Math 323 HW7

Minh Bui

May 25, 2017

Problem 5.2: Let $\mathbb{F}$ be an ordered field.

(a) Prove: For all $n \in \mathbb{N}$, if $a_1 \in \mathbb{F}; a_2 \in \mathbb{F}; ...a_n \in \mathbb{F}$, then $\sum\limits_{k=1}^{n} a_k^2 \geq 0$.

(b) Prove: For all $n \in \mathbb{N}$, if $a_1 \in \mathbb{F}; a_2 \in \mathbb{F}; ...a_n \in \mathbb{F}$, and $\sum\limits_{k=1}^{n} a_k^2 = 0$, then $a_1 = a_2 = a_3 = ... = a_n = 0$.

Solution.

(a) We will prove (a) using induction on n.

Proof. Assume $n \in \mathbb{N}$. We need to prove two claims.

i. We want to show that: If for $n = 1$, then if $a_1 \in \mathbb{F}; ...; a_n \in \mathbb{F}$, then $\sum\limits_{k=1}^{n} a_k^2 \geq 0$.
Assume $n = 1$. Assume $a_1 \in \mathbb{F}; ...; a_n \in \mathbb{F}$. So $a_1 \in \mathbb{F}$. Consider the expression

$$\sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{1} a_k^2 = a_1^2 \geq 0$$

So for $n = 1$, if $a_1, ..., a_n \in \mathbb{F}$, then $\sum\limits_{k=1}^{n} a_k^2 \geq 0$.

ii. We also want to show that: If for $n = n_0$, if $a_1 \in \mathbb{F}; ...; a_n \in \mathbb{F}$, then $\sum\limits_{k=1}^{n} a_k^2 \geq 0$ then for $n = n_0 + 1$, if $a_1, ..., a_n \in \mathbb{F}$, then $\sum\limits_{k=1}^{n} a_k^2 \geq 0$.
Assume for $n = n_0$. Assume if $a_1, ..., a_n \in \mathbb{F}$, then $\sum\limits_{k=1}^{n} a_k^2$. And so if $a_1, ..., a_{n_0} \in \mathbb{F}$, then $\sum\limits_{k=1}^{n_0} a_k^2 \geq 0$. Assume $\exists a_{n_0+1} \in \mathbb{F}$.
We know $\sum\limits_{k=1}^{n_0} a_k^2 \geq 0$ by our inductive hypothesis. We also know

$a_{n_0+1}^2 \geq 0$. So we can conclude that

$$\sum_{k=1}^{n_0} a_k^2 + a_{n_0+1}^2 = \sum_{k=1}^{n_0+1} a_k^2 \geq 0$$

So we have proved that: If for $n = n_0$, if $a_1 \in \mathbb{F}; ...; a_n \in \mathbb{F}$, then $\sum\limits_{k=1}^{n} a_k^2 \geq 0$ then for $n = n_0 + 1$, if $a_1, ..., a_n \in \mathbb{F}$, then $\sum\limits_{k=1}^{n} a_k^2 \geq 0$.

Proving (i) and (ii) thus completes our proof by induction on n. $\square$

(b) We will prove (b) using induction on n.

Proof. Assume $n \in \mathbb{N}$. We need to prove two claims.

i. We want to show that: If for $n = 1$, then if $a_1, ..., a_n \in \mathbb{F}$ and $\sum\limits_{k=1}^{n} a_k^2 = 0$, then $a_1 = a_2 = ... = a_n = 0$.
Assume $n = 1$. Assume $a_1, ..., a_n \in \mathbb{F}$. So $a_1 \in \mathbb{F}$. Assume $\sum\limits_{k=1}^{n} a_k^2 = 0$, meaning

$$\sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{1} a_k^2 = a_1^2 = 0$$

So $a_1^2 = 0$. By trichotomy, we can imply that $a_1 = 0$.
So we have proved that for $n = 1$, then if $a_1, ..., a_n \in \mathbb{F}$ and $\sum\limits_{k=1}^{n} a_k^2 = 0$, then $a_1 = a_2 = ... = a_n = 0$.

ii. We also want to show that: If for $n = n_0$, if $a_1, ..., a_n \in \mathbb{F}$ and $\sum\limits_{k=1}^{n} a_k^2 = 0$, then $a_1 = a_2 = ... = a_n = 0$, then for $n = n_0 + 1$, if $a_1, ..., a_n \in \mathbb{F}$ and $\sum\limits_{k=1}^{n} a_k^2 = 0$, then $a_1 = a_2 = ... = a_n = 0$.

