Math 323 HW16

Minh Bui

June 13, 2017

Problem 10.14: Consider the relation on $\mathbb{Z}$ given by $a \doteq b$ if and only if $a - b$ is divisible by 7.

(a) Prove that this is an equivalence relation.

Proof. We will need to prove 3 properties of equivalence relation.

1. $\doteq$ is reflexive.
   Assume $z \in \mathbb{Z}$. We want to show $a \doteq a$. In other words, we need to show $a - a$ is divisible by 7. But $a - a = 0$ and $0 = 0.7$. So $a - a$ is divisible by 7. Thus $\doteq$ is reflexive.
2. $\doteq$ is symmetric.
   We want to show: If $a, b \in \mathbb{Z}$ and $a \doteq b$, then $b \doteq a$.
   Assume $a \doteq b$. So $a-b$ is divisible by 7. So then $\exists q \in \mathbb{Z}$ s.t $a-b = 7q$. Since $a - b = 7q$, $b - a = -7q$. Since $q \in \mathbb{Z}$, $(-q) \in \mathbb{Z}$. So then $b - a$ is divisible by 7. Thus $\doteq$ is symmetric.
3. $\doteq$ is transitive.
   We want to show: If $a, b,$ and $c \in \mathbb{Z}$ and $a \doteq b$ and $b \doteq c$, then $a \doteq c$.
   Assume $a, b,$ and $c \in \mathbb{Z}$ and $a \doteq b$ and $b \doteq c$. Respectively this means

$$\exists q_1 \in \mathbb{Z} \text{ s.t } a - b = 7q_1$$
$$\exists q_2 \in \mathbb{Z} \text{ s.t } b - c = 7q_2$$

Consider $(a - b) + (b - c)$.

$$\begin{aligned} a - b + b - c &= 7q_1 + 7q_2 \\ a - c &= 7q_1 + 7q_2 \\ a - c &= 7(q_1 + q_2) \end{aligned}$$

Since q_1 and $q_2 \in \mathbb{Z}$, $q_1 + q_2 \in \mathbb{Z}$. So then $a - c$ is divisible by 7 and thus $a \doteq c$. $\doteq$ is transitive.

$\square$

(b) Describe [3] for this relation.

Solution. $[3] = \{..., 23, 17, 10, 3, -4, -11, -19, ...\}$

(c) Find $\mathbb{Z}/_{\simeq}$.

Solution. $\mathbb{Z}/_{\simeq} = \{[0], [1], [2], [3], [4], [5], [6]\}$.

(d) Prove that an addition on $\mathbb{Z}/_{\simeq}$ defined by $[n] + [m] = [n + m]$ is well defined.

Proof. Assume $n, m, a, b \in \mathbb{Z}$ so that $[n] = [a]$ and $[m] = [b]$. We want to show: $[n] + [m] = [a] + [b]$. Since $[n] = [a]$ and $[m] = [b]$, we have

$$\begin{gathered}
\exists k_1 \in \mathbb{Z} \text{ s.t } n = 7k_1 + a \\
\exists k_2 \in \mathbb{Z} \text{ s.t } m = 7k_2 + b \\
n + m = 7k_1 + a + 7k_2 + b \\
n + m = 7(k_1 + k_2) + (a + b) \\
(n + m) - (a + b) = 7(k_1 + k_2) \\
(n + m) \simeq (a + b) \text{ since } k_1 + k_2 \in \mathbb{Z}. \\
[n + m] = [a + b]
\end{gathered}$$

But we defined $[n] + [m] = [n + m]$ and $[n + m] = [a + b] = [a] + [b]$. Thus $[n] + [m] = [a] + [b]$. $\square$

Problem 10.16: Let $S = \mathbb{Z} \times \mathbb{N}$. Define a relation on S by $(n, m) \equiv (p, q)$ if $nq = mp$.

(a) Prove that this is an equivalence relation.

