SET

MAT 101 : LECTURE NOTE 9/4/2023

~~Null set~~ or Void

Empty set $\emptyset$ or $\{\ \}$: Example $A = \{a : a^2 = -4\}$, $X = \{x : x$ is a married bachelor$\}$

Properties of Sets

Equality of sets: Two sets are equal if every element of one set is also an element of the other set. E.g $A = \{a, e, i, o, u\} = \{u, o, a, e, i\}$.

(ii) $\{a, e, i, o, u\} = \{a, a, e, e, u, o, o, o, i\}$.

Complement of a set: Let X be a subset of a Universal set U. The complement of X is the set of those elements of U that are not in X.

$X' = \{x : x \in U \ \& \ x \notin A\}$.

Cardinality of a set: The number of elements the set contains $n(X)$. e.g $X = \{a, e, i, o, u\}$. $n(X) = 5$.

Operations Between Two or More Sets

Union of Two sets A and B ($A \cup B$): The union of two sets A and B is the set of elements belonging to either A or B (or both).

$A \cup B = \{x / x \in A$ or $x \in B$ or $x \in$ both A and B$\}$

if $X = \{a, b, c, d, e, f\}$, $Y = \{e, f, g, h\}$

$X \cup Y = \{a, b, c, d, e, f, g, h\}$.

Intersection of Two Sets A and B ($A \cap B$): is the set of elements belonging to both A and B. Thus $A \cap B = \{x / x \in A$ and $x \in B\}$

$X \cap Y = \{e, f\}$.

Difference of Two sets A and B, $A - B$ or A / B: is the set of elements belonging to A but not to B.

Example: let $X = \{a, b, c, d\}$, $B = \{c, d, e, f\}$

$X \backslash Y$ or $X - Y = \{a, b\}$.

VENN DIAGRAMS

A simple and useful way to representing sets by using diagrams.

Number of elements in a two set problem

if A and B are subsets of U, the universal set and $A \cap B \neq \emptyset$ then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. if $A \cap \emptyset = \emptyset$ then ~~$n(A\cup B) = n$~~ $n(A \cup B) = n(A) + n(B)$.

Number of elements in a three set problem

If A, B, and C are subset of a universal set U such that $A \cap B \cap C \neq \emptyset$
Then $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$.

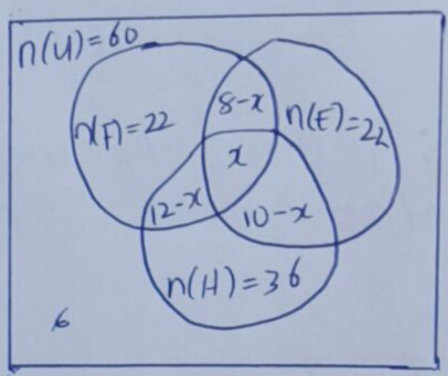
Example 6

In a class of 60 students, 22 offered French; 22 offered English, 36 offered History; 8 French and English; 10 English and History. 12 French and History while 6 did not offer any of the three subjects.

(a) Draw a Venn diagram to represent the data
(b) Use your diagram in (a) to find the number of students who offered (i) all three subjects (ii) History only.

Solutions

(a)



$n(F \cup E \cup H) = 60 - 6 = 54$
$n(F) = 22$, $n(E) = 22$, $n(H) = 36$
$n(F \cap E) = 8$, $n(E \cap H) = 10$
$n(F \cap H) = 12$, let $n(F \cap E \cap H) = x$.

b(i) $n(F \cup E \cup H) = n(F) + n(E) + n(H) - n(F \cap E) - n(E \cap H) - n(F \cap H) + n(F \cap E \cap H)$.

$\Rightarrow \quad 54 = 22 + 22 + 36 - 8 - 10 - 12 + x$.
$\quad ; \; x = 54 - 50 = 4$,

(ii) History only $= n(H) - n(F \cap H) - n(E \cap H) + n(F \cap E \cap H)$.
$= 36 - 12 - 10 + 4$
$=$

REAL NUMBERS: INTEGERS, RATIONAL AND IRRATIONAL NUMBERS

1. Real and Complex Numbers

Definition: A set is a collection of mathematical objects. An element is an object that is in a specific set. An interval is a collection of Real numbers.

Natural Numbers: The natural numbers are the numbers we use for counting $\{1, 2, 3, 4, \ldots\}$. The symbol: $\mathbb{N}$.

(i) The natural number do not include 0., (ii) the $\mathbb{N}$ do not include negative numbers.

· Whole Numbers: The set of natural number plus zero is the set of whole numbers: $\{0, 1, 2, 3, 4 \ldots\}$. The symbol: W

Whole number = 0 + Natural numbers, does not include 0, or negative numbers

Integers ($\mathbb{Z}$): The set of integers is the set of negative natural numbers plus the whole numbers: $\{\ldots, -2, -1, 0, 1, 2 \ldots\}$

Integers = negative natural numbers + whole numbers.

Rational Numbers ($\mathbb{Q} = \frac{m}{n}$) $n \neq 0$ and m, n are Integers

The set of rational numbers include fractions written as m/n. $\{x : x = \frac{m}{n}\}$. Rational numbers also include repeating and terminating decimals. $0.3333\ldots = 0.\dot{3}$ is a repeating decimal.

$0.25 = 1/4$ is a terminating Decimal.

Irrational Numbers: The set of irrational numbers is the set of numbers that are not Rational, are non-repeating & are non-terminating

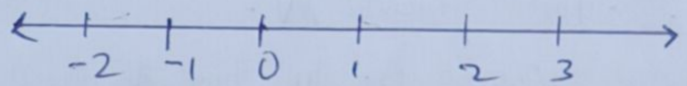
Example: $\pi = 3.1415926$ Non-repeating & non-terminating

$\sqrt{2} = 1.41421\ldots$ Non-repeating & non-terminating

Real Numbers ($\mathbb{R}$): The set of real numbers is the set of rational and Irrational numbers together.

