Math 323 HW14

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Problem 9.9b: Let A be a set and B_i with $i \in \mathcal{I}$ be a family of sets. Prove:

$$A \cup \left(\bigcup_{i \in \mathcal{I}} B_i \right) = \bigcup_{i \in \mathcal{I}} (A \cup B_i).$$

Proof. Let A be a set and B_i with $i \in \mathcal{I}$ be a family set. We will need to prove 2 claims.

1. $A \cup \left(\bigcup_{i \in \mathcal{I}} B_i \right) \subseteq \bigcup_{i \in \mathcal{I}} (A \cup B_i)$
   Assume $a \in A \cup \left(\bigcup_{i \in \mathcal{I}} B_i \right)$. We have two posibilities.
   i. $a \in A$.
   Assume $a \in A$. Then $a \in A \cup B_i$ for some $i \in \mathcal{I}$. So then $a \in \bigcup_{i \in \mathcal{I}} (A \cup B_i)$.
   ii. $a \in \bigcup_{i \in \mathcal{I}} B_i$.
   Assume $a \in \bigcup_{i \in \mathcal{I}} B_i$. So $\exists k \in \mathcal{I}$ s.t $a \in B_k$. Then $a \in B_k \cup A$. Thus $a \in \bigcup_{i \in \mathcal{I}} (A \cup B_i)$.

2. $A \cup \left(\bigcup_{i \in \mathcal{I}} B_i \right) \supseteq \bigcup_{i \in \mathcal{I}} (A \cup B_i)$
   Assume $a \in \bigcup_{i \in \mathcal{I}} (A \cup B_i)$. We have two possibilities.
   i. $a \in A$.
   Assume $a \in A$. Then $a \in A \cup \left(\bigcup_{i \in \mathcal{I}} B_i \right)$.
   ii. $a \notin A$.
   Assume $a \notin A$. Then $\exists k \in \mathcal{I}$ s.t $a \in B_k$. Then $a \in \bigcup_{k \in \mathcal{I}} B_i$. Thus
   $a \in \left(\bigcup_{k \in \mathcal{I}} B_i \right) \cup A$.

Proving 2 above claims entails: $A \cup \left(\bigcup_{i \in \mathcal{I}} B_i \right) = \bigcup_{i \in \mathcal{I}} (A \cup B_i)$.

$\square$

Problem 9.13: Prove that $\bigcap\limits_{n\in\mathbb{N}} (0, \frac{n+1}{n}) = (0, 1]$.

Proof. Since $n \in \mathbb{N}$, $\frac{n+1}{n} = 1 + \frac{1}{n}$. We will need to prove 2 claims.

1. $\bigcap\limits_{n\in\mathbb{N}} (0, 1 + \frac{1}{n}) \subseteq (0, 1]$.
   Precisely, we need to prove: if $x \in \bigcap\limits_{n\in\mathbb{N}} (0, 1 + \frac{1}{n})$, then $x \in (0, 1]$. We proceed by proving the contrapositive of the statement. So we want to show that: if $x \notin (0, 1]$, then $x \notin \bigcap\limits_{n\in\mathbb{N}} (0, 1 + \frac{1}{n})$.
   Assume $x \notin (0, 1]$. This means $x < 0$ or $x > 1$.
   1. $x < 0$.
      Assume $x < 0$. Then $\forall n \in \mathbb{N}, x \notin (0, 1+\frac{1}{n})$. So $x \notin \bigcap\limits_{n\in\mathbb{N}} (0, 1+\frac{1}{n})$.
   2. $x > 1$.
      Assume $x > 1$. Since $x \in \mathbb{R}$, $x - 1 > 0$ and $x - 1 \in \mathbb{R}$. By the corollary of the Archimedean principle, $\exists t \in \mathbb{N}$ s.t $\frac{1}{t} < x - 1$. So then $\frac{1}{t} + 1 < x$. This means $x \notin (0, 1 + \frac{1}{t})$ for some $t \in \mathbb{N}$. Then $x \notin \bigcap\limits_{n\in\mathbb{N}} (0, 1 + \frac{1}{n})$.

   Thus $0 < x \leq 1$ and hence $x \in (0, 1]$.
2. $\bigcap\limits_{n\in\mathbb{N}} (0, 1 + \frac{1}{n}) \supseteq (0, 1]$.
   Assume $x \in (0, 1]$. Then $0 < x \leq 1$. Then $\forall n \in \mathbb{N}$, $0 < x < 1 + \frac{1}{n}$. This means $x \in \bigcap\limits_{n\in\mathbb{N}} (0, 1 + \frac{1}{n})$.

So $\bigcap\limits_{n\in\mathbb{N}} (0, \frac{n+1}{n}) = (0, 1]$. $\square$

Problem 9.17: For each $s \in \mathbb{Q}$, let $E_s = \{1, \frac{1}{2}, s\}$.

(a) Find $\bigcup\limits_{t\in\mathbb{Q}} E_t$ and prove your answer is correct.

$\bigcup\limits_{t\in\mathbb{Q}} E_t = \mathbb{Q}$.

Proof. We will need to prove 2 things.

1. We want to show $\bigcup\limits_{t\in\mathbb{Q}} E_t \subseteq \mathbb{Q}$.
   Assume $q \in \bigcup\limits_{t\in\mathbb{Q}} E_t$. So then exactly one of these holds: $q = 1$, $q = \frac{1}{2}$, or $q = t$ for some $t \in \mathbb{Q}$. Since $\frac{1}{2} \in \mathbb{Q}$, $1 \in \mathbb{Q}$, and $t \in \mathbb{Q}$, $q \in \mathbb{Q}$.

