Vypočtěte:

1. $I = \iiint\limits_A x^2yz\,dx\,dy\,dz, \quad A :$ kvádr $< 1,3 > \times < 0,2 > \times < 1,2 >$.

Množina A je dána nerovnostmi $\begin{array}{ccccc} 1 & \leq & x & \leq & 3 \\ 0 & \leq & y & \leq & 2 \\ 1 & \leq & z & \leq & 2 \end{array}$

$$I = \int_1^3 \left(\int_0^2 \left(\int_1^2 x^2yz\,dz \right) dy \right) dx = \int_1^3 \left(\int_0^2 x^2y \left[\frac{z^2}{2} \right]_1^2 dy \right) dx$$

$$= \int_1^3 \left(\int_0^2 x^2y \left(\frac{4}{2} - \frac{1}{2} \right) dy \right) dx = \frac{3}{2} \int_1^3 \left(\int_0^2 x^2y\,dy \right) dx = \frac{3}{2} \int_1^3 x^2 \left[\frac{y^2}{2} \right]_0^2 dx$$

$$= \frac{3}{2} \int_1^3 x^2 \left(\frac{4}{2} - 0 \right) dx = \frac{3}{2} \cdot 2 \int_1^3 x^2\,dx = 3 \left[\frac{x^3}{3} \right]_1^3 = 3 \cdot \frac{1}{3} \left(3^3 - 1^3 \right) = 27 - 1 = 26.$$

2. $I = \iiint\limits_A x\,dx\,dy\,dz, \quad A : x = 0, y = 0, z = 0, x + y + z = 2.$

Množina A je dána nerovnostmi $\begin{array}{ccccc} 0 & \leq & x & \leq & 2 \\ 0 & \leq & y & \leq & 2-x \\ 0 & \leq & z & \leq & 2-x-y \end{array}$

$$I = \int_0^2 \left(\int_0^{2-x} \left(\int_0^{2-x-y} x\,dz \right) dy \right) dx = \int_0^2 \left(\int_0^{2-x} x\,[z]_0^{2-x-y}\,dy \right) dx = \int_0^2 \left(\int_0^{2-x} x\,(2-x-y)\,dy \right) d$$

$$= \int_0^2 \left(\int_0^{2-x} \left(2x - x^2 - xy \right) dy \right) dx = \int_0^2 \left[2xy - x^2y - x\frac{y^2}{2} \right]_0^{2-x} dx$$

$$= \int_0^2 \left[2x(2-x) - x^2(2-x) - x\frac{(2-x)^2}{2} \right] dx$$

$$= \int_0^2 \left(\frac{x^3}{2} - 2x^2 + 2x \right) dx = \left[\frac{x^4}{8} - 2 \cdot \frac{x^3}{3} + 2 \cdot \frac{x^2}{2} \right]_0^2 = \frac{2^4}{8} - 2 \cdot \frac{2^3}{3} + 2 \cdot \frac{2^2}{2} = \frac{2}{3}.$$

3. $I = \iiint\limits_A xyz\,dx\,dy\,dz, \quad A :$ I. oktant, $z = 0, z = xy, y = x, y = 1.$

Množina A je dána nerovnostmi $\begin{array}{ccccc} 0 & \leq & x & \leq & 1 \\ x & \leq & y & \leq & 1 \\ 0 & \leq & z & \leq & xy \end{array}$

$$I = \int_0^1 \left(\int_x^1 \left(\int_0^{xy} xyz\,dz \right) dy \right) dx = \int_0^1 \left(\int_x^1 xy \left[\frac{z^2}{2} \right]_0^{xy} dy \right) dx = \int_0^1 \left(\int_x^1 xy\frac{x^2y^2}{2}\,dy \right) dx$$

$$= \frac{1}{2} \int_0^1 \left(\int_x^1 x^3y^3\,dy \right) dx = \frac{1}{2} \int_0^1 x^3 \left[\frac{y^4}{4} \right]_x^1 dx = \frac{1}{2} \cdot \frac{1}{4} \int_0^1 x^3 \left(1^4 - x^4 \right) dx = \frac{1}{8} \int_0^1 \left(x^3 - x^7 \right) dx$$

$$= \frac{1}{8} \left[\frac{x^4}{4} - \frac{x^8}{8} \right]_0^1 = \frac{1}{8} \left(\frac{1}{4} - \frac{1}{8} \right) = \frac{1}{64}.$$