$\{x : x \text{ is a rational or Irrational number}\}$.

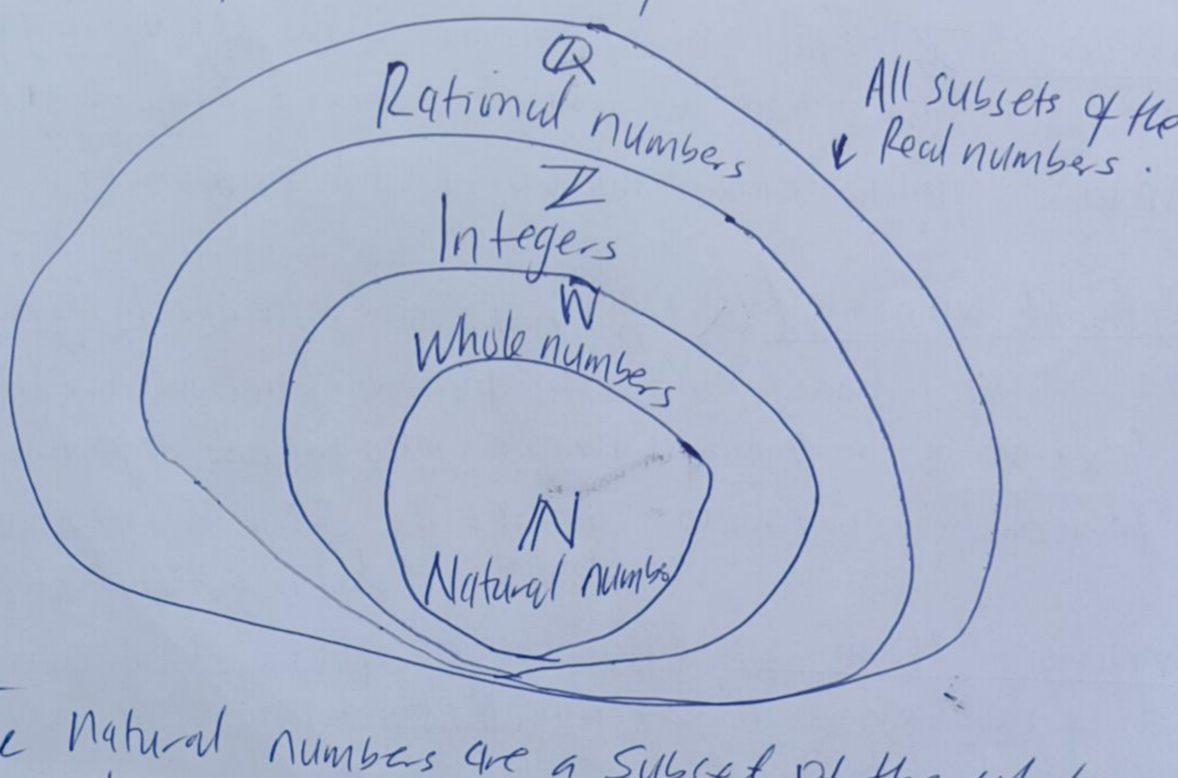
* A number Cannot be Both Rational and Irrational.
* The real numbers can be visualized on a Number line. (The Real Number line).



* The numbers that are NOT real numbers commonly occurs when we take the square root (or the even root) of a negative number and when we divide by 0 (You cannot divide by 0!).

Definition

let A and B be sets. Say that A is a subset of B, written $A \subseteq B$, if every element of A is an element of B.



(1) The natural numbers are a subset of the whole numbers because every natural number is also a whole number! $\{1, 2, 3, \ldots\} \subseteq \{0, 1, 2, 3, \ldots\}$.

(2) The whole numbers are NOT a subset of the natural numbers because 0 is not a natural number.

ARITHMETIC AND GEOMETRIC PROGRESSION / SERIES

What is a sequence? It is a set of numbers which are written in a definite or particular order. example 1, 3, 5, 7, 9 . . .
We have a sequence of odd numbers. ~~each~~ We start with the number 1, which is an odd number, and then each successive number is obtained by adding 2 to give the next odd number.

Arithmetic Progressions (A.P)

An arithmetic Progression is a sequence in which the difference between two consecutive terms like the n^{th} and $(n+1)^{th}$ term is a constant. This constant is called the common difference of the Progression. If we let a be the first term and d the common difference, then the A.P. is of the form

$$a, a+d, a+2d, \ldots \quad \ldots \quad (1)$$

where the n^{th} term is $a + (n-1)d$.

$$T_n = a+(n-1)d.$$

Example

The third term of an arithmetic progression is 10 and the seventh term is 34. Find the first term and the common difference.

Solution

Third term $T_3 = a+2d = 10$

$T_{10} = a+6d = 34$

$a+2d = 10 \quad -(i)$

$a+6d = 34 \quad -(ii)$

$(ii) - (i)$

$4d = 24$

$d = 6$

substitute $d = 6$ into (i)

$a+2(6) = 10$

$a+12 = 10 \quad \therefore \; a = 10-12 = -2$ //

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Sum of an Arithmetic Progression {Arithmetic Series}

The sum S_n of the first n terms of an AP with first term a and the common difference d is given by.

$$S_n = \frac{n}{2}\left(2a+(n-1)d\right)$$

Proof

$$\left.\begin{array}{lllll} S_n = a & +(a+d) & + \dots + & (a+(n-1)d) \\ S_n = (a+(n-1)d) & +(a+(n-2)d) & + \quad + & a \end{array}\right\} \text{by Writing in Reverse Order}$$

$\Rightarrow$

$$2S_n = (2a+(n-1)d) + (2a+(n-1)d) + \dots + (2a+(n-1)d)$$
$$= n(2a+(n-1)d)$$

Hence $S_n = \frac{n}{2}(2a+(n-1)d)$.

