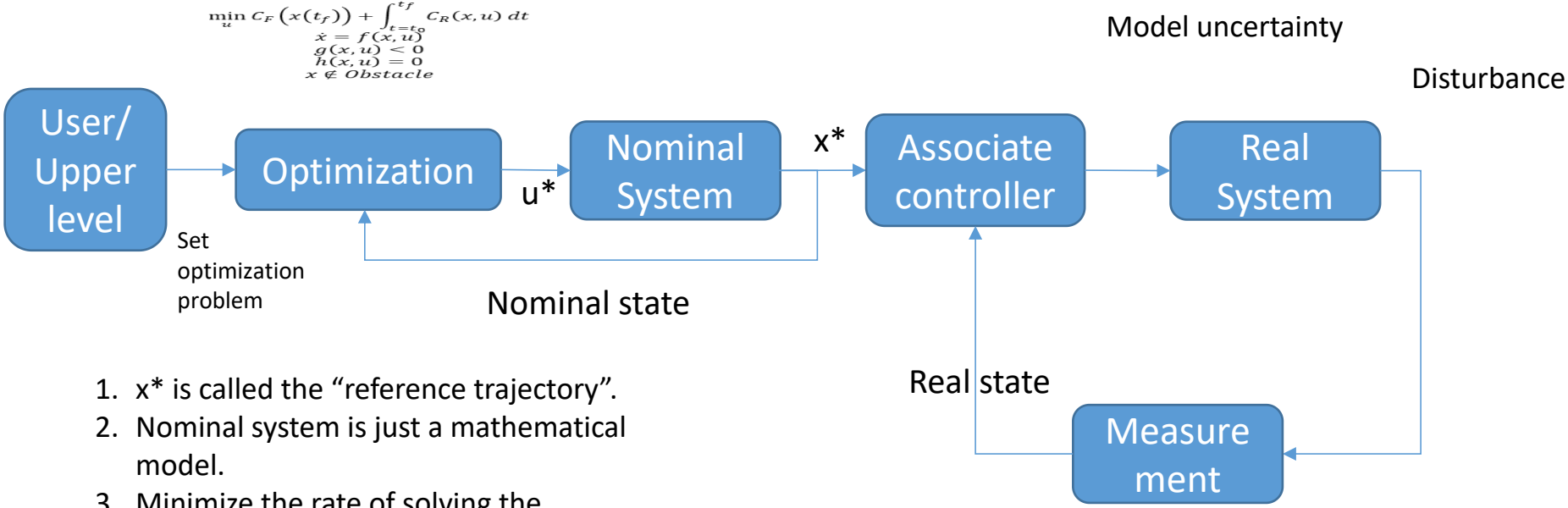
$$\min_{u} C_F\left(x(t_f)\right) + \int_{t=t_o}^{tf} C_R(x,u)\,dt$$
$$\dot{x} = f(x,u)$$
$$g(x,u) \leq 0$$
$$h(x,u) = 0$$
$$x \notin Obstacle$$




1. Set optimization problem.
2. Take in current state measurement and finalize the optimization problem.
3. Solve the optimization and get u*.
4. Apply u* for a short period of time.

# Introduction: Tube based MPC

$$\min_{u} C_F\left(x(t_f)\right) + \int_{t=t_0}^{t_f} C_R(x,u)\,dt$$
$$\dot{x} = f(x,u)$$
$$g(x,u) \leq 0$$
$$h(x,u) = 0$$
$$x \notin Obstacle$$




1. x* is called the “reference trajectory”.
2. Nominal system is just a mathematical model.
3. Minimize the rate of solving the optimization problem.
4. Leave the robust tracking problem to the associate controller.

*“Tube-Based MPC: a Contraction Theory Approach”*

# Introduction: Convenient sources

- Matlab MPC toolbox: https://www.mathworks.com/products/mpc.html
- µAO-MPC: http://ifatwww.et.uni-magdeburg.de/syst/muAO-MPC/
- Acado toolkit: https://acado.github.io/
- YANE: http://www.nonlinearmpc.com/
- Multi-Parametric Toolbox 3: https://www.mpt3.org/

# Linear MPC: basics

Continuous model

$$\dot{p} = v$$
$$\dot{v} = a$$
$$\dot{a} = j$$

$$p_i = p(i \cdot dt)$$
$$v_i = v(i \cdot dt)$$
$$a_i = a(i \cdot dt)$$

Time discretization

Discrete model

$$p_{i+1} = p_i + v_i dt + \frac{1}{2} a_i dt^2 + \frac{1}{6} j_i dt^3$$
$$v_{i+1} = v_i + a_i dt + \frac{1}{2} j_i dt^2$$
$$a_{i+1} = a_i + j_i dt$$
$$i = [0,2, \dots, 19]$$
$$dt = 0.2$$

# Linear MPC: basics

Discrete model

$$p_{i+1} = p_i + v_i dt + \frac{1}{2} a_i dt^2 + \frac{1}{6} j_i dt^3$$

$$v_{i+1} = v_i + a_i dt + \frac{1}{2} j_i dt^2$$

$$a_{i+1} = a_i + j_i dt$$

$$i = [0,2, \dots, 19]$$

$$dt = 0.2$$

Linear Matrix form

Prediction model

$$P = T_p J + B_p$$

$$V = T_v J + B_v$$

$$A = T_a J + B_a$$

$$P = [p_1, p_2, p_3, \dots, p_{20}]^T$$

$$V = [v_1, v_2, v_3, \dots, v_{20}]^T$$

$$A = [a_1, a_2, a_3, \dots, a_{20}]^T$$

$$J = [j_0, j_1, j_2, \dots, j_{19}]^T$$

# Linear MPC: basics

Prediction model

$$P = T_p J + B_p$$

$$V = T_v J + B_v$$

$$A = T_a J + B_a$$

```
function [Tp, Tv, Ta, Bp, Bv, Ba] = getPredictionMatrix(K,dt,p_0,v_0,a_0)
Ta=zeros(K);
Tv=zeros(K);
Tp=zeros(K);

for i = 1:K
    Ta(i,1:i) = ones(1,i)*dt;
end

for i = 1:K
    for j = 1:i
        Tv(i,j) = (i-j+0.5)*dt^2;
    end
end

for i = 1:K
    for j = 1:i
        Tp(i,j) = ((i-j+1)*(i-j)/2+1/6)*dt^3;
    end
end

Ba = ones(K,1)*a_0;
Bv = ones(K,1)*v_0;
Bp = ones(K,1)*p_0;

for i=1:K
    Bv(i) = Bv(i) + i*dt*a_0;
    Bp(i) = Bp(i) + i*dt*v_0 + i^2/2*a_0*dt^2;
end
```

# Linear MPC: basics

Problem model:

Target 1: Zero position, zero velocity and zero acceleration.
Target 2: Smooth trajectory.

Optimization target 1: $\min_J w_1 P^T P + w_2 V^\top V + w_3 A^\top A$

Optimization target 2: $\min_J w_4 J^\top J$

# Linear MPC: basics

Optimization:

The overall optimization target:

$$\min_J w_1 P^\top P + w_2 V^\top V + w_3 A^\top A + w_4 J^\top J$$

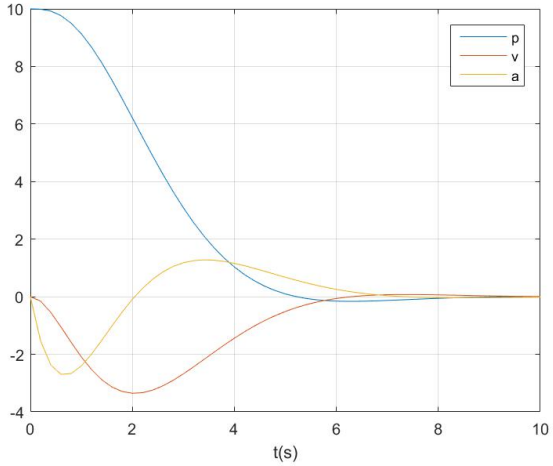
Combine with

$$P = T_p J + B_p \qquad V = T_v J + B_v \qquad A = T_a J + B_a$$

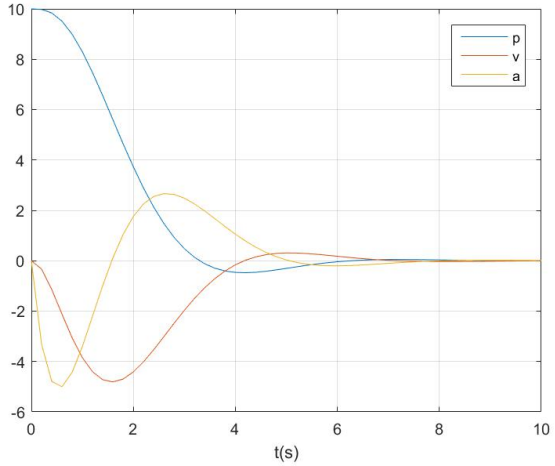
We have

$$\min_J J^\top \left(w_1 T_p^\top T_p + w_2 T_v^\top T_v + w_3 T_a^\top T_a + w_4 I\right) J +$$
$$2\left(w_1 B_p^\top T_p + w_2 B_v^\top T_v + w_3 B_a^\top T_a\right) J + \text{constant}$$

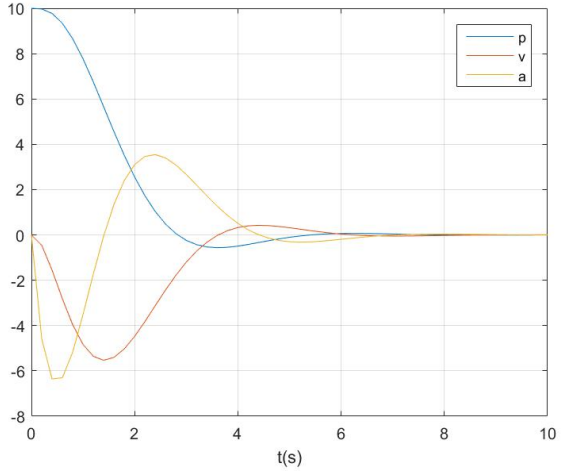
```
p_0 = 10;
v_0 = 0;
a_0 = 0;
K=20;
dt=0.2;
log=[0 p_0 v_0 a_0];
w1 = 1;
w2 = 1;
w3 = 1;
w4 = 1;
for t=0.2:0.2:10
    %% Construct the prediction matrix
    [Tp, Tv, Ta, Bp, Bv, Ba] = getPredictionMatrix(K,dt,p_0,v_0,a_0);

    %% Construct the optimization problem
    H = w4*eye(K)+w1*(Tp'*Tp)+w2*(Tv'*Tv)+w3*(Ta'*Ta);
    F = w1*Bp'*Tp+w2*Bv'*Tv+w3*Ba'*Ta;


    %% Solve the optimization problem
    J = quadprog(H,F,[],[]);

    %% Apply the control
    j = J(1);
    p_0 = p_0 + v_0*dt + 0.5*a_0*dt^2 + 1/6*j*dt^3;
    v_0 = v_0 + a_0*dt + 0.5*j*dt^2;
    a_0 = a_0 + j*dt;

    %% Log the states
    log = [log; t p_0 v_0 a_0];
end
```



