Math 323 HW4

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Problem 2.14: Prove the following: Let $P(n)$ be a statement that is either true or false (but not both) for each $n \in \mathbb{N}$. Let $m \in \mathbb{N}$. If the following two statements hold: If $n = m$, then $P(n)$ is true; and if for $n = n_0, P(n)$ is true, then for $n = n_0 + 1, P(n)$ is true, then for all $n \in \mathbb{N}$ where $n \geq m, P(n)$ is true.

Proof. Let $P(n)$ be a statement that is either true or false but not both for each $n \in \mathbb{N}$. Assume $m \in \mathbb{N}$ and $n \geq m$. Assume

(a) If $n = m$, then $P(n)$ is true.

(b) If for $n = n_0$, $P(n)$ is true, then for $n = n_0 + 1, P(n)$ is true.

Let A be the set such that

$$A = \{k \in \mathbb{N} \mid P(k) \text{ is false and } k \geq m\} \tag{1}$$

We consider two cases for this set A.

- Case 1: A is not empty.
  Assume by way of contradtiction A is not empty. Since set A is a set of natural numbers, by the Well Ordering principle, it has a minimum. Call it m_A. By definition, m_A has two properties:
  - $m_A \in A$.
  - If $s \in A$, then $s \geq m_A$.

  Since $m_A \in A$, we know $m_a \geq m$. By our assumption (a), $P(m)$ is true and so $m \notin A$. And so now we know $m_A > m$, meaning $\exists s \in \mathbb{N}$ s.t $s + m = m_A$ and so $k \geq 1$.
  Again, $P(m)$ is true. By our assumption (b), $P(m + 1)$ is also true. Because $P(m + 1)$ is true, $P(m + 2)$ is also true. Without loss of generality, $P(m + s)$ is also true. But this means that $m_A \notin A$ and this thus contradicts our assumption about the set A having a minimum. So the set of natural numbers A must be empty.
- Case 2: A is empty.
  A is empty meaning $n \notin \{k \in \mathbb{N} \mid P(k) \text{ is false and } k \geq m\}$
  By our assumptions,
  - $n \geq m$

- If $n = m$, then $P(n)$ is true.
- If for $n = n_0$, $P(n)$ is true, then for $n = n_0 + 1, P(n)$ is true.

We can conclude that $\forall n \in \mathbb{N}$ where $n \geq m, P(n)$ is true. □

Problem 2.7: Look up official mathematical definition of "factorial."

(a) Prove: $\forall n \in \mathbb{N}$ large enough, $n! \geq n + 200$.

(b) Prove: $\forall n \in \mathbb{N}$ large enough, $n! \geq 2^n$.

Proof. Assume $n \in \mathbb{N}$. Since there is no specific threshold for n. We will prove both (a) and (b) by picking thresholds for n.

(a) We will prove the following statement: $\forall n \in \mathbb{N}$ and $n \geq 6$, $n! \geq n+200$ using induction on n. Assume $n \in \mathbb{N}$ and $n \geq 6$, we will need to prove two claims.

i. If $n = 6$, then $n! \geq n + 200$.

Proof of claim (i). Assume $n = 6$. By the definition of factorial,

$$n! = 6! = 1.2.3.4.5.6 = 720. \text{ and } n + 200 = 6 + 200 = 206.$$

So for $n = 6$, $n! \geq n + 200$.

ii. If for $n = n_0$, $n! \geq n + 200$, then for $n = n_0 + 1, n! \geq n + 200$.

Proof of claim (ii). Assume for $n = n_0$, $n \geq n + 200$. Our inductive hypothesis means

$$\begin{aligned} n_0! &\geq n_0 + 200 \\ n_0!(n_0 + 1) &\geq (n_0 + 200)(n_0 + 1) \\ (n_0 + 1)! &\geq (n_0 + 200)(n_0 + 1) \\ (n_0 + 1)! &\geq n_0^2 + 201n_0 + 200 \\ (n_0 + 1)! &\geq n_0^2 - 1 + 200n_0 + n_0 + 1 + 200 > n_0 + 1 + 200 \\ (n_0 + 1)! &\geq n_0 + 1 + 200 \end{aligned}$$

We can make the two last claims because of our assumption $n \geq 6$ and so we know $n_0^2 - 1 + 200n_0 > 1$. So we have proved that for $n = n_0 + 1$, $n! \geq n + 200$.

Proving claim (i) and (ii) thus completes our proof by induction on n. □

(b) We will prove the following statement: $\forall n \in \mathbb{N}$ and $n \geq 4$, $n! \geq 2^n$ using induction on n. Assume $n \in \mathbb{N}$ and $n \geq 4$, we will prove two claims.

i. If $n = 4$, then $n! \geq 2^n$.

Proof of claim (i). Assume $n = 4$. By the definition of factorial,

$$n! = 4! = 1.2.3.4 = 24. \text{ and } 2^n = 2^4 = 16.$$

So for $n = 4$, $n! \geq 2^n$.

ii. If for $n = n_0$, $n! \geq 2^n$, then for $n = n_0 + 1$, $n! \geq 2^n$.
Proof of claim (ii). Assume for $n = n_0$, $n! \geq 2^n$, we have

$$\begin{aligned} n_0! &\geq 2^{n_0} \\ n_0!(n_0+1) &\geq 2^{n_0}(n_0+1) \\ (n_0+1)! &\geq 2^{n_0}(n_0+1) \\ (n_0+1)! &\geq 2^{n_0}n_0 + 2^{n_0} \end{aligned}$$

We know $n_0 \geq 4$ by our initial assumption. So

$$\begin{aligned} 2^{n_0}n_0 &> 2^{n_0}.2 \\ 2^{n_0}n_0 &> 2^{n_0+1} \\ 2^{n_0}n_0 + 2^{n_0} &> 2^{n_0+1} \end{aligned}$$

Combining (1) and (2):$(n_0+1)! \geq 2^{n_0+1}$.

Proving claim (i) and (ii) thus completes our proof by induction on n. $\square$