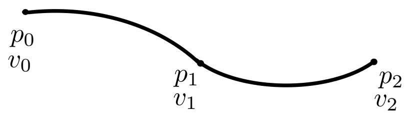
Continuous time trajectory optimization using Polynomials



$$p_1(t) = a_1 + b_1 \cdot t + c_1 \cdot t^2 + d_1 \cdot t^3$$

$$p_2(t) = a_2 + b_2 \cdot t + c_2 \cdot t^2 + d_2 \cdot t^3$$

$$p_1'(t) = b_1 + 2c_1 t + .3d_1 t^2$$

$$p_2'(t) = b_2 + 2c_2 t + 3d_2 t^2$$

Position and velocity constraints (fixed)

position at start of first segment

$$p_1(t_0) = p_0$$

position at end of second segment

$$p_2(t_1) = p_2$$

velocity at start of first segment

$$p_1'(t_0) = p_0$$

velpocity at end of second segment

$$p_2'(t_1) = p_2$$

We can also specify higher order constraints like acceleration and jerk by using higher degree polynomicals e.g quintic splines but for now cubic splines shall be discussed for simplicity.

Position and velocity constraints (free)

position at end of first segment should match start of second segment

$$p_1(t_1) = p_2(t_0) = p_1$$

velocity at end of first segment should match start of second segment

$$p_1'(t_1) = p_2'(t_0) = v_1$$

Writing the above constraints in matrix form we get

1st segment

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 0 & 1 & 2t_1 & 3t_1^2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} p_0 \\ v_0 \\ p_1 \\ v_1 \end{bmatrix}$$

- fixed

2nd segment

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 0 & 1 & 2t_1 & 3t_1^2 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ v_1 \\ p_2 \\ v_2 \end{bmatrix}$$

Using cubic splines and 2 segments we get to solve for 4 coefficients for each polynomial for total of 8 so we should have only 8 constraints.The cosntraints are so far are 4 including position and velocity constraints at start and end of trajectory and 2 constraints for equal position and velocity at the point joining two polynomials for total of 6 making it unsolvable directly thus requiring additional costs.

Rewriting the constranits as free and fixed derivatives

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 0 & 1 & 2t_1 & 3t_1^2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} d_{f1} \\ d_{f2} \\ d_{p1} \\ d_{p2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 0 & 1 & 2t_1 & 3t_1^2 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_{p1} \\ d_{p2} \\ d_{f3} \\ d_{f4} \end{bmatrix}$$

$$\begin{bmatrix} \begin{matrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 0 & 1 & 2t_1 & 3t_1^2 \end{matrix} & \\ & \begin{matrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 0 & 1 & 2t_1 & 3t_1^2 \end{matrix} \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_{f1} \\ d_{f2} \\ d_{p1} \\ d_{p2} \\ d_{p1} \\ d_{p2} \\ d_{f3} \\ d_{f4} \end{bmatrix}$$

Separating into fixed df and free dp notation. only optimize dp.

$$\underbrace{\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 0 & 1 & 2t_1 & 3t_1^2 \end{bmatrix}}_{A_1} \quad \underbrace{\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 0 & 1 & 2t_1 & 3t_1^2 \end{bmatrix}}_{A_2} \underbrace{\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}}_{p} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{C \text{ Constraint Reordering Matrix}} \underbrace{\begin{bmatrix} d_{f1} \\ d_{f2} \\ d_{f3} \\ d_{f4} \\ d_{p1} \\ d_{p2} \end{bmatrix}}_{d}$$

as
$$\begin{bmatrix} d_{f1} \\ d_{f2} \\ d_{p1} \\ d_{p2} \\ d_{p1} \\ d_{p2} \\ d_{f3} \\ d_{f4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_{f1} \\ d_{f2} \\ d_{f3} \\ d_{f4} \\ d_{p1} \\ d_{p2} \end{bmatrix}$$

We can re write the coefficients in terms of their end derivatives using this notation

$$\underbrace{\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}}_{p} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{C} \begin{bmatrix} d_{f1} \\ d_{f2} \\ d_{f3} \\ d_{f4} \\ d_{p1} \\ d_{p2} \end{bmatrix}$$

