Very High Order WENO Interpolation

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Abstract

In this paper we introduce very high order WENO interpolation schemes on uniform grids: all the WENO parameters (such as polynomial coefficients, smoothness indicators coefficients, optimal linear weights, etc) have been analytically evaluated for left and right interfaces of a cell, while detailed expressions are reported to evaluate these parameters in any point of the cell.

Keywords: Interpolation, Weighted Essentially Non-Oscillatory (WENO), Fortran

1. Introduction

Interpolation is the process of deriving a simple function from a set of discrete data points so that the function passes through all the given data points (i.e. reproduces the data points exactly) and can be used to estimate data points in-between the given ones.

Interpolation is also used to simplify complicated functions by sampling data points and then interpolating them using a simpler function. Polynomials are commonly used for interpolation because they are easier to evaluate, differentiate, and integrate. Unfortunately, interpolation of order greater than one can suffer of the Gibbs' phenomenon [1] next to discontinuities.

The original idea of WENO schemes [2] is to use a convex combination of all candidate stencils (instead of using only the smoothest one as in ENO schemes [3]) to obtain high order reconstruction: this approach can obviously be extended to interpolation process, leading to an high order oscillatory free interpolation.

Add interpolation background and citation to interpolation related works.

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2. Mathematical and Numerical Models

Assume we have a uniform mesh $x_1, x_2, \dots x_n$ with $\Delta x = x_{n+1} - x_n$ and that we know the values of a function u at all the grid points, that is $u_i = u(x_i)$ for all i. We would like to find an approximation of the function $u(x)$ at the point x^* other than the nodes x_i, with $x_{i-\frac{1}{2}} < x^* < x_{i+\frac{1}{2}}$, where $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$ are the cell interfaces.

For a r^{th} order accurate interpolation, there are r candidate stencils next to the target point x^*: we denote these stencil as S_k, where $k = 0, \dots, r-1$ labels the stencils from the leftmost stencil to the rightmost stencil in that order. Using the Lagrange form of the interpolation polynomial, the polynom $p_k(x)$ over the stencil S_k can be written as:

$$p_k(x^*) = \sum_{j=0}^{r-1} u_{i-r+k+j+1} \sum_{\substack{l=0 \\ l \neq j}}^{r-1} \frac{x^* - x_{i-r+k+l+1}}{x_{i-r+k+j+1} - x_{i-r+k+l+1}} = \sum_{j=0}^{r-1} a_{k,i-r+j+1} u_{i-r+k+j+1} \tag{1}$$

where $a_{k,i-r+j+1}$ are the Lagrange coefficients of the stencil S_k.

In table 1 are reported the polynomial coefficients from $r = 2$ to $r = 9$ for all the interpolating stencils, for $x^* = x_{i+\frac{1}{2}}$; polynomial coefficients for $x^* = x_{i-\frac{1}{2}}$ can be obtained by table 1 by symmetry.

If we consider the big stencil $S = \cup_{i=0}^{k} S_k$, we can obtain a $(2r-1)^{th}$ accurate interpolation and (1) becomes:

$$P(x^*) = \sum_{j=0}^{2r-2} u_{i-r+j+1} \sum_{\substack{l=0 \\ l \neq j}}^{2r-2} \frac{x^* - x_{i-r+l+1}}{x_{i-r+j+1} - x_{i-r+l+1}} = \sum_{j=0}^{2r-2} b_{i-r+j+1} u_{i-r+j+1} \tag{2}$$

where $b_{i-r+j+1}$ are the Lagrange coefficients of the stencil S.

Expression (2) can also be written as a linear convex combination of the r approximations of order r^{th} (1)

$$P(x^*) = \sum_{i=0}^{r-1} \gamma_i p_i(x^*), \text{ with } \sum_{i=0}^{r-1} \gamma_i = 1 \tag{3}$$

where γ_r are usually referred as the linear weights. The linear weights for the point x^* can be evaluated from the Lagrange coefficients $a_{k,i-r+j+1}$ and $b_{i-r+j+1}$ by means of:

$$\gamma_k(x^*) = \frac{b_{i-r+j+1} - \sum_{l=0}^{j-1} \gamma_l(x^*) a_{k,i-r+l+1}(x^*)}{a_{0,i-r+j+1}(x^*)}, \; j = 0, \dots, r-1 \tag{4}$$

In table 2 are reported linear weights from $r = 2$ to $r = 9$ for $x^* = x_{i+\frac{1}{2}}$; linear weights for $x^* = x_{i-\frac{1}{2}}$ can be obtained by table 2 by symmetry.

