Math 323 HW13

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Problem 7.14: Let $S = \{x \in \mathbb{R} \mid \exists n \in \mathbb{N} \text{ s.t } x = \frac{n+1}{n}\}$. Prove that 1 is the infimum of S.

Proof. We need to prove 2 claims.

1. If $s \in S$, then $1 \leq s$.
   Assume $s \in S$. $\exists m \in \mathbb{N}$ s.t $s = \frac{m+1}{m}$. Since $m \in \mathbb{N}, m > 0$. So we have $s = \frac{m+1}{m} = 1 + \frac{1}{m} \geq 1$ since we know $\frac{1}{m} \geq 0$. So 1 is the lower bound on the set S.
2. If $x \in \mathbb{R}$ and $x > 1$, then $\exists t \in S$ s.t $t < x$.
   Assume $x \in \mathbb{R}$ and $x > 1$. So $x - 1 > 0$. By the Archimedean principle, $\exists p \in \mathbb{N}$ s.t $0 < \frac{1}{p} < x - 1$. So $1 < \frac{1}{p} + 1 < x$. Let $t = \frac{1}{p} + 1$ and we know $t \in S$ and we are done.

Proving two claims above establishes that 1 is the infimum of S. □

Problem 8.1: Prove that the least upper bound of a set is unique.

Proof. Assume l_1 and l_2 is the least upper bound of a set S. Since l_1 and l_2 are both upper bounds and are both least upper bound of a set S. We have $l_1 \geq l_2$ and $l_2 \geq l_1$. By Trichotomy, $l_1 = l_2$. □

Problem 8.6: Prove that the set

$$S = \{x \in \mathbb{R} \mid \exists s \in \mathbb{R} \text{ s.t } x = s^6 + 19s^4 + 11s^2 + 14\}$$

has an infimum.

Proof. We will instead prove that: 14 is the minimum of the set

$$S = \{x \in \mathbb{R} \mid \exists s \in \mathbb{R} \text{ s.t } x = s^6 + 19s^4 + 11s^2 + 14\}$$

We will need to prove 2 claims.

1. $14 \in S$.
   Let $s \in S$. $\exists r \in \mathbb{R}$ s.t $s = r^6 + 19r^4 + 11r^2 + 14$. Let $r = 0$, then $s = 0 + 0 + 0 + 14 = 14$. So $14 \in S$.

2. If $s \in S$, then $14 \leq s$.
   Assume $s \in S$. So $\exists r \in \mathbb{R}$ s.t $s = r^6 + 19r^4 + 11r^2 + 14$. Since $r \in \mathbb{R}$, $r^6 \geq 0$ and $r^4 \geq 0$ and $r^2 \geq 0$. So $r^6 + 19r^4 + 11r^2 + 14 \geq 14$, which means $s \geq 14$. So 14 is a lower bound of the set S.

Since 14 is the minimum of the set S, we know 14 is the infimum of the set. And so we have proved that the set S has an infimum. $\square$