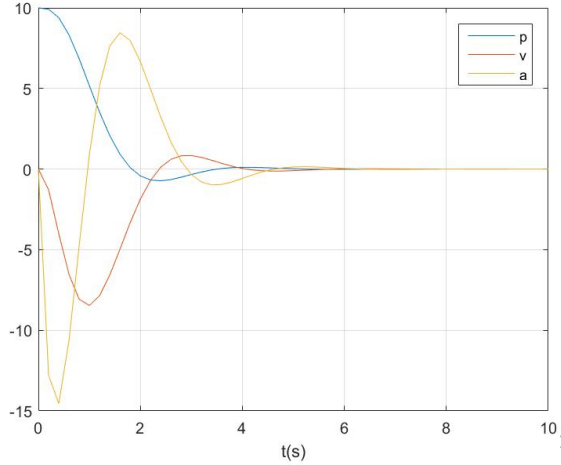

w1=1
w2=1
w3=1
w4=1




w1=5
w2=1
w3=1
w4=1




w1=10
w2=1
w3=1
w4=1




w1=100
w2=1
w3=1
w4=1

# Linear MPC: basics

Optimization problem

$$\min_J J^\top \left(w_1 T_p^\top T_p + w_2 T_v^\top T_v + w_3 T_a^\top T_a + w_4 I\right) J + 2\left(w_1 B_p^\top T_p + w_2 B_v^\top T_v + w_3 B_a^\top T_a\right) J + \text{constant}$$

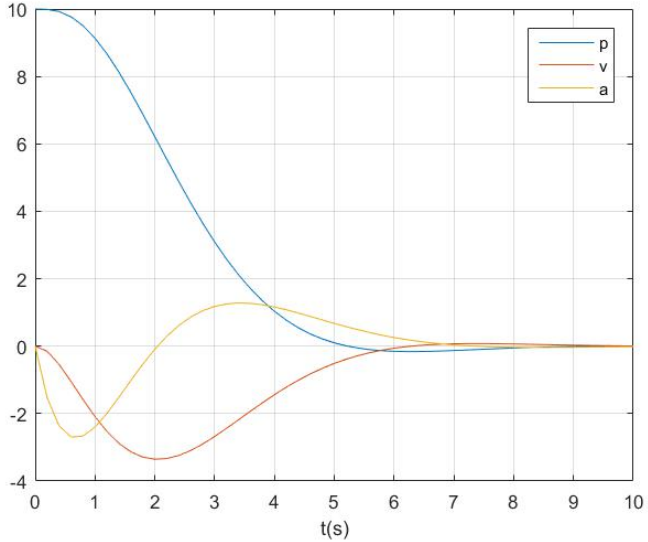
Let $H = \left(w_1 T_p^\top T_p + w_2 T_v^\top T_v + w_3 T_a^\top T_a + w_4 I\right)$

$$F = \left(w_1 B_p^\top T_p + w_2 B_v^\top T_v + w_3 B_a^\top T_a\right)$$

$$\min_J J^\top H J + 2FJ$$

$$\Downarrow$$

$$J = -H^{-1} F$$




```
%% Solve the optimization problem
J = -H\F';
```

# Linear MPC: hard constraints

Constraints

$$-1 \leq v_i \leq 1, \forall i \in \{1, 2, 3 \cdots, 20\}$$

$$-1 \leq a_i \leq 1, \forall i \in \{1, 2, 3, \ldots, 20\}$$

Matrix form:

$$-1_{20\times1} \leq V \leq 1_{20\times1}$$

$$-1_{20\times1} \leq A \leq 1_{20\times1}$$

$$V = T_v J + B_v$$

$$A = T_a J + B_a$$

Final form:

$$-1_{20\times1} \leq T_v J + B_v \leq 1_{20\times1}$$

$$-1_{20\times1} - B_v \leq T_v J \leq 1_{20\times1} - B_v$$

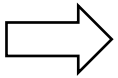
$$-1_{20\times1} - B_a \leq T_a J \leq 1_{20\times1} - B_a$$

Less equal form:

$$T_v J \leq 1_{20\times1} - B_v$$
$$-T_v J \leq 1_{20\times1} + B_v$$
$$T_a J \leq 1_{20\times1} - B_a$$
$$-T_a J \leqslant 1_{20\times1} + B_a$$

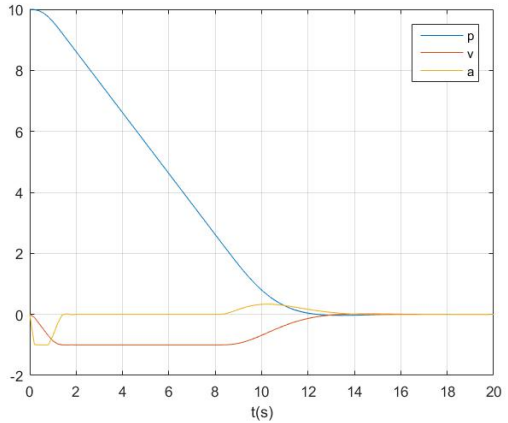
# Linear MPC: hard constraints

$$
\begin{aligned}
T_v J &\leq 1_{20\times 1} - B_v \\
-T_v J &\leq 1_{20\times 1} + B_v \\
T_a J &\leq 1_{20\times 1} - B_a \\
-T_a J &\leqslant 1_{20\times 1} + B_a
\end{aligned}
$$

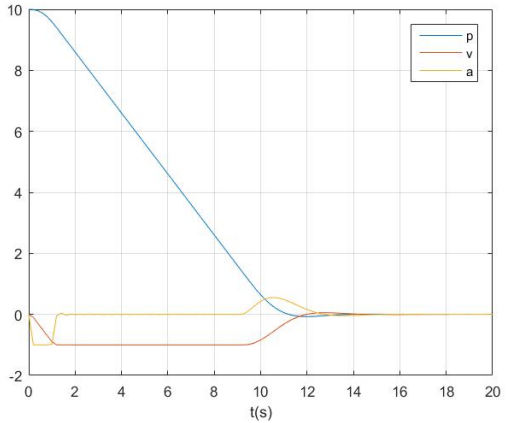


```
A = [Tv;-Tv;Ta;-Ta];
b = [ones(20,1)-Bv;ones(20,1)+Bv;ones(20,1)-Ba;ones(20,1)+Ba];

J = quadprog(H,F,A,b);
```

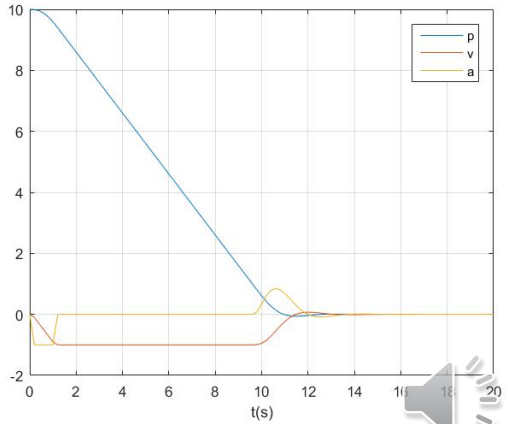
w1=1, w2=1, w3=1, w4=1




w1=10, w2=1, w3=1, w4=1




w1=100, w2=1, w3=1, w4=1



# Linear MPC: soft constraints

- What if the velocity and acceleration constraints are inevitably violated?

$$v_0 = 3m/s$$

$$-1 \leq v_i \leq 1, \forall i \in \{1, 2, 3 \cdots, 20\}$$

$$-1 \leq a_i \leq 1, \forall i \in \{1, 2, 3, \ldots, 20\}$$

The solver will report no solution!!

The controller won't know what to do.

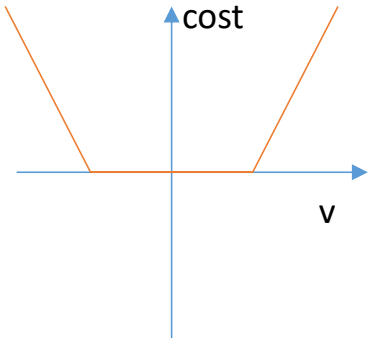
# Linear MPC: soft constraints

Add a penalty function to the optimization target

$$\min_J w_1 P^\top P + w_2 V^\top V + w_3 A^\top A + w_4 J^\top J + S(V)$$

$$S(V) = \sum_{i=1}^{20} s(v_i)$$

$$s(v_i) = \begin{cases} 0 & \text{if} \quad \|v_i\| \leqslant 1 \\ M \cdot (\|v_i\| - 1) & \text{else} \end{cases}$$




*M is just a large positive number*

# Linear MPC: soft constraints

Original constraints

$$T_v J \leq 1_{20\times 1} - B_v$$
$$-T_v J \leq 1_{20\times 1} + B_v$$
$$T_a J \leq 1_{20\times 1} - B_a$$
$$-T_a J \leqslant 1_{20\times 1} + B_a$$

New optimization target

$$\min_{J,L} w_1 P^T P + w_2 V^\top V + w_3 A^\top A + w_4 J^\top J + w_5 L^\top L$$

Set new programming variable as $J = \begin{bmatrix} J \\ L \end{bmatrix}$

Then all the H, F, A, b matrix shall be adjusted accordingly

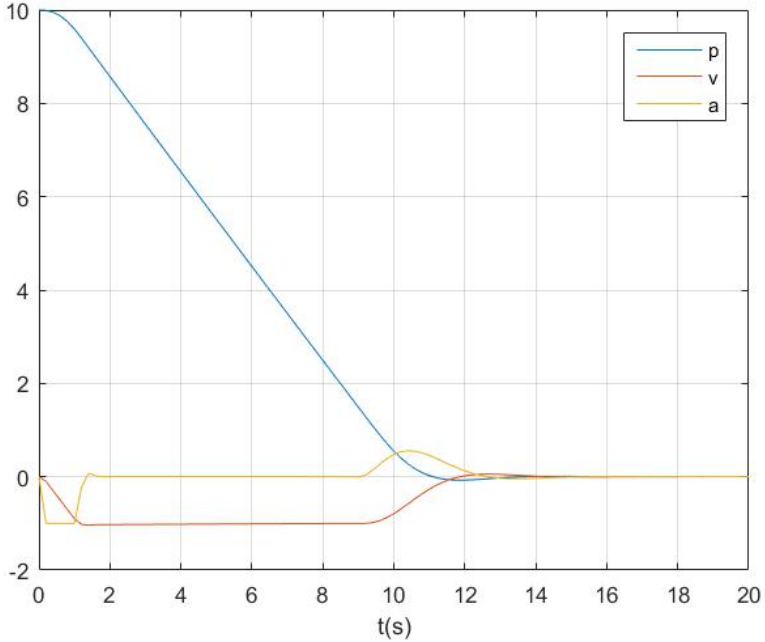
Add the slack variable L

$$-T_v J \leqslant 1_{20\times 1} + B_v + L$$
$$-L \leqslant 0$$
$$L = [l, l_2, l_3, \cdots, l_{20}]^\top$$

