Math 323 HW11

Minh Bui

June 5, 2017

Problem 7.5: Prove that $\forall n \in \mathbb{N}$ if

$$\sum_{k=1}^{n}(4k^3 - 6k^2 + 4k - 1) = n^4 - 1$$

then

$$\sum_{k=1}^{n+1}(4k^3 - 6k^2 + 4k - 1) = (n+1)^4 - 1$$

Proof. Assume $\forall n \in \mathbb{N}$, $\sum_{k=1}^{n}(4k^3 - 6k^2 + 4k - 1) = n^4 - 1$. Consider

$$\sum_{k=1}^{n}(4k^3 - 6k^2 + 4k - 1) = n^4 - 1$$

$$\begin{aligned}\sum_{k=1}^{n}(4k^3 - 6k^2 + 4k - 1) + (4(n+1)^3 - 6(n+1)^2 + 4(n+1) - 1)\\ = n^4 - 1 + (4(n+1)^3 - 6(n+1)^2 + 4(n+1) - 1)\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{n+1}(4k^3 - 6k^2 + 4k - 1) &= n^4 + 4(n+1)^3 - 6(n+1)^2 + 4(n+1) - 2\\ &= n^4 + 4(n^3 + 3n^2 + 3n + 1) - 6(n^2 + 2n + 1) + 4(n+1) - 2\\ &= n^4 + 4n^3 + 12n^2 + 12n + 4 - 6n^2 - 12n - 6 + 4n + 4 - 2\\ &= n^4 + 4n^3 + 6n^2 + 4n\\ &= n^4 + 3n^3 + n^3 + 3n^2 + 3n^2 + 3n + n + 1 - 1\\ &= n^4 + 3n^3 + 3n^2 + n + n^3 + 3n^2 + 3n + 1 - 1\\ &= n(n^3 + 3n^2 + 3n + 1) + (n^3 + 3n^2 + 3n + 1) - 1\\ &= n(n+1)^3 + (n+1)^3 - 1\\ &= (n+1)(n+1)^3 - 1\\ &= (n+1)^4 - 1\end{aligned}$$

$\square$

Problem 7.8: Let $\mathbb{F}$ be an ordered field. Let $x \in \mathbb{F}$ with $x \geq 0$. Consider the statement: if $\forall \epsilon > 0, x \leq \epsilon$, then $x = 0$.

(a) Write a contrapositive of this statement.

Solution. Let $\mathbb{F}$ be an ordered field. Let $x \in \mathbb{F}$ with $x \geq 0$. If $x > 0$, then $\exists \epsilon > 0$, $x > \epsilon$.

(b) Prove the statement.

Proof. Assume $x \in \mathbb{F}$ with $x > 0$. Since $x > 0$, $x + x > x$. So $2x > x$. And we have $2x > x > 0$. And $x > \frac{x}{2} > 0$. Let $\epsilon = \frac{x}{2}$. We see that $\epsilon > 0$ because $x > 0$. So this completes our proof. $\square$

(c) Why do some people call this result "The Average Theorem?"

Answer. Looking at the contrapositive of the statement above, we see that we can always find an $\epsilon \in \mathbb{F}$ such that $\epsilon = \frac{x}{2}$ which is the average between x and 0. In the general case where $a, b \in \mathbb{F}$ and $a < b$, $\exists \epsilon \in \mathbb{F}$ so that $\epsilon = \frac{a+b}{2}$ and so $a < \epsilon < b$ where ϵ is the average of a and b.

(d) It's easier to remember the original Average theorem though.

Problem 7.15: Use a calculator or computer to calculate $\sum\limits_{k=1}^{100}(4k^3 - 6k^2 + 4k - 1)$.

(a) Is the answer to part one $(100)^4 - 1$?

Answer. No. $\sum\limits_{k}^{100}(4k^3 - 6k^2 + 4k - 1) = 100^4$. Not $100^4 - 1$.

(b) Is the statement we proved in question 5 above true? Is the proof correct?

Answer. The statement we proved in question 5 is true since the hypothesis is wrong. The truth value of the conditional statement is always true if and only if the hypothesis is false. The proof is incorrect because our assumption $\forall n \in \mathbb{N}$, $\sum\limits_{k=1}^{n}(4k^3 - 6k^2 + 4k - 1) = n^4 - 1$ is false.