Math 323 HW14

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Problem 9.1: True or false

(a) $\emptyset \in \emptyset$. False.
(b) $\emptyset \subseteq \emptyset$. True.
(c) $\emptyset = \emptyset$. True.
(d) $\emptyset \in \{\emptyset\}$. True.
(e) $\emptyset \subseteq \{\emptyset\}$. True.
(f) $\emptyset = \{\emptyset\}$. False.
(g) $\{\emptyset\} \in \emptyset$. False.
(h) $\{\emptyset\} \subseteq \emptyset$. False.
(i) $\{\emptyset\} = \emptyset$. False.
(j) As a subset of $\mathbb{R}$: $\emptyset$ has a minimum. False
(k) As a subset of $\mathbb{R}$: $\emptyset$ has a lower bound. True.
(l) As a subset of $\mathbb{R}$: $\emptyset$ has an infimum. False.
(m) In the integers, $(1, 5) = \{2, 3, 4\}$. True.
(n) $(1, 5) \cap \mathbb{Z} = \{2, 3, 4\}$. True.
(o) $(-3, 3]$ has an upper bound. True.
(p) $(-3, 3]$ has an infimum. True.
(q) $\infty \in [-3, \infty]$. False.
(r) $(0, 4) \subseteq \mathbb{Q}$. False.
(s) $(0, \infty) \cap \mathbb{Z} \subseteq \mathbb{Q}$. True.
(t) $(0, \infty) \cup \mathbb{Z} \subseteq \mathbb{Q}$. False.
(u) $(0, 10) \setminus \mathbb{Z} \subseteq \mathbb{Q}$. False.
(v) $\mathbb{Z} \setminus (0, 10) \subseteq \mathbb{Q}$. True.

Problem 9.3c: Prove that for all sets A and B: $A \cap B = B$ if and only if $B \subseteq A$.

Proof. Let A and B be sets. We need to prove two statements.

1. If $A \cap B = B$, then $B \subseteq A$.

   Assume $A \cap B = B$. So $B \subseteq A \cap B$. So if $a \in B$, then $a \in A \cap B$. Assume $a \in B$, then $a \in A$ and $a \in B$. So if $a \in B$, then $a \in A$. This means $B \subseteq A$.

2. If $B \subseteq A$, then $A \cap B = B$.

   Assume $B \subseteq A$. We need to prove 2 claims.

   i. $A \cap B \subseteq B$.
   Assume $a \in (A \cap B)$. So $a \in A$ and $a \in B$. So $A \cap B \subseteq B$.

   ii. $B \subseteq A \cap B$.
   Assume $a \in B$. Since $B \subseteq A$, $a \in A$. So $a \in B \cap A$. So $B \subseteq (A \cap B)$.

   Thus if $B \subseteq A$, then $A \cap B = B$.

So we have proved that for all sets A and B: $A \cap B = B$ if and only if $B \subseteq A$. $\square$

Problem 9.4: An interval is supposed to be a subset of $\mathbb{R}$ with no gaps and no holes. Thus all the following sets are intervals:

- $\emptyset$ and $\mathbb{R}$.
- for $a \in \mathbb{R}$, $\{a\}, (a, \infty), [a, \infty), (-\infty, a)$ and $(-\infty, a]$;
- and for $a < b$, $(a, b), (a, b], [a, b),$ and $[a, b]$.

(a) Give a definition of "interval" based on the fact that an interval must contain all the real numbers that lie between any 2 numbers in the interval.

Definition 1. Let $a, b \in \mathbb{R}$ so that $a \leq b$. We say the the set of all real numbers between a and b is an *interval*. The inclusion of a or b or both depends on the notation of interval.

(b) Use the definition to prove, if $A \subseteq \mathbb{R}$ is a nonempty interval with a upper and lower bounds, then $\exists a, b \in \mathbb{R}$ s.t $(a, b) \subseteq A \subseteq [a, b]$.

Proof. Assume $A \subseteq \mathbb{R}$ is a nonempty interval with a upper bound and lower bound. We want to prove $\exists a, b \in \mathbb{R}$ s.t $(a, b) \subseteq A \subseteq [a, b]$. Since $A \in \mathbb{R}$, $A \neq \emptyset$, and A has an upper bound, by the Completeness Axiom, A has a least upper bound. Let it be b. We also know A has a lower bound, by the Completeness Axiom, A has a greatest lower bound. Let it be a. Let $r \in A$. We know $a \leq r \leq b$. So $r \in [a, b]$. So $A \subseteq [a, b]$.
Let $q \in (a, b)$. This means $a < q < b$, so $q \in A$. So $(a, b) \subseteq A$. Thus $(a, b) \subseteq A \subseteq [a, b]$ where a and b are the greatest lower bound and the least upper bound of the set A. $\square$

(c) Use the definition to prove, if $A \subseteq \mathbb{R}$ is a nonempty interval with a upper bound and no lower bound, then $\exists b \in \mathbb{R}$ s.t $(-\infty, b) \subseteq A \subseteq (-\infty, b]$.