Example 1

Determine the sum S_{20} of the first 20 terms of the arithmetic progression: $10, 6, 2, -2, \dots$

Solution

$a=10, d=-4, n=20$

$$\Rightarrow S_{20} = \frac{20}{2}(2\cdot 10+19\cdot(-4)) = 10(20-76) = -560$$

Example 2

The sum S_8 of the 8 terms of an AP is 90, and its first term is 6. what is the common difference?

Solution

$$90 = S_8 = \frac{8}{2}(2\cdot 6+7\cdot d)$$

$$\Rightarrow \frac{90}{4} - 12 = 7d$$

$$\Rightarrow d = \frac{42}{28} = \frac{3}{2} = 1.5$$

Example 3

How many terms of the AP 3, 6, 9, ... must be taken so that their Sum is 135?

Solution

$a = 3,\ d = 3\ \ S_n = 135\ \ n = ?$

$\Rightarrow 135 = S_n = \frac{n}{2}(2\cdot 3 + (n-1)\cdot 3)$ — cross multiply $\Rightarrow 2(135) = (6n + 3n^2 - 3n)$

$\Rightarrow 270 = 3n^2 + 3n$

$\Rightarrow n^2 + n - 90 = 0$

$\Rightarrow (n+10)(n-9) = 0$

$\Rightarrow n = 9$ (since n must be positive).

Geometric Progressions (finite or Infinite)

A Geometric progression (G.P) is a sequence of numbers

$$a_1, a_2, a_3, \dots$$

Such that the quotient of the consecutive terms is a Constant (also called the Common ratio) r.

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \dots$$

In general n^{th}

$$a_n = a \cdot r^{n-1} \quad \text{where } a = a_1$$

Example

- $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \ (r = \frac{1}{2})$
- $2, -6, 18, -54, \dots \ (r = -3)$

Theorem (sum of a GP)

The sum S_n of the first n terms of a GP with first term a and the common ratio r is given by

$$S_n = a \cdot \frac{1 - r^n}{1 - r}$$

in particular if $-1<r<1$, then the sum S_∞ of all (infinitely many) terms is $S_\infty = a \cdot \frac{1}{1-r}$ (The sum of an Infinite GP is also called geometric series)

Proof

$$(1-r)S_n = (1-r)(a+ar+\cdots+ar^{n-1})$$
$$= (a+ar+\cdots+ar^{n-1}) - (ar+ar^2+\cdots+ar^n)$$
$$= a - ar^n = a(1-r^n) \text{ (everything else cancels)}$$

Thus $S_n = a\frac{1-r^n}{1-r}$

Now if $-1<r<1$, then $\lim_{n\to\infty} r^n = 0$. so $S_\infty = \lim_{n\to\infty} S_n = a\frac{1}{1-r}$

Example 1

Determine the sum S_7 of the first 7 terms of the geometric progression $4, -8, 16, \ldots$

Solution.

$a=4, r=-2, n=7$

$$\Rightarrow S_7 = 4\,\frac{1-(-2)^7}{1-(-2)} = 4 \cdot \frac{129}{3} = 172$$

Example 2

A GP is given by $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$ Determine S_∞.

Solution

$a=\frac{1}{4}, r=\frac{1}{4}$

$$\Rightarrow S_\infty = \frac{1}{4} \cdot \frac{1}{1-\frac{1}{4}} = \frac{1}{4} \cdot \frac{1}{\frac{3}{4}} = \frac{1}{3}$$

Remainder and Factor Theorems

Polynomial: is an expression of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where n is a non negative Integer and the Integer n is called the degree of the polynomial. $a_0, a_1, \dots, a_{n-1}, a_n$ are constants, also called Coefficients.

Note: Its degree (or order) is the highest occuring power of x; $a_n \neq 0$.

example Polynomial	Its degree	also called
$x - 7$	1	linear polynomial
$3x^2 + 2$	2	quadratic polynomial
$4x^3 + 5x^2 + 3x$	3	Cubic polynomial
$-x^4 + x^2 - 2$	4	quartic polynomial.

Recall long division of numbers: {Improper fraction}

$$\begin{array}{r} 205 \text{ r. } 18 \\ 21 \overline{)4323} \\ -42 \\ \hline 123 \\ -105 \\ \hline 18 \end{array}$$

$\Rightarrow$ We can write this as

$$\frac{4323}{21} = 205 + \frac{18}{21}$$

or

$$4323 = 205 \cdot 21 + 18$$

In general:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

or

$$\text{divided} = \text{quotient} \cdot \text{divisor} + \text{remainder}$$

$$\frac{P(x)}{S(x)} = Q(x) + \frac{R(x)}{S(x)}$$

or

$$P(x) = Q(x)S(x) + R(x)$$

Note the degree of $R(x)$ is less than the degree of the divisor $S(x)$.

Q

Example

Divide the polynomial $P(x) = x^4 - 3x^3 - 17x^2 + 34x + 5$ by $S(x) = x - 5$.

$$
\begin{array}{r|l}
 & x^3 + 2x^2 - 7x - 1 \\
\hline
x-5 & x^4 - 3x^3 - 17x^2 + 34x + 5 \\
 & \underline{x^4 - 5x^3} \\
 & \quad 2x^3 - 17x^2 + 34x + 5 \\
 & \quad \underline{2x^3 - 10x^2} \\
 & \qquad -7x^2 + 34x + 5 \\
 & \qquad \underline{-7x^2 + 35x} \\
 & \qquad\qquad -x + 5 \\
 & \qquad\qquad \underline{-x + 5} \\
 & \qquad\qquad\quad 0
\end{array}
$$

Thus $x^4 - 3x^3 - 17x^2 + 34x + 5 = (x^3 + 2x^2 - 7x - 1)(x - 5)$

or $\dfrac{x^4 - 3x^3 - 17x^2 + 34x + 5}{x - 5} = x^3 + 2x^2 - 7x - 1$.