4. $I = \iiint\limits_A \frac{1}{x}\, dx\, dy\, dz, \quad A: x+y+z=4, x=1, x=2, y=0, y=x, z=0.$

Množina A je dána nerovnostmi $\begin{array}{ccccc} 1 & \le & x & \le & 2 \\ 0 & \le & y & \le & x \\ 0 & \le & z & \le & 4-x-y \end{array}$

$$I = \int_1^2 \left(\int_0^x \left(\int_0^{4-x-y} \frac{1}{x}\, dz \right) dy \right) dx = \int_1^2 \left(\int_0^x \frac{1}{x} \left[z \right]_0^{4-x-y} dy \right) dx = \int_1^2 \left(\int_0^x \frac{1}{x} (4-x-y)\, dy \right) dx$$

$$= \int_1^2 \left(\int_0^x \left(\frac{4}{x} - 1 - \frac{y}{x} \right) dy \right) dx = \int_1^2 \left[\frac{4}{x} \cdot y - y - \frac{1}{x} \cdot \frac{y^2}{2} \right]_0^x dx = \int_1^2 \left[\frac{4}{x} \cdot x - x - \frac{1}{x} \cdot \frac{x^2}{2} \right]_0 dx$$

$$= \int_1^2 \left(4 - x - \frac{x}{2} \right) dx = \int_1^2 \left(4 - \frac{3}{2} x \right) dx = \left[4x - \frac{3}{4} x^2 \right]_1^2 = 4 \cdot 2 - \frac{3}{4} \cdot 2^2 - \left(4 \cdot 1 - \frac{3}{4} \cdot 1^2 \right) = \frac{7}{4}.$$

5. $I = \iiint\limits_A e^y\, dx\, dy\, dz, \quad A: y=1, y=-x, y=x, z=0, z=y.$

Množina A je dána nerovnostmi $\begin{array}{ccccc} 0 & \le & y & \le & 1 \\ -y & \le & x & \le & y \\ 0 & \le & z & \le & y \end{array}$

$$I = \int_0^1 \left(\int_{-y}^y \left(\int_0^y e^y\, dz \right) dx \right) dy = \int_0^1 \left(\int_{-y}^y e^y \left[z \right]_0^y dx \right) dy = \int_0^1 \left(\int_{-y}^y y \cdot e^y\, dx \right) dy = \int_0^1 y \cdot e^y \cdot [x]_{-y}^y\, dy$$

$$= \int_0^1 y \cdot e^y \cdot [y+y]\, dy = \int_0^1 2y^2 e^y\, dy = \left| \begin{array}{ll} u = 2y^2 & u' = 4y \\ v' = e^y & v = e^y \end{array} \right| = \left[2y^2 e^y \right]_0^1 - \int_0^1 4y e^y\, dy$$

$$= \left| \begin{array}{ll} u = 4y & u' = 4 \\ v' = e^y & v = e^y \end{array} \right| = 2 \cdot 1^2 \cdot e^1 - 2 \cdot 0 \cdot e^0 - \left(\left[4y e^y \right]_0^1 - \int_0^1 4 e^y\, dy \right) = 2e - \left(4 \cdot 1 \cdot e^1 - 0 \right) + 4 \cdot \int_0^1 e^y\, dy$$

$$= -2e + 4 \cdot \left[e^1 - e^0 \right] = 2e - 4.$$

Integrujte transformací do válcových (cylindrických) souřadnic

$$\begin{array}{rcl} x & = & r \cos \varphi \\ y & = & r \sin \varphi \\ z & = & z \\ |J| & = & r. \end{array}$$

1. $I = \iiint\limits_A \left(x^2 + y^2 \right) dx\, dy\, dz, \quad A: x^2 + y^2 = 4, z = 0, z = 3.$

