Linear Algebra Chapter 1

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Section 1.2

1. Label the following statements as being true or false.
   i. Every vector space contains a zero vector.
      True by the property of vector space.
   ii. A vector space may have more than 1 zero vector.
      False by the uniqueness of the additive identity.
   iii. In any vector space $ax = bx$ implies that $a = b$.
      False. $ax - bx = 0$. So $x(a - b) = 0$. x could be a zero vector while $a \neq b$.
   iv. In any vector space $ax = ay$ implies $x = y$.
      False. What if $a = 0$?
   v. An element of F^n may be regarded as an element of $M_{n\times 1}(F)$.
      True. Since n-tuple can be written as a column vector.
   vi. An $m \times n$ matrix has m columns and n rows.
      False. An $m \times n$ matrix has m rows and n columns.
   vii. In the vector space $P(F)$ only polynomials of the same degree may be added.
      False.
   viii. If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.
      False. Consider $x + (-x) = 0$
   ix. If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n.
      True.
   x. A nonzero element of F may be considered to be an element of $P(F)$ having degree zero.
      True.
   xi. Two funstions in $\mathcal{F}(S, F)$ are equal if and only if they have the same values at each element of S.
      True.
2. Write the zero vector of $M_{3\times 4}(F)$.

Solution.
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3. If
$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$
what are M_{13}, M_{21}, and M_{22}?

   Solution. $M_{13} = 3$. $M_{21} = 4$. $M_{22} = 5$.

4. Perform the indicated operations.
   (a) $\begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 5 \\ -5 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 3 & 2 \\ -4 & 3 & 9 \end{pmatrix}$
   (b) $\begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix} + \begin{pmatrix} 7 & -5 \\ 0 & -3 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & -5 \\ 3 & 8 \end{pmatrix}$
   (c) $4\begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 8 & 10 & -12 \\ 4 & 0 & 28 \end{pmatrix}$
   (d) $-5\begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} 30 & -20 \\ -15 & 10 \\ -5 & -40 \end{pmatrix}$
   (e) $(2x^4 - 7x^3 + 4x + 3) + (8x^3 + 2x^2 - 6x + 7) = 2x^4 + x^3 + 2x^2 - 2x + 10$.
   (f) $(-3x^3 + 7x^2 + 8x - 6) + (2x^3 - 8x + 10) = -x^3 + 7x^2 + 4$
   (g) $5(2x^7 - 6x^4 + 8x^2 - 3x) = 10x^7 - 30x^4 + 40x^2 - 15x$
   (h) $3(x^5 - 2x^3 + 4x + 2) = 3x^5 - 6x^3 + 12x + 6$.

7. Let $S = \{0, 1\}$ and $F = R$. In $\mathcal{F}(S, R)$ show that $f = g$ and $f + g = h$, where $f(x) = 2x + 1, g(x) = 1 + 4x - 2x^2$, and $h(x) = 5^x + 1$.

   Solution. To show $f = g$, we need to show: if $s \in S$, then $f(s) = g(s)$.
   - For $s = 0$, $f(0) = 1$ and $g(0) = 1$. So $f(s) = g(s)$ for $s = 0$.
   - For $s = 1$, $f(1) = 3$ and $g(1) = 1 + 4 - 2 = 3$. So $f(s) = g(s)$ for $s = 1$.

   So $f = g$.
   To show $f + g = h$, we need to show: if $s \in S$, then $(f + g)(s) = h(s)$.
   - For $s = 0$, $(f + g)(s) = f(s) + g(s) = f(0) + g(0) = 2$. $h(s) = h(0) = 5^0 + 1 = 2$.
   - For $s = 1$, $(f + g)(s) = f(s) + g(s) = f(1) + g(1) = 6$. $h(s) = h(1) = 5^1 + 1 = 6$.

   So $f + g = h$.

8. In any vector space $\mathcal{V}$, show that $(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in \mathcal{V}$ and any $a, b \in F$.

Solution. Let $x, y \in \mathcal{V}$. Let $a, b \in F$. Consider

$$ax + ay = a(x + y) \text{ and } bx + by = b(x + y) \text{ by VS7}$$
$$ax + ay + bx + by = a(x + y) + b(x + y)$$
$$ax + ay + bx + by = (a + b)(x + y) \text{ by VS8}$$

9. Prove that

Corollary 0.0.1. *The vector $0 \in \mathcal{V}$ s.t $x + 0 = x$ for all $x \in \mathcal{V}$ is unique.*

Proof. Let $\mathcal{V}$ be a vector space in F. Assume $x \in \mathcal{V}$. Assume by way of contradiction, there is a $\mathcal{O} \in \mathcal{V}$ s.t $x + \mathcal{O} = x$ and $\mathcal{O} \neq 0$. Since $\mathcal{V}$ is a vector space, $\exists 0 \in \mathcal{V}$ s.t $x + 0 = x$.
Since $\mathcal{O}$ is an additive identity in $\mathcal{V}$, $(x + 0) + \mathcal{O} = x + 0$. So $x + (0 + \mathcal{O}) = x + 0$ by associativity. By the Cancellation Law for vector addition, $0 + \mathcal{O} = 0$.
Since 0 is an additive identity in $\mathcal{V}$, $(x + \mathcal{O}) + 0 = x + \mathcal{O}$. So $x + (0 + \mathcal{O}) = x + \mathcal{O}$ by associativity. By the Cancellation Law for vector addition, $0 + \mathcal{O} = \mathcal{O}$.
So $0 + \mathcal{O} = 0 = \mathcal{O}$ which contradicts our assumption of $\mathcal{O} \neq 0$. So $0 = \mathcal{O}$. Therefore the vector $0 \in \mathcal{V}$ is unique. □

Corollary 0.0.2. *For all $x \in \mathcal{V}$, there is a vector $y \in \mathcal{V}$ so that $x + y = 0$. Prove that this vector y is unique.*

Proof. Pretty much the same approach as the previous corollary. □

10. Let $\mathcal{V}$ denote the set of all differentiable real-valued functions defined on the real line. Prove that $\mathcal{V}$ is a vector space under the operation of addition and scalar multiplication defined by

$$(f + g)(s) = f(s) + g(s) \text{ and } (cf)(s) = c[f(s)]$$

Proof. Let f, g be differentiable real-valued functions in $\mathcal{V}$. Since $(f+g)(s) = f(s)+g(s)$ is differentiable on real numbers, addition is closed in $\mathcal{V}$. Since $(cf)(s) = c[f(s)]$ for some $c \in \mathbb{R}$ is differentiable on real numbers, scalar multiplication is closed in $\mathcal{V}$. And the function $f = 0$ would be the zero vector in the vector space $\mathcal{V}$. Thus, $\mathcal{V}$ is a vector space. □

11. Let $\mathcal{V} = \{0\}$ consist of a single vector 0 and define $0 + 0 = 0$ and $c0 = 0$ for each scalar $c \in \mathbb{F}$. Prove that $\mathcal{V}$ is a vector space over $\mathbb{F}$. Here $\mathcal{V}$ is called the *zero vector space.*

Proof. Since 0 is the only element of $\mathcal{V}$, all conditions and properties for vector space $\mathcal{V}$ can be easily checked. □