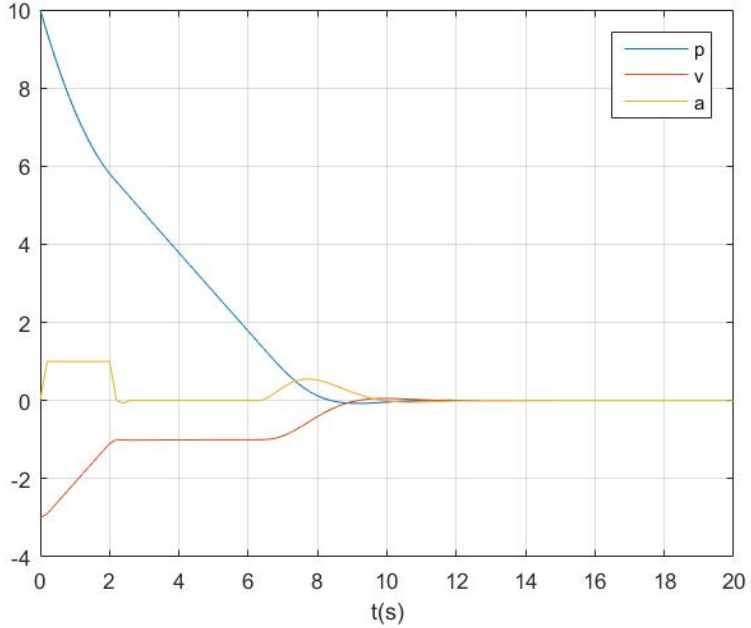
```
%% Construct the prediction matrix
[Tp, Tv, Ta, Bp, Bv, Ba] = getPredictionMatrix(K,dt,p_0,v_0,a_0);

%% Construct the optimization problem
H = blkdiag(w4*eye(K)+w1*(Tp'*Tp)+w2*(Tv'*Tv)+w3*(Ta'*Ta),w5*eye(K));
F = [w1*Bp'*Tp+w2*Bv'*Tv+w3*Ba'*Ta zeros(1,K)];

A = [Tv zeros(K);-Tv -eye(K);Ta zeros(K); -Ta zeros(K); zeros(size(Ta)) -eye(K)];
b = [ones(20,1)-Bv;ones(20,1)+Bv;ones(20,1)-Ba;ones(20,1)+Ba; zeros(K,1)];
%% Solve the optimization problem
J = quadprog(H,F,A,b);
```

# Linear MPC: soft constraints

w1=10, w2=1, w3=1, w4=1, w5 = 1e4




w1=10, w2=1, w3=1, w4=1, w5 = 1e4



# Linear MPC: soft constraints

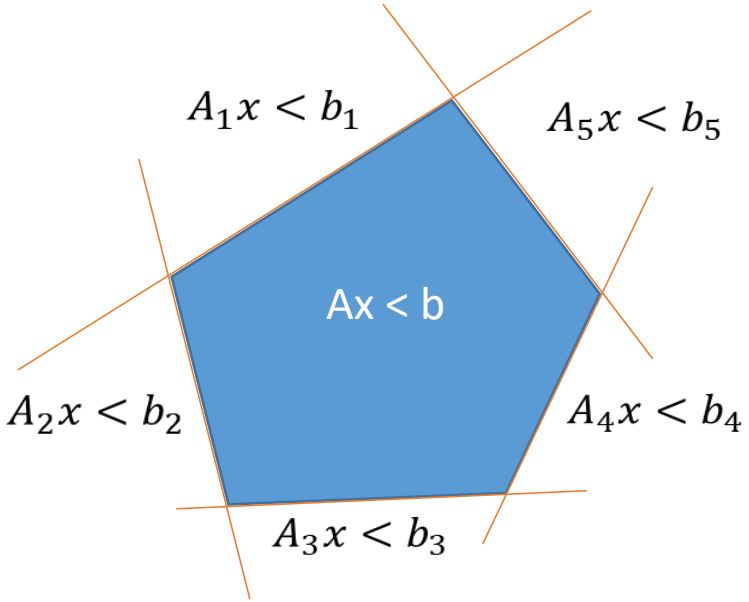
- State constraints -> use soft constraints
  - They are effected by measurement noise and disturbances

- Input constraints -> use hard constraints
  - They can be varied arbitrarily, and their violation might harm the physical system.

# Linear MPC: limitation

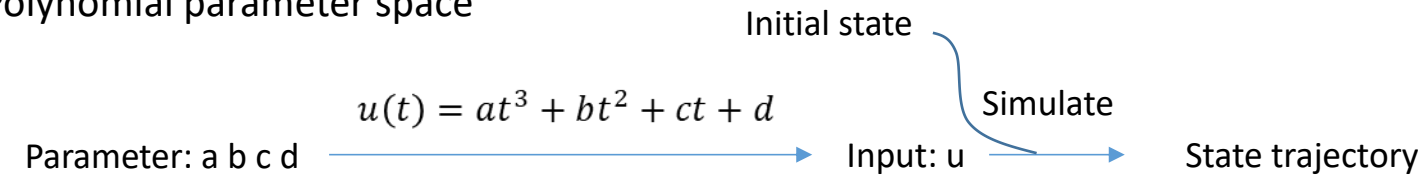
- It usually requires a linear model, or the model can be reasonably linearized (adaptive MPC).

- Obstacle constraints are usually non-convex by nature.

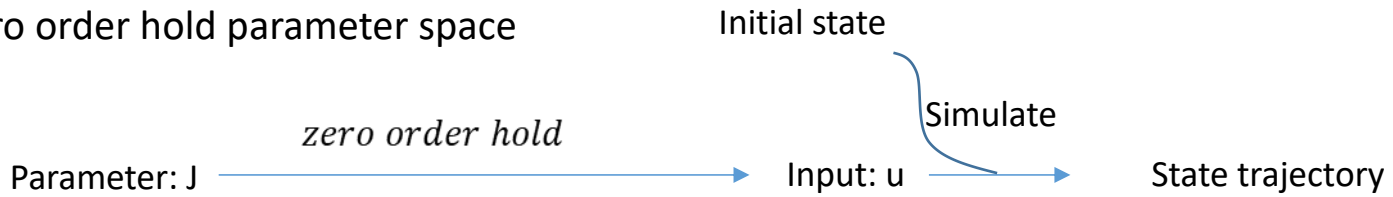


# Parameter space: Boundary constrained motion primitives (BSCP)

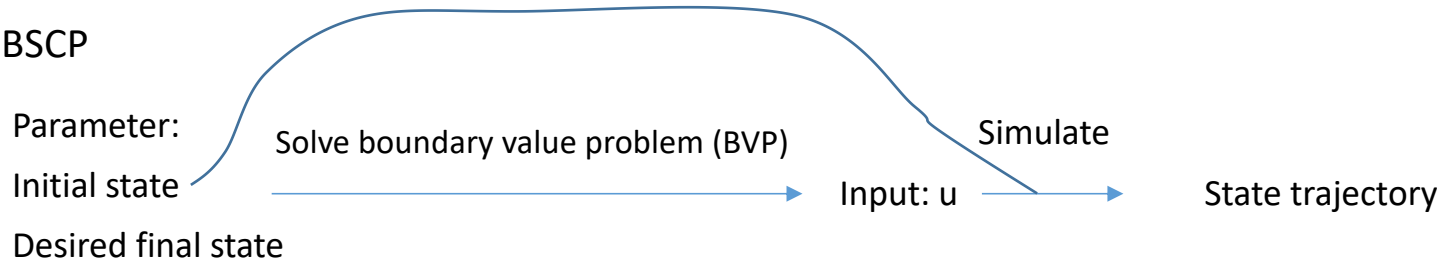
- Polynomial parameter space




- Zero order hold parameter space

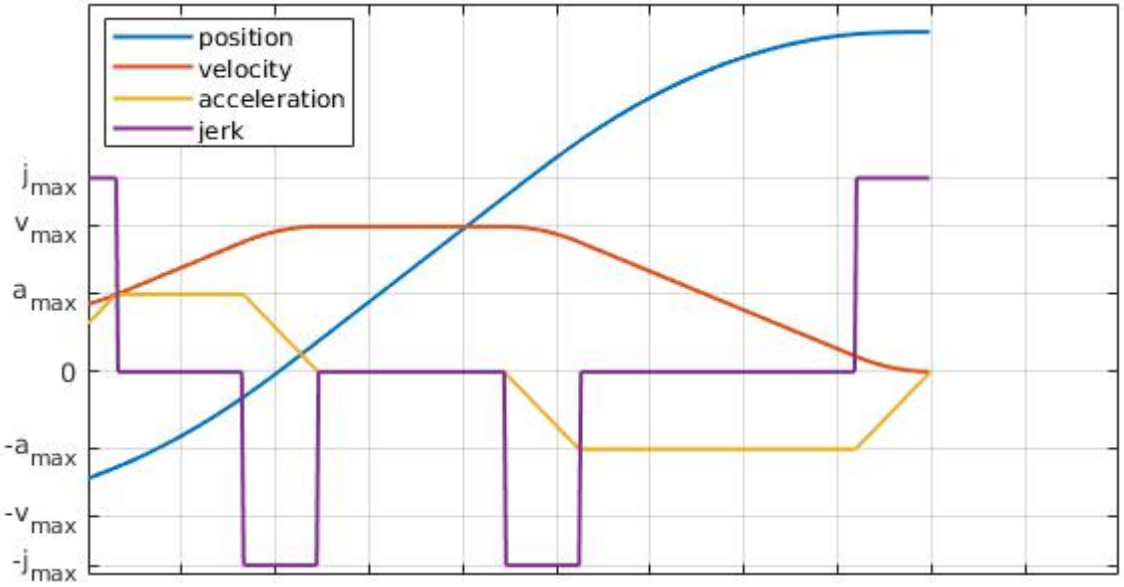



- BSCP



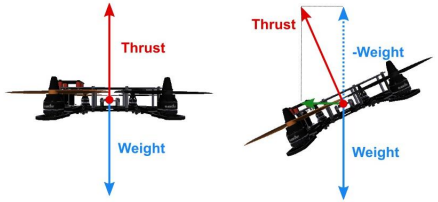
# Parameter space: Jerk limited trajectory

- Not only limited jerk, but also limited acceleration and velocity.
- We will refer it as JLT later.




The jerk limited trajectory for a single axis

# Parameter space: Jerk limited trajectory




**Inner loop constraints:**

The limited *body rate* $\omega_{max}$ and *total thrust* $f_{max}$

Sufficient condition

**Outer loop constraints:**

The limited acceleration and jerk

$$\sqrt{\dddot{x}^2 + \dddot{y}^2 + \dddot{z}^2} \leq (\ddot{z}_{\min} + g)\omega_{\max}$$

$$\sqrt{\ddot{x}^2 + \ddot{y}^2 + (\ddot{z} + g)^2} \leq \frac{f_{\max}}{m_v}$$

$$\ddot{z} \geq \ddot{z}_{\min} \geq \frac{f_{\min}}{m_v} - g$$

Decoupling

Sufficient condition

**Single axis constraints:**

The limited acceleration and jerk

$$-a_{\max} \leqslant a(t) \leqslant a_{\max},$$

$$-u_{j\max} \leqslant u_j(t) \leqslant u_{j\max},$$

Limit the velocity for safety

**Final constraints:**

The limited acceleration and jerk

$$-v_{\max} \leqslant v(t) \leqslant v_{\max},$$

$$-a_{\max} \leqslant a(t) \leqslant a_{\max},$$

$$-u_{j\max} \leqslant u_j(t) \leqslant u_{j\max},$$

# Parameter space: Jerk limited trajectory

The jerk limited trajectory problem considers a system characterized by

$$\begin{pmatrix} \dot{p} \\ \dot{v} \\ \dot{a} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} p \\ v \\ a \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_{\mathrm{j}}$$

with $-v_{\max} \leqslant v(t) \leqslant v_{\max}, \quad -a_{\max} \leqslant a(t) \leqslant a_{\max}, \quad -u_{\mathrm{j}\max} \leqslant u_{\mathrm{j}}(t) \leqslant u_{\mathrm{j}\max}$

The traditional Time Optimal Control (TOC) or bang-bang control problem considers

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ a \end{bmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} \tag{4.6}$$

