Mathematical Foundations
of
Computer Science

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Group Name: **Mogicians**

2 Partial Orderings

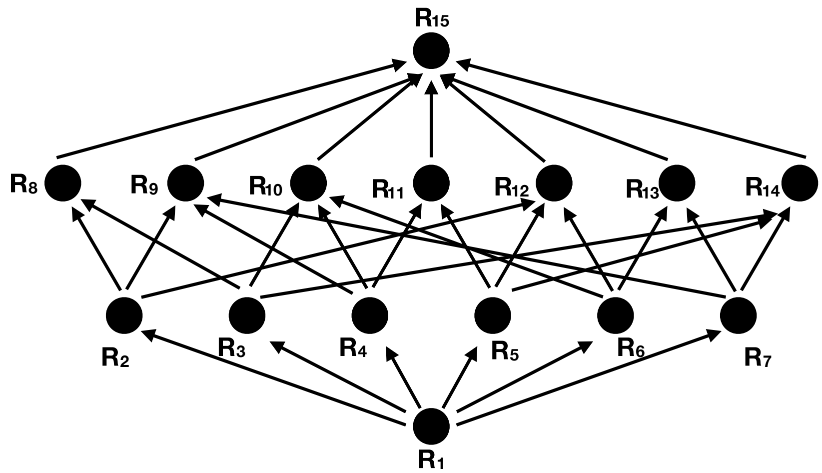
2.1 Equivalence Relations as a Partial Ordering

Exercise 2.1. Let E_4 be the set of all equivalence relations on $\{1, 2, 3, 4\}$. Note that E_4 is ordered by set inclusion, i.e.,

$$(E_4, \{(R_1, R_2) \in E_4 \times E_4 | R_1 \subset R_2\}) \tag{1}$$

is a partial ordering.
1. Draw the Hasse diagram of this partial ordering in a nice way.
2. What is the size of the largest chain?
3. What is the size of the largest antichain?

Proof. 1. The Hasse diagram of this partial ordering. (We use [ ] to stand for a full relation between the element in it.)
$R_1 = [[1][2][3][4]], R_2 = [[12][3][4]], R_3 = [[34][1][2]], R_4 = [[13][2][4]], R_5 = [[24][1][3]], R_6 = [[14][2][3]], R_7 = [[23][1][4]], R_8 = [[12][34]], R_9 = [[123][4]], R_{10} = [[134][2]], R_{11} = [[13][24]], R_{12} = [[124][3]], R_{13} = [[14][23]], R_{14} = [[1][234]], R_{15} = [[1234]]$
2. The size of the largest chain is 4.
3. The size of the largest antichain is 7. $\square$



2.2 Chains and Antichains

Define the partially ordered set $(\mathbb{N}_0^n, \leq)$ as follows: $x \leq y$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. For example, $(2,5,4) \leq (2,6,6)$ but $(2,5,4) \not\leq (3,1,1)$.

Exercise 2.2. Consider the infinite partially ordered set $(\mathbb{N}_0^n, \leq)$.

1. Which elements are minimal? Which are maximal?
2. Is there a minimum? A maximum?
3. Does it have an infinite chain?
4. Does it have arbitrarily large antichains? That is, can you find an antichain A of size $|A| = k$ for every $k \in \mathbb{N}$?

Answer and Proof

1. The minimal element is n-dimension zero vector $x = (0, 0, \ldots, 0)$. There's no maximal element.

   Proof. If $\exists y \in S : \ y \leq x$, then $\forall i \in [n], 0 \leq y_i \leq x_i = 0$, so $y = x$. As a result, $\nexists y \in S : y < x$.

   For $\forall y = (y_1, y_2, \ldots, y_n) \in S, \exists z = (y_1 + 1, y2 + 1, \ldots, y_n + 1) : y \leq z$, so there's no maximal element. $\square$

2. Similarly, the minimum is n-dimension zero vector $x = (0, 0, \ldots, 0)$. There's no maximum.

   Proof. Because of the definition, $\forall y \in S, \forall i \in [n], y_i \geq 0 = x_i$, then $x \leq y$. So x is the minimum.

   For $\forall y = (y_1, y_2, \ldots, y_n) \in S, \exists z = (y_1 + 1, y2 + 1, \ldots, y_n + 1) : y \leq z$, so there's no maximum. $\square$

3. Yes. For example, let $a_i = (i, i, \ldots, i) \in \mathbb{N}_0^n$, then $\{a_1, a_2, \ldots\}$ is a infinite chain. (Because obviously $a_i \leq a_{i+1}$)

4. Yes. For example, let $a_i = (i, k - i, 0, 0, \ldots, 0) \in \mathbb{N}_0^n,\ i = 1, 2, \ldots, k$, then $A = \{a_1, a_2, \ldots, a_k\}$ is an antichain of size k. (Because $\forall a_i, a_j \in A$, if $i < j$, then $k - i > k - j$, they're incomparable.)