Množina $F^{-1}(A)$ je dána nerovnostmi $\begin{array}{ccccc} 0 & \le & r & \le & 2 \\ 0 & \le & \varphi & \le & 2\pi \\ 0 & \le & z & \le & 3 \end{array}$

$$I = \int_0^{2\pi} \left(\int_0^2 \left(\int_0^3 \left(r^2 \cos^2 \varphi + r^2 \sin^2 \varphi \right) \cdot r\, dz \right) dr \right) d\varphi$$

$$= \int_0^{2\pi} \left(\int_0^2 \left(\int_0^3 r^2 \left(\cos^2 \varphi + \sin^2 \varphi \right) \cdot r\, dz \right) dr \right) d\varphi = \int_0^{2\pi} \left(\int_0^2 \left(\int_0^3 r^3\, dz \right) dr \right) d\varphi$$

$$= \int_0^{2\pi} \left(\int_0^2 r^3 \left[z \right]_0^3 dr \right) d\varphi = \int_0^{2\pi} \left(\int_0^2 r^3 \cdot 3\, dr \right) d\varphi = 3 \cdot \int_0^{2\pi} \left(\int_0^2 r^3\, dr \right) d\varphi = 3 \cdot \int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^2 d\varphi$$

$$= 3 \cdot \int_0^{2\pi} \frac{2^4}{4}\, d\varphi = 12 \cdot \int_0^{2\pi} d\varphi = 12 \cdot \left[\varphi \right]_0^{2\pi} = 12 \cdot 2\pi = 24\pi.$$

2. $I = \iiint\limits_A xz\,dx\,dy\,dz, \quad A: x^2+y^2+z^2 \le 4,$ I. oktant

Množina $F^{-1}(A)$ je dána nerovnostmi $\begin{array}{ccccc} 0 & \le & r & \le & 2 \\ 0 & \le & \varphi & \le & \frac{\pi}{2} \\ 0 & \le & z & \le & \sqrt{4-r^2} \end{array}$

$$I = \int_0^{\frac{\pi}{2}} \left(\int_0^2 \left(\int_0^{\sqrt{4-r^2}} r\cos\varphi \cdot z \cdot r\,dz \right) dr \right) d\varphi = \int_0^{\frac{\pi}{2}} \left(\int_0^2 \left(\int_0^{\sqrt{4-r^2}} r^2\cos\varphi \cdot z\,dz \right) dr \right) d\varphi$$

$$= \int_0^{\frac{\pi}{2}} \left(\int_0^2 r^2\cos\varphi \cdot \left[\frac{z^2}{2} \right]_0^{\sqrt{4-r^2}} dr \right) d\varphi = \frac{1}{2} \int_0^{\frac{\pi}{2}} \left(\int_0^2 r^2\cos\varphi \cdot \left(4-r^2\right) dr \right) d\varphi$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left(\int_0^2 \left(4r^2-r^4\right)\cos\varphi\,dr \right) d\varphi = \frac{1}{2} \int_0^{\frac{\pi}{2}} \left[4\cdot\frac{r^3}{3} - \frac{r^5}{5} \right]_0^2 \cos\varphi\,d\varphi = \frac{1}{2}\left(\frac{32}{3} - \frac{32}{5} \right) \int_0^{\frac{\pi}{2}} \cos\varphi\,d\varphi$$

$$= \frac{32}{15} \cdot [\sin\varphi]_0^{\frac{\pi}{2}} = \frac{32}{15}\cdot(1-0) = \frac{32}{15}.$$

3. $I = \iiint\limits_A \sqrt{x^2+y^2}\,dx\,dy\,dz, \quad A: z=0, y+z=4, x^2+y^2=16.$
Rovina $y+z=4 \Rightarrow z = 4-y = 4-r\sin\varphi$.
$x^2+y^2 = r^2 = 16 \Rightarrow r = 4.$

