

# Machine Learning Basics

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"The field of study that gives computers the ability to learn without being explicitly programmed."

/Arthur Samuel/

Notes:

$h$  - hypothesis

$w$  - weights

$x$  - input

$y$  - output

$\hat{y}$  - prediction

$m$  - total number of samples

$i$  - index of samples

$C$  - cost function

MSE - Mean Squared Error

$\mu$  - learning rate,  $0 < \mu \leq 1$   $\lambda$  - regularization

$$X_{3 \times 1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, Y_{3 \times 1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad (1)$$

$$h_w(x) = wx = \hat{y} \quad (2)$$

$$w = 0, C = 2.33 \quad (3)$$

$$w = 0.5, C = 0.58 \quad (4)$$

$$w = 1, C = 0 \quad (5)$$

$$C(w) = \frac{1}{2m} \sum_{i=1}^m (wx^i - y^i)^2 \quad (6)$$

$$w = w - \mu \frac{\partial}{\partial w} C(w) \quad (7)$$

$$\frac{\partial}{\partial w} C(w) = \frac{1}{m} \sum_{i=1}^m (wx^i - y^i) \cdot x^i \quad (8)$$

$$w_j^t := w_j^{t-1} - \mu \frac{\partial}{\partial w_j} C(w_0, w_1) + \Delta w_j^{t-1}$$

$$\begin{bmatrix} (0 \cdot 1 - 1) \cdot 1 \\ (0 \cdot 2 - 2) \cdot 2 \\ (0 \cdot 3 - 3) \cdot 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ -9 \end{bmatrix}, 0.1 \cdot \frac{-14}{3} = -0.46 \quad (9)$$

$$w = 0 - (-0.46) = 0.46 \quad (10)$$

$$\begin{bmatrix} (0.46 \cdot 1 - 1) \cdot 1 \\ (0.46 \cdot 2 - 2) \cdot 2 \\ (0.46 \cdot 3 - 3) \cdot 3 \end{bmatrix} = \begin{bmatrix} -0.53 \\ -2.13 \\ -4.8 \end{bmatrix}, 0.1 \cdot \frac{-7.46}{3} = -0.249 \quad (11)$$

$$w = 0.46 - (-0.249) = 0.71 \quad (12)$$

$$\begin{bmatrix} (0.71 \cdot 1 - 1) \cdot 1 \\ (0.71 \cdot 2 - 2) \cdot 2 \\ (0.71 \cdot 3 - 3) \cdot 3 \end{bmatrix} = \begin{bmatrix} -0.28 \\ -1.13 \\ -2.56 \end{bmatrix}, 0.1 \cdot \frac{-3.98}{3} = -0.132 \quad (13)$$

$$w = 0.71 - (-0.132) = 0.842 \quad (14)$$

# 1. Labor:

## Linear regression with one variable

Hypothesis:

$$h_w(x) = w_0 + w_1x \quad (15)$$

$$h_w(x) = w_0 + w_1x = \hat{y} \quad (16)$$

Cost function:

$$C = \frac{1}{2m} \sum_{i=1}^m (\hat{y}^i - y^i)^2 \quad (17)$$

$$C = \frac{1}{2m} \sum_{i=1}^m (h_w(x^i) - y^i)^2 \quad (18)$$

$$C(w_0, w_1) \quad (19)$$

$$C(w_0, w_1) = \frac{1}{2m} \sum_{i=1}^m (w_0 + w_1x^i - y^i)^2 \quad (20)$$

$$X_{m \times 1} = \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ \vdots \\ x^m \end{bmatrix}, W_{2 \times 1} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, Y_{m \times 1} = \begin{bmatrix} y^1 \\ y^2 \\ y^3 \\ \vdots \\ y^m \end{bmatrix} \quad (21)$$

$$X = \begin{bmatrix} 1 & x^1 \\ 1 & x^2 \\ 1 & x^3 \\ \vdots & \vdots \\ 1 & x^m \end{bmatrix} \Rightarrow X_{m \times 2} = \begin{bmatrix} x_0^1 & x_1^1 \\ x_0^2 & x_1^2 \\ x_0^3 & x_1^3 \\ \vdots & \vdots \\ x_0^m & x_1^m \end{bmatrix} \quad (22)$$