Note that $v = \dot{y}$ is the velocity of the system. Let the control input be constrained as follows:

$$| u(t) | \leq u_{\max}$$

# Parameter space: Jerk limited trajectory

**Problem:**

From arbitrary state to a desired velocity.
Subject to:

$$v(0) = v_0, \quad v(T) = v_{\text{ref}}$$

$$a(0) = a_0, \quad a(T) = 0$$

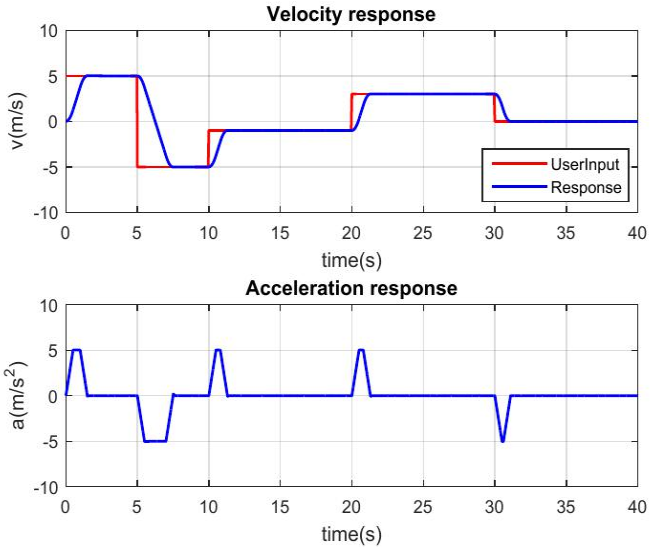
$$\dot{v}(t) = a(t)$$

$$\dot{a}(t) = u_{\text{j}}(t)$$

$$-a_{\max} \leqslant a(t) \leqslant a_{\max}, \qquad \forall t \in [0, T]$$

$$-u_{\text{j}\max} \leqslant u_{\text{j}}(t) \leqslant u_{\text{j}\max}, \quad \forall t \in [0, T]$$

**An Example:**




It is shown the trajectory consists of at most three segments with u = ± u_jmax and 0.

*On-Line Trajectory Generation in Robotic Systems*

# Parameter space: Jerk limited trajectory

**Intuition on the solution:** Covered area (acc) = Δ Velocity

Smaller area

Larger area

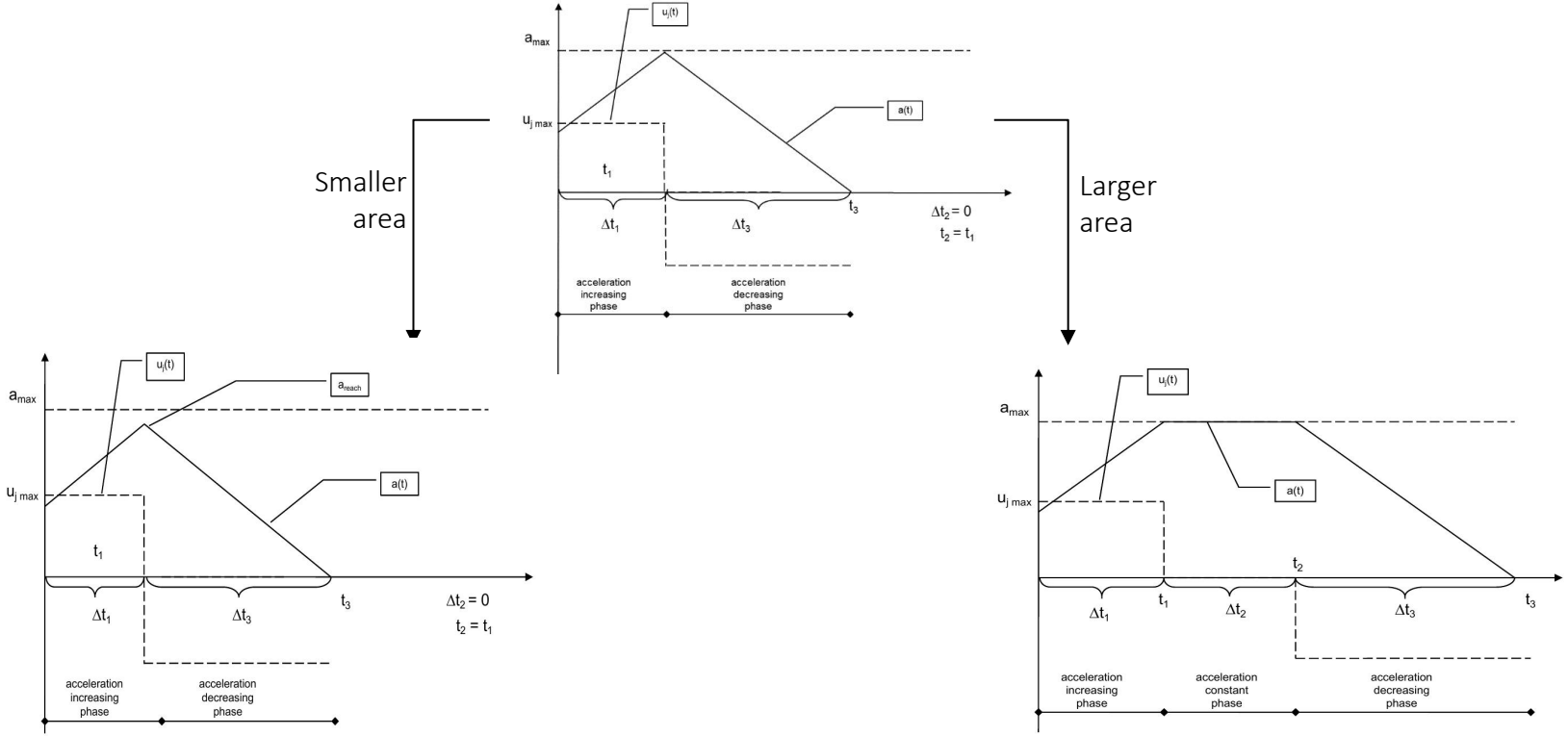



Figure 4.13: Wedge acceleration profile

Figure 4.12: Trapezoidal acceleration profile

# Parameter space: Jerk limited trajectory

**The Problem:**

From arbitrary state to a desired point subject to:

$$p(0) = p_0, \quad p(T) = p_{\text{ref}}$$
$$v(0) = v_0, \quad v(T) = 0$$
$$a(0) = a_0, \quad a(T) = 0$$
$$\dot{p}(t) = v(t)$$
$$\dot{v}(t) = a(t)$$
$$\dot{a}(t) = u_j(t)$$
$$-v_{\max} \leqslant v(t) \leqslant v_{\max}, \quad \forall t \in [0, T]$$
$$-a_{\max} \leqslant a(t) \leqslant a_{\max}, \quad \forall t \in [0, T]$$
$$-u_{\text{j}\max} \leqslant u_{\text{j}}(t) \leqslant u_{\text{j}\max}, \quad \forall t \in [0, T]$$

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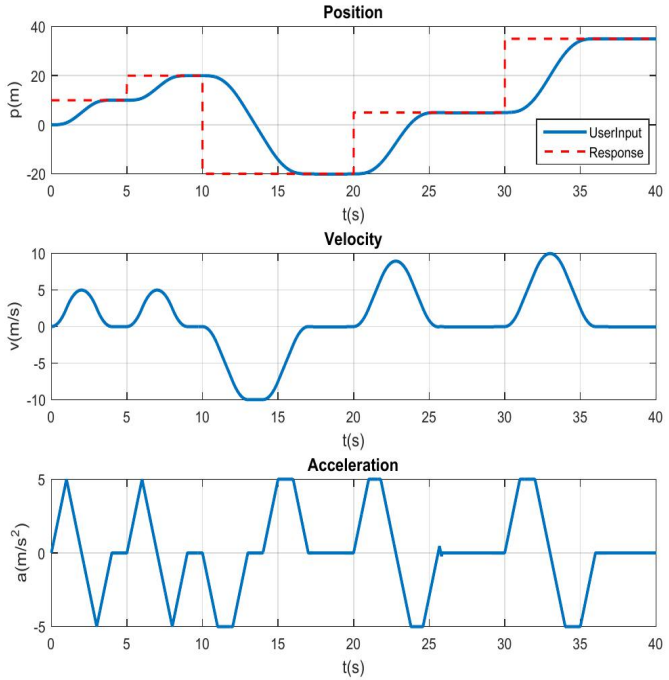
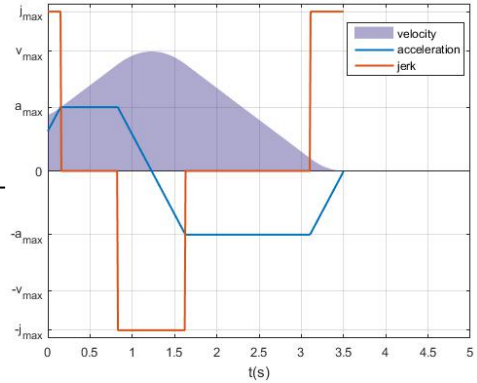


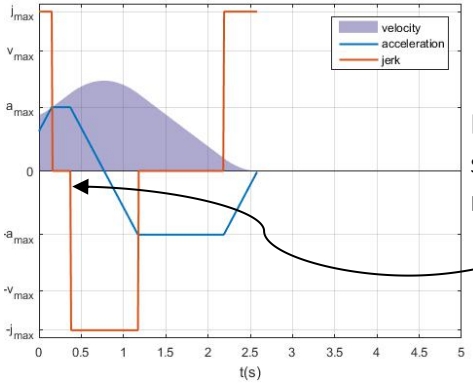
# Parameter space: Jerk limited trajectory

Intuition on the solution:

Covered area (vel) = Δ Position

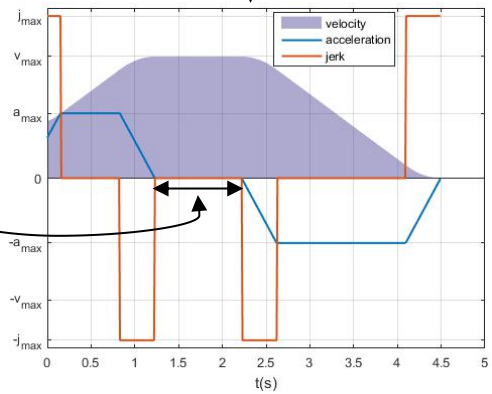



Need smaller area




Determine switching moment

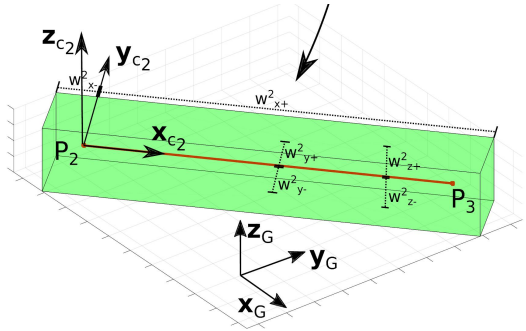
Need larger area

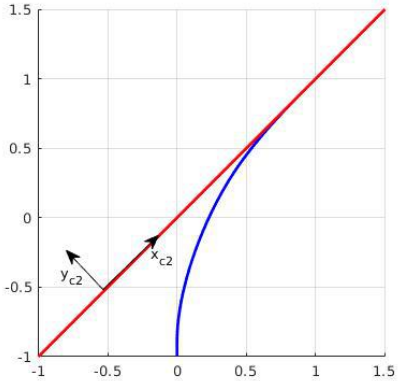



Determine cruise time

# Parameter space: Jerk limited trajectory

For the trajectory to "converge" to a given line segment.




On the y and z axes:

- Set the position set-point to zero.

