Math 323 HW9

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Problem 6.3: For any subset $S \subseteq \mathbb{R}$, let

$$LB(S) = \{l \in \mathbb{R} \mid l \text{ is a lower bound on the set } S\}$$

Prove that if $s \in S$, then s is an upper bound of $LB(S)$.

Proof. Assume $s \in S$. Assume $l \in LB(S)$. Since $LB(S)$ is the set of lower bound of S, $l \leq s$. This means s is an upper bound of $LB(S)$. $\square$

Problem 6.4: Let A be a set of real numbers.

(a) Prove that if A has an upper bound, then A has an upper bound that is a natural number.

Proof. Assume A is a set of real numbers. Assume A has an upper bound. Call it u. And so if $a \in A$, then $u \geq a$. By the Archimedean principle, $\forall a \in A$ and $a \in \mathbb{R}$, $\exists n \in \mathbb{N}$ so that $n > a$. And so $n \geq a$. This means n is an upper bound of A. Let $u = n$ and we have what we need. $\square$

(b) Prove that if A has a lower bound, then A has a lower bound that is an integer.

Proof. Assume A is a set of real numbers. Assume A has a lower bound. Call it l. And so if $a \in A$, then $l \leq a$. By the Archimedean principle, $\forall a \in A$ so that $-a \in \mathbb{R}$, $\exists n \in \mathbb{N}$ so that $n > -a$. And so $-n < a$. So $-n \leq a$. So $-n$ is a lower bound of A and $-n \in \mathbb{Z}$. Let $l = -n$ and we have what we need. $\square$

(c) Prove that if A has a lower bound and an upper bound, then there is a natural number n so that n is an upper bound and $-n$ is a lower bound of A.

Proof. Assume A is a set of real numbers. Assume A has a lower bound l and an upper bound u. So if $a \in A$, then $l \leq a \leq u$. Since $a \in \mathbb{R}$, by the Archimedean principle, $\exists n \in \mathbb{N}$ so that $n > a$. If $n \in \mathbb{N}$, in $\mathbb{Z}$, $n > -n$. So we have $-n < a < n$. And so $-n \leq a \leq n$. Let $l = -n$ and $u = n$ and we have what we need. $\square$

(d) Prove that A is bounded (above and below) if and only if $\exists n \in \mathbb{N}$ so that $\forall x \in A$, $-n \leq x \leq n$.

Proof. We need to prove 2 statements:

1. If A is bounded above and below then $\exists n \in \mathbb{N}$ so that $\forall x \in A$, $-n \leq x \leq n$.

   Proof. Assume A is bounded above and below. This means A has a lower bound l and an upper bound u. So if $a \in A$, then $l \leq a \leq u$. Since $a \in \mathbb{R}$, by the Archimedean principle, $\exists n \in \mathbb{N}$ so that $n > a$. If $n \in \mathbb{N}$, in $\mathbb{Z}$, $n > -n$. So we have $-n < a < n$. And so $-n \leq a \leq n$. Let $l = -n$ and $u = n$ and we have what we need. □

2. If $\forall x \in A$, $-n \leq x \leq n$, then A is bounded above and below.

   Proof. Assume $\forall x \in A$, $-n \leq x \leq n$. This means $-n \leq a$ and $a \leq n$. These statements, respectively mean $-n$ is a lower bound of A and n is an upper bound of A. So A is bounded above and below. □

□

(e) Prove that A is bounded (above and below) if and only if $\exists n \in \mathbb{N}$ so that $\forall x \in A$, $-n < x < n$.

Proof. We need to prove 2 statements:

1. If A is bounded above and below then $\exists n \in \mathbb{N}$ so that $\forall x \in A$, $-n < x < n$.

   Proof. Assume A is bounded above and below. This means A has a lower bound l and an upper bound u. So if $a \in A$, then $l \leq a \leq u$. Since $a \in \mathbb{R}$, by the Archimedean principle, $\exists n \in \mathbb{N}$ so that $n > a$. If $n \in \mathbb{N}$, in $\mathbb{Z}$, $n > -n$. So we have $-n < a < n$. Let $l = -n$ and $u = n$ and we have what we need. □

2. If $\forall x \in A$, $-n < x < n$, then A is bounded above and below.

   Proof. Assume $\forall x \in A$, $-n < x < n$. This means $-n < a$ and $a < n$. These statements, respectively mean $-n$ is a lower bound of A and n is an upper bound of A. So A is bounded above and below. □

□

Problem 6.5: Let $a, b \in \mathbb{R}$ with $a < b$. Prove that $\exists s \in \mathbb{R}$ such that $a < s < b$.

Proof. Assume $a, b \in \mathbb{R}$ with $a < b$. Since $\mathbb{R}$ is an ordered field, by the Average theorem, $\exists s \in \mathbb{R}$ so that $a < s < b$. $\square$