$$\hat{y} = h_w(x) = w_0 + w_1x^i = w_0x_0^i + w_1x_1^i \quad (23)$$

$$\begin{aligned}
X_{m \times 2} &= \begin{bmatrix} x_0^1 & x_1^1 \\ x_0^2 & x_1^2 \\ x_0^3 & x_1^3 \\ \vdots & \vdots \\ x_0^m & x_1^m \end{bmatrix} \quad W_{2 \times 1} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \\
&\quad \begin{bmatrix} w_0 x_0^1 + w_1 x_1^1 \\ w_0 x_0^2 + w_1 x_1^2 \\ w_0 x_0^3 + w_1 x_1^3 \\ \vdots \\ w_0 x_0^m + w_1 x_1^m \end{bmatrix} = \hat{Y}_{m \times 1} = XW \\
C &= \frac{\sum (XW - Y)^2}{2m} \tag{24}
\end{aligned}$$

## 1.1. Gradient descent

To solve:  $\min C(w_0, \dots, w_n)$

Algorithm:

repeat until convergence {  
 $w_j := w_j - \mu \frac{\partial}{\partial w_j} C(w_0, w_1)$   
}

Linear Regression Model

$$h_w(x) = w_0 + w_1 x \tag{25}$$

$$C(w_0, w_1) = \frac{1}{2m} \sum_{i=1}^m (h_w(x^i) - y^i)^2 \tag{26}$$

Gradient descent

repeat until convergence {  
 $w_j := w_j - \mu \frac{\partial}{\partial w_j} C(w_0, w_1)$   
}

$$\frac{\partial}{\partial w_j} C(w_0, w_1) = \frac{\partial}{\partial w_j} \frac{1}{2m} \sum_{i=1}^m (h_w(x^i) - y^i)^2 = \frac{\partial}{\partial w_j} \frac{1}{2m} \sum_{i=1}^m (w_0 + w_1 x^i - y^i)^2 \tag{27}$$

$$(j = 0) \quad \frac{\partial}{\partial w_j} C(w_0, w_1) = \frac{1}{m} \sum_{i=1}^m (w_0 + w_1 x^i - y^i) \cdot 1 = \frac{1}{m} \sum_{i=1}^m (h_w(x^i) - y^i) \cdot \mathbf{x}_0^i \quad (28)$$

$$(j = 1) \quad \frac{\partial}{\partial w_j} C(w_0, w_1) = \frac{1}{m} \sum_{i=1}^m (w_0 + w_1 x^i - y^i) \cdot x_1^i = \frac{1}{m} \sum_{i=1}^m (h_w(x^i) - y^i) \cdot x_1^i \quad (29)$$

Interpretation

$$w_0 = w_0 - \frac{\mu}{m} \sum_{i=1}^m (h_w(x^i) - y^i) \quad (30)$$

$$w_1 = w_1 - \frac{\mu}{m} \sum_{i=1}^m (h_w(x^i) - y^i) \cdot x^i \quad (31)$$

$$w_0 = w_0 - \frac{\mu}{m} \text{sum}(X * w - Y) \quad (32)$$

$$w_1 = w_1 - \frac{\mu}{m} \text{sum}(X * w - Y) \cdot X(:, 2) \quad (33)$$

## 2. Labor:

### Linear regression with multiple variable

$$X_{m \times (n+1)} = \begin{bmatrix} x_0^1 & x_1^1 & x_2^1 & \dots & x_n^1 \\ x_0^2 & x_1^2 & x_2^2 & \dots & x_n^2 \\ x_0^3 & x_1^3 & x_2^3 & \dots & x_n^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_0^m & x_1^m & x_2^m & \dots & x_n^m \end{bmatrix}, W_{(n+1) \times 1} = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, Y_{m \times 1} = \begin{bmatrix} y^1 \\ y^2 \\ y^3 \\ \vdots \\ y^m \end{bmatrix} \quad (34)$$

