Deep Reinforcement Learning

Overview of main articles
Part 2. Policy gradient algorithms

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MSU

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Basic policy gradient methods

Direct optimization

Recall RL goal:

$$\mathbb{E}_{\pi(\theta)}\mathbb{E}_{\mathcal{T}} R \to \max_{\theta}$$

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Options:

* Metaheurisics
* Log-derivative trick[1].

[1] aka REINFORCE

Stochastic estimators optimization via log-derivative trick

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$$= \int \pi(x,\theta) \nabla_\theta \log \pi(x,\theta) f(x) dx = \mathbb{E}_{x\sim\pi(x,\theta)} \nabla_\theta \log \pi(x,\theta) f(x)$$

From importance sampling point of view

Recall Importance Sampling. For arbitrary distribution $\phi(x)$:

$$\mathbb{E}_{x \sim \pi(x,\theta)} f(x) = \mathbb{E}_{x \sim \phi(x)} \frac{\pi(x,\theta)}{\phi(x)} f(x)$$

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Note: that is the same gradient as with log-derivative trick[2].

[2] really? Could it even happen otherwise?

REINFORCE

Let's apply log-derivative trick to our goal!

$$\nabla_\theta \mathbb{E}_{\pi(\theta)} \mathbb{E}_{\mathcal{T}} R = \mathbb{E}_{\pi(\theta)} \nabla_\theta \log \pi(\theta) \mathbb{E}_{\mathcal{T}} R$$

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$$\nabla_\theta \mathbb{E}_{\pi(\theta)} \mathbb{E}_{\mathcal{T}} R = \mathbb{E}_{\pi(\theta)} \nabla_\theta \log \pi(\theta) \mathbb{E}_{\mathcal{T}} R \approx$$

We can estimate this gradient using Monte-Carlo by playing, let's say, one game:

$$\approx \sum_t^T \nabla_\theta \log \pi(a_t \mid s_t, \theta) R$$

Problems of REINFORCE

- $\times$ Doesn't work.

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× Doesn't work.
  - **Reason:** *high variance* of Monte-Carlo gradient estimation.

Problems of REINFORCE

× Doesn't work.
  - **Reason:** *high variance* of Monte-Carlo gradient estimation.
  - you can play more than one game for one gradient step, but that doesn't help much.

Baseline

Proposition

For arbitrary distribution $\pi(\theta)$:

$$\mathbb{E}\nabla_\theta \log \pi(\theta) = \int \nabla_\theta \pi(\theta) = \nabla_\theta \int \pi(\theta) = \nabla_\theta 1 = 0$$

Baseline

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Adding $\mathbb{E}\nabla_\theta \log \pi(\theta) b$ for some b to gradient estimate will not lead to bias, but may change variance.

Lowest variance baseline

Theorem

$$b = \frac{\mathbb{E}(\nabla_\theta \log \pi(\theta))^2 R}{\mathbb{E}(\nabla_\theta \log \pi(\theta))^2}$$

is the baseline which leads to the lowest variance.

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$$b = \frac{\mathbb{E}(\nabla_\theta \log \pi(\theta))^2 R}{\mathbb{E}(\nabla_\theta \log \pi(\theta))^2}$$

is the baseline which leads to the lowest variance.

* similar to average reward, which is also a good baseline.

Careful REINFORCE

Strange thing: our gradient estimate depends on R, which includes reward in the first state $r(s_0)$, where we haven't performed any actions.[3]

[3] did we make any mistake?

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Let's untangle our goal:

$$\nabla_\theta \mathbb{E}_{p(s_1)} \left(r(s_1) + \mathbb{E}_{a_1 \sim \pi(s_1, \theta)} \mathbb{E}_{p(s_2 | s_1, a)} \left[r(s_2) + \dots \right] \right)$$

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After carefully applying log-derivative trick:

$$= \mathbb{E} \sum_t^T \nabla_\theta \log \pi(a_t \mid s_t, \theta) \left(\sum_{t' = t+1}^T r(s_{t'}) \right)$$

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✓ that's much better!

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Actor-critic

Note: $\sum_{t'=t+1}^{T} r(s_{t'})$ is estimation of $Q^{\pi}(s_t, a_t)$!

$$\nabla = \mathbb{E} \sum_{t}^{T} \nabla_\theta \log \pi(a_t \mid s_t, \theta) \left(\sum_{t'=t+1}^{T} r(s_{t'}) \right)$$

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Better estimation of $Q^{\pi}(s, a)$
should lead to lower variance.

