

Math 323 HW19

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Problem 12.8: Prove: If $f : A \rightarrow B$ is a function with domain A and T_i with $i \in \mathcal{I}$ is a family of sets where $\forall i \in \mathcal{I}, T_i \subseteq B$, then

$$f^{-1} \left(\bigcup_{i \in \mathcal{I}} T_i \right) = \bigcup_{i \in \mathcal{I}} f^{-1}(T_i)$$

Proof. Let $f : A \rightarrow B$ be a function and T_i with $i \in \mathcal{I}$ be a family of sets where $\forall i \in \mathcal{I}, T_i \subseteq B$. We want to prove 2 claims.

$$(a) \quad f^{-1} \left(\bigcup_{i \in \mathcal{I}} T_i \right) \subseteq \bigcup_{i \in \mathcal{I}} f^{-1}(T_i)$$

Assume $x \in f^{-1} \left(\bigcup_{i \in \mathcal{I}} T_i \right)$. By the theorem of image and pre-image, $f(x) \in \bigcup_{i \in \mathcal{I}} T_i$. So then $\exists i \in \mathcal{I}$ s.t. $f(x) \in T_i$. Since $f(x) \in T_i$ for some $i \in \mathcal{I}$, $x \in f^{-1}(T_i)$. Thus $x \in \bigcup_{i \in \mathcal{I}} f^{-1}(T_i)$.

$$(b) \quad f^{-1} \left(\bigcup_{i \in \mathcal{I}} T_i \right) \supseteq \bigcup_{i \in \mathcal{I}} f^{-1}(T_i)$$

Assume $x \in \bigcup_{i \in \mathcal{I}} f^{-1}(T_i)$. So then $\exists i \in \mathcal{I}$ s.t. $x \in f^{-1}(T_i)$. By the theorem of image and preimage, $f(x) \in T_i$ for some $i \in \mathcal{I}$. So then $f(x) \in \bigcup_{i \in \mathcal{I}} T_i$. Again, by the theorem of image and preimage, $x \in f^{-1} \left(\bigcup_{i \in \mathcal{I}} T_i \right)$.

$$\text{Thus } f^{-1} \left(\bigcup_{i \in \mathcal{I}} T_i \right) = \bigcup_{i \in \mathcal{I}} f^{-1}(T_i) \quad \square$$

Problem 12.10: Prove: Let $f : A \rightarrow B$ be a function with domain A . Prove: if $\forall S \subseteq A, S = f^{-1}(f(S))$, then $f(x)$ is injective.

Proof. Let $f : A \rightarrow B$ be a function with domain A .

Assume $\forall S \subseteq A, S = f^{-1}(f(S))$. So $S = \{x \in A \mid f(x) \in f(S)\}$. We wish to show: if $\forall x_1, x_2 \in A$ and $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Assume $x_1 \in f^{-1}(f(S))$ and $x_2 \in S$. So then $x_1, x_2 \in A$. Assume $f(x_1) = f(x_2)$. Since $x_1, x_2 \in S$, $f(x_1) \in f(S)$ and $f(x_2) \in f(S)$. \square

Problem 12.11: Let $f : A \rightarrow B$ be a function with domain A . Prove: if $\forall T \subseteq B, T = f(f^{-1}(T))$, then f is surjective.

Proof. Let $f : A \rightarrow B$ be a function with domain A .

Assume $\forall T \subseteq B, T = f(f^{-1}(T))$. We want to show: if $y \in B$, then $\exists x \in A$ so that $f(x) = y$.

Assume $y \in B$. So then $y \in T$, meaning $y \in f(f^{-1}(T))$. So then $\exists x \in A$ so that $x \in f^{-1}(T)$. Since $f^{-1}(T) \subseteq A$, $x \in A$. \square