Mathematical Foundations of Computer Science

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3 Basic Counting

A function $[m] \to [n]$ is *monotone* if $f(1) \le f(2) \le \cdots \le f(m)$. It is *strictly monotone* if $f(1) < f(2) < \cdots < f(m)$.

Exercise 3.1. Find and justify a closed formula for the number of strictly monotone functions from $[m]$ to $[n]$.

Answer The answer is $\binom{n}{m}$.
We can select m different elements in set $[n]$, then sort them in strict increments. Then we get a sequence a_n of n elements, $\forall 1 \le i < n$, $a_i < a_{i+1}$. Define $f(i) = a_i$, and we can see $f(i)$ is a strict monotone function. Therefore, there are $\binom{n}{m}$ different a_n, so there are $\binom{n}{m}$ different functions.

Exercise 3.2. Find and justify a closed formula for the number of monotone functions from $[m]$ to $[n]$.

Answer The answer is $\binom{n+m-1}{n-1}$.
Firstly, we select k different values to construct the value domain A. There are $\binom{n}{k}$ ways.
Secondly, we sort them in strict increments, then we get a sequence, and $\forall 1 \le i < k$, $a_i < a_{i+1}$.
Thirdly, we divide $[n]$ into k consecutive parts, the ith part we define it as

X_i. $\forall x \in X_i$, $f(x) = a_i$. We can see that $f(x)$ is a monotone function. Take a look at the following picture.

Figure 1: Dividing n elements into k parts

We need to divide the points into k consecutive parts that contain at least one point. It's like put k-1 clapboards in the n-1 gaps between two points. So there are $\binom{m-1}{k-1}$ different ways.
Therefore, for $0 < k \leq m$, there are $\binom{n}{k} \times \binom{m-1}{k-1}$ functions. In sum, there are $\sum_{k=1}^{m} \binom{n}{k} \times \binom{m-1}{k-1}$ different functions.
We know $\sum_{k=1}^{m} \binom{n}{k} \times \binom{m-1}{k-1} = \sum_{k=1}^{m} \binom{n}{n-k} \times \binom{m-1}{k-1}$. For $\sum_{k=1}^{m} \binom{n}{n-k} \times \binom{m-1}{k-1}$, we have an interpretation: we choose n-k people among n people, and choose k-1 people among another m-1 people. This is actually to choose n-1 people among n+m-1 people.
So we see $\sum_{k=0}^{m} \binom{n}{n-k} \times \binom{m-1}{k-1} = \binom{n+m-1}{n-1}$, which is the answer to the question.

Exercise 3.3. Prove that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$ for every $n \geq 0$ by finding a combinatorial interpretation.

Answer Suppose that we choose n items from $2n$ items. There are $\binom{2n}{n}$ situations. Another interpretation of choosing is that we first divide $2n$ items equally into two n elements item set. Then we choose k items from the first set and $n-k$ items from the second one. There are $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$ situations. Since $\binom{n}{k} = \binom{n}{n-k}$, $\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$.

Exercise 3.4. [From the textbook] Find a closed formula for $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k}$ and prove it combinatorially, i.e., by giving an interpretation.

Answer The equation: $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k}$ stand for all of the situation that we first choose k from n and then choose m from k.
We can list another situation: we first choose m from n, then the remain n-m menbers have two cases: be chosen to k or not. So the total status's number is 2^{n-m}. So there are $\binom{n}{m}2^{n-m}$ and we can achieve that:
$\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k} = \binom{n}{m}2^{n-m}$

Exercise 3.5. Let B_n be the number of partitions of the set $[n]$ (this is the same as the number of equivalence relations on $[n]$). This is called the Bell number, thus we denote it B_n. Prove that the following recursive formula for B_n is correct:

$$\begin{aligned} B_0 &= 1 \\ B_{n+1} &= \sum_{k=0}^{n} \binom{n}{k} B_k \ . \end{aligned}$$

Proof. Choose $n-k$ elements in $[n]$ to be in a same subset with $n+1$, then the remaining k elements have B_k partitions. So sum up these partition numbers of different choices of k, we get the formula:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{n-k} B_k = \sum_{k=0}^{n} \binom{n}{k} B_k$$

□

Exercise 3.6. Let P_n be the number of ways to write the natural number n as a sum $a_1 + a_2 + \cdots + a_k$ such that $1 \le a_1 \le a_2 \le \cdots \le a_k$. For example, 3 can be written as 3, $2+1$, and $1+1+1$, so $P_3 = 3$. Find a recursive formula for P_n.

Answer Let $f(n,m)$ be the number of ways to write the natural number n as a sum $a_1 + a_2 + \cdots + a_k$ such that $1 \le a_1 \le a_2 \le \cdots \le a_k \le m$. So $P_n = f(n,n)$.

For $f(n,m)$, it is obviously that $1 \le a_k \le m, n$. So $f(n,1) = 1$, and $f(n,m) = f(n,n)$ if $m > n$. If $1 < m < n$, the split of n depends on whether $a_k = m$. If $a_k = m$, the number of plans equals to $f(n-m,m)$, which means $a_1 + a_2 + \cdots + a_{k-1} = n - m$ such that $a_{k-1} \le m$. If $a_k \ne m$, the number of plans equals to $f(n, m-1)$, which means $a_1 + a_2 + \cdots + a_k = n$ such that $a_k \le m$.

In summary, $P_n = f(n,n)$.

$$f(n,m) = \begin{cases} 1 & m = 1 \\ f(n,n) & m > n \\ f(n-m,m) + f(n,m-1) & 1 < m < n \\ 1 + f(n,m-1) & n > 1, m = n \end{cases}$$

4 Questions

4.1

Since we do Exercise 3.6 by finding out a recurrence relation, we want to know whether it has an one-dimensional recursion relation. Furthermore, can it be solved by giving a general formula?