Constraint Reordering Matrix

Optimization problem

To minimize the trajectory length we can minimize the position magnitude squared over the segment length

$$\mathbf{min}\ _{a,b,c,d} \int_{t_0}^{t1} c \cdot p(t)^2$$

or better minimize velocity magnitude squared over the segment, for higher order splines we can also minimize acceleration magnitude over the segment length

$$\mathbf{min}\ _{a,b,c,d} \int_{t_0}^{t1} c \cdot p'(t)^2 dt$$

$$\int_{t_0}^{t1} c \cdot p'(t)^2 dt = \int_{t_0}^{t1} c \cdot (b_1 + 2c_1 t + 3d_1 t^2) \cdot (b_1 + 2c_1 t + 3d_1 t^2) dt$$

$$= \int_{t_0}^{t1} c \cdot (b_1^2 + 2b_1c_1t + 3b_1d_1t^2 + 2b_1c_1t + 3b_1d_1t^2 + 4c_1^2t^2 + 6c_1d_1t^3 + 6c_1d_1t^3 + 9d_1^2t^4)dt$$

with c =2 we get

$$= 2 \cdot \left(\frac{b_1^2 t}{1} + \frac{2b_1c_1t^2}{2} + \frac{3b_1d_1t^3}{3} + \frac{2b_1c_1t^2}{2} + \frac{3b_1d_1t^3}{3} + \frac{4c_1^2t^3}{3} + \frac{6c_1d_1t^4}{4} + \frac{6c_1d_1t^4}{4} + \frac{9d_1^2t^5}{5} \right)\Bigg|_{t=0}^{t=t_1}$$

$$= 2b_1^2t + 2b_1c_1t^2 + 2b_1d_1t^3 + 2b_1c_1t^2 + 2b_1d_1t^3 + \frac{8c_1^2t^3}{3} + 3c_1d_1t^4 + 3c_1d_1t^4 + \frac{18d_1^2t^5}{5}$$

writing in matrix form we get

$$= \underbrace{\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \end{bmatrix}}_{p^T} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2t & 2t^2 & 2t^3 \\ 0 & 2t^2 & 2.67t^3 & 3t^4 \\ 0 & 2t^3 & 3t^4 & 3.6t^5 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix}}_{p}$$

Jointly optimizing for p1 and p2

$$\mathbf{min}_{p^*}\ P^T \cdot Q \cdot P$$

$$\text{s.t}\quad A \cdot p_1 = d_1$$
$$A \cdot p_2 = d_2$$

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & a_2 & b_2 & c_2 & d_2 \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2t & 2t^2 & 2t^3 \\ 0 & 2t^2 & 2.67t^3 & 3t^4 \\ 0 & 2t^3 & 3t^4 & 3.6t^5 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2t & 2t^2 & 2t^3 \\ 0 & 2t^2 & 2.67t^3 & 3t^4 \\ 0 & 2t^3 & 3t^4 & 3.6t^5 \end{bmatrix}$$

Unconstrained quadratic problem

We got a Quadratic problem with linear constraints which can be solved with any quadratic solver but we can also form it into a more stable and unconstrained quadratic problem by writing the probelm such that instead of directly optimizing for coefficients p's, subsitute the value of p in terms of d's i.e the constraints. For example if we want the constraint that the values of a variable x be even, instead of solving for x with constraints xis even, sovle for y and reformulate it as 2x=y. Optimize y and the value of x would be its double thus satisfying the constraint.

substituting the value of p we get

$$\mathbf{min}_{d_{p1}^*,d_{p2}^*} f(\vec{d}) = \mathbf{min}_{d_{p1}^*,d_{p2}^*} d^T \cdot \underbrace{C^T \cdot A^{-T} \cdot Q \cdot A^{-1} \cdot C}_{R} \cdot d$$

$$\overset{d^T}{\begin{bmatrix} d_{f1} & d_{f2} & d_{f3} & d_{f4} & d_{p1}^* & d_{p2}^* \end{bmatrix}} \overset{C^T}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}} \overset{A^{-T}}{\begin{bmatrix} A_1^{-T} & 0 \\ 0 & A_2^{-T} \end{bmatrix}} \overset{Q}{\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}} \overset{A^{T}}{\begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{bmatrix}} \overset{C}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}} \overset{d}{\begin{bmatrix} d_{f1} \\ d_{f2} \\ d_{f3} \\ d_{f4} \\ d_{p1}^* \\ d_{p2}^* \end{bmatrix}}$$

For brevity wiriting as a single vector dF and dp. Similarly we group the new matrix R and partition it according to the indices of the fixed and free derivatives.

$$d_F = \begin{bmatrix} d_{f1} \\ d_{f2} \\ d_{f3} \\ d_{f4} \end{bmatrix} = \begin{bmatrix} p_0 \\ v_0 \\ p_2 \\ v_2 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^{-T}Q_1A_1^{-1} & 0 \\ 0 & A_2^{-T}Q_2A_1^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$d_P = \begin{bmatrix} d_{p1} \\ d_{p2} \end{bmatrix}$$

