# Math 323 HW8

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May 29, 2017

Problem 6.* Prove the *Alternate Completeness axiom*.
We first review our definition of a completed ordered field and the Alternate Completeness Axiom.

**Definition 1.** An ordered field $\mathbb{F}$ is completed when: if for any nonempty subset $S$ of $\mathbb{F}$ and $S$ has at least one lower bound, then $\exists s \in \mathbb{F}$ that is the greatest lower bound of $S$.

**Theorem 0.1.** The Alternate Completeness Axiom. *For a completed ordered field $\mathbb{F}$: If $S$ is a nonempty set from $\mathbb{F}$ and $S$ has at least one upper bound, then $\exists u \in \mathbb{F}$ that is the least upper bound of $S$.*

*Proof.* Assume $\mathbb{F}$ is a completed ordered field.
Assume $S$ is a nonempty set from $\mathbb{F}$.
Assume $S$ has at least one upper bound.
We define

$$UB(S) = \{u \in \mathbb{F} \mid u \text{ is an upper bound of } S\}$$

We first want to prove the following

**Lemma 0.2.** *If $s \in S$ then $s$ is a lower bound of $UB(S)$.*

*Proof.* Assume $s \in S$. We want to prove: if $u \in UB(S)$, then $u \geq s$.
Assume $u \in UB(S)$, since $UB(S)$ is a set of upper bounds of $S$, we know: if $t \in S$, then $u \geq t$. And so $s$ is a lower bound of $UB(S)$. □

We consider $UB(S)$. $UB(S)$ has at least one element by our assumption of $S$ having at least one upper bound. We know $UB(S)$ has at least one lower bound by our lemma. We also know $\mathbb{F}$ is completed. So by the definition of completeness, $\exists g \in \mathbb{F}$ that is the greatest lower bound of $UB(S)$. This means

1. If $u \in UB(S)$, then $u \geq g$.
2. If $x \in \mathbb{F}$ and $x > g$, then $\exists t \in UB(S)$ so that $t < x$.

We claim that $g$ is the least upper bound of $S$. This means we have to prove two properties:

1. If $s \in S$, then $g \geq s$.
   Assume $s \in S$. Since $g$ is the greatest lower bound of $UB(S)$, if $l$ is a lower bound on $UB(S)$, then $g \geq l$. By our lemma, we know if $s \in S$, then $s$ is a lower bound on $UB(S)$. And so $g \geq s$.
2. If $y \in \mathbb{F}$ and $y < g$, then $\exists v \in S$ so that $y < v$.
   We can try to rewrite this statement: If $u$ is an upper bound of $S$, then $g \leq u$. Assume $u \in UB(S)$. Since $g$ is the greatest lower bound of $UB(S)$, $u \geq g$.

So we have proved that: For a completed ordered field $\mathbb{F}$: If $S$ is a nonempty set from $\mathbb{F}$ and $S$ has at least one upper bound, then $\exists u \in \mathbb{F}$ that is the least upper bound of $S$.

□

Problem 6.1: Let $S = \{x \in \mathbb{R} \mid x^{-1} \in \mathbb{N}\}$. Prove that 0 is the greatest lower bound of $S$.

*Proof.* To prove the above statement, we have to establish two properties for 0.

1. If $x \in S$, $0 \leq x$.
   Since $x^{-1} \in \mathbb{N}$, $x = \frac{1}{n}$ where $n \in \mathbb{N}$. So $x = \frac{1}{n} \geq 0$. This means 0 is a lower bound of $S$.
2. If $x \in \mathbb{R}$ and $x > 0$, then $\exists t \in S$ so that $t < x$.
   Since $S$ is a nonempty set of real numbers and has 0 as a lower bound, by completeness, $S$ has a greatest lower bound. Assume $x \in \mathbb{R}$ and $x > 0$. We need to prove the following lemma.

   **Lemma 0.3.** *If $r \in \mathbb{R}$ and $r > 0$, then $\exists n \in \mathbb{N}$ so that $\frac{1}{n} < r$.*

   *Proof.* Assume $r \in \mathbb{R}$ and $r > 0$. Consider $r^{-1}$. Since $r \in \mathbb{R}$, $r^{-1} \in \mathbb{R}$. By the Archimedean principle, $\exists n \in \mathbb{N}$ so that $n > r^{-1}$. This means

$$\begin{aligned} n &> \frac{1}{r} \\ nr &> 1 \text{ because } r > 0 \\ r &> \frac{1}{n} \end{aligned}$$

□

   The lemma we just proved basically says: For any real number $r$ and $r > 0$, $\exists n$ such that $r > \frac{1}{n}$.
   By our lemma, assume $x \in \mathbb{R}$ and $x > 0$, then $\exists n_0 \in \mathbb{N}$ such that $r > \frac{1}{n_0}$. Let $t = \frac{1}{n_0}$. So $r > t$. But $t \in S$ so we are done.

$\square$

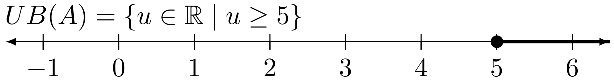
Problem 6.2: Using the notation

$$UB(S) = \{u \in \mathbb{R} \mid u \text{ is an upper bound on the set } S\}.$$

set in the proof of the proof of the alternate completeness axiom, find:

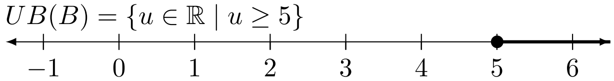
(a) $UB(A)$ for $A = \{x \in \mathbb{R} \mid x < 5\}$.

$UB(A) = \{u \in \mathbb{R} \mid u \geq 5\}$



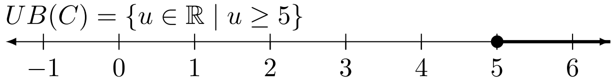

(b) $UB(B)$ for $B = \{x \in \mathbb{R} \mid x \leq 5\}$.

$UB(B) = \{u \in \mathbb{R} \mid u \geq 5\}$



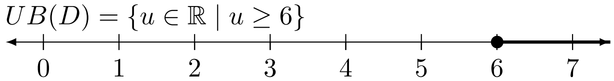

(c) $UB(C)$ for $C = \{5\}$

$UB(C) = \{u \in \mathbb{R} \mid u \geq 5\}$



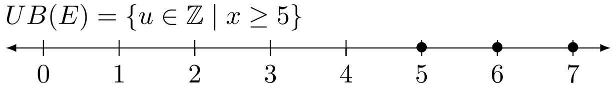

(d) $UB(D)$ for $D = \{4, 5, 6\}$

$UB(D) = \{u \in \mathbb{R} \mid u \geq 6\}$




(e) $UB(E)$ for $E = \{x \in \mathbb{Z} \mid x < 5\}$

$UB(E) = \{u \in \mathbb{Z} \mid x \geq 5\}$




(f) $UB(F)$ for $F = \{x \in \mathbb{Z} \mid x \leq 5\}$

$UB(F) = \{u \in \mathbb{Z} \mid u \geq 5\}$