# Math 323 HW17

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Problem 11.8: Define $f : [3, \infty) \to [-6, \infty)$ by $f(x) = x^2 - 6x + 3$.

(a) Find a function $g : [-6, \infty) \to [3, \infty)$ so that $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$.

*Solution.* $g(x) = 3 + \sqrt{6 + x}$ or $g(x) = 3 - \sqrt{6 + x}$.

(b) Prove that $f(x)$ is bijective.

*Proof.* We will need to prove 2 claims.

1. $f(x)$ is injective.
   Precisely, we want to show that: if $x_1, x_2 \in [3; \infty)$ and $f(x_1) = f(x_2)$, then $x_1 = x_2$.
   Assume $x_1, x_2 \in [3; \infty)$ and $f(x_1) = f(x_2)$. So then $f(x_1) = x_1^2 - 6x_1 + 3$ and $f(x_2) = x_2^2 + 6 - 6x_2 + 3$. Because $f(x_1) = f(x_2)$, $x_1^2 - 6x_1 + 3 = x_2^2 - 6x_2 + 3$. So

$$x_1^2 - 6x_1 = x_2^2 - 6x_2$$
$$x_1^2 - x_2^2 - 6x_1 + 6x_2 = 0$$
$$(x_1 + x_2)(x_1 - x_2) - 6(x_1 - x_2) = 0$$
$$(x_1 - x_2)(x_1 + x_2 + 6) = 0$$

   Since $x_1, x_2 \in [3; \infty)$, $x_1 + x_2 + 6 > 0$, so then $x_1 - x_2 = 0$. Thus $x_1 = x_2$.
2. $f(x)$ is surjective.
   We want to show that: if $b \in [-6, \infty)$, then $\exists a \in [3, \infty)$ s.t $b = a^2 - 6a + 3$.
   Assume $b \in [-6, \infty)$. Let $a = 3 \pm \sqrt{6 + b}$. We know $\sqrt{6 + b} \in \mathbb{R}$ for $b \in [-6, \infty)$. So $3 \pm \sqrt{6 + b} \in \mathbb{R}$ for $b \in [-6, \infty)$. We also observe that $a = 3 \pm \sqrt{6 + b} \in [3, \infty)$ for $b \in [-6, \infty)$. Thus $f(x)$ is surjective on $[3, \infty) \to [-6, \infty)$.

Since $f(x)$ is both injective and surjective, $f(x)$ is bijective. □

Problem 11.9: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 + 5x - 8$. Prove that $f(x)$ is injective. (Hint: Can't use calculus, prove that $f(a) - f(b) = (a - b) \cdot g(a, b)$)

*Proof.* Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 + 5x - 8$. We want to show that $f(x)$ is injective. Precisely: if $a_1, a_2 \in \mathbb{R}$ and $f(a_1) = f(a_2)$, then $a_1 = a_2$.
Assume $a_1, a_2 \in \mathbb{R}$ and $f(a_1) = f(a_2)$. So then

$$\begin{gathered}
a_1^3 + 5a_1 - 8 = a_2^3 + 5a_2 - 8 \\
a_1^3 + 5a_1 - a_2^3 - 5a_2 = 0 \\
(a_1^3 - a_2^3) + 5(a_1 - a_2) = 0 \\
((a_1 - a_2)^3 + 3a_1a_2(a_1 - a_2)) + 5(a_1 - a_2) = 0 \\
(a_1 - a_2)((a_1 - a_2)^2 + 3a_1a_2 + 5) = 0
\end{gathered}$$

We need to consider 2 cases:

- $a_1 - a_2 = 0$.
  Assume that is the case, so then $a_1 = a_2$ and we are done.
- $(a_2 - a_2)^2 + 3a_1a_2 + 5 = 0$.
  Assume BWOC, $(a_1 - a_2)^2 + 3a_1a_2 + 5 = 0$. We know $(a_1 - a_2)^2 \geq 0$. But then $(a_1 - a_2)^2 = -3a_1a_2 - 5$ and $-3a_1a_2 - 5 \geq 0$. The equality only holds when $a_1 = a_2 = 0$. So then $-3a_1a_2 - 5 = -5 \geq 0$, which is a contradiction. So $(a_1 - a_2)^2 + 3a_1a_2 + 5 \neq 0$.

$\square$

Problem 12.3: Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{x^2}$. Find the following.

(a) $f((0,3)) = (\frac{1}{9}, \infty)$
(b) $f([0,4) = [\frac{1}{16}, \infty)$
(c) $f(\mathbb{R}) = (0, \infty)$
(d) $f([-2,3]) = [\frac{1}{9}, \infty)$
(e) $f(\emptyset) = \emptyset$
(f) $f^{-1}((0,\infty)) = \mathbb{R} \setminus \{0\}$
(g) $f^{-1}((-1,1)) = [1,\infty) \cup (-\infty,-1]$
(h) $f^{-1}((\frac{1}{4}, 1]) = [1,2) \cup (-2,-1]$
(i) $f^{-1}(\mathbb{R}) = \mathbb{R} \setminus \{0\}$
(j) $f^{-1}(\emptyset) = \emptyset$

Problem 12.4: Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - x$. Find the following.

(a) $f(\{-1,0,1\}) = \{0\}$
(b) $f([-1,1]) = [\frac{-2\sqrt{3}}{9}, \frac{2\sqrt{3}}{9}]$
(c) $f((-1,1)) = [\frac{-2\sqrt{3}}{9}, \frac{2\sqrt{3}}{9}]$
(d) $f((-5,5)) = (-120, 120)$

(e) $f(\mathbb{R}) = \mathbb{R}$

(f) $f(\emptyset) = \emptyset$

(g) $f^{-1}(\{0\}) = \{-1, 0, 1\}$

(h) $f^{-1}((0, \infty)) = (1, \infty) \cup (-1, 0)$

(i) $f^{-1}((-120, 120)) = (-5, 5)$

(j) $f^{-1}(\mathbb{R}) = \mathbb{R}$

(k) $f^{-1}(\emptyset) = \emptyset$