Proof. Assume $A \subseteq \mathbb{R}$ is a nonempty with an upper bound and no lower bound. By the Completeness Axiom, A has a least upper bound. Let it be b. Let $r \in A$. We know $-\infty < r \leq b$, so $A \subseteq (-\infty, b]$. Let $q \in (-\infty, b)$. This means $-\infty < q < b$. But then that also means $q \in A$. So $(-\infty, b) \subseteq A$. $\square$

(d) Use the definition to prove, if $A \in \mathbb{R}$ is a nonempty interval with a lower bound and no upper bound, then $\exists a \in \mathbb{R}$ such that $(a, \infty) \subseteq A \subseteq [a, \infty)$.

Proof. Assume $A \subseteq \mathbb{R}$ is a nonempty with a lower bound and no upper bound. By the Completeness Axiom, A has a greatest lower bound. Let it be a. Let $r \in A$. We know $a \leq r < \infty$, so $A \subseteq [a, \infty)$. Let $q \in (a, \infty)$. This means $a < q < \infty$. But then that also means $q \in A$. So $(a, \infty) \subseteq A$. $\square$

(e) Use the definition to prove, if $A \subseteq \mathbb{R}$ is a nonempty interval with no lower bound and no upper bound, then $A = \mathbb{R}$.

Proof. Assume A is a nonempty interval with no lower bound and no upper bound. We will need prove 2 claims.

1. $A \subseteq \mathbb{R}$.
   Assume $a \in A$. Since A has no lower bound and no upper bound, $A = (-\infty, \infty)$. So $-\infty < a < \infty$. But then that means $a \in \mathbb{R}$. So $A \subseteq \mathbb{R}$.
2. $\mathbb{R} \subseteq A$. Assume $r \in \mathbb{R}$. So $-\infty < r < \infty$. Since A has no lower bound and no upper bound, $A = (-\infty, \infty)$. So $r \in A$. Thus $\mathbb{R} \subseteq A$.

$\square$

(f) Use the definition to prove, if $A \subseteq \mathbb{R}$ is an interval, then A has one of the forms

$$\emptyset, \{a\}, (a, b), (a, b], [a, b), [a, b]$$
$$(a, \infty), [a, \infty), (-\infty, b), (-\infty, b] \text{ or } \mathbb{R}$$

Proof. By Trichotomy and (b), (c), (d), and (e), A has to be one of the form: $(a, b), (a, b]), [a, b), [a, b], (a, \infty), [a, \infty), (-\infty, b), (-\infty, b]$ or $\mathbb{R}$. By the definition of interval, assume $a, b \in \mathbb{R}$ so that $a = b$, then by Trichotomy and interval notation, we have 2 cases:

1. If $r \in A$, then $a < r < a$. Assume $r \in A$, $a < r < a$. There's no r satisfy that condition. So $A = \emptyset$.

2. If $r \in A$, then $a \le r \le a$. Assume $r \in A$, $a \le r \le a$. By Trichotomy, $r = a$. So $A = \{a\}$.

$\square$

Problem 9.5a: Let A, B, and C be sets, prove that: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. Let A, B, and C be sets. We need to prove 2 claims.

1. $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
   Assume $a \in A \cap (B \cup C)$. So $a \in A$ and $a \in (B \cup C)$. Consider $a \in B \cup C$. We have two cases.

Case 1: $a \in B$. We also know $a \in A$. So $a \in A$ and $a \in B$. This means $a \in A \cap B$. Thus $a \in (A \cap B) \cup (A \cap C)$.

Case 2: $a \in C$. We also know $a \in A$. So $a \in A$ and $a \in C$. This means $a \in A \cap C$. Thus $a \in (A \cap B) \cup (A \cap C)$.

So $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

2. $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.
   Assume $a \in (A \cap B) \cup (A \cap C)$. We have 2 cases.

Case 1: $a \in A \cap B$. So $a \in A$ and $a \in B$. So $a \in B \cup C$. Since $a \in A$, $a \in A \cap (B \cup C)$.

Case 2: $a \in A \cap C$. So $a \in A$ and $a \in C$. So $a \in B \cup C$. Since $a \in A$, $a \in A \cap (B \cup C)$.

Thus $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

So $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. $\square$

Problem 9.7: Define a family of sets by: for $n \in \mathbb{Z}$, $S_n = \{0, \pm n\}$.

(a) $n = m$ implies $S_n = S_m$. True

(b) $S_n = S_m$ implies $n = m$. False. Take $S_{-1} = S_1$ but $1 \neq -1$.

(c) $n \neq m$ implies $S_n \neq S_m$. False. If $1 \neq -1$, then $S_1 \neq S_{-1}$. But $S_1 = S_{-1}$.

(d) $S_n \neq S_m$ implies $n \neq m$. True.

(e) Every set in the family has three elements. False. $S_0 = \{0\}$.

(f) If $S_n \subseteq S_m$, then $m = n$. False. $S_0 \subseteq S_1$ but $0 \neq 1$.

(g) $\forall n, m \in \mathbb{Z}$, $S_n \cap S_m = \{0\}$. True

(h) $\forall n, m \in \mathbb{Z}$, $S_n \cap S_m = \emptyset$. False.