Množina $F^{-1}(A)$ je dána nerovnostmi $\begin{array}{ccccc} 0 & \le & r & \le & 4 \\ 0 & \le & \varphi & \le & 2\pi \\ 0 & \le & z & \le & 4-r\sin\varphi \end{array}$

$$I = \int_0^{2\pi} \left(\int_0^4 \left(\int_0^{4-r\sin\varphi} \sqrt{r^2\cos^2\varphi + r^2\sin^2\varphi} \cdot r\,dz \right) dr \right) d\varphi = \int_0^{2\pi} \left(\int_0^4 \left(\int_0^{4-r\sin\varphi} r^2\,dz \right) dr \right) d\varphi$$

$$= \int_0^{2\pi} \left(\int_0^4 r^2\,[z]_0^{4-r\sin\varphi}\,dr \right) d\varphi = \int_0^{2\pi} \left(\int_0^4 \left(4r^2 - r^3\sin\varphi\right) dr \right) d\varphi = \int_0^{2\pi} \left[\frac{4}{3}r^3 - \frac{r^4}{4}\sin\varphi \right]_0^4 d\varphi$$

$$= \int_0^{2\pi} \left(\frac{256}{3} - 64\sin\varphi \right) d\varphi = \left[\frac{256}{3}\varphi - 64\,(-\cos\varphi) \right]_0^{2\pi} = \left[\frac{256}{3}\varphi + 64\cos\varphi \right]_0^{2\pi}$$

$$= \frac{256}{3}\cdot 2\pi + 64\cdot 1 - 0 - 64\cdot 1 = \frac{512}{3}\pi.$$

4. $I = \iiint\limits_A yz\,dx\,dy\,dz, \quad A: x=0, y=0, z=0, x^2+y^2+z^2=1, x^2+y^2=x.$
Kulová plocha $x^2+y^2+z^2=1$ ve válcových souřadnicích - horní mez pro z (dolní mez je $z=0$):

$$r^2\cos^2\varphi + r^2\sin^2\varphi + z^2 = r^2 + z^2 = 1 \Longrightarrow z^2 = 1-r^2 \Rightarrow z = \sqrt{1-r^2}$$

Válcová plocha $x^2+y^2=x$. Upravíme na čtverec: $\left(x-\frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$.
(průmětem válcové plochy do roviny xy je kružnice $\left(x-\frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$ se středem v bodě $\left(\frac{1}{2}, 0\right)$ a poloměrem $R = \frac{1}{2}$).
Válcová plocha $x^2+y^2=x$ ve válcových souřadnicích:

$$r^2\cos^2\varphi + r^2\sin^2\varphi = r\cos\varphi \Longrightarrow r = \cos\varphi.$$

Množina $F^{-1}(A)$ je dána nerovnostmi $\begin{array}{ccccc} o & \le & r & \le & \cos\varphi \\ 0 & \le & \varphi & \le & \frac{\pi}{2} \\ 0 & \le & z & \le & \sqrt{1-r^2} \end{array}$

$$I=\int_0^{\frac{\pi}{2}}\left(\int_0^{\cos\varphi}\left(\int_0^{\sqrt{1-r^2}} r\sin\varphi\cdot z\cdot r\,dz\right)dr\right)d\varphi=\int_0^{\frac{\pi}{2}}\left(\int_0^{\cos\varphi} r^2\sin\varphi\left[\frac{z^2}{2}\right]_0^{\sqrt{1-r^2}}dr\right)d\varphi$$
$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\int_0^{\cos\varphi} r^2\sin\varphi\cdot\left(1-r^2\right)dr\right)d\varphi=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\int_0^{\cos\varphi}\left(r^2-r^4\right)\sin\varphi\,dr\right)d\varphi$$
$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[\frac{r^3}{3}-\frac{r^5}{5}\right]_0^{\cos\varphi}\sin\varphi\,d\varphi=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\frac{\cos^3\varphi}{3}-\frac{\cos^5\varphi}{5}\right)\sin\varphi\,d\varphi$$
$$=\left|\begin{array}{rcrcrcl}\cos\varphi= & t & \varphi= & 0 & \Rightarrow & t & =\cos 0=1\\ -\sin\varphi\,d\varphi= & dt & \varphi= & \frac{\pi}{2} & \Rightarrow & t & =\cos\frac{\pi}{2}=0\end{array}\right|$$
$$=\frac{1}{2}\int_1^0\left(\frac{t^3}{3}-\frac{t^5}{5}\right)(-dt)=\frac{1}{2}\int_0^1\left(\frac{t^3}{3}-\frac{t^5}{5}\right)dt=\frac{1}{2}\left[\frac{1}{3}\cdot\frac{t^4}{4}-\frac{1}{5}\cdot\frac{t^6}{6}\right]_0^1=\frac{1}{2}\left(\frac{1}{12}-\frac{1}{30}\right)=\frac{1}{40}.$$

