Math 323 Definitions & important theorems

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The following is a list of learned definitions and theorems in introduction to pure math class.

1. Minimum

   Definition 1. Let S be a non empty set of numbers. We say m is a *minimum* of the set S if:

   1. $m \in S$.
   2. If $s \in S$, then $m \leq s$.

2. Maximum

   Definition 2. Let S be a non empty set of numbers. We say m is a *maximum* of the set S if:

   1. $m \in S$.
   2. If $s \in S$, then $m \geq s$.

3. Lower bound

   Definition 3. Let S be a set of numbers. We say l is a *lower bound* of the set S when: if $s \in S$, then $l \leq s$.

4. Lower bound

   Definition 4. Let S be a set of numbers. We say u is a *upper bound* of the set S when: if $s \in S$, then $l \geq s$.

5. Odd integer

   Definition 5. An integer z is *odd* if and only if $\exists k \in \mathbb{Z}$ s.t $z = 2k + 1$.

6. Even integer

   Definition 6. An integer z is *even* if and only if $\exists k \in \mathbb{Z}$ s.t $z = 2k$.

7. Trichotomy of an order

Definition 7. An order is said to have *trichotomy* if for 2 numbers a, b in that order exactly one of these holds: $a < b$, $a > b$, or $a = b$.

8. Transitivity of an order

Definition 8. An order is said to have *transitivity* if for 3 numbers a, b, c in that order, if $a < b$ and $b < c$, then $a < c$.

9. The Division Algorithm

Theorem 0.1. *Let $a, b \in \mathbb{Z}$ and $b > 0$, then $\exists q, r \in \mathbb{Z}$ s.t $a = bq + r$ with $0 \leq r < b$.*

10. The rational numbers $\mathbb{Q}$

Definition 9. Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, then

$$\mathbb{Q} = \{\frac{m}{n} \mid m \in \mathbb{Z} \text{ and } n \in \mathbb{Z}\}$$

where $\frac{m}{n}$ is a set of equivalence fractions.

11. Equality on $\mathbb{Q}$

Definition 10. Let $\frac{m}{n}, \frac{p}{q} \in \mathbb{Q}$. $\frac{m}{n} = \frac{p}{q}$ if $mq = np$.

12. Order on $\mathbb{Q}$

Definition 11. Let $\frac{m}{n}, \frac{p}{q} \in \mathbb{Q}$. $\frac{m}{n} < \frac{p}{q}$ if $mq < np$.

13. Addition on $\mathbb{Q}$

Definition 12. Let $\frac{m}{n}, \frac{p}{q} \in \mathbb{Q}$. We define $\frac{m}{n} + \frac{p}{q} = \frac{mq+np}{nq}$

14. Multiplication on $\mathbb{Q}$

Definition 13. Let $\frac{m}{n}, \frac{p}{q} \in \mathbb{Q}$. We define $\frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$

15. The Average Theorem

Theorem 0.2. *If $a, b \in \mathbb{F}$ with $a < b$, then $\exists r \in \mathbb{F}$ s.t $a < r < b$. In fact, $r = \frac{a+b}{2}$ is an example.*

16. Absolute value

Definition 14. Let $x \in \mathbb{F}$. We define the absolute value of x, denoted by $|x|$ as

$$|x| = \begin{cases} -x \text{ if } x < 0 \\ \;\;\, x \text{ if } x \geq 0 \end{cases}$$

17. Important theorems in absolute value

Theorem 0.3. *Let $\mathbb{F}$ be an ordered field.*

1. *If $a \in \mathbb{F}$, then $|a| \geq 0$.*
2. *If $a \in \mathbb{F}$, then $-|a| \leq a \leq |a|$.*
3. *Let $r \in \mathbb{F}$, $r \geq 0$. Consider x as a variable in $\mathbb{F}$. Then $|x - a| \leq r$ if and only if $a - r \leq x \leq a + r$.*
4. *Let $r \in \mathbb{F}$, $r > 0$. Consider x as a variable in $\mathbb{F}$. Then $|x - a| < r$ if and only if $a - r < x < a + r$.*
5. *If $a, b \in \mathbb{F}$, then $|ab| = |a| \cdot |b|$.*
6. The Triangle Inequalities. *Let $a, b \in \mathbb{F}$, then $|a + b| \leq |a| + |b|$.*

18. Infimum (Greatest lower bound)

    Definition 15. Let S be a set of real numbers and $g \in \mathbb{R}$, g is an infimum of S when

    1. If $s \in \mathbb{S}$, then $g \leq s$. And
    2. If $x \in \mathbb{R}$ and $x > g$, then $\exists t \in S$ s.t $t < x$.

