Math 323 HW23

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Problem 16.1: Prove that for all $A \subseteq \mathbb{R}$:

(a) $Int(A) \subseteq A$

Proof. Assume $x \in Int(A)$. So then $\exists \epsilon > 0$ so that $N(x, \epsilon) \subseteq A$. We know $x \in N(x, \epsilon)$. But $N(x, \epsilon) \subseteq A$. So then $x \in A$. Thus $Int(A) \subseteq A$. □

(b) $A \subseteq Cl(A)$

Proof. Assume $x \in A$. Want to show $x \in Cl(A)$, which means $x \in Int(A) \cup \partial(A)$. By (a), we know if $a \in Int(A)$, then $a \in A$. So then we have 2 cases:

i. $x \in Int(A)$. Then $x \in Int(A) \cup \partial(A)$ and thus $x \in Cl(A)$.
ii. $x \notin Int(A)$. In short, $x \in A$ and $x \notin Int(A)$.

□

(c) $A^\circ \subseteq A$

Proof. Assume $x \in A^\circ$. So then $\exists \epsilon > 0$ so that $N(x, \epsilon) \cap A = \{x\}$. So then $x \in A$. Thus $A^\circ \subseteq A$. □

(d) $A^\circ \subseteq \partial(A)$

Proof. Assume $a \in A^\circ$. So $\exists \epsilon > 0$ so that $N(a, \epsilon) \cap A = \{a\}$. So then $N(a, \epsilon) \cap A \neq \emptyset$. By the definition of ϵ-neighborhood, $N(a, \epsilon) = \{x \in \mathbb{R} \mid |x - a| < \epsilon\}$. We also know $N(x, \epsilon) \cap \mathbb{R} \setminus A \neq \emptyset$.So then $a \in \partial(A)$. □

(e) $Int(A) \subseteq A'$.

Problem 16.4: Let $A \subseteq \mathbb{R}$ with $A \neq \emptyset$. Prove that if A is bounded below, then $Inf(A) \in \partial(A)$.

Proof. Let $A \subseteq \mathbb{R}$ with $A \neq \emptyset$. Assume A is bounded below. By the completeness axiom, A has an infimum, denoted by $Inf(A)$. $Inf(A)$ has the following properties:

1. If $a \in A$, then $Inf(A) \leq a$.
2. If $x \in \mathbb{R}$ and $x > Inf(A)$, then $\exists \alpha \in A$ s.t $\alpha < x$.

We want to prove that $\forall \epsilon > 0, N(Inf(A), \epsilon) \cap (\mathbb{R} \setminus A) \neq \emptyset$ and $N(Inf(A), \epsilon) \cap A \neq \emptyset$.
Let $\epsilon > 0$. We know $Inf(A) - \epsilon \notin A$. So then $(Inf(A) - \epsilon, Inf(A) + \epsilon) \cap (\mathbb{R} \setminus A) \neq \emptyset$. This means $N(Inf(A), \epsilon) \cap (\mathbb{R} \setminus A) \neq \emptyset$.
Let $x = Inf(A) + \epsilon$. By (2), $\exists \alpha \in A$ s.t $\alpha < Inf(A) + \epsilon$.So then $\alpha \in (Inf(A) - \epsilon, Inf(A) + \epsilon)$, which means $\alpha \in N(Inf(A), \epsilon)$.Thus $N(Inf(A), \epsilon) \cap A \neq \emptyset$. □

Problem 16.6a: Let $A \subseteq B \subseteq \mathbb{R}$. Prove the following: $Int(A) \subseteq Int(B)$.

Proof. Let $A \subseteq B \subseteq \mathbb{R}$. Assume $x \in Int(A)$. So then $\exists \epsilon > 0$ such that $N(x, \epsilon) \subseteq A$. But we know $N(x, \epsilon) = \{r \in \mathbb{R} \mid |r - x| < \epsilon\}$, which means $x \in N(x, \epsilon)$. Since $N(x, \epsilon) \subseteq A$ and we also know $A \subseteq B$, $N(x, \epsilon) \subseteq B$. This means $x \in Int(B)$. □

Problem 16.7a: Let $A \subseteq B \subseteq \mathbb{R}$. Why can't we use this to prove the following results? $\partial(A) \subseteq \partial(B)$.

Attempted answer. Let $A \subseteq B \subseteq \mathbb{R}$. Let $x \in \partial(A)$ and let $\epsilon > 0$. So then $N(x, \epsilon) \cap A \neq \emptyset$ and $N(x, \epsilon) \cap (\mathbb{R} \setminus A) \neq \emptyset$. Since $A \subseteq B$ and $N(x, \epsilon) \cap A \neq \emptyset$, $N(x, \epsilon) \cap B \neq \emptyset$. Knowing $N(x, \epsilon) \cap (\mathbb{R} \setminus A) \neq \emptyset$ is not enough to say $N(x, \epsilon) \cap (\mathbb{R} \setminus B) \neq \emptyset$.

Example. Consider set $A = (1, 2)$ and $B = (0, 3)$. It's true that $A \subseteq B \subseteq \mathbb{R}$. But $\partial(A) = \{1, 2\}$ and $\partial(B) = \{0, 3\}$ and $\partial(A) \not\subseteq \partial(B)$.