On the x axis:

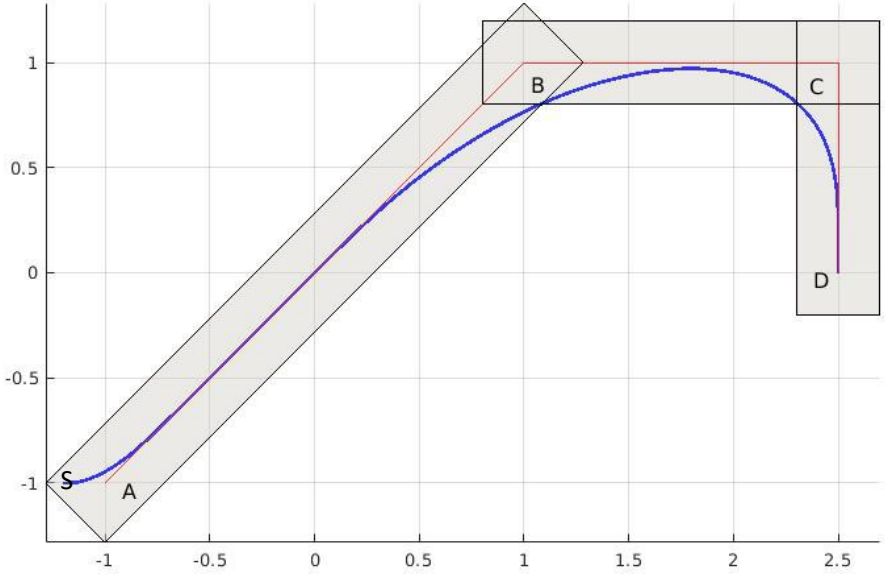
- Set the position set-point to $|P_3 - P_2|$

Result:

- The trajectory converge to the desired line-segment path.

# Parameter space: Jerk limited trajectory

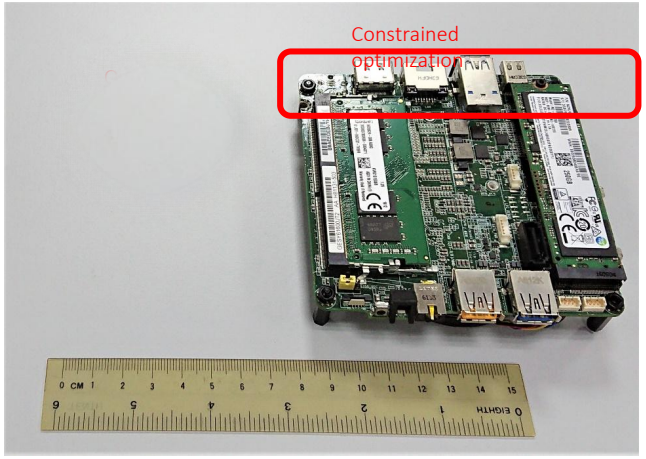
Online & incremental,
sample & throw

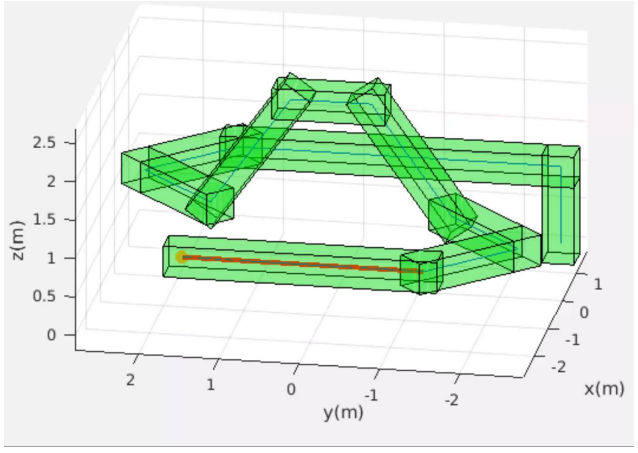


# Parameter space: Jerk limited trajectory

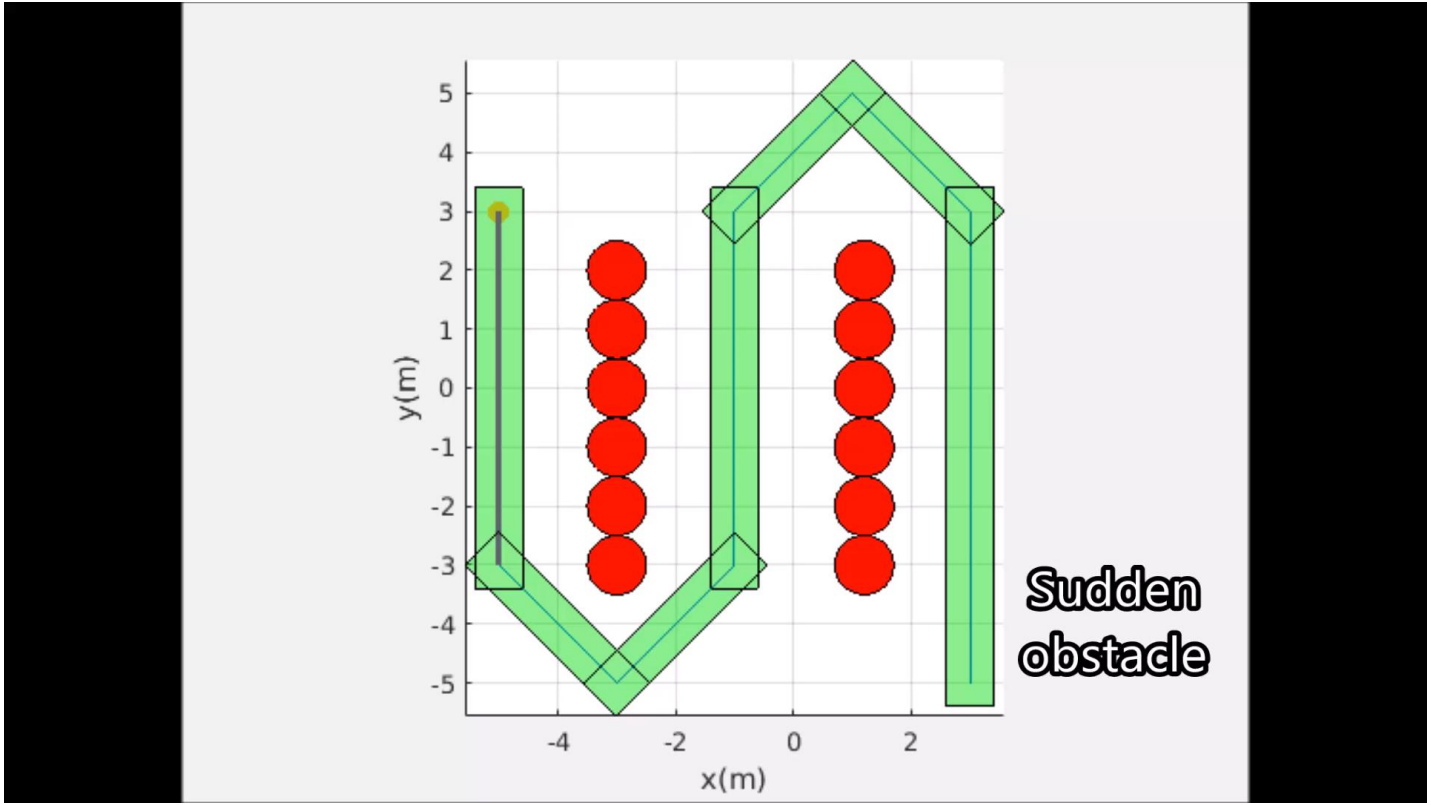
- An ***efficient*** algorithm to generate ***3D*** velocity, acceleration and jerk-limited trajectory.

- Instant reaction to changes (environment/mission)



# Parameter space: Jerk limited trajectory



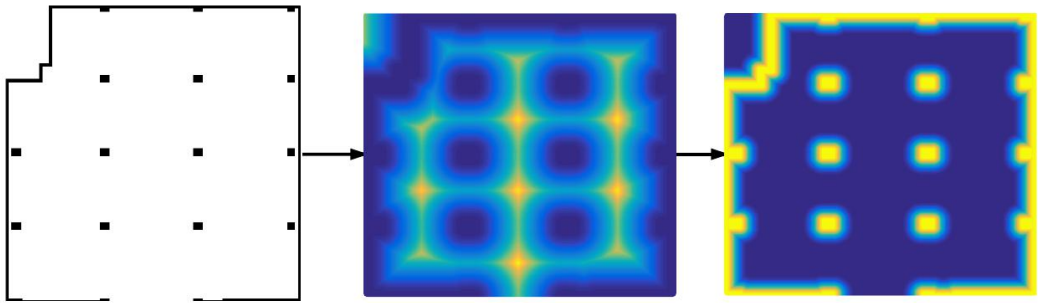
# Parameter space: Jerk limited trajectory

# JLT: Nonlinear-MPC

**Environment perception**

Pixelized environment

- **Occupancy map**
  - Whether a pixel is occupied or not
- **Cost map**
  - The cost of stepping into certain pixel
  - The information on distance to nearest obstacle



Occupancy Map: Black means occupied, white means unoccupied

EDT: The darker the color, the closer to an obstalce.

Cost Map: The lighter the color, the higher the cost.

# JLT: Nonlinear-MPC

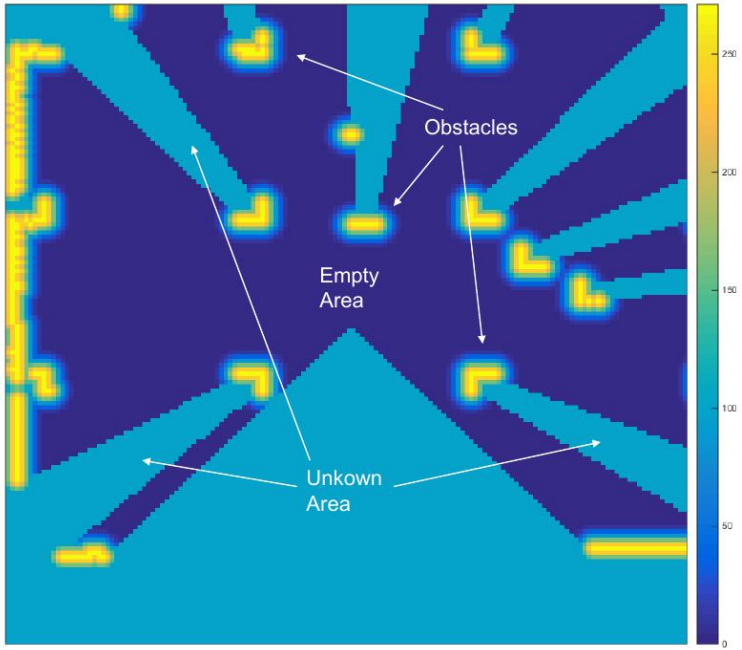
**Environment perception**

Sensor

- **Laser scanner**

Mapping procedure

1. **Read sensor data**
2. **Update the occupancy map**
    a. Add in new obstacle
    b. Clean LOS area
3. **Perform EDT**
4. **Perform Cost assigning**



# JLT: Nonlinear-MPC

**Two level guidance**

Cut the problem into a series of TPBVP.

