Problem 5.2b: Let $\mathbb{F}$ be an ordered field. Prove: For all $n \in \mathbb{N}$, if $a_1 \in \mathbb{F}; a_2 \in \mathbb{F}; ...a_n \in \mathbb{F}$, and $\sum_{k=1}^{n} a_k^2 = 0$, then $a_1 = a_2 = a_3 = ... = a_n = 0$.

Proof. Assume $n \in \mathbb{N}$. We need to prove two claims.

i. We want to show that: If for $n = 1$, then if $a_1, ..., a_n \in \mathbb{F}$ and $\sum_{k=1}^{n} a_k^2 = 0$, then $a_1 = a_2 = ... = a_n = 0$.

Assume $n = 1$. Assume $a_1, ..., a_n \in \mathbb{F}$. So $a_1 \in \mathbb{F}$. Assume $\sum_{k=1}^{n} a_k^2 = 0$, meaning

$$\sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{1} a_k^2 = a_1^2 = 0$$

So $a_1^2 = 0$. By trichotomy, we can imply that $a_1 = 0$.

So we have proved that for $n = 1$, then if $a_1, ..., a_n \in \mathbb{F}$ and $\sum_{k=1}^{n} a_k^2 = 0$, then $a_1 = a_2 = ... = a_n = 0$.

ii. We also want to show that: If for $n = n_0$, if $a_1, ..., a_n \in \mathbb{F}$ and $\sum_{k=1}^{n} a_k^2 = 0$, then $a_1 = a_2 = ... = a_n = 0$, then for $n = n_0 + 1$, if $a_1, ..., a_n \in \mathbb{F}$ and $\sum_{k=1}^{n} a_k^2 = 0$, then $a_1 = a_2 = ... = a_n = 0$.

Assume for $n = n_0$, $a_1, ..., a_{n_0} \in \mathbb{F}$ and $\sum_{k=1}^{n} a_k^2 = 0$ then $a_1 = a_2 = ... = a_{n_0} = 0$. We want to prove that for $n = n_0+1$, if $b_1, b_2, ..., b_n \in \mathbb{F}$ and $\sum_{k=1}^{n} b_k^2 = 0$ then $b_1 = b_2 = ... = b_n = 0$.

Assume $b_1, b_2, ...b_{n_0+1} \in \mathbb{F}$ and $\sum_{k=1}^{n_0+1} b_k^2 = 0$. We let $a_1 = b_1 = a_2 = b_2 = ... = a_{n_0} = b_{n_0} = 0$. And so we have $\sum_{k=1}^{n_0} b_k^2 = 0$. Consider

$$\sum_{k=1}^{n_0} b_k^2 + b_{n_0+1}^2 = b_{n_0+1}^2$$
$$\sum_{k=1}^{n_0+1} b_k^2 = b_{n_0+1}^2$$

But our assumption says $\sum_{k=1}^{n_0+1} b_k^2 = 0$. So $b_{n_0+1}^2 = 0$. So $b_{n_0+1} = 0$. And so we have $b_1 = b_2 = ... = b_{n_0} = b_{n_0+1} = 0$.

Proving both claim (i) and (ii) completes our proof by induction on n. $\square$