Math 323 HW10

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Problem 6.6: Let $a, b \in \mathbb{R}$ with $a < b$. Prove $\exists s \notin \mathbb{Q}$ such that $a < s < b$.

Proof. Assume $a, b \in \mathbb{R}$ with $a < b$. Assume $c, d \in \mathbb{R}$ so that $c = \frac{a}{\sqrt{2}}$ and $d = \frac{b}{\sqrt{2}}$. Since $a < b$, $c < d$. By the Density theorem, we know $\exists q \in \mathbb{Q}$ so that $c < q < d$. So we have

$$c < q < d$$
$$\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$$
$$a < q\sqrt{2} < b$$

But $q\sqrt{2} \in \mathbb{R}$. So for $a, b \in \mathbb{R}$, $\exists s \notin \mathbb{Q}$ such that $a < s < b$. □

Problem 6.7: Prove if $a, b \in \mathbb{R}$, with $a < b$, then $\forall n \in \mathbb{N}$, there are numbers $x_1, x_2, ..., x_{n-1}, x_n \in \mathbb{Q}$ all different so that for all $i = 1, 2, 3, ..., n$, we have $a < x_i < b$.

Proof. Assume $a, b \in \mathbb{R}$ with $a < b$. We will prove the above statement using induction on n. We will need to prove two claims.

1. We want to show: If $n = 1$, then if $\exists x_1, x_2, ..., x_n \in \mathbb{Q}$ all different so that for all $i = 1, 2, 3, ..., n$, we have $a < x_i < b$.
   Assume $n = 1$. Assume $\exists x_1, ..., x_n \in \mathbb{Q}$ all different. Since $n = 1$, $\exists x_1 \in \mathbb{Q}$. Since $a, b \in \mathbb{R}$, by the Density theorem, $\exists q \in \mathbb{Q}$ so that $a < q < b$. Let $q = x_1$ and so $a < x_1 < b$.
   So we have shown that for $n = 1$, if $\exists x_1, x_2, ..., x_n \in \mathbb{Q}$ all different so that for all $i = 1, 2, 3, ..., n$, we have $a < x_i < b$.
2. We also want to show: If for $n = n_0$, if $\exists x_1, x_2, ..., x_n \in \mathbb{Q}$ all different so that for all $i = 1, 2, 3, ..., n$, we have $a < x_i < b$, then for $n = n_0+1$, if $\exists x_1, x_2, ..., x_n \in \mathbb{Q}$ all different so that for all $i = 1, 2, 3, ..., n$, we have $a < x_i < b$.
   Assume $n = n_0$. Assume $\exists x_1, x_2, ..., x_n \in \mathbb{Q}$ so that for all $i = 1, 2, 3, ..., n$, we have $a < x_i < b$. Our inductive hypothesis says $\exists x_1, x_2, ..., x_{n_0} \in \mathbb{Q}$ so that for all $i = 1, 2, 3, ..., n$, we have $a < x_i < b$. Consider $a < x_{n_0} < b$. Since $x_{n_0} \in \mathbb{Q}$, it has to be in $\mathbb{R}$. So $x_{n_0} \in \mathbb{R}$.

We also know $b \in \mathbb{R}$ and $b > x_{n_0}$ by our inductive hypothesis. So then by the Density theorem, $\exists x_{n_0+1} \in \mathbb{Q}$ so that $x_{n_0} < x_{n_0+1} < b$. So then for all $i = 1, 2, ..., n_0, n_0 + 1$, we have $a < x_i < b$.

Proving claim (1) and claim (2) thus completes the proof by induction on n. $\square$

In problems 8-13, consider the sets:

$$A = \{r \in \mathbb{Z} \mid 5 < 2r\}$$
$$B = \{r \in \mathbb{Z} \mid 5 \leq 2r\}$$
$$C = \{r \in \mathbb{Q} \mid 5 < r\}$$
$$D = \{r \in \mathbb{Q} \mid 5 \leq 2r\}$$
$$E = \{r \in \mathbb{R} \mid 5 < 2r\}$$
$$F = \{r \in \mathbb{R} \mid 5 \leq 2r\}$$

Problem 6.8: For which of the sets A through F is there a rational number that is a lower bound on the set?

Solution. A, B, C, D, E, F.

Problem 6.9: For which of the sets A through F is there a rational number that is a greatest lower bound on the set?

Solution. A, B, D, F.

Problem 6.10: For which of the sets A through F is there a rational number that is a minimum of the set?

Solution. A, B, D, F.

Problem 6.11: For which of the sets A through F is there a real number that is a lower bound on the set?

Solution. A, B, C, D, E, F.

Problem 6.12: For which of the sets A through F is there a real number that is a greatest lower bound on the set?

Solution. A, B, C, D, E, F.

Problem 6.13: For which of the sets A through F is there a real number that is a minimum of the set?

Solution. A, B, D, F.

In problem 14-19, consider the sets:

$$
\begin{aligned}
A' &= \{r \in \mathbb{Z} \mid 5 < 2r^3\} \\
B' &= \{r \in \mathbb{Z} \mid 5 \leq 2r^3\} \\
C' &= \{r \in \mathbb{Q} \mid 5 < r^3\} \\
D' &= \{r \in \mathbb{Q} \mid 5 \leq 2r^3\} \\
E' &= \{r \in \mathbb{R} \mid 5 < 2r^3\} \\
F' &= \{r \in \mathbb{R} \mid 5 \leq 2r^3\}
\end{aligned}
$$

Problem 6.14: For which of the sets A' through F' is there a rational number that is a lower bound on the set?

Solution. A', B', C', D', E', F'

Problem 6.15: For which of the sets A' through F' is there a rational number that is a greatest lower bound on the set?

Solution. A', B',

Problem 6.16: For which of the sets A' through F' is there a rational number that is a minimum of the set?

Solution. A', B',

Problem 6.17: For which of the sets A' through F' is there a real number that is a lower bound on the set?

Solution. A', B', C', D', E', F'

Problem 6.18: For which of the sets A' through F' is there a real number that is a greatest lower bound on the set?

Solution. A', B', C', D', E', F'

Problem 6.19: For which of the sets A' through F' is there a real number that is a minimum of the set?

Solution. A', B', D', F'