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Better estimation of $Q^\pi(s, a)$
should lead to lower variance.

* π is an *actor*
* estimate of $Q^\pi(s, a)$ is a *critic*

Advantage Actor Critic

Let's insert some baseline:

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Critic set up

Critic can be a second neural net!

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  ✓ the least complex one! [5]

[4] can we just use Q-learning for this?
[5] why?

Critic learning

For given state s we can calculate a target $y = V^{\pi}(s) \approx \sum_{t'=t+1}^{T} r(s_{t'})$.

At the end of the game, make a step of gradient descent to teach critic.

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Alternative: $y = V^{\pi}(s) \approx r(s') + V^{\pi}(s')$

A2C: Complete algorithm

Advantage Actor-Critic (A2C) Algorithm:

- get (s, a, r, s')

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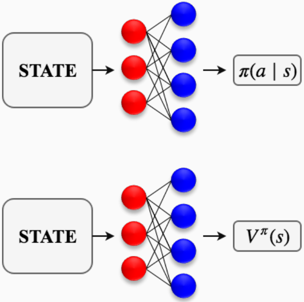
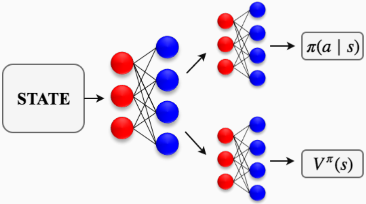
- evaluate $\hat{A}(s, a) = r + \hat{V}(s') - \hat{V}(s)$
- update policy using estimate of gradient $\nabla_\theta \log \pi(a \mid s, \theta)\hat{A}(s, a)$

Dealing with two networks

Option 1: just two neural nets



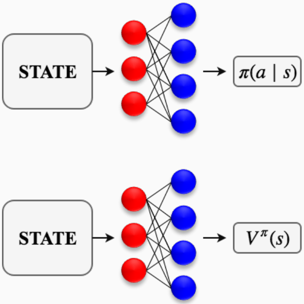


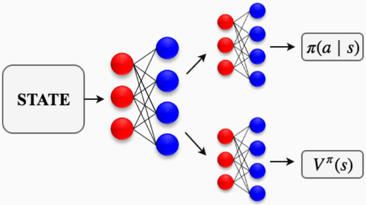
Option 2: shared feature extractor

Dealing with two networks

Option 1: just two neural nets

× obviously redundant





Option 2: shared feature extractor

× may be unstable

A2C: Resume

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Check out this comic about A2C!

Generalized Advantage Estimation (GAE) (2018)

Playing with Q and V...

$$\nabla = \mathbb{E} \sum_{t}^{T} \nabla_\theta \log \pi(a_t \mid s_t, \theta) \left(Q^\pi(s_t, a_t) - V^\pi(s_t) \right)$$

In practice we may use separate approximations for $Q^\pi(s_t, a_t)$ and baseline $b = V^\pi(s_t)$ and play with different ways to do that:

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In practice we may use separate approximations for $Q^\pi(s_t, a_t)$ and baseline $b = V^\pi(s_t)$ and play with different ways to do that:

$$\nabla = \mathbb{E} \sum_t^T \nabla_\theta \log \pi(a_t \mid s_t, \theta) \Psi_t$$

Ψ_t	**bias**	**variance**
$\sum_t^T r(s_t)$	0	very high
$Q^\pi(s_t, a_t)$	tolerant	high
$A^\pi(s_t, a_t)$	tolerant	low enough
$\sum_t^T r(s_t) - V^\pi(s_t)$	0	low

Eligibility trace

We may use critic only for baseline:

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Or use a compromise (for simplicity $\gamma = 1$):

$$\nabla = \mathbb{E} \sum_{t}^{T} \nabla_\theta \log \pi(a_t \mid s_t, \theta) \left(\sum_{t'=t+1}^{t+N} r(s_{t'}) + V^\pi(s_{t+N}) - V^\pi(s_t) \right)$$

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× new hyperparameter N

✓ regulates trade-off between variance and bias

GAE

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Generalized Advantage Estimaton (2018):

> Create an ensemble out of them!