***Exercise 2.3.** *Does every infinite subset $S \subseteq \mathbb{N}_0^n$ contain an infinite chain?*

The answer is yes.

Lemma Let U, V be infinite sets with total ordering relation $(U, \leq), (V, \leq)$. Define $X = U \times V$ and partial ordering $(X, \{((u_1, v_1), (u_2, v_2)) \mid u_1 \leq u_2\ v1 \leq v2,\ (u_1, v_1), (u_2, v_2) \in X\})$: every infinite subset $S \subseteq X$ contain an infinite chain A.

Proof. Because U, V are totally ordered, $\forall p_0 = (x_0, y_0) \in S$,we can divide the set S into 4 parts based on p_0:

$$A^{(1)} = \{(x, y) | x_0 \leq x, y_0 \leq y\} \cap S$$

$$A^{(2)} = \{(x, y) | x < x_0, y_0 \leq y\} \cap S$$
$$A^{(3)} = \{(x, y) | x < x_0, y < y_0\} \cap S$$
$$A^{(4)} = \{(x, y) | x \leq x_0, y < y_0\} \cap S$$

To visualize the concept of this division, we take $U = V = \mathbb{N}$ for example(Figure 1)

Note that every element in $A^{(1)}$ is greater than p_0 and every element in $A^{(3)}$ is less than p_0.

Then we assume that there's no infinite chain in S, then we can find:

1. $|A^{(3)}| < (x_0 - 1)(y_0 - 1)$, so it's an finite set.

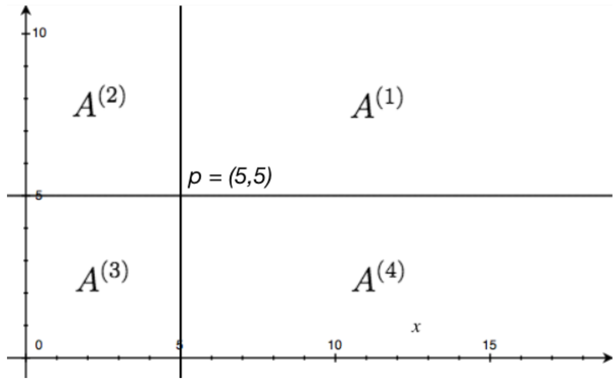


Figure 1: The division of S

2. For $A^{(2)}$, define:

$$A_i^{(2)} = \{(x, y) \mid x = i,\ (x, y) \in A^{(2)}\},\ i = 1, 2, \dot{s}, x_0 - 1$$

Because the elements in each $A_i^{(2)}$ is equal on the x-axis, so the all elements in each $A_i^{(2)}$ can form a chain. Considering the assumption that there's no infinite chain in S, $A_i^{(2)}$ are all finite sets. So $A_i^{(2)}$ is finite.Similarly, $A^{(4)}$ is a finite set.

So we have a conclusion that $A^{(1)}$ is an infinite set, i.e. $\forall p \in S,\ \exists p' \in S,\ s.t.\ p < p'$. With this conclusion, we can construct an infinite chain $(p_0, p_1, p_2, \dots)$ because you can always find a greater element p_{i+1} in the $A^{(1)}$ set of p_i, which contradicts to the assumption.

So in conclusion, there are always infinite chains in S. $\square$

Theorem Every infinite subset $S \subseteq \mathbb{N}_0^n$ contain an infinite chain.

Proof. 1. Let $X = \mathbb{N}_0^2 = \mathbb{N} \times \mathbb{N}$ and directly apply the lemma, we can find the theorem is valid.

2. If the theorem is valid on $X_k = \mathbb{N}_0^k = \mathbb{N}_0 \times \mathbb{N}_0^{k-1}$, i.e. for every infinite subset S of X_k, ther's always an infinite chain A.

Then talk about $X_{k+1} = \mathbb{N}_0^{k+1} = \mathbb{N}_0 \times \mathbb{N}_0^k = \mathbb{N}_0 \times X_k$.

For all infinite set $S \subseteq X_{k+1}$, we assume that there's no infinite chain in S. Define:

$$T = \{y \mid (x, y) \in S\} \subseteq X_k$$

$$T_i = \{(x, y) \mid y = i\}, \forall i \in T$$

$$|S| = \sum_{i \in T} |T_i|$$

T_i are all finite set, or there would be an infinite chain composed of all elements in T_i which contradicts to the assumption. So T is an infinite set because S is infinite.