Proof. We need to prove 3 properties.

i. $\equiv$ is reflexive.
Assume $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. We want to show that $(n, m) \equiv (n, m)$. Since $nm = mn$ in $\mathbb{Z}$ by commutativity, $(n, m) \equiv (n, m)$.

ii. $\equiv$ is symmetric.
Assume $m, p \in \mathbb{Z}$ and $n, q \in \mathbb{N}$. We want to show that: If $(m, n) \equiv (p, q)$, then $(p, q) \equiv (m, n)$.
Assume $(m, n) \equiv (p, q)$. Then $mq = np$ in $\mathbb{Z}$. By commutativity in $\mathbb{Z}$, $pn = qm$ in $\mathbb{Z}$. This means $(p, q) \equiv (m, n)$.

iii. $\equiv$ is transitive.
Assume $m, n, p \in \mathbb{Z}$ and $q, r, s \in \mathbb{N}$. We want to show that: If $(m, q) \equiv (n, r)$ and $(n, r) \equiv (p, s)$, then $(m, q) \equiv (p, s)$.
Assume $(m, q) \equiv (n, r)$ and $(n, r) \equiv (p, s)$. Respectively these means

$$\begin{gathered}
mr = qn \text{ in } \mathbb{Z} \\
ns = rp \text{ in } \mathbb{Z}
\end{gathered}$$

Consider $mr = qn$.

$$mrs = qns \text{ since we know } s \in \mathbb{N}$$
$$mrs = qrp \text{ because } ns = rp$$
$$ms = qp \text{ since we know } r \in \mathbb{N}$$

This means $(m, q) \equiv (p, s)$.

$\square$

(b) The set $S/_{\equiv}$ has a much better and more familiar name, what is it?

Answer. Equivalence of two fractions.

(c) Define an addition on $S/_{\equiv}$ by $(n, m) \oplus (p, q) = (nq+mp, mq)$. Prove it is well defined.

Proof. Let $n, p, a, b \in \mathbb{Z}$ and $m, q, c, d \in \mathbb{N}$ so that $(n, m) = (a, c)$ and $(p, q) = (b, d)$. We want to show that $(n, m) \oplus (p, q) = (a, c) \oplus (b, d)$. Since $(n, m) = (a, c)$, $nc = ma$ in $\mathbb{Z}$. Since $(p, q) = (b, d)$, $pd = qb$ in $\mathbb{Z}$. Consider

$$nc = ma$$
$$ncd = mad \text{ since } d \in \mathbb{N}$$
$$ncdq = madq \text{ since } q \in \mathbb{N}$$

And

$$pd = qb$$
$$mpd = mqb \text{ since } m \in \mathbb{N}$$
$$cmpd = cmqp \text{ since } c \in \mathbb{N}$$

Now consider

$$ncdq + cmpd = madq + cmpd$$
$$nqcd + mpcd = mqad + mqcp$$
$$(nq + mp)cd = mq(ad + bc)$$
$$(nq + mp, mq) = (ad + bc, cd)$$
$$(n, m) \oplus (p, q) = (a, c) \oplus (b, d).$$

$\square$

(d) We cannot define an addition on $S/_{\equiv}$ by $(n, m) \boxplus (p, q) = (n+p, m+q)$. Why not? (In the book it was $(m+p, m+q)$ but I believe it's a typo)

Answer. Because the operation is not well defined. (Do I really need to prove this?)

Proof. Assume BWOC, the addition operation on $S/_{\equiv}$ by $(n,m) \boxplus (p,q) = (n+p, m+q)$ is well defined. Mathematically speaking, if $(n,m) = (a,b)$ and $(p,q) = (c,d)$, then $(n,m) \boxplus (p,q) = (a,b) \boxplus (c,d)$. We know these:

- Since $(n,m) = (a,b)$, $nb = ma$ in $\mathbb{Z}$. So then $n = \frac{ma}{b}$ in $\mathbb{Q}$ because $b \in \mathbb{N}$.
- Since $(p,q) = (c,d), pd = qc$ in $\mathbb{Z}$. So then $p = \frac{qc}{d}$ in $\mathbb{Q}$ because $d \in \mathbb{N}$.
- We also have $(n,m) \boxplus (p,q) = (a,b) \boxplus (c,d)$. This means $(n+p, m+q) = (a+c, b+d)$. So $(n+p)(b+d) = (m+q)(a+c)$. So $(n+p)(b+d) = (m+q)(a+c)$.