GAE

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Create an ensemble out of them!

Let $A^{\pi}_{(N)}(s_t, a_t)$ be a N-step advantage estimator:

$$A^{\pi}_{(N)} = \sum_{t'=t+1}^{t+N} r(s_{t'}) + V^{\pi}(s_{t+N}) - V^{\pi}(s_t)$$

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$$A^{\pi}_{(N)} = \sum_{t'=t+1}^{t+N} r(s_{t'}) + V^{\pi}(s_{t+N}) - V^{\pi}(s_t)$$

Let's take exponentially-weighted average:

$$A^{\pi}_{(\mathrm{GAE})}(s_t, a_t) = (1 - \lambda)(A^{\pi}_{(1)} + \lambda A^{\pi}_{(2)} + \lambda^2 A^{\pi}_{(3)} + \dots)$$

GAE in practice

Move convenient formula:

$$A^{\pi}_{(\mathrm{GAE})}(s_t, a_t) = \sum_{i=0}^{\infty} (\lambda\gamma)^i (r(s_{t+i}) + \gamma V^{\pi}(s_{t+i+1}) - V^{\pi}(s_{t+i}))$$

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* $\lambda = 0$: A2C algorithm
* $\lambda = 1$: infinite eligibility trace algorithm

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* $\lambda = 0$: A2C algorithm
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* the balance is in between... 

NEXT: TRPO

Trust Region Policy Optimization (TRPO) (2017)

Utilizing data

Problem: Actor-Critic algorithm is *on-policy*.

- $\times$ we throw away obtained data after one optimization step.

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× we throw away obtained data after one optimization step.

! but we have to do this!

$$\nabla(\theta) = \mathbb{E}_{\mathcal{T} \sim \pi(\theta)} \sum_{t}^{T} \nabla_\theta \log \pi(a_t \mid s_t, \theta) A^\pi(s_t, a_t)$$

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Use importance sampling!

Off-policy Actor-Critic

Let denote $P(\mathcal{T} \mid \pi)$ a probability of trajectory under policy π:

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$\times$ though transition probability reduce, this *importance sampling weight* tends to be very close to 0.

TRPO foundations

May be if π is close to $\tilde{\pi}$, this
weight is practically acceptable

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Trust-Region Policy Optimization (2017) hints:

- a lot of theory on relative performance of two close policies
- attempt to build policy optimization procedure with guarantees of optimizing the objective.[6]
- practical application of natural policy gradients.

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× doesn't provide enthusiastic results on practice...

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Relative Policy Performance Identity

Let's denote $J(\pi)$ a performance of policy π, i.e. our objective:

$$J(\pi) \stackrel{\text{def}}{=} \mathbb{E}_{\mathcal{T} \sim \pi} \sum_{t=0} \gamma^t r(s_t) = \mathbb{E}_{s_0} V^\pi(s_0)$$

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Theorem (Kakade & Langford, 2002):

$$J(\tilde{\pi}) - J(\pi) = \mathbb{E}_{\mathcal{T} \sim \tilde{\pi}} \sum_{t=0} \gamma^t A^\pi(s_t, a_t)$$

Relative Policy Performance Identity: Proof

$$J(\tilde{\pi}) - J(\pi) = \mathbb{E}_{\mathcal{T} \sim \tilde{\pi}} \sum_{t=0} \gamma^t r(s_t) - J(\pi) =$$

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Applying importance sampling

Denote $d_\pi(s)$ a *discounted state-visitation probability* for policy π:

$$d_\pi(s) = (1-\gamma)\sum_{t=0}\gamma^t \mathcal{P}(s_t = s)$$

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Suppose π is current policy and $\tilde{\pi}$ is a policy after one optimization step.

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Theorem:

If ε is the approximation error:

$$|\varepsilon| \leq \text{Const}\, KL^{\max}(\tilde{\pi} \parallel \pi)$$

The familiar gradients...

