Math 323 HW22

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1. Prove that the integers is countable.

   Proof. To prove that the set of integers is countable, we need to find an injection (or one-to-one relationship) that maps $\mathbb{Z}$ to $\mathbb{N}$: $i : \mathbb{Z} \to \mathbb{N}$.
   Let
   $$i(x) = \begin{cases} 2x+1 & \text{if } x \geq 0, \\ -2x & \text{if } x < 0. \end{cases}$$
   We need to show that $i(x)$ is an injection.
   Assume $x_1, x_2 \in \mathbb{Z}$ and $i(x_1) = i(x_2)$. We have 3 cases.

   1. $2x_1 + 1 = 2x_2 + 1$
      Assume that is the case. We have $2x_1 = 2x_2$ and $x_1 = x_2$.
   2. $-2x_1 = -2x_2$
      Assume that is the case. We have $x_1 = x_2$.
   3. $2x_1 + 1 = -2x_2$
      Assume BWOC that is the case. Then $x_1 \geq 0$ and $x_2 < 0$. From the equation, $x_1 = \frac{-2x_2-1}{2}$. But we know $x_1, x_2 \in \mathbb{Z}$ and $\frac{-2x_2-1}{2} \notin \mathbb{Z}$. So this case can't happen.

   Thus, $i(x)$ is an injection. And so the set of integers $\mathbb{Z}$ is countable. □

2. Let A and B be countable sets. Prove that $A \cup B$ is countable.

   Proof. Let A and B be countable sets. This respectively means,

   $$\text{There is an injection } f : A \to \mathbb{N}$$
   $$\text{There is an injection } g : B \to \mathbb{N}$$

   We define $i_3 : A \cup U \to \mathbb{N}$

   $$h(x) = \begin{cases} 2(f(x)) \text{ if } x \in A \\ 2(g(x)) + 1 \text{ if } x \in B \end{cases}$$

   Want to show: There is a function $h : A \cup B \to \mathbb{N}$ such that h is an injection.
   Assume $x_1, x_2 \in A \cup B$ and $h(x_1) = h(x_2)$. We have 3 cases to consider.

1. $x_1, x_2 \in A$.
   Assume that is the case. Then we have $2(f(x_1)) = 2(f(x_2))$. So then $f(x_1) = f(x_2)$. Since f is an injection, $x_1 = x_2$.
2. $x_1, x_2 \in B$.
   Assume that is the case. Then we have $2(g(x_1)) + 1 = 2(g(x_2)) + 1$. So then $g(x_1) = g(x_2)$. Since g is an injection, $x_1 = x_2$.
3. $x_1 \in A$ and $x_2 \in B$.
   Assume BWOC that is the case. Then we have $2(f(x_1)) = 2(g(x)) + 1$. Since $f : A \to \mathbb{N}$ and so is g, $2(f(x_1))$ and $2(g(x)) + 1$ are in $\mathbb{R}$. So then $2(f(x_1))$ is even and $2(g(x)) + 1$ is odd and they are equal. This is a contradition, this case cannot happen.

Thus, if A and B are countable sets, then $A \cup B$ is countable. $\square$

3. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions that are both continuous at 1. Prove that $f + g$ is continuous at 1.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions that are both continuous at 1. So

$$\forall \epsilon > 0, \exists \delta_1 > 0 \text{ s.t } |x - 1| < \delta_1 \text{ implies } |f(x) - f(1)| < \epsilon$$
$$\forall \epsilon > 0, \exists \delta_2 > 0 \text{ s.t } |x - 1| < \delta_2 \text{ implies } |g(x) - g(1)| < \epsilon$$

We want to show that $|f(x) + g(x) - f(1) - g(1)| < \epsilon$. Let $\epsilon > 0$. So $\frac{\epsilon}{2} > 0$. Assume there is $\delta_1 > 0$ so that $|x - 1| < \delta_1$. Let $x \in \mathbb{R}$. Assume there is $\delta_2 > 0$ so that $|x - 1| < \delta_2$. So then $|x - 1| < min\{\delta_1, \delta_2\}$. Since $|f(x) - f(1)| < \epsilon$ for all $\epsilon > 0$, $|f(x) - f(1)| < \frac{\epsilon}{2}$. Same thing happens to $|g(x) - g(1)|$. Then consider

$$|f(x) - f(1)| < \frac{\epsilon}{2} \text{ and } |g(x) - g(1)| < \frac{\epsilon}{2}$$
$$|f(x) - f(1)| + |g(x) - g(1)| < \epsilon$$
$$|f(x) - f(1) + g(x) - g(1)| \leq |f(x) - f(1)| + |g(x) - g(1)| < \epsilon$$
$$|f(x) + g(x) - f(1) - g(1)| < \epsilon$$

Thus $f + g$ is continuous at 1. $\square$