Math 323 HW6

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Problem 4.6: Prove that addition in $\mathbb{Q}$ is well-defined.

Proof. We will prove the following statement: If $\frac{m}{n}, \frac{p}{q}, \frac{r}{s}, \frac{t}{u} \in \mathbb{Q}$ such that $\frac{m}{n} = \frac{r}{s}$ and $\frac{p}{q} = \frac{t}{u}$, then $\frac{m}{n} + \frac{p}{q} = \frac{r}{s} + \frac{t}{u}$.
Assume $\frac{m}{n}, \frac{p}{q}, \frac{r}{s}, \frac{t}{u} \in \mathbb{Q}$ such that $\frac{m}{n} = \frac{r}{s}$ and $\frac{p}{q} = \frac{t}{u}$. So $ms = rn$ and $pu = qt$. Consider $ms = rn$ when we multiply both sides by the natural number qu.

$$ms = rn \tag{1}$$
$$ms(qu) = rn(qu) \tag{2}$$

Similarly, consider $pu = qt$ when we multiply both sides by the natural number sn.

$$pu = qt \tag{3}$$
$$pu(sn) = qt(sn) \tag{4}$$

Revisiting equation (2) again

$$\begin{gathered}
ms(qu) = rn(qu) \\
ms(qu) + pu(sn) = rn(qu) + pu(sn) \\
ms(qu) + pu(sn) = rn(qu) + qt(sn) \text{ by (4)} \\
sumq + supn = nqru + nqst \text{ by commutativity and associativity} \\
su(mq + pn) = nq(ru + st) \\
\frac{(mq + pn)}{nq} = \frac{(ru + st)}{su} \\
\frac{m}{n} + \frac{p}{q} = \frac{r}{s} + \frac{t}{u}
\end{gathered}$$

And so we have proved that addition in $\mathbb{Q}$ is well-defined. $\square$

Problem 4.12: Prove that multiplication in $\mathbb{Q}$ is commutative.

Proof. We need to prove: If $\frac{m}{n}, \frac{p}{q} \in \mathbb{Q}$, then $\frac{m}{n} \cdot \frac{p}{q} = \frac{p}{q} \cdot \frac{m}{n}$.
Assume $\frac{m}{n}, \frac{p}{q} \in \mathbb{Q}$, consider the expression

$$\begin{aligned}
\frac{m}{n} \cdot \frac{p}{q} &= \frac{mp}{nq} \\
&= \frac{pm}{nq} \text{ multiplication is commutative in } \mathbb{Z} \\
&= \frac{pm}{qn} \text{ multiplication is commutative in } \mathbb{N} \\
&= \frac{p}{q} \cdot \frac{m}{n}
\end{aligned}$$

So multiplication is commutative in $\mathbb{Q}$. $\square$

Problem 4.17: Prove that there is no $r \in \mathbb{Q}$ so that $r^2 = 6$.

Proof. Assume by way of contradiction, $\exists r \in \mathbb{Q}$ so that $r^2 = 6$. Since $r \in \mathbb{Q}$, $\exists i \in \mathbb{Z}$ and $\exists j \in \mathbb{N}$ so that $r = \frac{i}{j}$. Let D be a set of natural numbers such that

$$D = \{k \in \mathbb{N} \mid \exists l \in \mathbb{Z} \text{ so that } r = \frac{l}{k}\} \tag{5}$$

Since $r = \frac{i}{j}$, we know $j \in D$ and so $D \neq \emptyset$. By the Well Ordering principle, D has a minimum. Call it m_D. So $\exists m_N \in \mathbb{Z}$ so that $r = \frac{m_N}{m_D}$
Consider $r = \frac{m_N}{m_D}$.

$$\begin{aligned}
r^2 = \frac{m_N^2}{m_D^2} &= 6 \\
m_N^2 &= 6m_D^2 \\
m_N^2 &= 2.3.m_D^2
\end{aligned}$$

This means m_N^2 is even. And so is m_N, which means $\exists n \in \mathbb{Z}$ so that $m_N = 2n$. We have

$$\begin{aligned}
m_N^2 &= 6m_D^2 \\
(2n)^2 &= 6m_D^2 \\
4n^2 &= 6m_D^2 \\
2n^2 &= 3m_D^2
\end{aligned}$$

This means $3m_D^2$ is even. And so is m_D^2. And so is m_D, which means $\exists m \in \mathbb{Z}$ so that $m_D = 2m$. So now we can rewrite r as $r = \frac{m_N}{m_D} = \frac{2n}{2m} = \frac{n}{m}$.
And so $m \in D$.
But our assumption says that m_D is the minimum of D. And now we have $m_D = 2m$ and $m \in D$. This contradicts our assumption that m_D is the minimum of D. $\Rightarrow\Leftarrow$
So $\nexists r \in \mathbb{Q}$ such that $r^2 = 6$. $\square$