5. $I=\iiint\limits_A dx\,dy\,dz,\quad A: x^2+y^2+z^2\le 2, x^2+y^2\le z$

Průsečná křivka kulové plochy $x^2+y^2+z^2=2$ a parabolické plochy $x^2+y^2=z:$

$$z+z^2=2\Rightarrow z^2+z-2=0\Rightarrow z_1=1, z_2=-2\quad\text{(nevyhovuje)}\quad\Rightarrow x^2+y^2=1.$$

Dosadíme válcové souřadnice: $r^2\cos^2\varphi+r^2\sin^2\varphi=1\Rightarrow r=1.$
Dolní mez pro $z: z=x^2+y^2=r^2\cos\varphi+r^2\sin^2\varphi=r^2$
Horní mez pro $z: x^2+y^2+z^2=r^2+z^2=2\Rightarrow z^2=2-r^2\Rightarrow z=\sqrt{2-r^2}$

Množina $F^{-1}(A)$ je dána nerovnostmi $\begin{array}{ccccc}0&\le&\varphi&\le&2\pi\\ 0&\le&r&\le&1\\ r^2&\le&z&\le&\sqrt{2-r^2}\end{array}$

$$I=\int_0^{2\pi}\left(\int_0^1\left(\int_{r^2}^{\sqrt{2-r^2}} r\,dz\right)dr\right)d\varphi=\int_0^{2\pi}\left(\int_0^1 r\cdot[z]_{r^2}^{\sqrt{2-r^2}}dr\right)d\varphi$$
$$=\int_0^{2\pi}\left(\int_0^1 r\cdot\left(\sqrt{2-r^2}-r^2\right)dr\right)d\varphi=\int_0^{2\pi}\left(\int_0^1\left(r\sqrt{2-r^2}-r^3\right)dr\right)d\varphi$$
$$=\int_0^{2\pi}\left(\int_0^1 r\sqrt{2-r^2}\,dr\right)d\varphi-\int_0^{2\pi}\left(\int_0^1 r^3\,dr\right)d\varphi=I_1-I_2$$
$$I_1=\int_0^{2\pi}\left(\int_0^1 r\sqrt{2-r^2}\,dr\right)d\varphi=\left|\begin{array}{rcrl}2-r^2= & t & r= & 0\Rightarrow t=2-0=2\\ -2r\,dr= & dt & r= & 1\Rightarrow t=2-1=1\end{array}\right|$$
$$=\int_0^{2\pi}\left(\int_2^1\sqrt{t}\left(\frac{-dt}{2}\right)\right)d\varphi=\frac{1}{2}=\int_0^{2\pi}\left(\int_1^2\sqrt{t}\,dt\right)d\varphi=\frac{1}{2}\int_0^{2\pi}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^2 d\varphi$$
$$=\frac{1}{2}\cdot\frac{2}{3}\int_0^{2\pi}\left(2^{\frac{3}{2}}-1\right)d\varphi=\frac{1}{3}\cdot\left(2\sqrt{2}-1\right)\int_0^{2\pi}d\varphi=\frac{1}{3}\cdot\left(2\sqrt{2}-1\right)\cdot 2\pi=\frac{2}{3}\left(2\sqrt{2}-1\right)\pi$$

$$I_2=\int_0^{2\pi}\left(\int_0^1 r^3\,dr\right)d\varphi=\int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 d\varphi=\frac{1}{4}\int_0^{2\pi}d\varphi=\frac{1}{4}\cdot 2\pi=\frac{\pi}{2}$$

$$I=I_1-I_2=\frac{2}{3}\left(2\sqrt{2}-1\right)\pi-\frac{\pi}{2}=\frac{\pi}{6}\left(8\sqrt{2}-7\right).$$