Now, consider $(n+p)(b+d) = (m+q)(a+c)$

$$\begin{aligned}
(n+p)(b+d) &= (m+q)(a+c) \\
(\frac{ma}{b} + \frac{qc}{d})(b+d) &= (m+q)(a+c) \\
ma + \frac{qc}{d}b + \frac{ma}{b}d + qc &= ma + qa + mc + qc \\
\frac{qc}{d}b + \frac{ma}{b}d &= qa + mc \text{ in } \mathbb{Q} \\
\frac{qb}{d}c + \frac{md}{b}a &= mc + qa \text{ in } \mathbb{Q} \\
c(\frac{qb}{d} - m) &= a(q - \frac{md}{b})
\end{aligned}$$

And now I'm stuck. :(☐

Problem 10.19: Consider the set $S = \{a,b,c,d,e,f\}$ and the relation $\mathcal{R} \subseteq S \times S$ given by.

$$\mathcal{R} = \{(a,a),(a,b),(a,d),(b,a),(b,b),(b,d),(c,c),\\ (c,f),(d,a),(d,b),(d,d),(e,e),(f,c),(f,f)\}$$

As it happens $\mathcal{R}$ is an equivalence relation on S. Find $S/\mathcal{R}$.

Solution. We observe $[a] = [b] = [d], [c] = [f]$, and $[e]$ stands alone. So $S/_{\mathcal{R}} = \{[a],[e],[c]\}$

Problem 11.2: Let $A = \{a,b,c,d,e\}$. Define a function $f : A \to A$ using a table of values:

x	$f(x)$
a	b
b	b
c	d
d	c
e	e

Define a function $g : A \to A$ and using a table of values:

x	$g(x)$
a	b
b	c
c	d
d	e
e	a

(a) Is either $f(x)$ or $g(x)$ injective? $f(x)$ is not injective but $g(x)$ is injective.

(b) Is either $f(x)$ or $g(x)$ surjective? $f(x)$ is not surjective but $g(x)$ is surjective.

(c) Find a table for $(f \circ g)(x)$.

x	$(f \circ g)(x)$
a	b
b	b
c	d
d	c
e	e

(d) Find a table for $(f \circ f)(x)$.

x	$(f \circ f)(x)$
a	b
b	b
c	d
d	c
e	e

Problem 11.3: Consider the functions $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^{-1}$, and $g : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ given by $g(x) = x^{-1}$.

(a) Prove $f(x)$ is injective.

Proof. We want to prove: Let $a_1, a_2 \in \mathbb{R}$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$.
We need to consider 2 cases:

1. $a_1 = 0$ and $a_2 = 0$. Then $a_1 = a_2 = 0$ and the conditional statement is true by default.
2. Assume $a_1 \neq 0$, $a_2 \neq 0$ and $f(a_1) = f(a_2)$. So $a_1^{-1} = a_2^{-1}$. So then $\frac{1}{a_1} = \frac{1}{a_2}$. Since both $a_1, a_2 \in \mathbb{R}$, $a_1 = a_2$.

$\square$

(b) Prove $g(x)$ is injective.

Proof. We want to prove: if $a_1, a_2 \in \mathbb{R} \setminus 0$ $\square$

(c) Prove $f(x)$ is not surjective.

Proof. We just need to pick at least one $b \in \mathbb{R}$ so that $\nexists a \in \mathbb{R}$ s.t $b = f(a)$. Let $b = 0$ and have found it. $\square$

(d) Prove $g(x)$ is surjective.

Proof. We want to show: if $b \in \mathbb{R} \setminus \{0\}$, then $\exists a \in \mathbb{R} \setminus \{0\}$ s.t $g(a) = b$. Assume $b \in \mathbb{R} \setminus \{0\}$. Then $\frac{1}{b} \in \mathbb{R} \setminus \{0\}$. Let $a = \frac{1}{b}$ so $a \in \mathbb{R} \setminus \{0\}$ and we are done. $\square$

(e) Carefully evaluate the two functions $(f \circ f)(x)$ and $(g \circ g)(x)$. Be completely precise about these 2 results.

Solution.

i. $(f \circ f)(x) : \mathbb{R} \to \mathbb{R}$. $(f \circ f)(x) = f(f(x)) = \frac{1}{\frac{1}{x}}$

ii. $(g \circ g)(x) : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$. $(g \circ g)(x) = g(g(x)) = \frac{1}{\frac{1}{x}} = x$.