Math 323 HW12

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Problem 7.10: Prove that $\forall n, m \in \mathbb{Z}$ if nm is even, then either n is even or m is even.

Proof. Let $n, m \in \mathbb{N}$. We will prove the contrapositive statement: If n is odd and m is odd, then nm is odd.
Assume n is odd and m is odd. Respectively,

$$\exists k \in \mathbb{Z} \text{ s.t } n = 2k + 1$$
$$\exists l \in \mathbb{Z} \text{ s.t } m = 2l + 1$$

Consider nm,

$$\begin{aligned} nm &= (2k+1)(2l+1) \\ &= 4kl + 2l + 2k + 1 \\ &= 2(2kl + l + k) + 1 \end{aligned}$$

We know $2kl + l + k \in \mathbb{Z}$, so nm is odd. This completes the proof by contraposition. $\square$

Problem 7.12: Let $S = \{x \in \mathbb{R} \mid x^{-1} \in \mathbb{N}\}$. Prove that 0 is the infimum of S.

Proof. We will need to establish 2 claims.

1. If $s \in \mathbb{S}$, then $0 \leq s$.
   Assume $s \in \mathbb{S}$, so $s = n^{-1} = \frac{1}{n}$ where $n \in \mathbb{N}$. Since $n \in \mathbb{N}$, we know $\frac{1}{n} > 0$. So $\frac{1}{n} \geq 0$. This means 0 is a lower bound on set S.
2. If $x \in \mathbb{R}$ and $x > 0$, then $\exists t \in S$ s.t $t < x$.
   Assume $x \in \mathbb{R}$ and $x > 0$. By the Archimedean principle, $\exists n \in \mathbb{N}$ so that $0 < \frac{1}{n} < x$. Let $t = \frac{1}{n}$ so we know $t \in S$.

Thus 0 is the infimum of the set $S = \{x \in \mathbb{R} \mid x^{-1} \in \mathbb{N}\}$. $\square$

Problem 7.*: Prove the following statement: Let $\epsilon \in \mathbb{R}$ and let $\epsilon > 0$. $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Proof. Assume $\epsilon \in \mathbb{R}$ and $\epsilon > 0$. Since $\epsilon \in \mathbb{R}$, $\frac{1}{\epsilon} \in \mathbb{R}$. By the Archimedean principle, $\exists m \in \mathbb{N}$ s.t $\frac{1}{\epsilon} < m$. Since $\epsilon > 0$, $1 < \epsilon m$. So $\frac{1}{m} < \epsilon$ since $m \in \mathbb{N}$. Let $n = m$ and we are done. $\square$