Therefore, T has an infinite chain A because it's a subset of X_k. Note that A is total ordered. Define:

$$S' = S \cap (\mathbb{N}_0 \times A) = \bigcup_{i \in A} T_i \ \subseteq \mathbb{N}_0 \times A$$

S' is infinite because A is infinite too. Considering the linearity of $\mathbb{N}_0$ and A, with the lemma, we can conclude that S' has an infinite chain A',which is also a chain of S.

By induction, we come to the conclusion that the theorem is valid to all $\mathbb{N}_0^n$.

$\square$

Exercise 2.4. Show that $(\mathbb{N}_0^n, \leq)$ has no infinite antichain. **Hint.** Use the previous exercise.

Proof. If there's an infinite antichain A, it is also an infinite subset of $\mathbb{N}_0^n$, we can conclude that A has an infinite chain using the result of exercise 2.3, which contradicts to the assumption. So $(\mathbb{N}_0^n, \leq)$ has no infinite antichain. $\square$

Consider the induced ordering on $\{0,1\}^n$. That is, for $x, y \in \{0,1\}^n$ we have $x \leq y$ if $x_i \leq y_i$ for every coordinate $i \in [n]$.

Exercise 2.5. Draw the Hasse diagrams of $(\{0,1\}^n, \leq)$ for $n = 2, 3$.

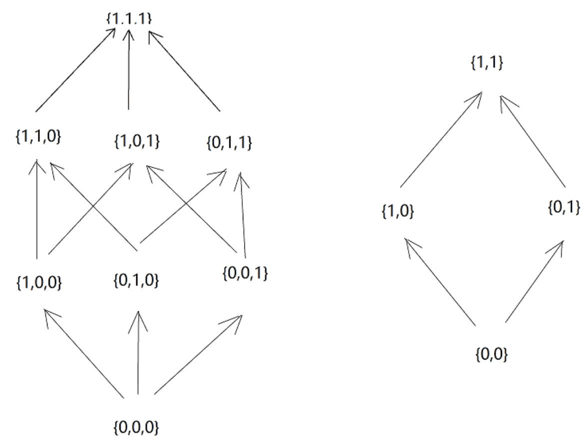


Figure 2: the Hasse diagrams

Exercise 2.6. Determine the maximum, minimum, maximal, and minimal elements of $\{0,1\}^n$.

Answer The maximum element and the maximal element of $\{0,1\}^n$ are both $\{\underbrace{1,1,\cdots 1}_{\text{n ones}}\}$.
The minimum element and the minimal element of $\{0,1\}^n$ are both $\{\underbrace{0,0,\cdots 0}_{\text{n ones}}\}$.

Exercise 2.7. What is the longest chain of $\{0,1\}^n$?

Answer The longest chain of $\{0,1\}^n$ is that the beginning of the chain is $\{\underbrace{0,0,\cdots 0}_{\text{n ones}}\}$, the tail of the chain is $\{\underbrace{1,1,\cdots 1}_{\text{n ones}}\}$, every node of the chain will change a 0 into 1 compared with the immediate predecessor. The length of this chain is n+1.

****Exercise 2.8.** *What is the largest antichain of $\{0,1\}^n$?*

Proof. The largest antichain is the set of all the elements with [n/2] 1s. The size is $C_n^{[n/2]}$. The proof is as follows.

Here we introduce some kind of layer structure to the question.

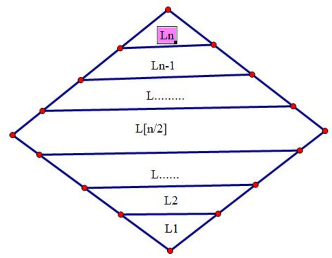


Figure 3: Layer Structure

We define that L_0 is the minimal elements of $\{0,1\}^n$, the all-0 vector. Then L_i is then minimal elements of $\{0,1\}^n - \sum_{j=0}^{i-1} L_j$. Apparently, L_i is the set of vectors containing i 1s. Therefore, $L_{[n/2]}$ is the set of vectors containing $[n/2]$ 1s, which is an antichain, and $|L_{[n/2]}| = C_n^{[n/2]}$.

Then, we will prove $L_{[n/2]}$ is the largest antichain.

Now, we assume that there exists an antichain A, and $|A| > |L_{[n/2]}|$. Apparently, $\exists x \in A, x \notin L_{[n/2]}$.

For $k \leq [n/2]$, we define $A_k = A \cap L_k$ and $B_{k+1} = \{x | x \in L_{k+1}; \exists y \in A_k, x \geq y\}$.

Now we assume $|A| = a$, and k is the smallest k that $|A_k| \neq 0$. For any element $x \in A_k$, there are n-k elements $y \in B_{k+1}$ and $y \geq x$. For any element $y \in B_{k+1}$, there are k+1 elements $x \in A_k$ and $x \leq y$. Therefore, the least possible $|B_{k+1}| = a * (n-k)/(k+1)$. Because $k \leq [n/2], k+1 \leq n-k$. Therefore, $|B_{k+1}| \geq |A_k|$.