19. Supremum (Least upper bound)

    Definition 16. Let S be a set of real numbers and $l \in \mathbb{R}$, l is a supremum of S when

    1. If $s \in \mathbb{S}$, then $l \geq s$. And
    2. If $x \in \mathbb{R}$ and $x < l$, then $\exists t \in S$ s.t $t > x$.

20. Complete ordered field.

    Definition 17. An ordered field $\mathbb{F}$ is complete if for any nonempty subset S of $\mathbb{F}$ and S has at least one lower bound, then $\exists g \in \mathbb{F}$ that is the infimum of S.

21. The Well-Ordering Principle

    Theorem 0.4. *Let U be a set with total order. U is well ordered if $A \subseteq U \neq \emptyset$, then A has a minimum.*

22. The theorem of Induction

    Theorem 0.5. *For all $n \in \mathbb{N}$. Let $P(n)$ be a statement that is either true or false but not both. If the following conditions hold*

    1. *If $n = 1$, then $P(n)$ is true.*
    2. *If for $n = n_0$, $P(n)$ is true, then for $n = n_0 + 1$, $P(n)$ is also true.*

    then $P(n)$ is true.

23. The Alternate completeness axiom

    Theorem 0.6. *For a completed ordered field $\mathbb{R}$: If S is a nonempty subset of $\mathbb{R}$ and S has at least 1 upper bound, then there is an $l \in \mathbb{R}$ that is the supremum of the set S.*

    Corollary 0.6.1. *1. If $r \in \mathbb{R}$, then there is an $n \in \mathbb{Z}$ s.t $n < r$.*

    2. If $x \in \mathbb{R}$ and $x > 0$, then there is an $n \in \mathbb{N}$ s.t $0 < \frac{1}{n} < x$.

24. The Archimedean principle

    Theorem 0.7. *Let $r \in \mathbb{R}$, then there is $n \in \mathbb{N}$ s.t $n > r$.*

    Corollary 0.7.1. *The following are equivalent to the Archimedean principle.*

    1. *If $r \in \mathbb{R}$, then there is an $n \in \mathbb{Z}$ s.t $n < r$.*
    2. *If $x \in \mathbb{R}$ and $x > 0$, then there is an $n \in \mathbb{N}$ s.t $0 < \frac{1}{n} < x$.*

25. The Density theorem

    Theorem 0.8. *Let $a, b \in \mathbb{R}$ so that $a < b$, then there is $q \in \mathbb{Q}$ so that $a < q < b$.*

26. Subset

    Definition 18. Let A and B be setes. We say A is a subset of B when if $x \in A$, then $x \in B$. We write this as $A \subseteq B$.

27. Equality of sets

    Definition 19. Let A and B be sets. We say $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

28. Union of the family of sets.

    Definition 20. Let S_i where $i \in \mathcal{I}$ be a family of sets. Then the union of the family is

    $$\bigcup_{i \in \mathcal{I}} S_i = \{x \mid \exists i \in \mathcal{I} \text{ s.t } x \in S_i\}$$

29. Intersection of the family of sets.

    Definition 21. Let S_i where $i \in \mathcal{I}$ be a family of sets. Then the intersection of the family is:

    $$\bigcap_{i \in \mathcal{I}} S_i = \{x \mid \forall i \in \mathcal{I}, x \in S_i\}$$

30. Definition of union, intersection, and takeaway of 2 sets. Let A and B be sets.

    1. The intersection of A and B is

    $$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

    2. The union of A and B is

    $$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

    3. The set A takeaway B is

    $$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

31. Theorems for family of sets.

    Theorem 0.9. *Let A be set and B_i with $i \in \mathcal{I}$ be a family of sets. Then*

    1. $A \cup (\bigcap_{i \in \mathcal{I}} B_i) = \bigcap_{i \in \mathcal{I}} (A \cup B_i)$
    2. $A \cup (\bigcup_{i \in \mathcal{I}} B_i) = \bigcup_{i \in \mathcal{I}} (A \cup B_i)$
    3. $A \setminus (\bigcap_{i \in \mathcal{I}} B_i) = \bigcup_{i \in \mathcal{I}} (A \setminus B_i)$
    4. $A \setminus (\bigcup_{i \in \mathcal{I}} B_i) = \bigcap_{i \in \rangle} (A \setminus B_i)$

32. Ordered pair

    Definition 22. An ordered pair is a set of the form $\{a, \{a, b\}\}$. We write it as (a, b).

