Math 323 HW18

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June 15, 2017

Problem 11.10: Define $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

(a) Prove that $f(x)$ is bijective.

Proof. We need to prove $f(x)$ is both injective and surjective.

1. $f(x)$ is injective.
   We want to show: if for $a, b \in \mathbb{R}$ and $f(a) = f(b)$, then $a = b$.
   Assume $a, b \in \mathbb{R}$ and $f(a) = f(b)$. We consider

$$f(a) = \begin{cases} a^2 & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a^2 & \text{if } a < 0 \end{cases}$$

$$f(b) = \begin{cases} b^2 & \text{if } b > 0 \\ 0 & \text{if } b = 0 \\ -b^2 & \text{if } b < 0 \end{cases}$$

   But $f(a) = f(b)$ because of our assumption. We have 3 cases for a (or b).

   i. a and b have the same sign. Assume $a < 0$ and $b < 0$. Then $f(a) = f(b) = -a^2 = -b^2$. $(a - b)(a + b) = 0$. Since a and b have the same sign, only $(a - b) = 0$ can happen. So $a = b$.
   ii. $a = 0$ and $b = 0$. Then $a = b = 0$.
   iii. a and b do not have the same sign. Assume $a > 0$ and $b < 0$. Then $f(a) = f(b) = a^2 = -b^2$. So $a^2 + b^2 = 0$. But we know $a^2 > 0$ and $b^2 > 0$ for $a \neq 0$ and $b \neq 0$. So $a^2 + b^2 > 0$. This case can't happen.

   Note that we actually have 5 cases but we only need to consider 3 cases because the value of a and b are interchangable. Thus, $f(x)$ is injective.

2. $f(x)$ is surjective.
   We want to show: If $y \in \mathbb{R}$, then $\exists x \in \mathbb{R}$ s.t $y = f(x)$.
   Assume $y \in \mathbb{R}$. We consider 3 cases for y.
   i. $y < 0$. Assume $y < 0$. Then $-y > 0$. Let $x = \sqrt{-y}$ and we are done.
   ii. $y = 0$. Let $x = 0$.
   iii. $y > 0$. Let $x = \sqrt{y}$.
   So $f(x)$ is surjective.

Thus $f(x)$ is bijective. □

(b) Find $f^{-1}(x)$.

Solution.

$$f^{-1}(x) = \begin{cases} \sqrt{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$$

(c) Consider $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x|x|$. Compare this to $f(x)$.

Using the definition of absolute value, we observe $g(x)$.

$$g(x) = x|x| = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ x|x| = x(-x) = -x^2 & \text{if } x < 0 \end{cases}$$

So $f(x) = g(x)$.

Problem 12.6: Let $f : A \to B$ be a function. Let $S_1, S_2 \subseteq A$ and $T_1, T_2 \subseteq B$.

(a) Prove that if $S_1 \subseteq S_2$, then $f(S_1) \subseteq f(S_2)$.

Proof. Let $f : A \to B$ be a function. Let $S_1, S_2 \subseteq A$.
Assume $S_1 \subseteq S_2$. We want to show: if $y \in f(S_1)$, then $y \in f(S_2)$.
By definition

$$f(S_1) = \{y \in B \mid \exists x \in S_1 \text{ s.t } y = f(x)\}$$
$$f(S_2) = \{y \in B \mid \exists x \in S_2 \text{ s.t } y = f(x)\}$$

Assume $y \in f(S_1)$. So then $\exists x_1 \in S_1$ s.t $y = f(x_1)$. But since $S_1 \subseteq S_2$, $x_1 \in S_2$. But then that means $\exists x_2 = x_1 \in S_2$ s.t $y = f(x_2)$.
And so $y \in f(S_2)$. □

(b) Prove that if $T_1 \subseteq T_2$, then $f^{-1}(T_1) \subseteq f^{-1}(T_2)$.

Proof. Let $f : A \to B$ be a function. Let $T_1, T_2 \subseteq B$.
Assume $T_1 \subseteq T_2$. We want to show: If $x \in f^{-1}(T_1)$, then $x \in f^{-1}(T_2)$. By definition

$$f^{-1}(T_1) = \{x \in A \mid f(x) \in T_1\}$$
$$f^{-1}(T_2) = \{x \in A \mid f(x) \in T_2\}$$

Assume $x \in f^{-1}(T_1)$. This means $f(x) \in T_1$. But since $T_1 \subseteq T_2$, $f(x) \in T_2$. But then that also means $x \in f^{-1}(T_2)$. Thus $f^{-1}(T_1) \subseteq f^{-1}(T_2)$. $\square$

Problem 12.7: Prove: If $f : A \to B$ is a function with domain A and S_i with $i \in \mathcal{I}$ is a family of sets where $\forall i \in \mathcal{I}$, $S_i \subseteq A$, then

$$f\left(\bigcap_{i \in \mathcal{I}} S_i\right) \subseteq \bigcap_{i \in \mathcal{I}} f(S_i)$$

Proof. Assume $f : A \to B$ is a function with domain A. Let S_i with $i \in \mathcal{I}$ is a family of sets where $\forall i \in \mathcal{I}, S_i \subseteq A$.
Assume $a \in f\left(\bigcap_{i \in \mathcal{I}} S_i\right)$. This means

$$a \in \{y \in B \mid \exists x \in \bigcap_{i \in \mathcal{I}} S_i \text{ s.t } y = f(x)\}$$

By this definition, $\exists \alpha \in \bigcap_{i \in \mathcal{I}} S_i$ s.t $a = f(\alpha)$. So then $\forall i \in \mathcal{I}, \alpha \in S_i$. So $\forall i \in \mathcal{I}$, $f(\alpha) \in f(S_i)$ in which $f(S_i) \subseteq B$. This also means $f(\alpha) \in \bigcap_{i \in \mathcal{I}} f(S_i)$ by our definition of intersection of family of sets. But we have $a = f(\alpha)$. So $a \in \bigcap_{i \in \mathcal{I}} f(S_i)$.

$\square$