Integrujte transformací do sférických (kulových) souřadnic

$$\begin{array}{rcl} x &=& r\cos\varphi\sin\psi \\ y &=& r\sin\varphi\sin\psi \\ z &=& r\cos\psi \\ |J| &=& r^2\sin\psi. \end{array}$$

1. $I = \iiint\limits_A (x^2+y^2)\,dx\,dy\,dz, \quad A: x^2+y^2+z^2 \le 4.$
   Dosadíme sférické souřadnice:

$$x^2+y^2+z^2 = r^2\cos^2\varphi\sin^2\psi + r^2\sin^2\varphi\sin^2\psi + r^2\cos^2\psi$$

$$= r^2\sin^2\psi\left(\cos^2\varphi+\sin^2\varphi\right) + r^2\cos^2\psi = r^2\sin^2\psi + r^2\cos^2\psi = r^2\left(\sin^2\psi+\cos^2\psi\right) = r^2 \le 4$$

Množina $F^{-1}(A)$ je dána nerovnostmi $\begin{array}{ccccc} 0 & \le & r & \le & 2 \\ 0 & \le & \varphi & \le & 2\pi \\ 0 & \le & \psi & \le & \pi \end{array}$

$$I = \int_0^{2\pi}\left(\int_0^2\left(\int_0^\pi \left(r^2\cos^2\varphi\sin^2\psi + r^2\sin^2\varphi\sin^2\psi\right)\cdot r^2\sin\psi\,d\psi\right)dr\right)d\varphi$$

$$= \int_0^{2\pi}\left(\int_0^2\left(\int_0^\pi r^2\sin^2\psi\left(\cos^2\varphi+\sin^2\varphi\right)\cdot r^2\sin\psi\,d\psi\right)dr\right)d\varphi$$

$$= \int_0^{2\pi}\left(\int_0^2\left(\int_0^\pi r^4\sin^3\psi\,d\psi\right)dr\right)d\varphi = \int_0^{2\pi}\left(\int_0^2\left(\int_0^\pi r^4\sin^2\psi\sin\psi\,d\psi\right)dr\right)d\varphi$$

$$= \int_0^{2\pi}\left(\int_0^2\left(\int_0^\pi r^4\left(1-\cos^2\psi\right)\sin\psi\,d\psi\right)dr\right)d\varphi$$

$$= \left|\begin{array}{rcl} \cos\psi = & t & \psi = \; 0 \;\Rightarrow\; t=\cos 0 = 1 \\ -\sin\psi\,d\psi = & dt & \psi = \; \pi \;\Rightarrow\; t = \cos\pi = -1 \end{array}\right|$$

$$= \int_0^{2\pi}\left(\int_0^2\left(\int_1^{-1} r^4\left(1-t^2\right)(-dt)\right)dr\right)d\varphi = \int_0^{2\pi}\left(\int_0^2\left(\int_{-1}^1 r^4\left(1-t^2\right)\,dt\right)dr\right)d\varphi$$

$$= \int_0^{2\pi}\left(\int_0^2 r^4\left[t-\frac{t^3}{3}\right]_{-1}^1 dr\right)d\varphi = \int_0^{2\pi}\left(\int_0^2 r^4\left[1-\frac{1}{3}-\left(-1-\frac{-1}{3}\right)\right]dr\right)d\varphi$$

$$= \frac{4}{3}\int_0^{2\pi}\left(\int_0^2 r^4\,dr\right)d\varphi = \frac{4}{3}\int_0^{2\pi}\left[\frac{r^5}{5}\right]_0^2 d\varphi = \frac{4}{3}\cdot\frac{32}{5}\int_0^{2\pi}d\varphi = \frac{128}{15}\cdot 2\pi = \frac{256}{15}\pi.$$