The Remainder Theorem

Consider the polynomial $P(x) = 2x^3 - 7x^2 + 5x - 6$ and divide $P(x)$ by $(x-2)$

$$
\begin{array}{r|l}
 & 2x^2 - 3x - 1 \\
\hline
x-2 & 2x^3 - 7x^2 + 5x - 6 \\
 & \underline{2x^3 - 4x^2} \\
 & \quad -3x^2 + 5x \\
 & \quad \underline{-3x^2 + 6x} \\
 & \qquad -x - 6 \\
 & \qquad \underline{-x + 2} \\
 & \qquad\quad -8 \text{ (remainder)}
\end{array}
$$

we have $2x^3 - 7x^2 + 5x - 6 = (x-2)(2x^2 - 3x - 1) - 8$ and the remainder is -8.

The divisor above is $(x-2)$. if we put $x = 2$ in $P(x)$ we obtain $P(2) = 2(2)^3 - 7(2)^2 + 5(2) - 6 = -8$ which is a remainder. This illustrate an Important result called the Remainder theorem.

Theorem. (Remainder Theorem): The remainder left when a polynomial $P(x)$ is divided by $x-\alpha$ is equal to $P(\alpha)$. In particular, $x-\alpha$ is a factor of $P(x)$ if an only if $P(\alpha)=0$.

Example 1

Find the remainder when $3x^4+x^3+4x-5$ is divided by $x+2$.

Solution

$P(x) \equiv 3x^4+x^3+4x-5$ and $\alpha=-2$. By the remainder theorem the remainder is

$$P(-2) = 3(-2)^4+(-2)^3+4(-2)-5$$
$$= 27$$

Example 2

Find the value of k if the remainder of x^3+3x^2+kx-2 divided by $(x-3)$ is 100

Solution

$P(x) \equiv x^3+3x^2+kx-2$ and $\alpha=3$, By the remainder theorem

$100 = P(3)$

$100 = 3^3+3(3)^2+3k-2$

$100 = 52+3k$

Thus, $3k=48$, or $k=16$

The Factor Theorem

On dividing a polynomial $P(x)$ by $(x-\alpha)$, we have, by the remainder theorem

$$P(x) \equiv (x-\alpha)\varphi(x) + P(\alpha).$$

An easy consequence of this identity

$$P(\alpha)=0 \Leftrightarrow P(x) \equiv (x-\alpha)(\varphi(x)).$$ is known as the factor theorem.

Theorem (The Factor Theorem)

Let $P(x)$ be a polynomial and α a real constant. Then $(x-\alpha)$ is a factor of $P(x)$ if and only if $P(\alpha)=0$.

Example 1

Prove that $(x-1)$ and $(x+1)$ are factors of the polynomial
$$x^8 - x^5 + x^3 - 1$$

Proof

let $P(x) \equiv x^8 - x^5 + x^3 - 1$

Notice that $P(1) = 1^8 - 1^5 + 1^3 - 1$
$= 0$

$P(-1) = (-1)^8 - (-1)^5 + (-1)^3 - 1$
$= 0$

Thus, by the factor theorem, $(x-1)$ and $(x+1)$ are factors of $P(x)$.

Partial Fractions and the Binomial Theorem

Partial Fractions

A rational function is a quotient of two polynomials, i.e an expression of the form $\frac{P_1(x)}{P_2(x)}$, where both $P_1(x)$ and $P_2(x)$ are polynomials, \$ $P_2(x) \neq 0$

A fraction $\frac{P(x)}{Q(x)}$ is proper if the degree of $P(x)$ is less than the degree of $Q(x)$; it is Improper if the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$.

Note If a given fraction $\frac{P(x)}{Q(x)}$ is Improper, then we obtain by long division

$$\frac{P(x)}{Q(x)} \equiv \underbrace{R(x)}_{\text{Polynomial}} + \underbrace{\frac{S(x)}{Q(x)}}_{\text{Proper fraction}}$$

Distinct Linear Factors

$$\frac{Px+Q}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

Example

$f(x) = \dfrac{2x+3}{(x+1)(x-1)(x-2)}$ in partial Fractions

solutions

$$\frac{2x+3}{(x+1)(x-1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$2x+3 \equiv A(x-1)(x-2) + B(x+1)(x-2) + C(x+1)(x-1).$$

Put $x=-1$, we get $1 = A(-2)(-3)$ or $A = 1/6$

Put $x=1$, we get $5 = B(2)(-1)$. or $B = -5/2$

put $x=2$, we get $7 = C(3)(1)$ or $C = 7/3$

Hence $\dfrac{2x+3}{(x+1)(x-1)(x-2)} = \dfrac{1}{6(x+1)} - \dfrac{5}{2(x-1)} + \dfrac{7}{3(x-2)}$

Repeated Linear Factors

$$\frac{Px+Q}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$$

Resolve $f(x) = \dfrac{x^2+6x+9}{(x-3)^2(x+5)}$ into partial fractions.

Solution

let $\dfrac{x^2+6x+9}{(x-3)^2(x+5)} = \dfrac{A}{x-3} + \dfrac{B}{(x-3)^2} \dfrac{C}{x+5}$

Then $x^2+6x+9 = A(x-3)(x+5) + B(x+5) + C(x-3)^2$

Putting $x=3$, we get $36 = B(8)$ or $B = \frac{9}{2}$

Putting $x=-5$, we get $4 = C(64)$ or $C = 1/16$

Equating Coefficients of x^2, we have $1 = A + C$ or $A = 15/16$

Hence $\dfrac{x^2+6x+9}{(x-3)^2(x+5)} = \dfrac{15}{16(x-3)} + \dfrac{9}{2(x-3)^2} + \dfrac{1}{16(x+5)}$.