Let π's parameters be θ_k (fixed), $\tilde{\pi}$'s parameters be θ.
To optimize θ, let's find $\nabla_\theta L(\tilde{\pi}(\theta))|_{\theta_k}$:

$$\nabla_\theta L(\tilde{\pi}(\theta))|_{\theta_k} = \nabla_\theta \left[\mathbb{E}_{\mathcal{T}\sim\pi} \frac{\tilde{\pi}_\theta(a_t \mid s_t)}{\pi(a_t \mid s_t)} A^\pi(s_t, a_t) \right] \Bigg|_{\theta_k} =$$

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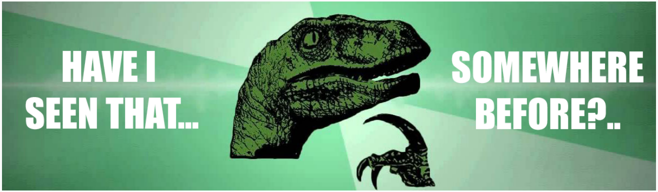

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✓ procedure guarantees to improve $J(\pi)$!

Harsh reality

$$\pi_{k+1} = \underset{\tilde{\pi}}{argmax} \left[\mathbb{E}_{\mathcal{T} \sim \pi_k} \frac{\tilde{\pi}(a_t \mid s_t)}{\pi_k(a_t \mid s_t)} A^{\pi_k}(s_t, a_t) - C \, KL^{\max}(\tilde{\pi} \parallel \pi_k) \right]$$

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 TRPO: Trust-Region optimization scheme!

Trust-Region optimization

$$\begin{cases} \pi_{k+1} = \mathbb{E}_{\mathcal{T} \sim \pi_k} \frac{\tilde{\pi}(a_t|s_t)}{\pi_k(a_t|s_t)} A^{\pi_k}(s_t, a_t) \to \max\limits_{\tilde{\pi}} \\ \text{s.t.} \quad \mathbb{E}_{s \sim d_{\pi_k}} KL(\tilde{\pi} \parallel \pi_k)[s] \le \delta \end{cases}$$

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✓ respects distance in policy space!

- also known in theory as *natural gradient*. In previous policy gradient methods we implicitly used the constrain

$$\|\tilde{\theta} - \theta_k\|_2^2 \leq \alpha$$

where α was learning rate of optimizer.

Natural Policy Gradient

Metric in most general form may depend from current coordinates:

$$\rho(x, x+d) = d^T G(x) d$$

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Theorem:

For space of policies, *Fisher information matrix* is metric tensor:

$$H(\theta) = \mathbb{E}_{a \sim \pi_\theta} \left[\nabla_\theta \log \pi_\theta(a \mid s) \log \pi_\theta(a \mid s)^T \right]$$

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Main natural gradient property (parametrization invariance)

For any parametrization π_θ

$$H^{-1} \nabla_\theta \pi_\theta$$

is the same vector in policies space.

Practical application

Recalling standard optimization methods to solve constraint task:

$$\begin{cases} L(\theta) \to \max\limits_{\theta} \\ \text{s.t.} \quad \mathbb{E}_{s \sim d_{\pi(\theta_k)}} KL(\pi(\theta) \parallel \pi(\theta_k))[s] \leq \delta \end{cases}$$

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Linear approximation of optimized objective:

$$L(\theta) \approx L(\theta_k) + g^T(\theta - \theta_k) \qquad \text{where} \quad g = \nabla_\theta L(\theta)|_{\theta_k}$$

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Quadratic approximation of constraint [7]:

$$\mathbb{E}_s \, KL(\pi(\theta) \parallel \pi(\theta_k))[s] \approx (\theta - \theta_k)^T H(\theta - \theta_k)$$
$$\text{where} \quad H = \mathbb{E}_s \, \nabla^2_\theta \, KL(\pi(\theta) \parallel \pi(\theta_k))[s]\Big|_{\theta_k}$$

[7] where is linear term?

Trust-Region optimization procedure

Theorem:

$\nabla^2_\theta KL(\pi(\theta) \parallel \pi(\theta_k))[s]\Big|_{\theta_k}$ is Fisher information matrix.

✓ that's why solving this task is equivalent to gradient ascent with natural policy gradient!

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Solution (derived with K.K.T. theorem):

$$\theta_{k+1} = \theta_k + \sqrt{\frac{2\delta}{g_k^T H_k^{-1} g_k}} H_k^{-1} g_k$$

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Trust-Region optimization procedure

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× **Problem:** how to compute H_k^{-1} on practice? For neural networks with N parameters inversion complexity is $\mathcal{O}(N^3)$!..

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- PPO (2017): coming soon.
- ACKTR (2017): coming soon.