The basic idea of WENO schemes is to use a nonlinear combination of the r interpolations to obtain a $(2r-1)^{th}$ order interpolation in smooth regions and handle stencil with discontinuities: the nonlinear weights, infact, are close to the linear weights if the function in the stencil is smooth and close to 0 if in that stencil is contained a discontinuity.

$$u(x^*) = \sum_{i=0}^{r-1} w_i p_i(x^*) \tag{5}$$

Following the work of Jiang and Shu [4], the nonlinear weights are evaluated as:

$$w_k = \frac{\gamma_k}{(\epsilon + \beta_k)^2} \tag{6}$$

where ϵ is a parameter to avoid division by zero and β_k are the smoothness indicators of the function u on the stencil l:

$$\beta_k = \sum_{j=1}^{r-1} \Delta x^{2j-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\frac{d^j p_k(x)}{dx^j} \right)^2 dx \tag{7}$$

Table 1: Polynomial coefficients from $r = 2$ to $r = 9$ for $x^* = x_{i+\frac{1}{2}}$

r	k	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$	$j = 8$
9	0	$\frac{6435}{32768}$	$-\frac{7293}{4096}$	$\frac{58905}{8192}$	$-\frac{69615}{4096}$	$\frac{425425}{16384}$	$-\frac{109395}{4096}$	$\frac{153153}{8192}$	$-\frac{36465}{4096}$	$\frac{109395}{32768}$
	1	$-\frac{429}{32768}$	$\frac{495}{4096}$	$-\frac{4095}{8192}$	$\frac{5005}{4096}$	$-\frac{32175}{16384}$	$\frac{9009}{4096}$	$-\frac{15015}{8192}$	$\frac{6435}{4096}$	$\frac{6435}{32768}$
	2	$\frac{99}{32768}$	$-\frac{117}{4096}$	$\frac{1001}{8192}$	$-\frac{1287}{4096}$	$\frac{9009}{16384}$	$-\frac{3003}{4096}$	$\frac{9009}{8192}$	$\frac{1287}{4096}$	$-\frac{429}{32768}$
	3	$-\frac{45}{32768}$	$\frac{55}{4096}$	$-\frac{495}{8192}$	$\frac{693}{4096}$	$-\frac{5775}{16384}$	$\frac{3465}{4096}$	$\frac{3465}{8192}$	$-\frac{165}{4096}$	$\frac{99}{32768}$
	4	$\frac{35}{32768}$	$-\frac{45}{4096}$	$\frac{441}{8192}$	$-\frac{735}{4096}$	$\frac{11025}{16384}$	$\frac{2205}{4096}$	$-\frac{735}{8192}$	$\frac{63}{4096}$	$-\frac{45}{32768}$
	5	$-\frac{45}{32768}$	$\frac{63}{4096}$	$-\frac{735}{8192}$	$\frac{2205}{4096}$	$\frac{11025}{16384}$	$-\frac{735}{4096}$	$\frac{441}{8192}$	$-\frac{45}{4096}$	$\frac{35}{32768}$
	6	$\frac{99}{32768}$	$-\frac{165}{4096}$	$\frac{3465}{8192}$	$\frac{3465}{4096}$	$-\frac{5775}{16384}$	$\frac{693}{4096}$	$-\frac{495}{8192}$	$\frac{55}{4096}$	$-\frac{45}{32768}$
	7	$-\frac{429}{32768}$	$\frac{1287}{4096}$	$\frac{9009}{8192}$	$-\frac{3003}{4096}$	$\frac{9009}{16384}$	$-\frac{1287}{4096}$	$\frac{1001}{8192}$	$-\frac{117}{4096}$	$\frac{99}{32768}$
	8	$\frac{6435}{32768}$	$\frac{6435}{4096}$	$-\frac{15015}{8192}$	$\frac{9009}{4096}$	$-\frac{32175}{16384}$	$\frac{5005}{4096}$	$-\frac{4095}{8192}$	$\frac{495}{4096}$	$-\frac{429}{32768}$
8	0	$-\frac{429}{2048}$	$\frac{3465}{2048}$	$-\frac{12285}{2048}$	$\frac{25025}{2048}$	$-\frac{32175}{2048}$	$\frac{27027}{2048}$	$-\frac{15015}{2048}$	$\frac{6435}{2048}$	
	1	$\frac{33}{2048}$	$-\frac{273}{2048}$	$\frac{1001}{2048}$	$-\frac{2145}{2048}$	$\frac{3003}{2048}$	$-\frac{3003}{2048}$	$\frac{3003}{2048}$	$\frac{429}{2048}$	
	2	$-\frac{9}{2048}$	$\frac{77}{2048}$	$-\frac{297}{2048}$	$\frac{693}{2048}$	$-\frac{1155}{2048}$	$\frac{2079}{2048}$	$\frac{693}{2048}$	$-\frac{33}{2048}$	