Distinct Quadratic Factors

$$\frac{Px^2+Qx+R}{(ax+b)^2(cx+d)} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{cx+d}$$

See example from other sources

Repeated Quadratic Factors

$$\frac{Px^2+Qx+R}{(ax^2+bx+c)(dx+e)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{dx+e}$$

Binomial Theorem

The expansion of the expression $(x+y)^n$, where n is a nonnegative Integer.

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$
$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5.$$

Notice that the Coefficients of terms in the above expansions from what is known as Pascal's triangle.

n	
$n=0$	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

In general, If n is a positive Integer, the expansion of $(x+y)^n$ is given by

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n-1}xy^{n-1} + y^n$$

$$= x^n + nx^{n-1}y + \frac{n(n-1)}{2\cdot 1}x^{n-2}y^2 + \cdots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)\ldots 3\cdot 2\cdot 1}x^{n-r}y^r + nxy^{n-1} + y^n$$

This is called the Binomial Theorem for any positive Integral value of n.

Example 1

Expand $(2x+3y)^5$ and simplify each term.

Solution

$$(2x+3y)^5 = (2x)^5 + 5(2x)^4(3y) + 10(2x)^3(3y)^2 + 10(2x)^2(3y)^3 + 5(2x)(3y)^4 + (3y)^5$$

$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4$$

Example 2

Expand $\left(2x - \frac{1}{x}\right)^4$ and simplify each term.

Solution

$$\left(2x-\frac{1}{x}\right)^4 = (2x)^4 - \binom{4}{1}(2x)^3\left(\frac{1}{x}\right) + \binom{4}{2}(2x)^2\left(\frac{1}{x}\right)^2 - \binom{4}{3}(2x)\left(\frac{1}{x}\right)^3 +$$

$$= 16x^4 - 32x^2 + 24 - \frac{8}{x^2} + \frac{1}{x^4}.$$

Binomial approximation

Binomial approximation: If $-1 < x < 1$, then $(1+x)^n \simeq 1 + nx$

Example 3: Approximate $\sqrt{1.05}$

Solution

$$\sqrt{1.05} = (1+0.05)^{\frac{1}{2}} \simeq 1 + \frac{1}{2}\cdot 0.05 = 1.025.$$

Example 4

$$(1.01)^5 = (1+0.01)^5$$

$$= 1 + 5\times(0.01) + 10\times(0.01)^2 + 10\times(0.01)^3 + 5\times(0.01)^4 + (0.0$$

$$\simeq 1 + 0.0500 + 0.0010 = \underline{1.0510}$$

MATHEMATICAL INDUCTION

The Basic Principle: An analogy of the Principle of Mathematical Induct is the game of dominoes. Suppose the dominoes are lined up properly, s that when one falls, the successive one will also fall. Now by pushin the first domino the second will fall; when the second falls the third will fall; and so on. We can see all dominoes will ultimately fall.

Theorem **. (Principle of Mathematical Induction)

Let $S(n)$ denote a statement involving a variable n. Suppose

(1) $S(1)$ is true;

(2) if $S(k)$ is true for some positive Integer k, then $S(k+1)$ is also true.

Then $S(n)$ is true for all positive Integers n.

Example 1

Prove that $1+3+5+\cdots+(2n-1)=n^2$ for all natural number n.

Solution

We shall prove the statement using Mathematical Induction ($S(n)$)

Clearly, the statement holds when $n=1$ since $1=1^2$.

Suppose the statement holds for some positive Integer k. That is,

$1+3+5+\cdots+(2k-1)=k^2$.

Consider the case $n=k+1$

By the above assumption (which we shall call the Induction hypothesis), we have

$$
\begin{aligned}
1+3+5+\cdots+[2(k+1)-1] &= [1+3+5+\cdots+(2k-1)]+(2k+1)\\
&= k^2+(2k+1)\\
&= (k+1)^2
\end{aligned}
$$

That is, the statement holds for $n=k+1$ provided that it holds for $n=k$.

By Principle of Mathematical Induction, we conclude $1+3+5+\cdots+(2n-1)=n^2$ for all natural number

The principle of mathematical induction can be used to prove a wide range of statements involving variables that take discrete values.

Example 2.

Prove that $23^n - 1$ is divisible by 11.

solution:

Clearly $23^1 - 1 = 22$ is divisible by 11

Suppose $11 \mid 23^k - 1$ for some positive integer k

Then $23^{k+1} - 1 = 23 \cdot 23^k - 1 = 11 \cdot 2 \cdot 23^k + (23^k - 1)$ which is also divisible by 11.

It follows that $23^n - 1$ is divisible by 11 for all positive integers n.

Example 3

Prove by Induction that

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

for all positive integers n.

solution

When $n = 1$ $\quad L.H.S = \frac{1}{1 \cdot 2} = \frac{1}{2}$ \& $R.H.S = \frac{1}{2}$

Thus the statement is true when $n = 1$

Assume that

$$\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$$

Consider the case when $n = k+1$. observe that

$$L.H.S = \sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$ (by Induction hypothesis).

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{k+1}{(k+1)+1} = R.H.S$$ which shows that the statement is true for $n = k+1$. Hence, the given statement is true for all positive Integers n.

PERMUTATIONS AND COMBINATIONS

Permutation: First we Introduce $n! = n.(n-1).(n-2)\ldots 3.2.1$

A Permutation of a given number of Objects is an arrangement of these objects either taking some or all of them, in a definite order.

$$^nP_r = \frac{n!}{(n-r)!}$$

Remark: This formula is used when a Counting problem involves both:
(1) Choosing a subset of r elements from a set of n elements; and
(2) Arranging the Choosen elements.

The formula for Permutation of n different Objects, taken all at a time.

$$_nP_n = n!$$

Example

Seven different books are to be arranged in 4 empty spaces on a bookshelf. In how many different ways can this be done?