- **Global planner**

  Provides a series of connected line segments

- **Trajectory planner**

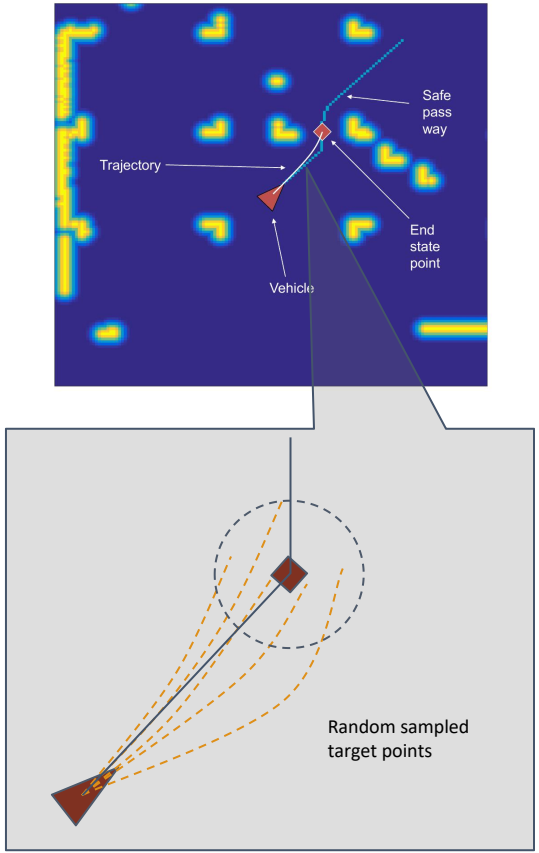
  Solve TPBVP for jerk limited trajectory.

  The trajectory leads the vehicle towards the first sharp turning point on line segment path.

  It propose multiple trajectories for evaluator.

- **Evaluator**

  Evaluate the quality of trajectories.



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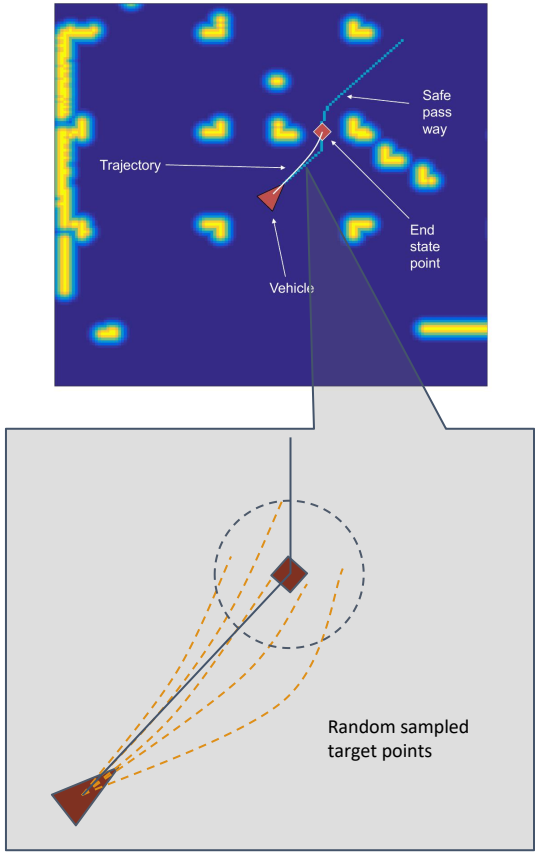
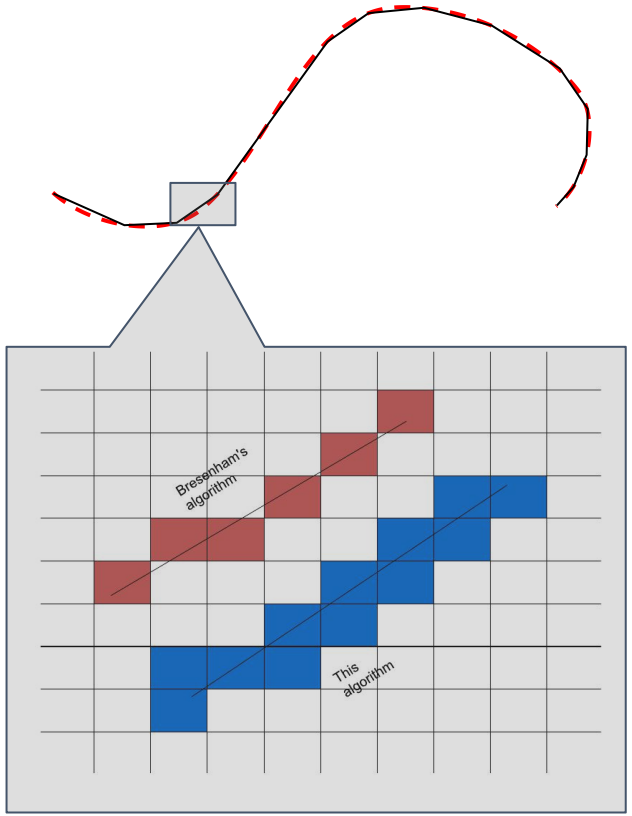


# JLT: Nonlinear-MPC

**Evaluator**

Check the quality of each trajectory

- Represent original trajectory using connected line segments.
- Perform DDA marching to exam each pixel covered by trajectory.
- Find the pixel with highest cost value.
- Evaluate the trajectory based on its time duration and clearness.

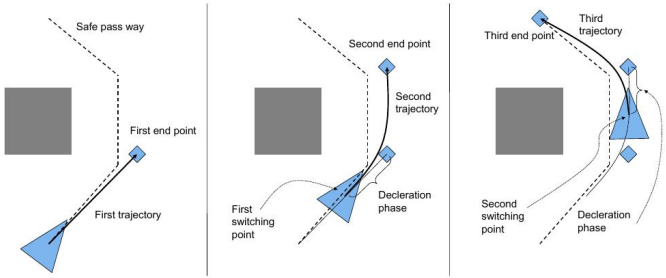


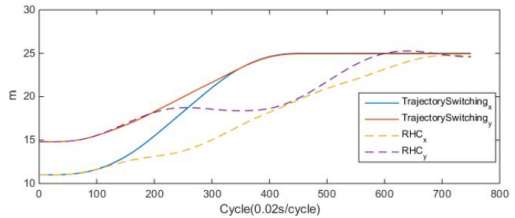
# JLT: Nonlinear-MPC

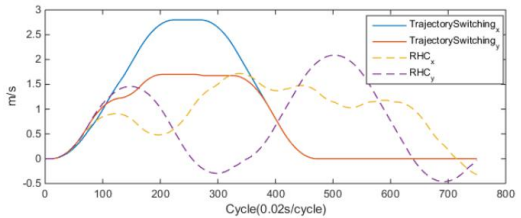
**Event Manager**

Determine when to use new trajectory

- **Trajectory governor**
  - Constantly monitoring whether the current trajectory is safe.
- **Event handler**
  - Fire a replanning event if the current trajectory is about to end
  - Fire emergency replanning event if the current trajectory is unsafe
- **Saving computational power and smoother flight**



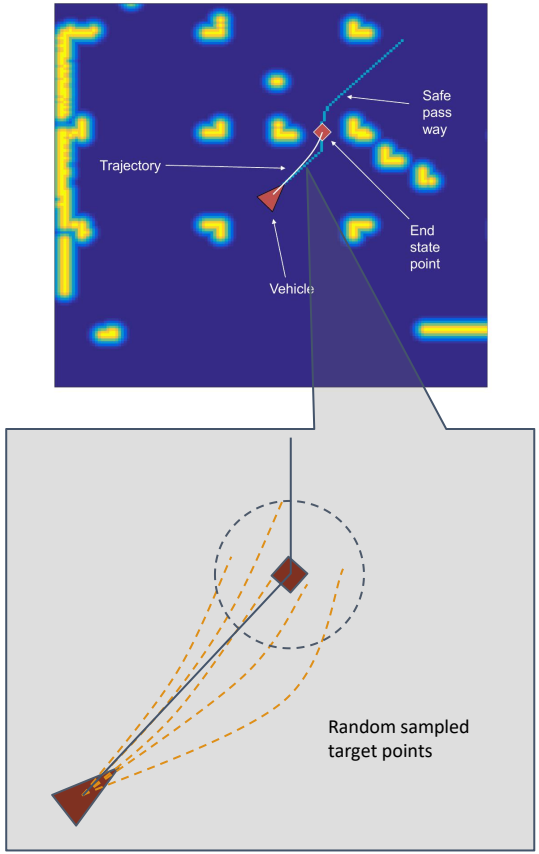



# JLT: Nonlinear-MPC

$$\min C_F\left(x(t_f)\right) + \int_{t=t_0}^{t_f} C_R(x,u)\,dt$$

$$\dot{x} = f(x,u)$$
$$g(x,u) < 0$$
$$h(x,u) = 0$$
$$x \notin Obstacle$$

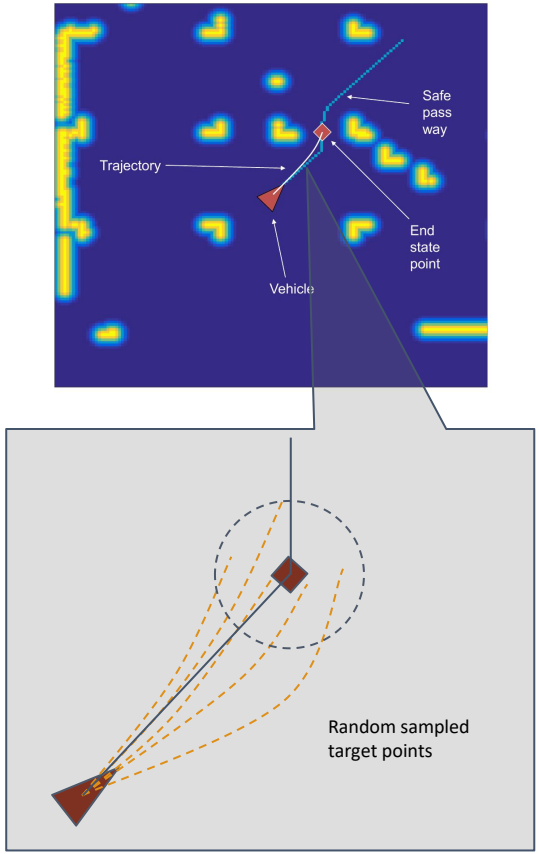


# JLT: Nonlinear-MPC

$$\min C_F\left(x(t_f)\right) + \int_{t=t_0}^{t_f} C_R(x,u)\,dt$$

Hard constraint: $x \notin Obstacle$

Soft constraint: $s(x) = \begin{cases} 0, if\, x \notin Obstacle \\ M, otherwise \end{cases}$