Hypothesis:

$$h_w(x) = w_0 + w_1x_1 + w_2x_2 + \dots + w_nx_n \quad (35)$$

$$h_w(x) = w_0x_0 + w_1x_1 + w_2x_2 + \dots + w_nx_n \quad (36)$$

$$X_{m \times (n+1)} = \begin{bmatrix} x_0^1 & x_1^1 & x_2^1 & \dots & x_n^1 \\ x_0^2 & x_1^2 & x_2^2 & \dots & x_n^2 \\ x_0^3 & x_1^3 & x_2^3 & \dots & x_n^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_0^m & x_1^m & x_2^m & \dots & x_n^m \end{bmatrix} \quad W_{(n+1) \times 1} = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

$$= \begin{bmatrix} w_0x_0^1 + w_1x_1^1 + w_2x_2^1 + \dots + w_nx_n^1 \\ w_0x_0^2 + w_1x_1^2 + w_2x_2^2 + \dots + w_nx_n^2 \\ w_0x_0^3 + w_1x_1^3 + w_2x_2^3 + \dots + w_nx_n^3 \\ \vdots \\ w_0x_0^m + w_1x_1^m + w_2x_2^m + \dots + w_nx_n^m \end{bmatrix}$$

Cost function:

$$C(W) = C(w_0, w_1, \dots, w_n) = \frac{1}{2m} \sum_{i=1}^m (h_w(x^i) - y^i)^2 \quad (37)$$

Gradient Descent:

$$w_j := w_j - \mu \frac{1}{2m} \sum_{i=1}^m (h_w(x^i) - y^i) \cdot x_j^i \quad (38)$$

$$x = \frac{x - mean(x)}{std(x)} \tag{39}$$

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$$v = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$v' = \begin{bmatrix} v_0 & v_1 & v_2 & \dots & v_n \end{bmatrix} \quad \begin{bmatrix} (v_0)^2 + (v_1)^2 + (v_2)^2 + \dots + (v_n)^2 \end{bmatrix} \quad \rightarrow v'v = sum(v.^2)$$



### 3. Labor:

#### Logistic regression Linear case

$$y \in \{0, 1\} \quad (40)$$

0: Negative class

1: Positive class

$$h_w(x) = XW \quad (41)$$

Threshold classifier output  $h_w(x)$  at 0.5:

If  $h_w(x) \geq 0.5$ , predict "y=1"

If  $h_w(x) < 0.5$ , predict "y=0"

$h_w(x)$  can be  $>1$  or  $<0$

Logistic regression:  $0 \leq h_w(x) \leq 1$

**Logistic Regression Model:**

Want  $0 \leq h_w(x) \leq 1$

$h_w(x) = g(Xw)$

$$g(z) = \frac{1}{1 + e^{-z}} \quad (42)$$

$h_w(x) \Rightarrow$  estimated probability that  $y = 1$  on input  $x$

$h_w(x) = P(y = 1|x, W)$

sigmoid function

Example I.

$$h_w(x) = g(w_0 + w_1x_1 + w_2x_2) \quad (43)$$

$w = [-3 \ 1 \ 1]$

Predict:  $y = 1$  if  $-3 + x_1 + x_2 \geq 0$

$x_1 + x_2 \geq 3$

Example II.

$$h_w(x) = g(w_0 1 + w_1 x_1 + w_2 x_2 + w_3 x_1^2 + w_4 x_2^2) \quad (44)$$

$$w = [-1 \ 0 \ 0 \ 1 \ 1]$$

Predict:  $y = 1$  if  $-1 + x_1^2 + x_2^2 \geq 0$

$$x_1^2 + x_2^2 \geq 1$$

$$\begin{aligned} &g(z) \geq 0.5 \\ &\text{when } z \geq 0 \end{aligned}$$

$$Cost(h_w(x), y) = \begin{cases} -\log(h_w(x)), & \text{if } y = 1 \\ -\log(1 - h_w(x)), & \text{if } y = 0 \end{cases} \quad (45)$$

$$Cost(h_w(x), y) = -y \cdot \log(h_w(x)) - (1 - y) \cdot \log(1 - h_w(x)) \quad (46)$$

$$C(W) = -\frac{1}{m} \sum_{i=1}^m y^i \cdot \log(h_w(x^i)) + 2(1 - y^i) \cdot \log(1 - h_w(x^i)) \quad (47)$$

Want  $\min \min_W \{C(W)\}$

Algorithm:

$$\begin{aligned} &\text{repeat until convergence } \{ \\ &\quad W_j := W_j - \mu \frac{\partial}{\partial W_j} C(W) \\ &\} \end{aligned}$$

$$\frac{\partial}{\partial W_j} C(W) = \frac{1}{m} \sum_{i=1}^m (h_w(x^i) - y^i) \cdot x_j^i \quad (48)$$

