Math 323 HW19

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Problem 12.8: Prove: If $f : A \to B$ is a function with domain A and T_i with $i \in \mathcal{I}$ is a family of sets where $\forall i \in \mathcal{I}$, $T_i \subseteq B$, then

$$f^{-1}\left(\bigcup_{i\in\mathcal{I}} T_i\right) = \bigcup_{i\in\mathcal{I}} f^{-1}(T_i)$$

Proof. Let $f : A \to B$ be a function and T_i with $i \in \mathcal{I}$ be a family of sets where $\forall i \in \mathcal{I}, T_i \subseteq B$. We want to prove 2 claims.

(a) $f^{-1}\left(\bigcup\limits_{i\in\mathcal{I}} T_i\right) \subseteq \bigcup\limits_{i\in\mathcal{I}} f^{-1}(T_i)$

Assume $x \in f^{-1}\left(\bigcup\limits_{i\in\mathcal{I}} T_i\right)$. By the theorem of image and pre-image, $f(x) \in \bigcup\limits_{i\in\mathcal{I}} T_i$. So then $\exists i \in \mathcal{I}$ s.t $f(x) \in T_i$. Since $f(x) \in T_i$ for some $i \in \mathcal{I}$, $x \in f^{-1}(T_i)$. Thus $x \in \bigcup\limits_{i\in\mathcal{I}} f^{-1}(T_i)$.

(b) $f^{-1}\left(\bigcup\limits_{i\in\mathcal{I}} T_i\right) \supseteq \bigcup\limits_{i\in\mathcal{I}} f^{-1}(T_i)$

Assume $x \in \bigcup\limits_{i\in\mathcal{I}} f^{-1}(T_i)$. So then $\exists i \in \mathcal{I}$ s.t $x \in f^{-1}(T_i)$. By the theorem of image and preimage, $f(x) \in T_i$ for some $i \in \mathcal{I}$. So then $f(x) \in \bigcup\limits_{i\in\mathcal{I}} T_i$. Again, by the theorem of image and preimage, $x \in f^{-1}(\bigcup\limits_{i\in\mathcal{I}} T_i)$.

Thus $f^{-1}\left(\bigcup\limits_{i\in\mathcal{I}} T_i\right) = \bigcup\limits_{i\in\mathcal{I}} f^{-1}(T_i)$ $\square$

Problem 12.10: Prove: Let $f : A \to B$ be a function with domain A. Prove: if $\forall S \subseteq A, S = f^{-1}(f(S))$, then $f(x)$ is injective.

Proof. Let $f : A \to B$ be a function with domain A.
Assume $\forall S \subseteq A, S = f^{-1}(f(S))$. So $S = \{x \in A \mid f(x) \in f(S)\}$. We wish to show: if $\forall x_1, x_2 \in A$ and $f(x_1) = f(x_2)$, then $x_1 = x_2$.

The contraposition of this statement says: if $\exists x_1, x_2 \in A$ s.t $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.
Assume $\exists x_1, x_2 \in A$, $x_1 \neq x_2$. □

Problem 12.11: Let $f : A \to B$ be a function with domain A. Prove: if $\forall T \subseteq B, T = f(f^{-1}(T))$, then $f(x)$ is surjective.

Proof. Let $f : A \to B$ be a function with domain A.
Assume $\forall T \subseteq B, T = f(f^{-1}(T))$. We want to show: if $y \in B$, then $\exists x \in A$ so that $f(x) = y$.
Assume $y \in B$. So then $y \in T$, meaning $y \in f(f^{-1}(T))$. So then $\exists x \in A$ so that $x \in f^{-1}(T)$. Since $f^{-1}(T) \subseteq A$, $x \in A$. □