2. $I = \iiint\limits_A \frac{1}{x^2+y^2+z^2}\,dx\,dy\,dz, \quad A: 9 \le x^2+y^2+z^2 \le 81, 3x^2+3y^2 \le z^2, z \ge 0.$
   Kuželová plocha $3x^2+3y^2=z^2$ ve sférických souřadnicích:

$$3r^2\cos^2\varphi\sin^2\psi + 3r^2\sin^2\varphi\sin^2\psi = 3r^2\sin^2\psi = (z^2) = r^2\cos^2\psi$$

$$\Rightarrow \tan^2\psi = \frac{1}{3} \Rightarrow \tan\psi = \frac{1}{\sqrt{3}}\;(-\frac{1}{\sqrt{3}}\text{ nevyhovuje, neboť } z\ge 0) \Rightarrow \quad \psi = \frac{\pi}{6}$$

Množina $F^{-1}(A)$ je dána nerovnostmi $\begin{array}{ccccc} 3 & \le & r & \le & 9 \\ 0 & \le & \varphi & \le & 2\pi \\ 0 & \le & \psi & \le & \frac{\pi}{6} \end{array}$

$$I=\int_0^{2\pi}\left(\int_3^9\left(\int_0^{\frac{\pi}{6}}\frac{1}{r^2}\cdot r^2\sin\psi\,d\psi\right)\,dr\right)\,d\varphi=\int_0^{2\pi}\left(\int_3^9\left[-\cos\psi\right]_0^{\frac{\pi}{6}}\,dr\right)\,d\varphi$$

$$=\int_0^{2\pi}\left(\int_3^9\left[-\cos\frac{\pi}{6}-(-\cos 0)\right]\,dr\right)\,d\varphi=\int_0^{2\pi}\left(\int_3^9\left(-\frac{\sqrt{3}}{2}+1\right)\,dr\right)\,d\varphi$$

$$=\left(1-\frac{\sqrt{3}}{2}\right)\cdot\int_0^{2\pi}[r]_3^9\,d\varphi=\left(1-\frac{\sqrt{3}}{2}\right)\cdot 6\cdot\int_0^{2\pi}\,d\varphi=\frac{2-\sqrt{3}}{2}\cdot 6\cdot 2\pi=6\left(2-\sqrt{3}\right)\pi.$$

3. $I=\iiint\limits_A\sqrt{z}\,dx\,dy\,dz,\quad A: x^2+y^2+z^2=16, y=\frac{\sqrt{3}}{3}x, y=x, z=0$, I. oktant.

Dolní mez pro $\varphi: y=\frac{\sqrt{3}}{3}x\Rightarrow\tan\varphi=\frac{\sqrt{3}}{3}\Rightarrow\varphi=\frac{\pi}{6}$

Horní mez pro $\varphi: y=x\Rightarrow\tan\varphi=1\Rightarrow\varphi=\frac{\pi}{4}$

Množina $F^{-1}(A)$ je dána nerovnostmi $\begin{array}{ccccc}0&\le&r&\le&4\\ \frac{\pi}{6}&\le&\varphi&\le&\frac{\pi}{4}\\ 0&\le&\psi&\le&\frac{\pi}{2}\end{array}$

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\int_0^4\left(\int_0^{\frac{\pi}{2}}\sqrt{r\cos\psi}\cdot r^2\sin\psi\,d\psi\right)\,dr\right)\,d\varphi=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\int_0^4\left(\int_0^{\frac{\pi}{2}}r^{\frac{5}{2}}\sqrt{\cos\psi}\sin\psi\,d\psi\right)\,dr\right)\,d\varphi$$

$$=\left|\begin{array}{rclcrcl}\cos\psi=&t&\psi=&0&\Rightarrow&t=\cos 0&=1\\ -\sin\psi\,d\psi=&dt&\psi=&\frac{\pi}{2}&\Rightarrow&t=\cos\frac{\pi}{2}&=0\end{array}\right|$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\int_0^4\left(\int_1^0 r^{\frac{5}{2}}\sqrt{t}\,(-dt)\right)\,dr\right)\,d\varphi=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\int_0^4\left(\int_0^1 r^{\frac{5}{2}}t^{\frac{1}{2}}\,dt\right)\,dr\right)\,d\varphi$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\int_0^4 r^{\frac{5}{2}}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1\,dr\right)\,d\varphi=\frac{2}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\int_0^4 r^{\frac{5}{2}}\,dr\right)\,d\varphi=\frac{2}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left[\frac{r^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^4\,d\varphi=\frac{2}{3}\cdot\frac{2}{7}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}4^{\frac{7}{2}}\,d\varphi$$