## 4. Labor:

### Logistic regression Non Linear case

Using Polynomial Features

$$x_1 \ x_2 \Rightarrow 1 \ x_1 \ x_2 \ x_1^2 \ x_1 x_2 \ x_2^2 \ x_1^3 \ x_1^2 x_2 \ x_1 x_2^2 \ x_2^3$$

$$w_0 + w_1 x$$

"Underfit"

"High Bias"

$$w_0 + w_1 x + w_2 x^2$$

Just Right

$$w_0 + w_1 x + w_2 x^2 + w_3 x^3 + w_4 x^4$$

"Overfit"

"High Variance"

Regularization:

$$C(w) = \frac{1}{2m} \sum_{i=1}^m (h_w(x^i) - y^i)^2 + \lambda \sum_{j=1}^n w_j^2 \quad (49)$$

If  $\lambda$  large: algorithm result in underfitting  
(fails to fit even the training set)

Regularized Logistic Regression:

Repeat{

$$w_0 := w_0 - \mu \frac{1}{m} \sum_{i=1}^m (h_w(x^i) - y^i) \cdot x_0^i$$

$$w_j := w_j - \mu \left[ \frac{1}{m} \sum_{i=1}^m (h_w(x^i) - y^i) \cdot x_j^i \right] + \frac{\lambda}{m} w_j$$

}

Cost function and derivative:

$$C(w) = [-\frac{1}{m} \sum_{i=1}^m y^i \cdot \log(h_w(x^i)) + (1 - y^i) \cdot \log(1 - h_w(x^i))] + \frac{\lambda}{2m} \sum_{j=1}^n w_j^2 \quad (50)$$

$$\frac{\partial}{\partial w_0} C(w) = \frac{1}{m} \sum_{i=1}^m (h_w(x^i) - y^i) \cdot x_0^i + \mathbf{0} \quad (51)$$

$$\frac{\partial}{\partial w_j} C(w) = \frac{1}{m} \sum_{i=1}^m (h_w(x^i) - y^i) \cdot x_j^i + \frac{\lambda}{m} w_j \quad (52)$$

## 5. Labor:

### Multi Class Classification

## 6. Labor:

Neural Networks Basics

## 7. Labor:

### Neural Network Train

To the picture

$$x_1$$

$$x_2$$

$$w_{11}^{(1)}$$

$$w_{12}^{(1)}$$

$$w_{13}^{(1)}$$

$$w_{21}^{(1)}$$

$$w_{22}^{(1)}$$

$$w_{23}^{(1)}$$

$$w_{11}^{(2)}$$

$$w_{12}^{(2)}$$

$$w_{13}^{(2)}$$

$$s^2$$

$$s^3$$

$$a^2$$

$$a^3$$

$$s = \sum_{i=1}^n w_i \cdot x_i \tag{53}$$

$$y = a(s) \tag{54}$$

$$xw^{(1)} = s^{(2)} \quad (55)$$

$$a^{(2)} = f(s^{(2)}) = \text{sigmoid}(s^{(2)}) \quad (56)$$

$$s^{(3)} = a^{(2)}w^{(2)} \quad (57)$$

$$\hat{y} = f(s^{(3)}) = \text{sigmoid}(s^{(3)}) \quad (58)$$

$$C = \sum \left\{ \frac{1}{2} (y - \hat{y})^2 \right\} \quad (59)$$

$$C = \sum \left\{ \frac{1}{2} (y - \textcolor{red}{a}^{(3)})^2 \right\} \quad (60)$$

$$C = \sum \left\{ \frac{1}{2} (y - f(\textcolor{red}{s}^{(3)}))^2 \right\} \quad (61)$$

$$C = \sum \left\{ \frac{1}{2} (y - f(\textcolor{red}{a}^{(2)}w^{(2)}))^2 \right\} \quad (62)$$

$$C = \sum \left\{ \frac{1}{2} (y - f(f(\textcolor{red}{s}^{(2)})w^{(2)}))^2 \right\} \quad (63)$$

$$C = \sum \left\{ \frac{1}{2} (y - f(f(xw^{(1)})w^{(2)}))^2 \right\} \quad (64)$$