2. We also want to show $\mathbb{Q} \subseteq \bigcup_{t\in\mathbb{Q}} E_t$.

   Assume $q \in \mathbb{Q}$. Then $q \in E_q$. Then $q \in \bigcup_{t\in\mathbb{Q}} E_t$.

Thus $\bigcup_{t\in\mathbb{Q}} E_t = \mathbb{Q}$. □

(b) Find $\bigcap_{t\in\mathbb{Q}} E_t$ and prove your answer is correct.

$\bigcap_{t\in\mathbb{Q}} E_t = \{1, \frac{1}{2}\}$.

Proof. We will need to show that.

1. $\bigcap_{t\in\mathbb{Q}} E_t \subseteq \{1, \frac{1}{2}\}$.
   Assume $q \in \bigcap_{t\in\mathbb{Q}} E_t$. So then $\forall t \in \mathbb{Q}, q \in E_t$. So then exactly one of these holds: $q = 1$ or $q = \frac{1}{2}$ which means $q \in \{1, \frac{1}{2}\}$.
2. $\{1, \frac{1}{2}\} \subseteq \bigcap_{t\in\mathbb{Q}} E_t$.
   Assume $q \in \{1, \frac{1}{2}\}$. Then $q \in \{1, \frac{1}{2}, t\} \forall t \in \mathbb{Q}$. Thus $q \in \bigcap_{t\in\mathbb{Q}} E_t$.

Hence, $\bigcap_{t\in\mathbb{Q}} E_t = \{1, \frac{1}{2}\}$. □

(c) Is the statement: If $E_s = E_r$, then $s = r$ true or false? Prove your answer.

True.

Proof. Assume $E_s = E_r$. This means

1. $E_s \subseteq E_r$. Assume $q \in E_s$. Then exactly one of these has to be true: $q = 1, q = \frac{1}{2}$, or $q = s$. But $s = r$. So then exactly one of these holds: $q = 1, q = \frac{1}{2}$, or $q = r$. This means $q \in E_r$.
2. $E_r \subseteq E_s$. Assume $q \in E_r$. Then exactly one of these has to be true: $q = 1, q = \frac{1}{2}$, or $q = r$. But $r = s$. So then exactly one of these holds: $q = 1, q = \frac{1}{2}$, or $q = s$. This means $q \in E_s$.

Thus: If $E_s = E_r$, then $s = r$. □

Problem 10.4: Consider the relation on $\mathbb{Z}$ defined by $n\mathcal{R}m$ if $n + m$ is even.

(a) Is $\mathcal{R}$ reflexive? Yes.

Proof. Let $n \in \mathbb{Z}$. We need to show $n\mathcal{R}n$ if $n+n$ is even. Regardless of whether n is even or n is odd, $n + n = 2n$ is always even. □

(b) Is $\mathcal{R}$ symmetric? Yes

Proof. Let $n, m \in \mathbb{Z}$. We need to show: if $n\mathcal{R}m$ then $m\mathcal{R}n$. Assume $n\mathcal{R}m$. This means $n+m$ is even. But then $n+m = m+n$, which is also even. So $m+n$ is even. Thus $m\mathcal{R}n$. □

(c) Is $\mathcal{R}$ transitive? Yes.

Proof. Let $a, b,$ and $c \in \mathbb{Z}$. We need to show: If $a\mathcal{R}b$ and $b\mathcal{R}c$, then $a\mathcal{R}c$.
Assume $a\mathcal{R}b$ and $b\mathcal{R}c$. Respectively we have,

$$\exists k \in \mathbb{Z} \text{ s.t } a+b=2k$$
$$\exists l \in \mathbb{Z} \text{ s.t } b+c=2l$$

Consider $a+b+b+c$

$$a+b+b+c = 2k+2l$$
$$a+2b+c = 2k+2l$$
$$a+c = 2k+2l-2b$$
$$a+c = 2(k+l-b)$$

We know $(k+l-b) \in \mathbb{Z}$. So the relation R is transitive. □

(d) Does $\mathcal{R}$ have trichotomy? No since $\mathcal{R}$ is symmetric.

Problem 10.5: Consider the relation on $\mathbb{R}$ defined by $n \backsimeq m$ if $n-m \in \mathbb{Z}$.

(a) Is $\backsimeq$ reflexive? Yes

Proof. Assume $n \in \mathbb{R}$. We want to show $n \backsimeq n$. This means we want $n-n \in \mathbb{Z}$. Since $n-n = 0 \in \mathbb{Z}$, the relation $\backsimeq$ is reflexive. □

(b) Is $\backsimeq$ symmetric? Yes

Proof. Assume $n, m \in \mathbb{R}$. We want to show: if $n \backsimeq m$, then $m \backsimeq n$. Assume $n \backsimeq m$. This means $\exists k \in \mathbb{Z}$ s.t $n-m=k$. Multiply both sides by -1: $m-n=-k$. We know $-k \in \mathbb{Z}$ so the relation $\backsimeq$ is symmetric. □

(c) Is $\backsimeq$ transitive? Yes

Proof. Assume $m, n, p \in \mathbb{R}$. We want to show: if $n \backsimeq m$ and $m \backsimeq p$, then $n \backsimeq p$.
Assume $n \backsimeq m$ and $m \backsimeq p$. Respectively we have,

$$\exists k \in \mathbb{Z} \text{ s.t } n-m=2k$$
$$\exists l \in \mathbb{Z} \text{ s.t } m-p=2l$$

Consider $(n-m)+(m-p)$

$$\begin{aligned} n-m+m-p &= 2k+2l \\ n-p &= 2k+2l \end{aligned}$$

We know $2k+2l \in \mathbb{Z}$. So $n \backsimeq p$. Thus the relation $\backsimeq$ is transitive. $\square$

(d) Does $\backsimeq$ have trichotomy? No because $\backsimeq$ is symmetric.