	3	$\frac{5}{2048}$	$-\frac{45}{2048}$	$\frac{189}{2048}$	$-\frac{525}{2048}$	$\frac{1575}{2048}$	$\frac{945}{2048}$	$-\frac{105}{2048}$	$\frac{9}{2048}$	
	4	$-\frac{5}{2048}$	$\frac{49}{2048}$	$-\frac{245}{2048}$	$\frac{1225}{2048}$	$\frac{1225}{2048}$	$-\frac{245}{2048}$	$\frac{49}{2048}$	$-\frac{5}{2048}$	
	5	$\frac{9}{2048}$	$-\frac{105}{2048}$	$\frac{945}{2048}$	$\frac{1575}{2048}$	$-\frac{525}{2048}$	$\frac{189}{2048}$	$-\frac{45}{2048}$	$\frac{5}{2048}$	
	6	$-\frac{33}{2048}$	$\frac{693}{2048}$	$\frac{2079}{2048}$	$-\frac{1155}{2048}$	$\frac{693}{2048}$	$-\frac{297}{2048}$	$\frac{77}{2048}$	$-\frac{9}{2048}$	
	7	$\frac{429}{2048}$	$\frac{3003}{2048}$	$-\frac{3003}{2048}$	$\frac{3003}{2048}$	$-\frac{2145}{2048}$	$\frac{1001}{2048}$	$-\frac{273}{2048}$	$\frac{33}{2048}$	
7	0	$\frac{231}{1024}$	$-\frac{819}{512}$	$\frac{5005}{1024}$	$-\frac{2145}{256}$	$\frac{9009}{1024}$	$-\frac{3003}{512}$	$\frac{3003}{1024}$		
	1	$-\frac{21}{1024}$	$\frac{77}{512}$	$-\frac{495}{1024}$	$\frac{231}{256}$	$-\frac{1155}{1024}$	$\frac{693}{512}$	$\frac{231}{1024}$		
	2	$\frac{7}{1024}$	$-\frac{27}{512}$	$\frac{189}{1024}$	$-\frac{105}{256}$	$\frac{945}{1024}$	$\frac{189}{512}$	$-\frac{21}{1024}$		
	3	$-\frac{5}{1024}$	$\frac{21}{512}$	$-\frac{175}{1024}$	$\frac{175}{256}$	$\frac{525}{1024}$	$-\frac{35}{512}$	$\frac{7}{1024}$		
	4	$\frac{7}{1024}$	$-\frac{35}{512}$	$\frac{525}{1024}$	$\frac{175}{256}$	$-\frac{175}{1024}$	$\frac{21}{512}$	$-\frac{5}{1024}$		
	5	$-\frac{21}{1024}$	$\frac{189}{512}$	$\frac{945}{1024}$	$-\frac{105}{256}$	$\frac{189}{1024}$	$-\frac{27}{512}$	$\frac{7}{1024}$		
	6	$\frac{231}{1024}$	$\frac{693}{512}$	$-\frac{1155}{1024}$	$\frac{231}{256}$	$-\frac{495}{1024}$	$\frac{77}{512}$	$-\frac{21}{1024}$		
6	0	$-\frac{63}{256}$	$\frac{385}{256}$	$-\frac{495}{128}$	$\frac{693}{128}$	$-\frac{1155}{256}$	$\frac{693}{256}$			
	1	$\frac{7}{256}$	$-\frac{45}{256}$	$\frac{63}{128}$	$-\frac{105}{128}$	$\frac{315}{256}$	$\frac{63}{256}$			
	2	$-\frac{3}{256}$	$\frac{21}{256}$	$-\frac{35}{128}$	$\frac{105}{128}$	$\frac{105}{256}$	$-\frac{7}{256}$			
	3	$\frac{3}{256}$	$-\frac{25}{256}$	$\frac{75}{128}$	$\frac{75}{128}$	$-\frac{25}{256}$	$\frac{3}{256}$			
	4	$-\frac{7}{256}$	$\frac{105}{256}$	$\frac{105}{128}$	$-\frac{35}{128}$	$\frac{21}{256}$	$-\frac{3}{256}$			
	5	$\frac{63}{256}$	$\frac{315}{256}$	$-\frac{105}{128}$	$\frac{63}{128}$	$-\frac{45}{256}$	$\frac{7}{256}$			
5	0	$\frac{35}{128}$	$-\frac{45}{32}$	$\frac{189}{64}$	$-\frac{105}{32}$	$\frac{315}{128}$				
	1	$-\frac{5}{128}$	$\frac{7}{32}$	$-\frac{35}{64}$	$\frac{35}{32}$	$\frac{35}{128}$				
	2	$\frac{3}{128}$	$-\frac{5}{32}$	$\frac{45}{64}$	$\frac{15}{32}$	$-\frac{5}{128}$				
	3	$-\frac{5}{128}$	$\frac{15}{32}$	$\frac{45}{64}$	$-\frac{5}{32}$	$\frac{3}{128}$				
	4	$\frac{35}{128}$	$\frac{35}{32}$	$-\frac{35}{64}$	$\frac{7}{32}$	$3-\frac{5}{128}$				
4	0	$-\frac{5}{16}$	$\frac{21}{16}$	$-\frac{35}{16}$	$\frac{35}{16}$					
	1	$\frac{1}{16}$	$-\frac{5}{16}$	$\frac{15}{16}$	$\frac{5}{16}$					
	2	$-\frac{1}{16}$	$\frac{9}{16}$	$\frac{9}{16}$	$-\frac{1}{16}$					
	3	$\frac{5}{16}$	$\frac{15}{16}$	$-\frac{5}{16}$	$\frac{1}{16}$					
3	0	$\frac{3}{8}$	$-\frac{5}{4}$	$\frac{15}{8}$						