Back propagation:

$$\frac{\partial C}{\partial w^{(2)}} = \frac{\partial \sum \frac{1}{2} (y - \hat{y})^2}{\partial w^{(2)}} = \sum \left( \frac{\partial \frac{1}{2} (y - \hat{y})^2}{\partial w^{(2)}} \right) \quad (65)$$

$$\begin{aligned} \frac{\partial \frac{1}{2} (y - \hat{y})^2}{\partial w^{(2)}} &= (y - \hat{y}) \left( -\frac{\partial \hat{y}}{\partial w^{(2)}} \right) \\ &= -(y - \hat{y}) \cdot \frac{\partial \hat{y}}{\partial s^{(3)}} \cdot \frac{\partial s^{(3)}}{\partial w^{(2)}} \\ &= -(y - \hat{y}) \cdot f'(s^{(3)}) \cdot \frac{\partial a^{(2)}w^{(2)}}{\partial w^{(2)}} \\ &= \textcolor{red}{\delta}^{(3)} \cdot \textcolor{red}{a}^{(2)} \end{aligned} \quad (66)$$



Dimension check:

$$(a^{(2)})^T \delta^{(3)} \quad (67)$$

$$-(y - \hat{y}) \cdot f'(s^{(3)}) = \delta^{(3)} \quad (68)$$

$$(a^{(2)})^T \delta^{(3)} = \frac{\partial C}{\partial w^{(2)}} \quad (69)$$

$$\delta^{(3)} \cdot (w^{(2)})^T \cdot f'(s^{(2)}) = \delta^{(2)} \quad (70)$$

$$x^T \delta^{(2)} = \frac{\partial C}{\partial w^{(1)}} \quad (71)$$

$$w^{(1)} = w^{(1)} - \mu \frac{\partial C}{\partial w^{(1)}} + \textit{regularization} \quad (72)$$

$$w^{(2)} = w^{(2)} - \mu \frac{\partial C}{\partial w^{(2)}} + \textit{regularization} \quad (73)$$

$$\delta_1^{(3)}$$

$$\delta_1^{(2)}$$

$$\delta_2^{(2)}$$

$$\delta_3^{(2)}$$

## 8. Labor:

### Regularization

## 9. Labor:

### Support Vector Machine

Logistic regression:

$$h_w(x) = \frac{1}{1 + e^{-Xw}} \quad (74)$$

$$h_w(x) = g(Xw) \quad (75)$$

$$h_w(x) = g(\textcolor{red}{z}) \quad (76)$$

Cost function:

$$C = -(y \cdot \log(h_w(x)) + (1 - y) \cdot \log(1 - h_w(x))) \quad (77)$$

$$C = -y \cdot \log(h_w(x)) - (1 - y) \cdot \log(1 - h_w(x)) \quad (78)$$

If  $y = 1$ , we want  $Xw \geq 1$  (not just  $\geq 0$ )

If  $y = 0$ , we want  $Xw < -1$  (not just  $< 0$ )

$$h_w(x) = \frac{1}{1 + e^{-Xw}} = \frac{1}{1 + e^{-z}} = g(z) \quad (79)$$

$$\text{cost}_1(z)$$

$$\text{cost}_0(z)$$

$$\min_w \frac{1}{m} \left[ \sum_{i=1}^m y^{(i)} \cdot (-\log(h_w(x^{(i)}) + (1 - y^{(i)})) \cdot (-\log(1 - h_w(x^{(i)}))) \right] + \frac{\lambda}{2m} \sum_{j=1}^n w_j^2 \quad (80)$$

$$\min_w \textcolor{red}{C} \left[ \sum_{i=1}^m y^{(i)} \cdot \textcolor{red}{cost}_1(h_w(x^{(i)}) + (1 - y^{(i)})) \cdot \textcolor{red}{cost}_0(1 - h_w(x^{(i)})) \right] + \frac{1}{2} \sum_{j=1}^n w_j^2 \quad (81)$$

## 10. Labor:

### Spam Email

## 11. Labor:

### K-Means

## 12. Labor:

### Principal Component Analysis

## 13. Labor:

### Anomaly Detection

## 14. Labor:

### Recommender System