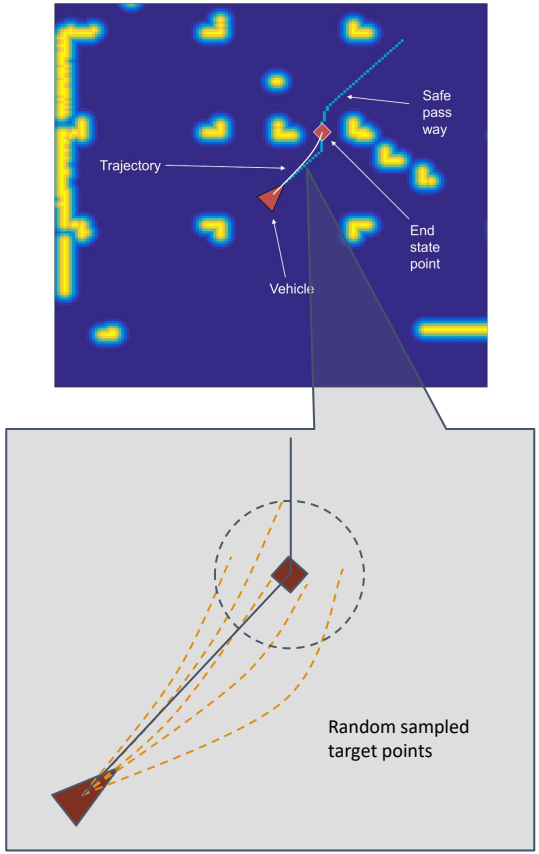
# JLT: Nonlinear-MPC

$$\min C_F\left(x(t_f)\right) + \int_{t=t_0}^{t_f} C_R(x,u)\,dt$$

$$C_R(x,u) = (x - x_d)^T Q(x - x_d) + u^T R u + s(x)$$

$$C_F(x) = (x - x_d)^T W(x - x_d) + s(x)$$

$$s(x) = \begin{cases} 0, if\, x \notin Obstacle \\ M, otherwise \end{cases}$$



# JLT: Nonlinear-MPC

$x_0, x_f \xrightarrow{JLT} x(t)$

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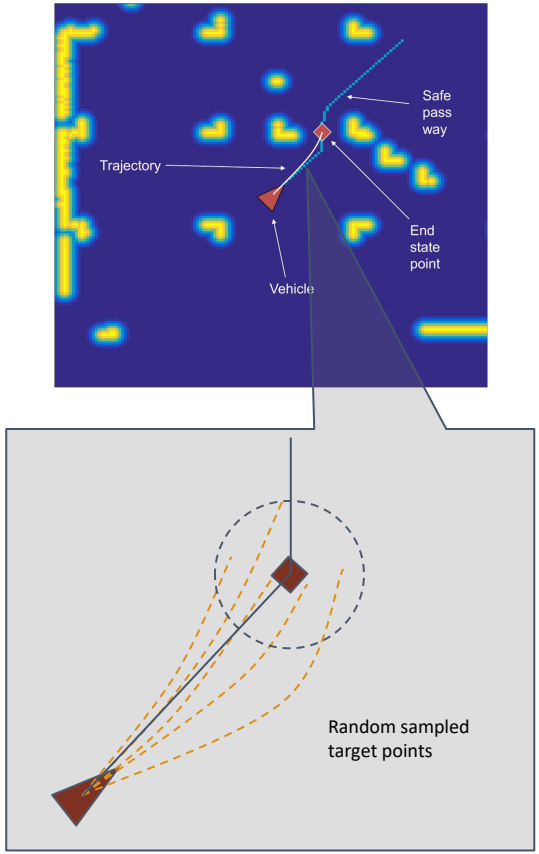

# JLT: Nonlinear-MPC

**Algorithm 1:** Particle Swarm Optimization.

**Input:** $\mathrm{x}_{\mathrm{ini}}$, $\mathcal{M}$(map)
**Output:** $\theta^*$ (best end state constraint)

$\Theta \leftarrow$ Particle_Initialization();
$c_i^* \leftarrow \infty$, $\theta_i^* \leftarrow \theta_i$, $\delta_i \leftarrow$ rand, $\forall i \in [1, \mathrm{size}(\Theta)]$
**for** $m = 1$ to MAX_ITERS **do**
  **for each** $\theta_i \in \Theta$ **do**
    $[\mathrm{x}(t), \mathrm{u}(t)] = \mathcal{S}_{\mathrm{NN}}(\mathbf{x}_{\mathrm{ini}}, \theta_i)$
    $c_i = J(\mathrm{x}(t), \mathrm{u}(t), \mathcal{M})$
    **if** $c_i < c_i^*$ **then**
      $c_i^* = c_i$
      $\theta_i^* = \theta_i$
  $i^* = \underset{i}{\mathrm{argmin}}(c_i^*)$
  $\theta^* = \theta_{i^*}$
  **for each** $\theta_i \in \Theta$ **do**
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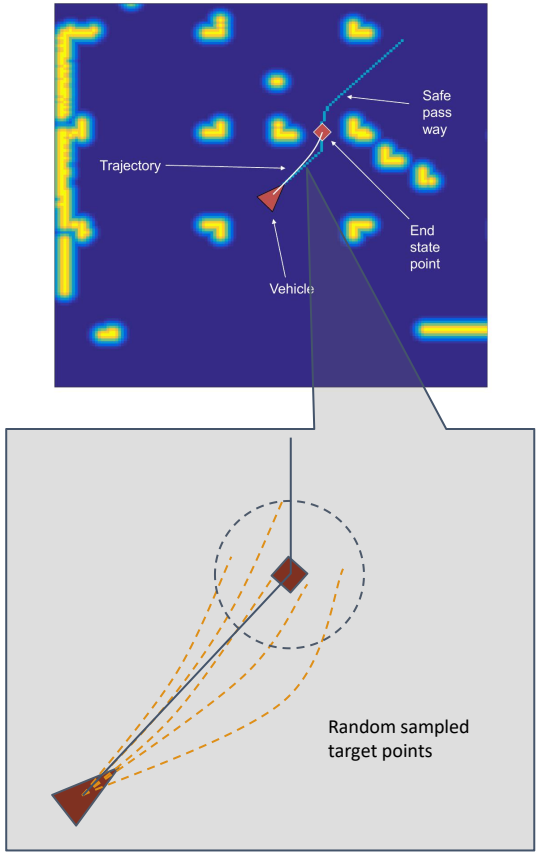
    $\theta_i = \theta_i + \delta_i$



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# JLT: Nonlinear-MPC

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# Parameter space: General BSCP

A general BVP problem:

$$\min_{\mathbf{u}(t),\mathbf{x}(t),t_f} \quad G(\mathbf{x}(t),\mathbf{u}(t),t_f)$$

$$g(\mathbf{x}(t_f),\boldsymbol{\theta}) = 0$$

$$h(\mathbf{x},\mathbf{u}) = 0,\ \tilde{h}(\mathbf{x},\mathbf{u}) \leq 0$$

$$\dot{\mathbf{x}} = f(\mathbf{x},\mathbf{u}),$$

Often requires a numerical optimization.

Some times can be solved through a controller:

$$\|g(\mathbf{x}(t_f),\boldsymbol{\theta})\| < \epsilon,\ \forall t \geq t_f,$$

$$\mathcal{S}:\ \langle \mathbf{x}_0,\boldsymbol{\theta}\rangle \rightarrow \left\langle \hat{\mathbf{u}}(t),\hat{\mathbf{x}}(t),\hat{t}_f\right\rangle.$$

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*Model Predictive Local Motion Planning With Boundary State Constrained Primitives*

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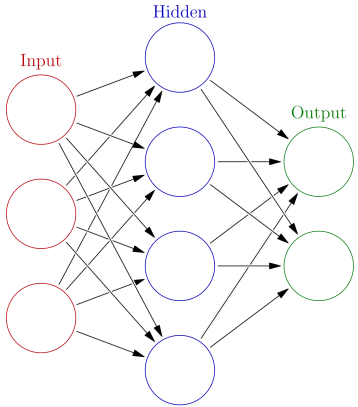
$$\mathcal{S}:\ \langle \mathbf{x}_0,\boldsymbol{\theta}\rangle \rightarrow \langle \hat{\mathbf{u}}(t),\hat{\mathbf{x}}(t),\hat{t}_f\rangle\,.$$

$$\min_{\theta} L_F(\theta) + \int_{t=t_0}^{t_f} L_R(\theta)\,dt$$

*Model Predictive Local Motion Planning With Boundary State Constrained Primitives*

# Parameter space: General BSCP

$$\mathcal{S}: \langle \mathbf{x}_0, \boldsymbol{\theta} \rangle \rightarrow \langle \hat{\mathbf{u}}(t), \hat{\mathbf{x}}(t), \hat{t}_f \rangle .$$




$$\min C_F\left(x(t_f)\right) + \int_{t=t_0}^{t_f} C_R(x,u)\,dt$$

Update $x_0$.

Optimize $\min_{\theta} L_F(\theta) + \int_{t=t_0}^{t_f} L_R(\theta)\,dt$ and get $\theta^*$.

1) Use the NN approximate input u.
2) Solve the original BVP

$$\min_{\mathbf{u}(t),\mathbf{x}(t),t_f} \quad G(\mathbf{x}(t),\mathbf{u}(t),t_f)$$

$$g(\mathbf{x}(t_f),\boldsymbol{\theta}) = 0 \qquad \dot{\mathbf{x}} = f(\mathbf{x},\mathbf{u}),$$

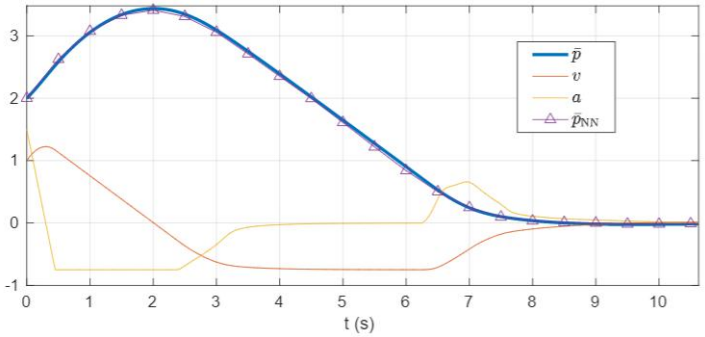
$$h(\mathbf{x},\mathbf{u}) = 0,\ \tilde{h}(\mathbf{x},\mathbf{u}) \leq 0$$

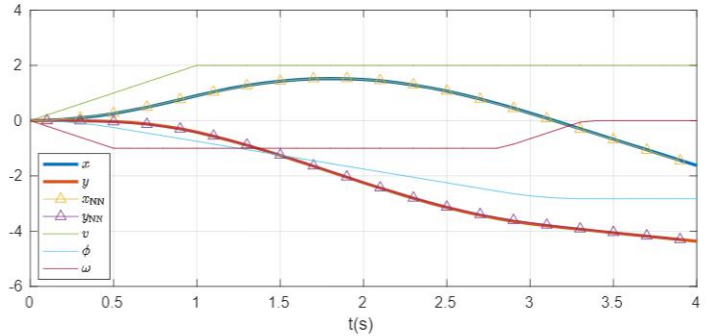
# Parameter space: General BSCP

Network size and training details.

| | NN structure | Training set size | Testing set size | Batch size | Epoch | Average MSE | Maximum MSE |
|---|---|---|---|---|---|---|---|
| Quadrotor horizontal | 64-128-128-41 MLP | 800,000 | 200,000 | 20 | 12 | $4.72 \times 10^{-4}$ | 0.071 |
| Quadrotor vertical | 64-128-128-41 MLP | 800,000 | 200,000 | 10 | 15 | $1.2 \times 10^{-3}$ | 0.088 |
| 2nd order unicycle | 80-256-128-81 MLP | 860,000 | 130,000 | 10 | 20 | $3.4 \times 10^{-3}$ | 0.094 |

* MSE: Mean Squared Error. MLP: Multi-Layer Perceptron.