Writing the cost function for optimizing derivative costs here 1st derivative (velocity) costs as compact form of the previous equation

$$f(\vec{d}) = \begin{bmatrix} d_F & d_P^* \end{bmatrix}^T \begin{bmatrix} R_{FF} & R_{FP} \\ R_{PF} & R_{PP} \end{bmatrix} \begin{bmatrix} d_F \\ d_P^* \end{bmatrix}$$

$$= d^T R_{FF} d_F + d_F^T R_{FP} d_P + d_p^T R_{PF} d_F + d_P^T R_{PP} d_P$$

Differentiating f(d) wrt dp and equating to zero yields the vector of optimal values for the free derivatives in terms of the fixed/specified derivatives and the cost matrix.

$$d_P^* = -R_{PP}^{-1} R_{FP}^T d_F$$

Here dF are the fixed constraints like position and velocity at the start and end of trajectory. using those values we can directly get the values for dp* (optimal) values for that specific dF.

we can also add other costs to the objective functions such as collision cost and instead iteratively optimize by adding soft constraints.
Note the costs are cauculated wrt d's not p's.

$$\mathbf{min}_{d_{p1}^*,d_{p2}^*} f(\vec{d}) + c(\vec{d})$$

Calculate R Matrix

$$A = \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 0 & 1 & 2t_1 & 3t_1^2 \end{bmatrix} \qquad Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2t & 2t^2 & 2t^3 \\ 0 & 2t^2 & 2.67t^3 & 3t^4 \\ 0 & 2t^3 & 3t^4 & 3.6t^5 \end{bmatrix}$$

Using segment start t0 =1 and end t1=1, we get

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \qquad Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2.67 & 3 \\ 0 & 2 & 3 & 3.6 \end{bmatrix}$$

Calculating A inverse

We make use of the Schur-complement by exploiting the structure of A

$$A = \begin{bmatrix} A_{\text{diag}} & B = 0 \\ C & D \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} A_{\text{diag}}^{-1} & 0 \\ D^{-1} \cdot C \cdot A_{\text{diag}}^{-1} & D^{-1} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix}$$

$$A^{-T} = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^{-T} Q_1 A_1^{-1} & 0 \\ 0 & A_2^{-T} Q_2 A_1^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Using the Values of A1, A2 inverse and Q1, Q2. As both segments are polynomials of same oder so they only difer in their coefficients thus A and Q is same for all polynomials.

$$A_1^{-T} Q_1 A_1^{-1} = A_2^{-T} Q_2 A_2^{-1} = \begin{bmatrix} 2.43 & 0.22 & -2.43 & 0.21 \\ 0.22 & 0.28 & -0.22 & -0.06 \\ -2.43 & -0.22 & 2.43 & -0.21 \\ -3.81 & -0.74 & 3.81 & 0.93 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2.43 & 0.22 & -2.43 & 0.21 & 0 & 0 & 0 & 0 \\ 0.22 & 0.28 & -0.22 & -0.06 & 0 & 0 & 0 & 0 \\ -2.43 & -0.22 & 2.43 & -0.21 & 0 & 0 & 0 & 0 \\ -3.81 & -0.74 & 3.81 & 0.93 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.43 & 0.22 & -2.43 & 0.21 \\ 0 & 0 & 0 & 0 & 0.22 & 0.28 & -0.22 & -0.06 \\ 0 & 0 & 0 & 0 & -2.43 & -0.22 & 2.43 & -0.21 \\ 0 & 0 & 0 & 0 & -3.81 & -0.74 & 3.81 & 0.93 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$R = \left[\begin{array}{cccc|cc} \multicolumn{4}{c|}{R_{FF}} & \multicolumn{2}{c}{R_{FP}} \\ 2.4 & 0.2 & 0 & 0 & -2.4 & 0.2 \\ 0.2 & 0.26 & 0 & 0 & -0.2 & -0.06 \\ 0 & 0 & 2.4 & -0.2 & -2.4 & -0.2 \\ 0 & 0 & -0.2 & 0.267 & 0.2 & -0.06 \\ \hline -2.4 & -0.2 & -2.4 & 0.2 & 4.8 & 0 \\ 0.2 & -0.06 & -0.2 & 0.06 & 0 & 0.53 \\ \multicolumn{4}{c|}{R_{PF}} & \multicolumn{2}{c}{R_{PP}} \end{array}\right]$$

$$d_P^* = -R_{PP}^{-1} R_{FP}^T d_F$$

For a two dimensional space, we can use separate polynomials for x and y each consiting of two segments. Using the following values of initial and final position and velocity for X and Y dimensions, we get