Assume for $n = n_0$, $a_1, ..., a_n \in \mathbb{F}$. Assume $\sum\limits_{k=1}^{n} a_k^2 = 0$. By our inductive hypothesis, we have $a_1 = a_2 = ... = a_{n_0} = 0$. Assume $\exists a_{n_0+1} \in \mathbb{F}$. By trichotomy, exactly one of these holds: $a_{n_0+1} > 0, a_{n_0+1} = 0$, or $a_{n_0+1} < 0$. Whatever case that is, we know $a_{n_0+1}^2 \geq 0$. Now, we consider $\sum\limits_{k=1}^{n_0} a_k^2$.

$$\sum_{k=1}^{n_0} a_k^2 + a_{n_0+1}^2 = \sum_{k=1}^{n_0+1} a_k^2 \geq 0 \tag{1}$$

Our inductive hypothesis says that $\sum\limits_{k=1}^{n_0} a_k^2 = 0$.
And we know $a_{n_0+1}^2 \geq 0$. So the equality of the equation (1)

holds if and only if $a_{n_0+1}^2 = 0$. This means that it must be the case that $a_{n_0+1} = 0$.
And so $a_1 = a_2 = ... = a_{n_0} = a_{n_0+1} = 0$.
So we have proved that: If for $n = 1$, then if $a_1, ..., a_n \in \mathbb{F}$ and $\sum\limits_{k=1}^{n} a_k^2 = 0$, then $a_1 = a_2 = ... = a_n = 0$.

Proving both claim (i) and (ii) completes our proof by induction on n. ☐

Problem 5.7: Let $\mathbb{F}$ be any ordered field with $a, b, c \in \mathbb{F}$, prove that $|a-c| \leq |a-b|+|c-b|$.

Proof. Assume $a, b, c \in \mathbb{F}$. To prove the above statement, we need to prove a lemma.

Lemma 0.1. *Let $\mathbb{F}$ be any ordered field with $a, b \in \mathbb{F}$: $|a - b| = |b - a|$.*

Proof. Assume $a, b \in \mathbb{F}$. By the definition of absolute value we have

$$|a-b| = \begin{cases} -(a-b) = b-a & \text{if } (a-b) < 0 \Leftrightarrow a < b, \\ 0 & \text{if } a-b = 0 \Leftrightarrow a = b, \\ (a-b) & \text{if } a-b > 0 \Leftrightarrow a > b. \end{cases}$$

And

$$|b-a| = \begin{cases} -(b-a) = a-b & \text{if } (b-a) < 0 \Leftrightarrow a > b, \\ 0 & \text{if } b-a = 0 \Leftrightarrow a = b, \\ (b-a) & \text{if } b-a > 0 \Leftrightarrow a < b. \end{cases}$$

So we can conclude that $|a - b| = |b - a|$ for $a, b \in \mathbb{F}$. ☐

Now we are ready to prove our main statement. Consider $|a - b| + |c - b|$

$$|a-b| + |c-b| = |a-b| + |b-c| \text{ by lemma 0.1}$$
$$|a-b| + |b-c| \geq |(a-b) + (b-c)| \text{ by the Triangle Inequality}$$
$$|a-b| + |b-c| \geq |a-b+b-c|$$
$$|a-b| + |b-c| \geq |a-c|$$
$$\text{So } |a-b| + |c-b| \geq |a-c|$$

☐

Problem 5.8a: Let $\mathbb{F}$ be any ordered field. Prove that if $a, b \in \mathbb{F}$ so that $a < b$, then $\forall n \in \mathbb{N}$, there are numbers $x_i \in \mathbb{F}$ so that $a < x_1 < x_2 < ... < x_n < b$.

Proof. Assume $a, b \in \mathbb{F}$ so that $a < b$. Since we have $n \in \mathbb{N}$, we can attempt to prove the above proposition using a proof by induction on n. We need to prove two claims.

(i) We want to prove that: If for $n = 1$, then $a < x_1 < x_2 < ... < x_n < b$.
Assume $n = 1$. Since $a, b \in \mathbb{F}$ and $a < b$, by the Average theorem, $\exists x_1 \in \mathbb{F}$ such that $a < x_1 < b$. And so for $n = 1$, $a < x_1 < x_2 < .. < x_n < b$.

(ii) We also want to prove: If for $n = n_0$, $a < x_1 < x_2 < ... < x_n < b$, then for $n = n_0 + 1$, $a < x_1 < x_2 < ... < x_n < b$.
Assume for $n = n_0$, $a < x_1 < x_2 < ... < x_n < b$, meaning

$$a < x_1 < x_2 < ... < x_{n_0-1} < x_{n_0} < b \text{ where } x_1, x_2, ..., x_{n_0} \in \mathbb{F} \quad (2)$$

From (1) we know $x_{n_0} < b$. Again, by the Average theorem, $\exists x_{n_0+1} \in \mathbb{F}$ such that $x_{n_0} < x_{n_0+1} < b$. In conclusion, we know

$$a < x_1 < x_2 < ... < x_{n_0-1} < x_{n_0} < x_{n_0+1} < b,$$

which is what we want to prove.

Proving both claim (i) and (ii) thus completes our proof by induction on n. $\square$