Math 323 HW3

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Problem 2.4: Prove: $\forall n, m \in \mathbb{N}$, with $n < m$ $\sum\limits_{k=n}^{m} k = \frac{(n+m)(m-n+1)}{2}$

(a) Use previously proved theorems.

Proof. Assume $n, m \in \mathbb{N}$ with $n < m$. In class we have proven the claim

Claim 1. $\forall l \in \mathbb{N}$, $\sum\limits_{k=1}^{l} k = \frac{l(l+1)}{2}$

Claim 1 basically says

$$\sum_{k=1}^{l} k = 1 + 2 + ... + l - 1 + l = \frac{l(l+1)}{2}$$

By the definition of summation, consider

$$\begin{aligned}
\sum_{k=n}^{m} k &= n + n + 1 + n + 2 + ... + m - 1 + m \\
&= (1 + 2 + ... + m - 1 + m) - (1 + 2 + ... + n - 1) \\
&= \sum_{k=1}^{m} k - \sum_{k=1}^{n-1} k \\
&= \frac{m(m+1)}{2} - \frac{(n-1)n}{2} \\
&= \frac{m^2 + m - n^2 + n}{2} \\
&= \frac{(m-n)(m+n) + (m+n)}{2} \\
&= \frac{(m+n)(m-n+1)}{2} \quad \square
\end{aligned}$$

(b) Use induction on m.

Proof. Assume $n, m \in \mathbb{N}$ and $n < m$. We will prove the above statement using a proof by induction on m. Since $n, m \in \mathbb{N}$ and $n < m$, it can be implied that $m \geq 2$. We will need to prove two claims.

i. If $m = 2$, then $\forall n < m \sum\limits_{k=n}^{m} k = \frac{(n+m)(m-n+1)}{2}$

Proof. Assume $m = 2$, since $n < m, n = 1$. We have

$$\sum_{k=n}^{m} k = 1 + 2 = 3 \text{ and}$$

$$\frac{(n+m)(m-n+1)}{2} = \frac{(1+2)(2-1+1)}{2} = 3$$

ii. If for $m = m_0$, $\forall n < m \sum\limits_{k=n}^{m} k = \frac{(n+m)(m-n+1)}{2}$, then for $m = m_0 + 1$, $\forall n < m \sum\limits_{k=n}^{m} k = \frac{(n+m)(m-n+1)}{2}$

Proof. Assume $m = m_0$, $\forall n < m \sum\limits_{k=n}^{m} k = \frac{(m+n)(m-n+1)}{2}$, meaning

$$\sum_{k=n}^{m_0} k = \frac{(m_0+n)(m_0-n+1)}{2}$$

$$\sum_{k=n}^{m_0} k + m_0 + 1 = \frac{(m_0+n)(m_0-n+1)}{2} + m_0 + 1$$

$$\begin{aligned}
\sum_{k=n}^{m_0+1} k &= \frac{(m_0+n)(m_0-n+1)}{2} + m_0 + 1 \\
&= \frac{(m_0+n)(m_0-n+1) + 2(m_0+1)}{2} \\
&= \frac{m_0^2 - m_0 n + m_0 + n m_0 - n^2 + n + 2m_0 + 2}{2} \\
&= \frac{m_0^2 - m_0 n + m_0 + n m_0 - n^2 + 2n - n + 2m_0 + 2}{2} \\
&= \frac{m_0^2 + m_0 n + m_0 - n m_0 - n^2 + -n + 2n + 2m_0 + 2}{2} \\
&= \frac{m_0(m_0+n+1) - n(m_0+n+1) + 2(n+m_0+1)}{2} \\
&= \frac{(m_0+n+1)(m_0-n+2)}{2} \\
&= \frac{((m_0+1)+n)((m_0+1)-n+1)}{2}
\end{aligned}$$

So we have proved if for $m = m_0$, $\forall n < m \sum\limits_{k=n}^{m} k = \frac{(n+m)(m-n+1)}{2}$ then for $m = m_0 + 1$, $\forall n < m \sum\limits_{k=n}^{m} k = \frac{(n+m)(m-n+1)}{2}$.

Proving claim (i) and (ii) completes our proof by induction on m. $\square$

Problem 2.10: Prove that the sum of two odd integers is even.

Proof. Let $n, m \in \mathbb{Z}$. Assume n, m are odd numbers, meaning

$$\exists k \in \mathbb{Z} \text{ s. t } n = 2k + 1$$
$$\exists l \in \mathbb{Z} \text{ s. t } m = 2l + 1$$

Consider the expression $n + m$

$$\begin{aligned} n + m &= (2k + 1) + (2l + 1) \\ &= 2k + 2l + 2 \\ &= 2(k + l + 1) \end{aligned}$$

We know $(k + l + 1) \in \mathbb{Z}$ and so the sum of two odd integers is even. $\square$

Problem 2.13: Prove that if $n, m \in \mathbb{Z}$ and nm is even, then either n is even or m is even.

Proof. Assume $n, m \in \mathbb{Z}$. We will prove the contrapositive of this statement: If both n and m are odd, then nm is odd. Assume both n and m are odd natural numbers, meaning

$$\exists k \in \mathbb{Z} \text{ s. t } n = 2k + 1$$
$$\exists l \in \mathbb{Z} \text{ s. t } m = 2l + 1$$

We consider the product nm

$$\begin{aligned} nm &= (2k + 1)(2l + 1) \\ &= 4kl + 2l + 2k + 1 \\ &= 2(2kl + l + k) + 1 \end{aligned}$$

Since $k, l \in \mathbb{Z}$, we know $(2kl + l + k) \in \mathbb{N}$. This completes the proof by contraposition. $\square$