$$p_{0_x} = 10 \;\; p_{0_y} = 10; \quad p_{2_x} = 20 \;\; p_{2_y} = 25$$

$$v_{0_x} = 1 \;\; v_{0_y} = 50; \quad v_{2_x} = 0 \;\; v_{2_y} = 0$$

$$d_{F_x} = \begin{bmatrix} 10 \\ 1 \\ 20 \\ 0 \end{bmatrix} \qquad d_{F_y} = \begin{bmatrix} 10 \\ 50 \\ 25 \\ 0 \end{bmatrix} \qquad d_{F_x} = \begin{bmatrix} d_{f1} \\ d_{f2} \\ d_{f3} \\ d_{f4} \end{bmatrix}_x = \begin{bmatrix} p_0 \\ v_0 \\ p_2 \\ v_2 \end{bmatrix}_x$$

$$d_{P_x}^* = \begin{bmatrix} 15.04 \\ 3.87 \end{bmatrix} \qquad d_{P_y}^* = \begin{bmatrix} 19.58 \\ 11.87 \end{bmatrix}$$

We can retrieve the coefficients of polynomials using the equations given above

$$p_x = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}_x = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_{f1} \\ d_{f2} \\ d_{f3} \\ d_{f4} \\ d_{p1} \\ d_{p2} \end{bmatrix}_x$$

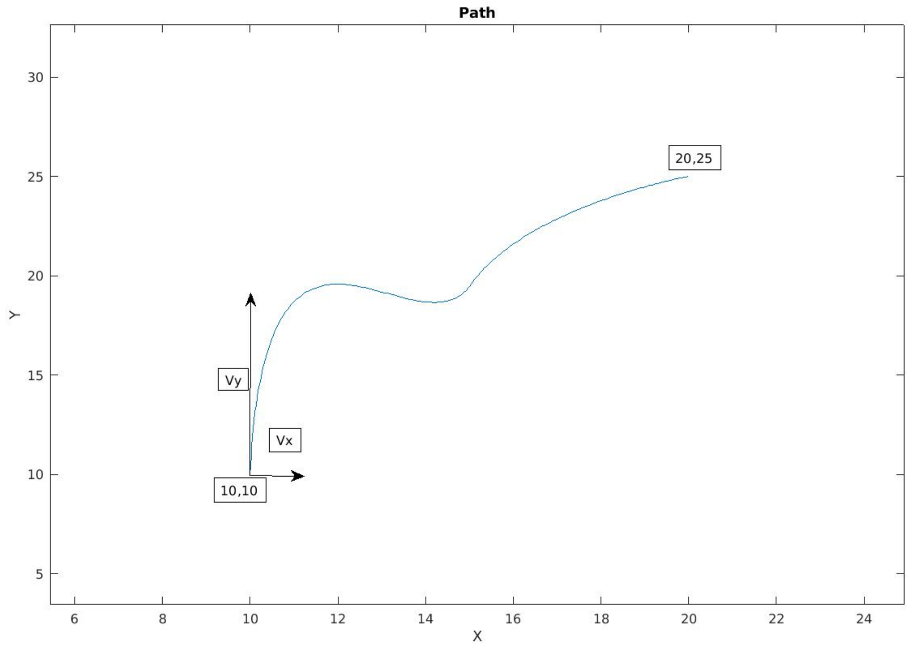
$$= \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix} & \\ & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \\ 20 \\ 0 \\ 15.04 \\ 3.87 \end{bmatrix}_x$$

$$p_x = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}_x = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & -2 & 0 & 0 & 3 & -1 \\ 2 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & -1- & 3 & -2 \\ 0 & 0 & -2 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \\ 20 \\ 0 \\ 15.04 \\ 3.87 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 9.25 \\ -5.20 \\ 15.04 \\ 3.87 \\ 7.12 \\ -6.04 \end{bmatrix}$$

and similarly we get py

$$p_y = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}_y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & -2 & 0 & 0 & 3 & -1 \\ 2 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & -1- & 3 & -2 \\ 0 & 0 & -2 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 50 \\ 25 \\ 0 \\ 19.58 \\ 11.87 \end{bmatrix} = \begin{bmatrix} 10 \\ 50 \\ -83.12 \\ 42.70 \\ 19.58 \\ 11.87 \\ -7.5 \\ 1.04 \end{bmatrix}$$

A plot using the coefficients above minimizing velocity over segment length.



Similarly we can have additional polynomials for z dimension and calculate position, velocity and accelerations as a continuous function of time.