

# Math 323 HW22

Minh Bui

June 20, 2017

1. Prove that the integers is countable.

*Proof.* To prove that the set of integers is countable, we need to find an injection (or one-to-one relationship) that maps  $\mathbb{Z}$  to  $\mathbb{N}$ :  $i : \mathbb{Z} \rightarrow \mathbb{N}$ .

Let

$$i(x) = \begin{cases} 2x + 1 & \text{if } x \geq 0, \\ -2x & \text{if } x < 0. \end{cases}$$

We need to show that  $i(x)$  is an injection.

Assume  $x_1, x_2 \in \mathbb{Z}$  and  $i(x_1) = i(x_2)$ . We have 3 cases.

1.  $2x_1 + 1 = 2x_2 + 1$

Assume that is the case. We have  $2x_1 = 2x_2$  and  $x_1 = x_2$ .

2.  $-2x_1 = -2x_2$

Assume that is the case. We have  $x_1 = x_2$ .

3.  $2x_1 + 1 = -2x_2$

Assume BWOC that is the case. Then  $x_1 \geq 0$  and  $x_2 < 0$ . From the equation,  $x_1 = \frac{-2x_2-1}{2}$ . But we know  $x_1, x_2 \in \mathbb{Z}$  and  $\frac{-2x_2-1}{2} \notin \mathbb{Z}$ . So this case can't happen.

Thus,  $i(x)$  is an injection. And so the set of integers  $\mathbb{Z}$  is countable.  $\square$

2. Let  $A$  and  $B$  be countable sets. Prove that  $A \cup B$  is countable.

*Proof.* Let  $A$  and  $B$  be countable sets. This respectively means,

There is an injection  $i_1 : A \rightarrow \mathbb{N}$

There is an injection  $i_2 : B \rightarrow \mathbb{N}$

Want to show: There is an injection  $i_3 : A \cup B \rightarrow \mathbb{N}$ . We define

$$i_3(x) = \begin{cases} i_1(x) & \text{if } x \in A \text{ and } x \notin B, \\ i_2(x) & \text{if } x \in B \text{ and } x \notin A, \\ x & \text{if } x \in A \text{ and } x \in B. \end{cases}$$

$\square$

3. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be functions that are both continuous at 1. Prove that  $f + g$  is continuous at 1.

*Proof.* Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be functions that are both continuous at 1. So

$$\forall \epsilon > 0, \exists \delta_1 > 0 \text{ s.t. } |x - 1| < \delta_1 \text{ implies } |f(x) - f(1)| < \epsilon$$

$$\forall \epsilon > 0, \exists \delta_2 > 0 \text{ s.t. } |x - 1| < \delta_2 \text{ implies } |g(x) - g(1)| < \epsilon$$

We want to show that  $|f(x) + g(x) - f(1) - g(1)| < \epsilon$ . Let  $\epsilon > 0$ . So  $\frac{\epsilon}{2} > 0$ . Assume there is  $\delta_1 > 0$  so that  $|x - 1| < \delta_1$ . Assume there is  $\delta_2 > 0$  so that  $|x - 1| < \delta_2$ . Consider

$$|f(x) - f(1)| < \frac{\epsilon}{2} \text{ and } |g(x) - g(1)| < \frac{\epsilon}{2}$$

$$|f(x) - f(1)| + |g(x) - g(1)| < \epsilon$$

$$|f(x) - f(1) + g(x) - g(1)| \leq |f(x) - f(1)| + |g(x) - g(1)| < \epsilon$$

$$|f(x) + g(x) - f(1) - g(1)| < \epsilon$$

□