Solution

$$= \frac{7!}{3!} \text{ that is } \frac{7!}{(7-4)!} = 840$$

Example

How many 5-letter Code-words can be formed from the letters of the word, MATRICES?

Solution

The number of permutations of the 8 different letters of the word MATRICES, taken 5 at a time $= {}_8P_5$

$$= 8 \times 7 \times 6 \times 5 \times 4$$
$$= 6720$$

Permutation of n objects, not all distinct

In a set of n objects, if there are n_1 of them of the first kind, n_2 of them of the second kind & so on for k kinds of objects then the number of permutations of these n Objects taken all at a time is

$$\frac{n!}{n_1!\,n_2!\ldots n_k!}$$

Example

How many 11-letter code-words can be formed from the letters of the word, INDEPENDENT?

Solution

$$= \frac{11!}{2!\,3!\,3!\,1!\,1!\,1!} = 554400$$

2'D, 3E, 3N, 1T, 1P

Circular Permutations

$$\frac{n!}{n} = (n-1)!$$

Combination

Combination of n objects, taken r at a time

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example

In how many ways can 5 basketball players be selected from a team of 12 players to participate in a friendly game?

Solution

The number of ways of selecting 5 players from 12 $= \binom{12}{5}$

$$= \frac{12 \times 11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4 \times 5} = 792$$

(1) QUADRATIC EXPRESSIONS AND EQUATIONS

A polynomial of degree two is called quadratic expression and is of the form.

$$f(x) = ax^2 + bx + c \qquad (a \neq 0).$$

$$ax^2 + bx + c = 0$$
$$ax^2 + bx = -c$$
$$x^2 + \frac{b}{a}x = -\frac{c}{a} \text{ (using } a \neq 0)$$
$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

This trick (adding $\left(\frac{b}{2a}\right)^2$ to both sides) is called "completing the square".

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{(2a)^2}$$
$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad b^2 - 4ac \geq 0$$

$\therefore\ x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is called the root of the quadratic equation which can also be obtained by factoring.

Example

Find the roots of

(i) $x^2 + 3x + 2 = 0$

solution:

$a = 1, b = 3, c = 2$ & $b^2 - 4ac = 1$

the roots of the equation.

$x^2 + 3x + 2 = 0$ are

$$x_1 = \frac{-3 + \sqrt{1}}{2} = -1$$
$$x_2 = \frac{-3 - \sqrt{1}}{2} = -2$$

(ii) $x^2 + 4x + 4 = 0$

Solution

$a = 1, b = 4, c = 4$ & $b^2 - 4ac = 0$

the roots of the equation

$$\alpha_1 = \frac{-3 + \sqrt{-3}}{2} \text{ \& } \cancel{\beta} = \frac{-3 + \sqrt{-3}}{2}$$

Quadratic equation without determining the nature of the roots

By the Factor Theorem, an equivalent equation for the quadratic equation

$$(x-\alpha)(x-\beta)=0$$

or

$$x^2-(\alpha+\beta)x+\alpha\beta=0$$

NB

$x^2-(\text{sum of roots})x+(\text{Product of roots})=0$.

for

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

This requires that $x^2-(\alpha+\beta)x+\alpha\beta \equiv x^2+\frac{b}{a}x+\frac{c}{a}$ which gives $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

Example 1

If α, β are the roots of $2x^2-x+4=0$, find the values of (i) $\alpha^2+\beta^2$ (ii) $\alpha^3+\beta^3$ (iii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$.

Solution

$$x^2-\frac{1}{2}x+2=0$$

$$\equiv$$

$$x^2-(\alpha+\beta)x+\alpha\beta=0$$

$\alpha+\beta=\frac{1}{2}$ & $\alpha\beta=2$

(i) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$$=\frac{1}{4}-4=\frac{-15}{4}$$

(ii) $\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$

$$=(\alpha+\beta)[(\alpha^2+\beta^2)-\alpha\beta]$$

$$=\frac{1}{2}\left(-\frac{15}{4}-2\right)$$

$$=-\frac{23}{8}$$

(iii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$

$$=\left(\frac{-15}{4}\right)\cdot\frac{1}{2}$$

$$=\frac{-15}{8}$$

COMPLEX NUMBERS $\mathbb{C} = \{z : z = x + yi,\ x, y \in \mathbb{R}\}$

The Complex Number system

$\mathbb{C}$ = The set of all complex numbers

The quadratic equation $x^2+1=0$ has no real roots since there is no real number x such that $x^2 = -1$. If we Introduce a number i, defined by $i^2 = -1$ (or $i = \sqrt{-1}$ then the solution to $x^2+1=0$ is then $x = i$ or $x = -i$.

Example

$x^2 - 8x + 20 = 0$

Observe that $b^2 - 4ac < 0 = -16$ the determinant.

applying the quadratic formula.

$$x = \frac{8 \pm \sqrt{64-80}}{2}$$

$= 4 \pm 2\sqrt{-1}$ since $\sqrt{-1} = i$

$x = 4+2i$ or $x = 4-2i$, and these are called complex numbers.

$Z = x+yi$, where Real part $Re(z) = x$

Imaginary part $Im(z) = y$

Operations on Complex number

Addition and subtraction

$(a+bi) + (c+di) = (a+c) + (b+d)i$

$(a+bi) - (c+di) = (a-c) + (b-d)i$

Examples

subtract $6i - (4-i)$.

solution

$-4+(6i+i) = \underline{-4+7i}$

Example 2

(i) $(2+3i)+(7-2i) = (2+7)+(3-2)i = \underline{\underline{9+i}}$

(ii) $(3+2i)-(7+3i) = \underline{\underline{-4-i}}$

Multiplication of Two Complex Numbers

The product of two complex number is itself a complex number and is defined by

$$z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i) = x_1 x_2 + x_1 y_2 i + x_2 y_1 i + y_1 y_2 i^2$$
$$= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i \quad \text{since } i^2 = -1$$