33. The Cartesian product

    Definition 23. Let A and B be sets. The Cartesian product $A \times B$ is the set.

    $$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

34. Relation

    Definition 24. Let A and B be sets. A relation between A and B is a subset $\mathcal{R} \subseteq A \times B$. If $A = B$, we say it is a relation on A and B. We write

    $$(a, b) \in \mathcal{R} \text{ as } a\mathcal{R}b$$

35. Reflexive relation

    Definition 25. Let $\mathcal{R}$ be a relation on set A. We say the relation is reflexive when, for all $a \in A$, $a\mathcal{R}a$.

36. Symmetric relation

    Definition 26. Let $\mathcal{R}$ be a relation on set A. We say the relation is symmetric if $a, b \in A$ and $a\mathcal{R}b$, then $b\mathcal{R}a$.

36. Transitive

    Definition 27. Let $\mathcal{R}$ be a relation on set A. We say the relation is transitive if $a, b, c \in A$ and $a\mathcal{R}b$ and $b\mathcal{R}c$, then $a\mathcal{R}c$.

37. Trichotomy

    Definition 28. Let $\mathcal{R}$ be a relation on the set A. We say that the relation has trichotomy when, $\forall a, b \in A$, exactly 1 of the following holds: $a\mathcal{R}b$, $b\mathcal{R}a$, or $a = b$.

38. Total order

    Definition 29. Let A be set. A relation on A is a total order when it is transitive and has trichotomy.

39. Equivalence relation

    Definition 30. Let A be a set. A relation on A is an equivalence relation when it is reflexive, symmetric, and transitive.

40. Equivalence class

    Definition 31. Let A be a set with an equivalence relation $\equiv$. For any $a \in A$, the equivalence class of a is a set

    $$[a] = \{x \in A \mid x \equiv A\} \subseteq A$$

41. Theorems for equivalence relation.

    Theorem 0.10. *Let A be a set with an equivalence relation $\equiv$. Assume that $a, b \in A$.*

    1. $a \in [a]$
    2. *If* $a \in [b]$*, then* $b \in [a]$.
    3. *If* $a \in [b]$*, then* $[a] = [b]$.
    4. *If* $[a] \cap [b] \neq \emptyset$*, then* $[a] = [b]$.

42. Modulo equivalence

    Definition 32. Let A be a set with an equivalence relation $\equiv$. We define a new set called "A modulo equivalence" or "A mod $\equiv$" as

    $$A_{/\equiv} = \{[a] \subseteq A \mid a \in A\}$$

43. Function

Definition 33. Let A and B be sets. A function from A to B is a pair (f, B) where $f \subseteq A \times B$ s.t if $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$.

44. Domain of function

Definition 34. Let $f : A \to B$ be a function. The domain of f is

$$Domain(f) = \{x \in A \mid \exists y \in B \text{ s.t } y = f(x)\}$$

45. Range of a function

Definition 35. Let $f : A \to B$ be a function. The range of f is

$$Range(f) = \{y \in B \mid \exists x \in A \text{ s.t } y = f(x)\}$$

46. Codomain of a function

Definition 36. Let $f : A \to B$ be a function. The co-domain of f is

$$CoDomain(f) = B$$

47. Injective function

Definition 37. Let $f : A \to B$ be a function. We say f is an injective function if: if $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$, then $a_1 = a_2$

48. Surjective function

Definition 38. Let $f : A \to B$ be a function. We say f is surjective if: if $y \in \mathbb{B}$, then $\exists x \in A$ s.t $f(x) = y$.

49. Bijective function.

Definition 39. A function is bijective when it is both injective and surjective.