Math 323 HW21

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Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let a be in $\mathbb{R}$. We say f is continuous at a if $\forall \epsilon > 0, \exists \delta > 0$ s.t $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

1. Prove that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$ is continuous at 1.

   Proof. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$. We want to show $\forall > 0, \exists \delta > 0$ s.t $|x - 1| < \delta$ implies $|2x - 2| < \epsilon$.
   Let $\epsilon > 0$. Let $\delta = \frac{\epsilon}{2}$. By the Average theorem, we know $\frac{\epsilon}{2} > 0$. Assume $|x - 1| < \delta$.

   $$\begin{aligned} |x-1| &< \frac{\epsilon}{2} \\ 1 - \frac{\epsilon}{2} < x &< 1 + \frac{\epsilon}{2} \\ 2 - \epsilon < 2x &< 2 + \epsilon \\ |2x - 2| &< \epsilon \end{aligned}$$

   Thus $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$ is continuous at 1. □

2. Prove that the sign function is not continuous at 0.

   Proof. Let $g(x)$ be a sign function of x.

   $$g(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

   We want to show: $\exists \epsilon > 0$ s.t $\forall \delta > 0$, $|x - 0| < \delta$ and $|g(x) - g(0)| \geq \epsilon$.
   Let $\epsilon = \frac{1}{2}$. Let $\delta > 0$. We choose $x = \frac{\delta}{2}$. So we have $|x| < \delta$ since $\frac{\delta}{2} < \delta$ with $\delta > 0$. And we also have $|g(\frac{\delta}{2} - g(0)| = |1 - 0| = 1 \geq \epsilon$.
   Thus the sign function is not continuous at 0. □