Math 323 HW5

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Problem 3.2: Give an example of a set of integers with at least one element but no minimum.

Solution. Let $A = \{k \in \mathbb{Z} \mid k < 0\}$ is a nonempty set of integers that does not have a minimum.

Problem 3.4: Consider $A = \{m \in \mathbb{Z} \mid m > 17\}$.

(a) Is 17 a lower bound of A?

(b) Is 12 a lower bound of A?

(c) Is 20 a lower bound of A?

(d) Is 17 a minimum of A?

(e) Is 12 a minimum of A?

(f) Is 20 a minimum of A?

(g) Is 18 a lower bound of A?

(h) Is 18 a minimum of A?

Solution: We review our definition of lower bound and minimum from Prof. Madden's book.

Definition 1. In an ordered number system the number l is said to be a lower bound on a set S when: if $s \in S$, then $l \leq s$.

Definition 2. Let S be a set of numbers. We say m is a minimum of the set S when

1. $m \in S$
2. If $s \in S$, then $m \leq s$

(a) 17 is a lower bound of A.

(b) 12 a lower bound of A.

(c) 20 is not a lower bound of A since $20 \in A$, $18 \in A$ but $20 > 18$.

(d) 17 is not a minimum of A since it's not in the set A.

(e) 12 is not a minimum of A since it's not in the set A.

(f) 20 is not a minimum of A since $18 \in A$ but $20 > 18$.

(g) 18 is a lower bound of A since for $m \in \mathbb{Z}$ and $m > 17$, $m \geq 18$.

(h) 18 is a minimum of A since $18 \in A$ and $\forall m \in A$, $m \geq 18$.

Problem 3.12: Let A and B be sets of integers so that every element of A is also an element of B. Let $r, s \in \mathbb{Z}$ with $r < s$. Are the following true or false?

(a) If r is a lower bound on A, then s is a lower bound on A.

False. If there is an integer k such that $r < k < s$ and if $k \in A$ then s is not a lower bound of A.

(b) If s is a lower bound on A, then r is a lower bound on A.

True since s is smaller than every elements in A and $r < s$.

(c) If r is a lower bound on A, then s is a lower bound on B.

False. We know $A \subseteq B$. So there is a case that $A = B$. Assume set A or B has a minimum and assume such minimum is actually r. Assume $s \in B$. We know $s > r$. And so s cannot be a lower bound on B.

(d) If s is a lower bound on A, then r is a lower bound on B.

False. We know $A \subseteq B$, but what if $\exists k \in B$ and $k \notin A$ such that $k < r$?

(e) If r is a lower bound on B, then s is a lower bound on A.

False. s is not a lower bound on A if $\exists a \in A$ such that $r < a < s$.

(f) If s is a lower bound on B, then r is a lower bound on A.

True.

Problem 3.13: Let A be a set of integers and $m \in \mathbb{Z}$. Define the terms:

(a) m is a maximum of A.

(b) m is an upper bound of A.

Solution.

Definition 3. Let A be a set of integers and $m \in \mathbb{Z}$. We say m is a maximum of A if and only if

1. $m \in A$
2. If $a \in A$, then $a \leq m$

Definition 4. Let A be a set of integers and $m \in \mathbb{Z}$. We say m is an upper bound of A if and only if it satisfies this condition: if $a \in A$, then $m \geq a$.

Problem 3.11: Let $a, b \in \mathbb{Z}$. Prove: If $a \neq b$, then $ab \neq 1$.

Proof. Assume $a, b \in \mathbb{Z}$ and $a \neq b$. Assume by way of contradiction, $ab = 1$. By our assumption, $a \neq 0$ and $b \neq 0$. Since $a, b \in \mathbb{Z}$,

$$a^2 > 0 \text{ and } b^2 > 0$$
$$a^2 \geq 1 \text{ and } b^2 \geq 1$$
$$a^2b^2 \geq b^2 \text{ and } b^2a^2 \geq a^2 \text{ but } ab = 1$$
$$1 \geq b^2 \text{ and } 1 \geq a^2 \Rightarrow\!\Leftarrow$$

So $ab \neq 1$ and thus this completes the proof by contradiction. $\square$