$$=\frac{2}{3}\cdot\frac{2}{7}\cdot 2^7\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\,d\varphi=\frac{2}{3}\cdot\frac{2}{7}\cdot 128\cdot\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\frac{2}{3}\cdot\frac{2}{7}\cdot 128\cdot\frac{1}{12}\pi=\frac{128}{63}\pi.$$

4. $I=\iiint\limits_A\,dx\,dy\,dz,\quad A: x^2+y^2+z^2\le 4z, x^2+y^2\le z^2, z\ge 0.$

Úprava na úplný čtverec: $x^2+y^2+(z-2)^2\le 4$

(Koule se středem $S=(0,0,2)$ a poloměrem $R=2$).

Průsečíky $x^2+y^2+z^2=4z$ a $z^2=x^2+y^2$:

$$z^2+z^2=4z\Rightarrow z^2=2z\Rightarrow z_1=0, z_2=2$$

Průsečná křivka je kružnice $x^2+y^2=(2^2)=4$.

Dosadíme sférické souřadnice do rovnice kulové plochy $x^2+y^2+z^2=4z$:

$$r^2=4r\cos\psi\Rightarrow r=4\cos\psi.$$

Dosadíme do rovnice kuželové plochy $x^2+y^2\le z^2$:

$$r^2\cos^2\varphi\sin^2\psi+r^2\sin^2\varphi\sin^2\psi=r^2\sin^2\psi=r^2\cos^2\psi$$

$$\Rightarrow\tan^2\psi=1\Rightarrow\tan\psi=1(-1\,\text{nevyhovuje})\Rightarrow\psi=\frac{\pi}{4}$$

Množina $F^{-1}(A)$ je dána nerovnostmi $\begin{array}{ccccc} 0 & \leq & r & \leq & 4\cos\psi \\ 0 & \leq & \varphi & \leq & 2\pi \\ 0 & \leq & \psi & \leq & \frac{\pi}{4} \end{array}$

$$I = \int_0^{2\pi} \left(\int_0^{\frac{\pi}{4}} \left(\int_0^{4\cos\psi} r^2 \sin\psi \, dr \right) d\psi \right) d\varphi = \int_0^{2\pi} \left(\int_0^{\frac{\pi}{4}} \sin\psi \cdot \left[\frac{r^3}{3} \right]_0^{4\cos\psi} d\psi \right) d\varphi$$

$$= \frac{64}{3} \int_0^{2\pi} \left(\int_0^{\frac{\pi}{4}} \sin\psi \cos^3\psi \, d\psi \right) d\varphi = \left| \begin{array}{rccccl} \cos\psi = & t & \psi = & 0 & \Rightarrow & t = \cos 0 \quad = 1 \\ -\sin\psi \, d\psi = & dt & \psi = & \frac{\pi}{4} & \Rightarrow & t = \cos\frac{\pi}{4} \quad = \frac{\sqrt{2}}{2} \end{array} \right|$$

$$= \frac{64}{3} \int_0^{2\pi} \left(\int_1^{\frac{\sqrt{2}}{2}} t^3 \, (-dt) \right) d\varphi = \frac{64}{3} \int_0^{2\pi} \left(\int_{\frac{\sqrt{2}}{2}}^1 t^3 \, dt \right) d\varphi = \frac{64}{3} \int_0^{2\pi} \left[\frac{t^4}{4} \right]_{\frac{\sqrt{2}}{2}}^1 d\varphi$$

$$= \frac{64}{3} \cdot \frac{1}{4} \int_0^{2\pi} \left[1 - \left(\frac{\sqrt{2}}{2} \right)^4 \right] d\varphi = \frac{64}{3} \cdot \frac{1}{4} \cdot \left(1 - \frac{1}{4} \right) \int_0^{2\pi} d\varphi = 4 \cdot 2\pi = 8\pi.$$