Table 2: Linear weights from $r = 2$ to $r = 9$ for $x^* = x_{i+\frac{1}{2}}$

r	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$	$j = 8$
9	$\frac{1}{65536}$	$\frac{17}{8192}$	$\frac{595}{16384}$	$\frac{1547}{8192}$	$\frac{12155}{32768}$	$\frac{2431}{8192}$	$\frac{1547}{16384}$	$\frac{85}{8192}$	$\frac{17}{65536}$
8	$\frac{1}{16384}$	$\frac{105}{16384}$	$\frac{1365}{16384}$	$\frac{5005}{16384}$	$\frac{6435}{16384}$	$\frac{3003}{16384}$	$\frac{455}{16384}$	$\frac{15}{16384}$	
7	$\frac{1}{4096}$	$\frac{39}{2048}$	$\frac{179}{1024}$	$\frac{429}{1024}$	$\frac{1287}{4096}$	$\frac{143}{2048}$	$\frac{13}{4096}$		
6	$\frac{1}{1024}$	$\frac{55}{1024}$	$\frac{165}{512}$	$\frac{231}{512}$	$\frac{165}{1024}$	$\frac{11}{1024}$			
5	$\frac{1}{256}$	$\frac{9}{64}$	$\frac{63}{128}$	$\frac{21}{64}$	$\frac{9}{256}$				
4	$\frac{1}{64}$	$\frac{21}{64}$	$\frac{35}{64}$	$\frac{7}{64}$					
3	$\frac{1}{16}$	$\frac{5}{8}$	$\frac{5}{16}$						
2	$\frac{1}{4}$	$\frac{3}{4}$							

This is clearly just a scaled sum of the square L2 norms of all the derivatives of the relevant interpolation polynomial $p_k(x)$ in the relevant interval $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, where the interpolating point is located. The scaling factor Δ_x^{2l-2} is to make sure that the final explicit formulas for the smoothness indicators do not depend on the mesh size Δx.

Substitution of (1) for any $k = 0, \ldots, r-1$ into (7) yelds to:

$$\beta_k = \sum_{j=0}^{r-1} \sum_{l=0}^{j} \sigma_{k,j,l} u_{i+k-j} u_{i+k-l} \tag{8}$$

The coefficients $\sigma_{k,j,l}$ are reported in table 3.