Example

$$(2+3i)(4-8i) = 2(4-8i) + 3i(4-8i)$$
$$= 8 - 16i + 12i - 24i^2$$
$$= 32 - 4i$$

Conjugate Complex numbers

Let $z = x + yi$, the conjugate z^* or $\bar{z} = x - yi$

$$zz^* = (x+yi)(x-yi)$$
$$= (x^2 - y^2 i^2) + (xy - xy)i$$
$$= x^2 + y^2$$

Example

$$(3-2i)(3+2i)^* = (9+4) = 13$$

Example

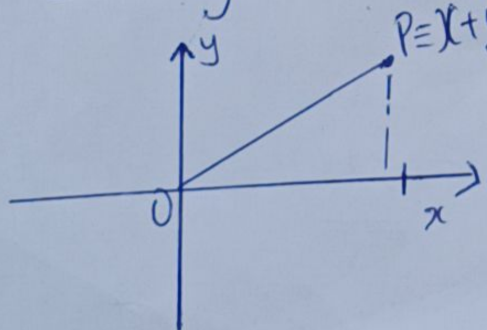
Express $\frac{1}{1+3i}$ in the form $a+bi$ where $a, b \in \mathbb{R}$

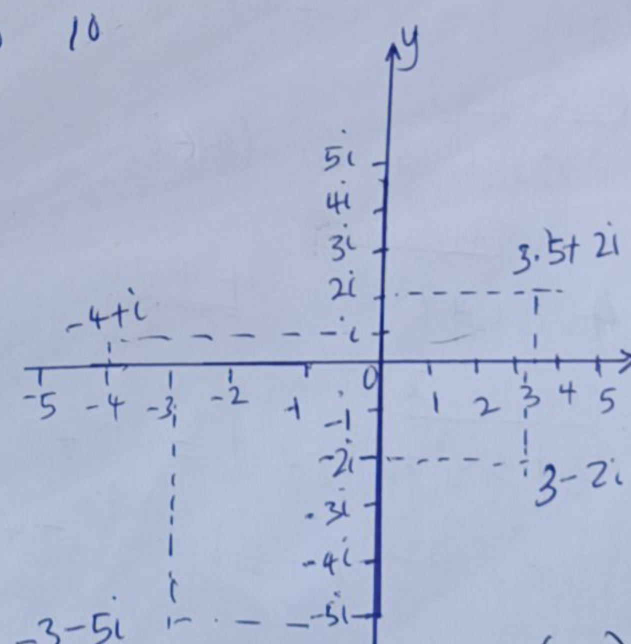
solution

Multiply the numerator and denominator by the conjugate $(1+3i)$

$$\frac{1}{1+3i} = \frac{1}{(1+3i)} \frac{(1-3i)}{(1-3i)} = \frac{1-3i}{1^2+3^2} = \frac{1}{10} - \frac{3}{10} i$$

The Argand Diagram





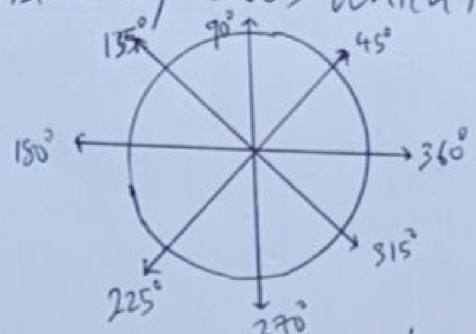
TRIGONOMETRY

Radian and Degree Measures of an Angle

Two kinds of units commonly used for measuring angles are radian and degree, this is because Trigonometry focuses on relationships of sides and angles of a triangle.

Degrees

A Circle is comprised of 360°, which is called one revolution



Degrees are primarily used to describe the size of an angle.

Conversions from degrees to minutes and seconds

Example: Convert 18.478° to minutes and seconds.

Solution

$18.478^\circ = 18^\circ + (0.478 \times 60)'$; we multiply the fractional part of the degree by 60

$= 18^\circ + 28.68'$

$= 18^\circ + 28' + (0.68 \times 60)''$

$= 18^\circ + 28' + 40.8''$

$= \underline{18^\circ 28' 41'}$

Conversions from minutes, seconds to degrees

Example: Convert 45°36'18" to degree

Solution

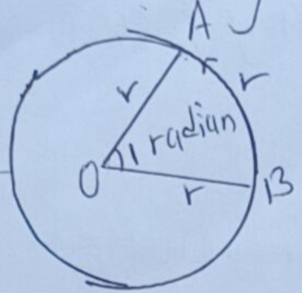
$45^\circ 36' 18'' = 45^\circ \times \frac{36}{60} \times \frac{18}{3600}$

$= (45 + 0.6 + 0.005)^\circ$

$= \underline{45.605^\circ}$

Radian

The radian measure is employed almost in advanced mathematics and in many branches of science.



The Circumference of a Circle is equal to $2\pi r$, It subtends a Centre angle of 2π radians

$$2\pi \text{ radians} = 360^\circ$$

$$\pi = 180^\circ$$

The following Conversion is fundamental relation between radians and degrees.