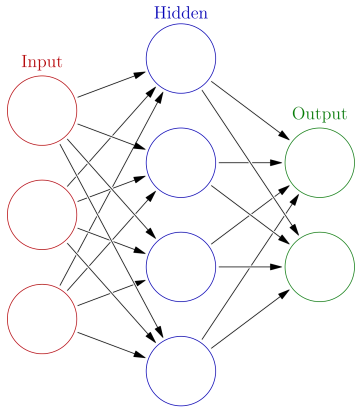



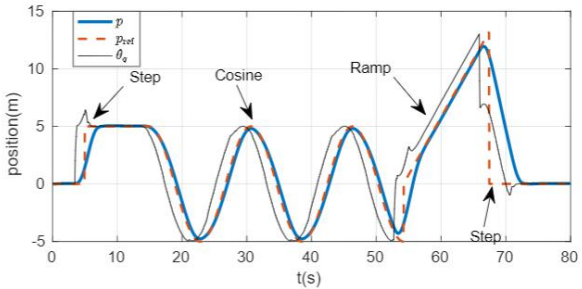
# Parameter space: General BSCP

Approximated gradient

$$input = \theta, x_0$$

$$output = x(t), u(t)$$




$$\frac{dx}{d\theta}, \frac{du}{d\theta}$$

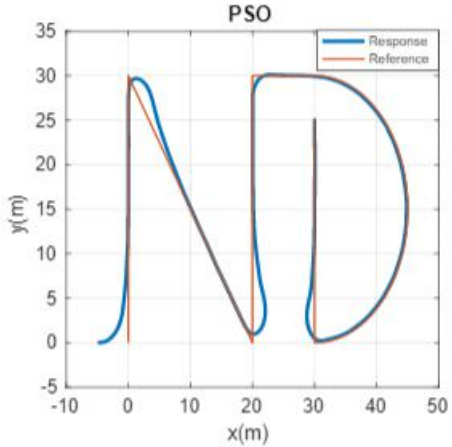
$$\frac{dJ}{dx}, \frac{dJ}{du}$$

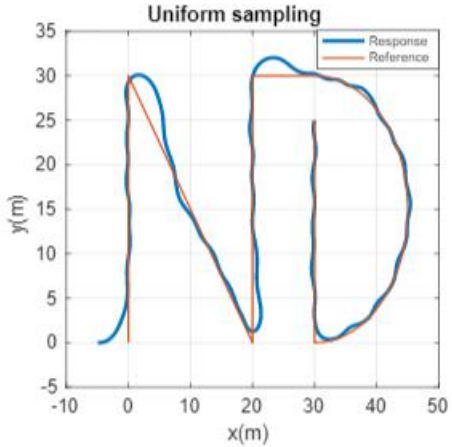
$$J = C_F\left(x(t_f)\right) + \int_{t=t_0}^{t_f} C_R(x,u)\,dt$$

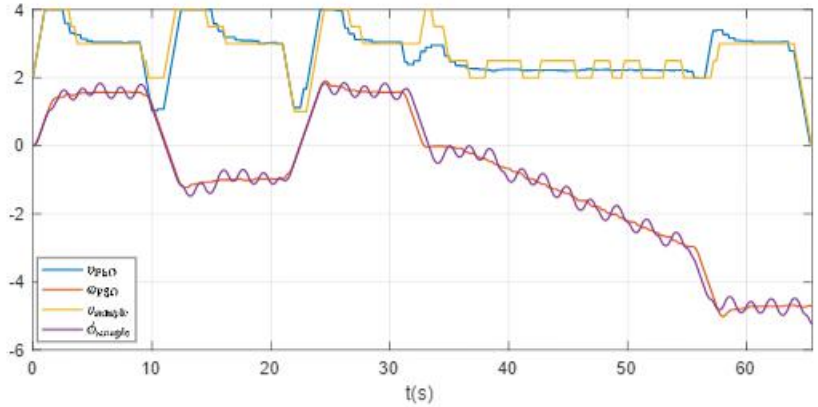
$$\frac{dJ}{d\theta}$$

# Parameter space: General BSCP

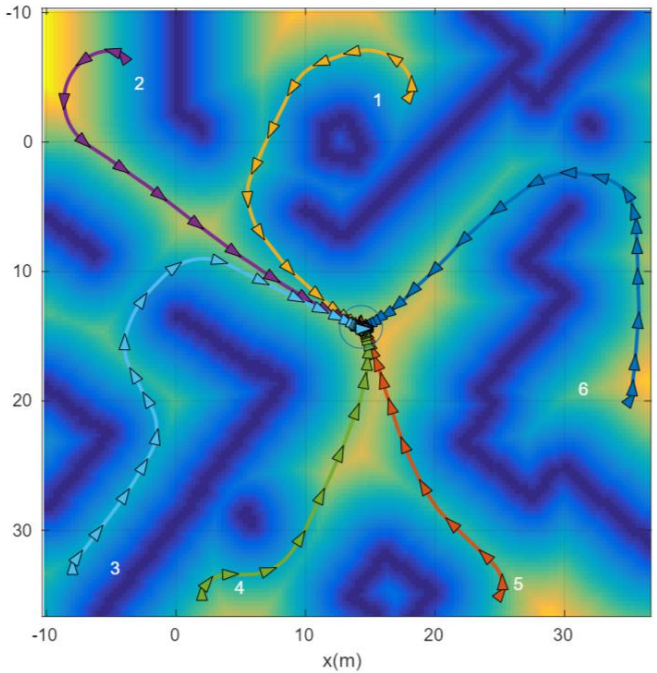
PSO: 10 particle, 10 iterations. VS Uniform sampling: 100 particles 1 iteration.



# Parameter space: General BSCP




$$\min C_F\left(x(t_f)\right) + \int_{t=t_0}^{t_f} C_R(x,u)\,dt$$

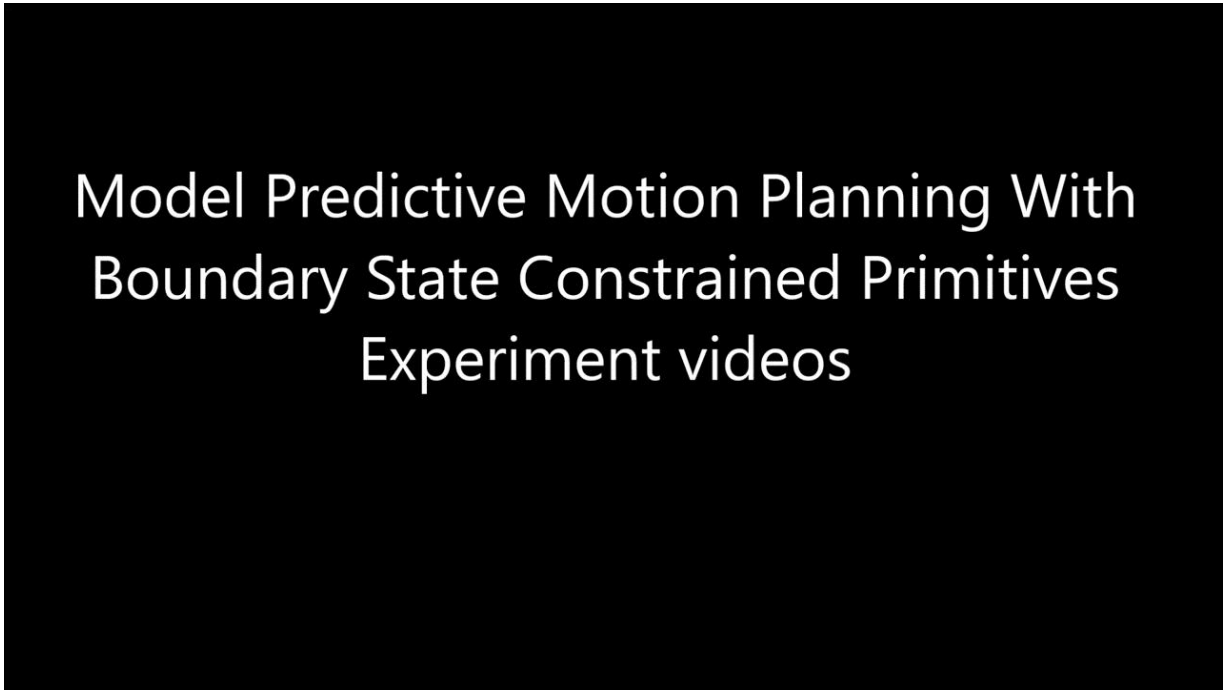
$$C_R(x,u) = (x - x_d)^T Q(x - x_d) + u^T R u + s(x)$$

$$C_F(x) = (x - x_d)^T W(x - x_d) + s(x)$$

$$s(x) = \begin{cases} 0, if x \notin Obstacle \\ M, otherwise \end{cases}$$

cost_to_goal(x), through method like Dijkstra.

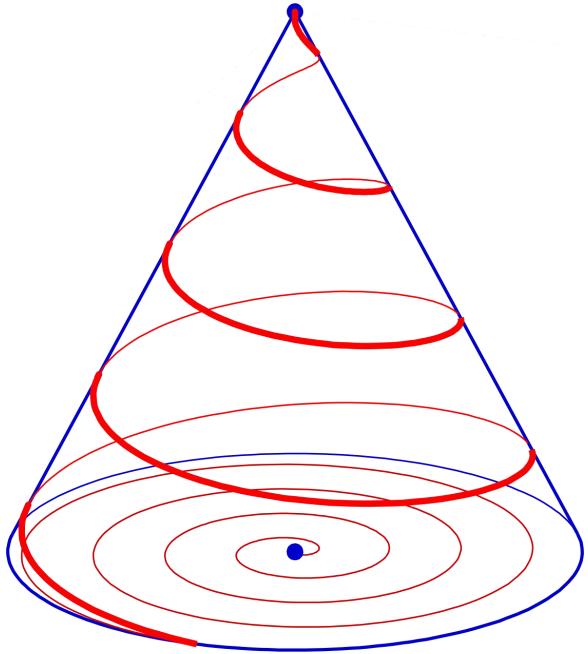
# Parameter space: General BSCP



# Homework

Previously, we have discussed how to design a quadratic programming based MPC to allow a single-axis triple integrator to travel from an arbitrary state to the centre of the state space, a.k.a with zero position, velocity and acceleration.

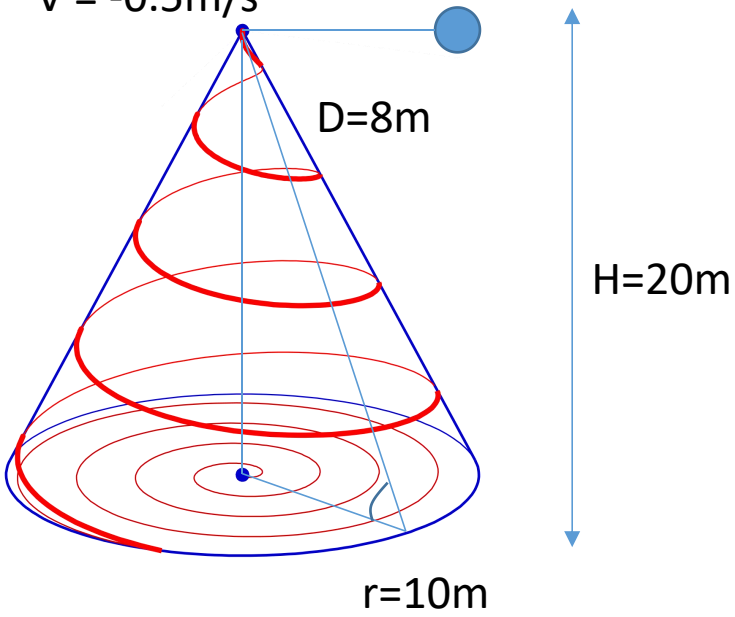
Now please design a quadratic programming based MPC to track conical spiral for a 3-axis triple integrator (basically a quadrotor model).



# Homework

W = 0.08 rad /s

V = -0.5m/s




$$-6 \leq v_{x,y} \leq 6 \qquad -1 \leq v_z \leq 6$$

$$-3 \leq a_{x,y} \leq 3 \qquad -1 \leq a_z \leq 3$$

$$-3 \leq j_{x,y} \leq 3 \qquad -2 \leq j_z \leq 2$$

# 感谢各位聆听!

Thanks for Listening