For example, if $r = 3$, (1) can be applied to the leftmost stencil $S_0 = \{x_{i-2}, x_{i-1}, x_i\}$ to obtain the polynomial p_0:

$$p_0(x_{i+\frac{1}{2}}) = \frac{3}{8}u_{i-2} - \frac{5}{4}u_{i-1} + \frac{15}{8}u_i \tag{9}$$

and this approximation is third order accurate if the function $u(x)$ is smooth in the stencil S_0. If we choose a different stencil $S_1 = \{x_{i-1}, x_i, x_{i+1}\}$ we obtain the polynomial p_1:

$$p_1(x_{i+\frac{1}{2}}) = -\frac{1}{8}u_{i-1} + \frac{3}{4}u_i + \frac{3}{8}u_{i+1} \tag{10}$$

that is also third order accurate. The last stencil that can be used is the stencil $S_2 = \{x_i, x_{i+1}, x_{i+2}\}$ to obtain the third order accurate interpolating polynom p_2:

$$p_2(x_{i+\frac{1}{2}}) = \frac{3}{8}u_i + \frac{3}{4}u_{i+1} - \frac{1}{8}u_{i+2} \tag{11}$$

When $r = 3$, using (2) on the stencil $S = S_0 \cup S_1 \cup S_2$, we obtain a fifth order accurate approximation of the function u at the point $x_{i+\frac{1}{2}}$:

$$P(x_{i+\frac{1}{2}}) = \frac{3}{128}u_{i-2} - \frac{5}{32}u_{i-1} + \frac{45}{64}u_i + \frac{15}{32}u_{i+1} - \frac{5}{128}u_{i+2} \tag{12}$$

For $r = 3$, $\gamma_0 = \frac{1}{16}$, $\gamma_1 = \frac{5}{8}$, $\gamma_2 = \frac{5}{16}$.

Table 3: Smoothness indicators coefficients from $r = 2$ to $r = 9$

$r = 2$					
j	l	$k = 0$	$k = 1$		
1	1	-2	-2		
	0	1	1		
0	0	1	1		
$r = 3$					
j	l	$k = 0$	$k = 1$	$k = 2$	
2	2	$\frac{11}{3}$	$\frac{5}{3}$	$\frac{11}{3}$	
	1	$-\frac{31}{3}$	$-\frac{13}{3}$	$-\frac{19}{3}$	
	0	$\frac{10}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	
1	1	$-\frac{19}{3}$	$-\frac{13}{3}$	$-\frac{31}{3}$	
	0	$\frac{25}{3}$	$\frac{13}{3}$	$\frac{25}{3}$	
0	0	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{10}{3}$	
$r = 4$					
j	l	$k = 0$	$k = 1$	$k = 2$	$k = 3$
3	3	$-\frac{11389}{1440}$	$-\frac{2989}{1440}$	$-\frac{2989}{1440}$	$-\frac{11389}{1440}$
	2	$\frac{14369}{480}$	$\frac{1283}{160}$	$\frac{3169}{480}$	$\frac{9449}{480}$
	1	$-\frac{6383}{160}$	$-\frac{5069}{480}$	$-\frac{3229}{480}$	$-\frac{2623}{160}$
	0	$\frac{25729}{2880}$	$\frac{6649}{2880}$	$\frac{3169}{2880}$	$\frac{6649}{2880}$
2	2	$\frac{9449}{480}$	$\frac{3169}{480}$	$\frac{1283}{160}$	$\frac{14369}{480}$
	1	$-\frac{35047}{480}$	$-\frac{11767}{480}$	$-\frac{11767}{480}$	$-\frac{35047}{480}$
	0	$\frac{44747}{960}$	$\frac{13667}{960}$	$\frac{11147}{960}$	$\frac{28547}{960}$
1	1	$-\frac{2623}{160}$	$-\frac{3229}{480}$	$-\frac{5069}{480}$	$-\frac{6383}{160}$
	0	$\frac{28547}{960}$	$\frac{11147}{960}$	$\frac{13667}{960}$	$\frac{44747}{960}$
0	0	$\frac{6649}{2880}$	$\frac{3169}{2880}$	$\frac{6649}{2880}$	$\frac{25729}{2880}$

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