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ$$

$$\simeq 57.3^\circ \ (\text{or } 57^\circ\, 17'\, 45'')$$

and

$$1^\circ = \left(\frac{\pi}{180}\right) \text{radians}$$

$$\simeq 0.01745 \text{ radians.}$$

Example

$\frac{\pi}{2}$ radians $= 90^\circ$, $\frac{\pi}{3}$ radians $= 60^\circ$, $\frac{4\pi}{5}$ radians $= 144^\circ$, $\frac{7\pi}{6}$ rads $= 210^\circ$

$$45^\circ = 45 \times \frac{\pi}{180} \text{ rad} = \frac{\pi}{4} \text{ radians}$$

$$120^\circ = 120 \times \frac{\pi}{180} \text{ rad} = \frac{2\pi}{3} \text{ rad.}$$

Example

Convert the following radians to degree (i) $\frac{3\pi}{2}$ (ii) $-\frac{7\pi}{3}$

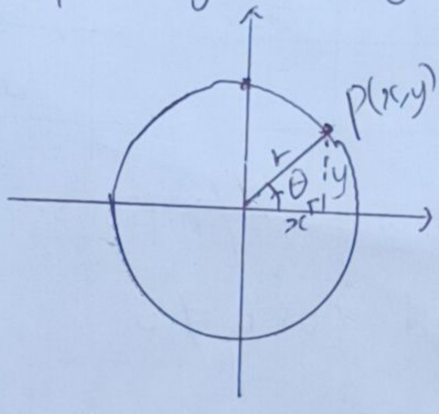
Solution

(i) $\frac{3\pi}{2}$ (1 radian) $\frac{180}{\pi} = \frac{3\pi}{2} \cdot \frac{180}{\pi} = \frac{540\pi}{2\pi} = 270^\circ$

(ii) $-\frac{7\pi}{3} \cdot \frac{180}{\pi} = \frac{1260}{3} = 420^\circ$

The Trigonometric Functions

In Trigonometry there are six trigonometric ratios that relate the angle measures of a right triangle to the length of its sides.



$\sin\theta = \frac{y}{r} = \frac{opp}{hyp}$

$\cos\theta = \frac{x}{r} = \frac{adj}{hypo}$

$\tan\theta = \frac{y}{x} = \frac{opp}{adj}$

Reciprocal functions

$\csc\theta = \frac{r}{y}$ which is the reciprocal of $\sin\theta$

$\sec\theta = \frac{r}{x}$ which is the reciprocal of $\cos\theta$

$\cot\theta = \frac{x}{y}$, which is the reciprocal of $\tan\theta$

Angle θ		$\sin\theta$	$\cos\theta$	$\tan\theta$	
Degrees	Radians				
0°	0	0	1	0	
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	
45°	$\pi/4$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	
60°	$\pi/3$	$\frac{\sqrt{3}}{2}$	$1/2$	$\sqrt{3}$	
90°	$\pi/2$	1	0	∞	
120°	$2\pi/3$	$\frac{\sqrt{3}}{2}$	$-1/2$	$-\sqrt{3}$	
135°	$3\pi/4$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	
150°	$5\pi/6$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	
180°	π	0	-1	0	
210°	$7\pi/6$	$-1/2$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	
240°	$4\pi/3$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	
270°	$3\pi/2$	-1	0	∞	

300°	$5\pi/3$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
315°	$7\pi/4$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
330°	$11\pi/6$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
360°	2π	0	1	0

Bearing in mind the following definitions:

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ Provided $\cos\theta \neq 0$

$\sec\theta = \dfrac{1}{\cos\theta}$ " " " "

$\operatorname{Cosec}\theta = \dfrac{1}{\sin\theta}$ Provided $\sin\theta \neq 0$

$\cot\theta = \dfrac{1}{\tan\theta}$ Provided $\tan\theta = 0.$

We have

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \operatorname{Cosec}^2\theta$$

$\sin(\pi-\theta) = \sin\theta$

$\cos(\pi-\theta) = -\cos\theta$

$\tan(\pi-\theta) = -\tan\theta$

$\sin(2\pi-\theta) = \cos\theta$

$\tan(2\pi-\theta) = -\tan\theta$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

$\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$

$\sin(\pi+\theta) = -\sin\theta$

$\cos(\pi+\theta) = -\cos\theta$

$\sin(2\pi+\theta) = \sin\theta$

$\cos(2\pi+\theta) = \cos\theta$

Example *

(i) $\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$

Solution

$= \sin\left(\frac{\pi}{2} - (-\theta)\right)$

$= \cos(-\theta)$

$= \cos\theta$

Example

Show that

(i) $\frac{\cos\theta}{1+\sin\theta} - \frac{1-\sin\theta}{\cos\theta} = 0$

Solution

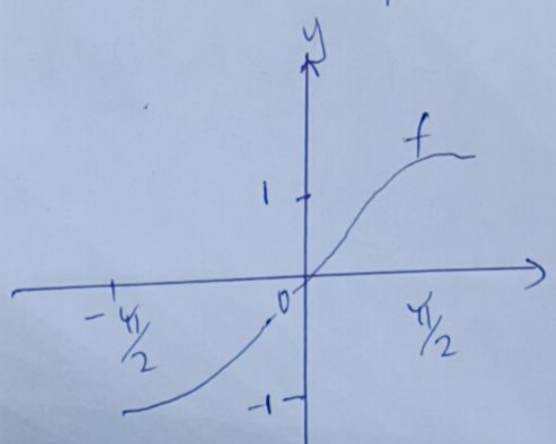
$$= \frac{\cos^2\theta - (1-\sin\theta)(1+\sin\theta)}{\cos\theta(1+\sin\theta)}$$

$$= \frac{\cos^2\theta - 1 + \sin^2\theta}{\cos\theta(1+\sin\theta)}$$

$= 0.$

THE INVERSE TRIGONOMETRICAL FUNCTIONS

Every One-One and Onto function has an Inverse. The sine function $f: x \longrightarrow \sin x$ where $-\pi/2 \le x \le \pi/2$ onto $[-1, 1]$



The Inverse function is defined by $f^{-1}: x \longrightarrow y$ where $x = \sin y$

Example *

If $\alpha = \sin^{-1}\frac{1}{2}$, find the principal value of α and hence obtain the values of $\cos\alpha$ and $\tan\alpha$.

solution.

$$\alpha = \sin^{-1}\frac{1}{2} \Rightarrow \sin\alpha = \frac{1}{2} \text{ and } \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow \alpha = \pi/6.$$

$$\cos\alpha = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\tan\alpha = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

Fundamental Identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$$

$$\tan(\theta-\phi) = \frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}$$

Double Angle Formulae

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$