Now we can create a new set $A' = (A - A_k) \cap B_{k+1}$, $|A'| \geq |A|$. Because A is an antichain, $\forall x \in A_k$, $\forall y \geq x$, $y \notin A$. Therefore, $\forall x \in B_{k+1}$, $\forall y > x$, because $\exists z \in A_k$, $z < x < y$, $y \notin A$; $\forall y < x$, because $y \in L_j$ and $j < k$, since $|A_j| = 0$, $y \notin A$. So the set A' turns out to be a bigger antichain, and $\forall j \leq k$, $|A'_j| = 0$.

The way to create a bigger antichain is also true when $k \geq [n/2]$, and the only thing we need to do is to change k+1 to k-1.

Then we can find an algorithm to create a bigger antichain based on A, which goes as follows:

```
set makeBigger(set X):
  for i=1 to [n/2]-1
    X = (A - A_k) ∩ B_{k+1}
  for i=n to [n/2]+1
    X = (A - A_k) ∩ B_{k-1}
  return X
```

According to the description, after every loop, we get a new antichain, with a larger or unchanged cardinality.

Define A'=makeBigger(A). We can see A' is an antichain, $|A'| \geq |A|$, and $A' \subseteq L_{[n/2]}$. Then we find $|A| \leq |A'| \leq |L_{[n/2]}|$. However, in the assumption, $|A| > |L_{[n/2]}|$). Here comes a contradiction, so the assumption doesn't make sense. In other words, there is no larger antichain than $L_{n/2}$.

In conclusion, the largest antichain is the set of all the elements with [n/2] 1s. The size is $C_n^{[n/2]}$. $\square$

2.3 Infinite Sets

In the lecture (and the lecture notes) we have showed that $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$, i.e., there is a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. From this, and by induction, it follows quite easily that $\mathbb{N}^k \cong \mathbb{N}$ for every k.

Exercise 2.9. Consider $\mathbb{N}^*$, the set of all finite sequences of natural numbers, that is, $\mathbb{N}^* = \{\epsilon\} \cup \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \cup \ldots$. Here, ϵ is the empty sequence. Show that c by defining a bijection $\mathbb{N} \to \mathbb{N}^*$.

Proof. We can sort the elements in $\mathbb{N}^*$ by the sum of value. For example, (1), (1,1), (2), (1,1,1), (1,2), (2,1), (3), (1,1,1,1), (1,1,2), (1,3), (2,1,1), $\cdots$ If two elements have the same sum, we sort them by lexicographic order. In this way, each element in $\mathbb{N}^*$ can be mapped to a unique natural number, and each natural number can be mapped to a unique element in $\mathbb{N}^*$. So $\mathbb{N} \to \mathbb{N}^*$ is a bijection, $\mathbb{N} \cong \mathbb{N}^*$. $\square$

Exercise 2.10. Show that $\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$. **Hint:** Use the fact that $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$ and thus show that $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

Proof. For every element in $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$, let it be $r_1 \times r_2$, Define:

$r_1 = (a_1, a_2, \cdots, a_n, \cdots)$, $r_2 = (b_1, b_2, \cdots, b_n, \cdots)$, $a_i, b_i = 0$ or 1

$r = (a_1, b_1, a_2, b_2, \cdots, a_n, b_n, \cdots)$

It is easy to proof that each element $r_1 \times r_2$ in $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ can be mapped to an element r in $\{0,1\}^{\mathbb{N}}$, and different $r_1 \times r_2$ is mapped to different r. So $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.Combine $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$, we get the conclusion $\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$. $\square$

Exercise 2.11. Consider $\mathbb{R}^{\mathbb{N}}$, the set of all infinite sequences $(r_1, r_2, r_3, \ldots)$ of real numbers. Show that $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. **Hint:** Again, use the fact that $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$.

Proof. Known that $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$, we suppose that $x \in \mathbb{R}^{\mathbb{N}}$, $x \to \begin{bmatrix} r_1 \\ r_2 \\ \cdots \\ r_n \\ \cdots \end{bmatrix}$, where $r_i = (a_{i1}, a_{i2}, \cdots, a_{in}, \cdots)$, $a_{ij} = 0$ or 1. Then $x \to \begin{bmatrix} a_{11}, a_{12}, \cdots, a_{1n}, \cdots \\ a_{21}, a_{22}, \cdots, a_{2n}, \cdots \\ \cdots \\ a_{n1}, a_{n2}, \cdots, a_{nn}, \cdots \\ \cdots \end{bmatrix}$.

Define: $r = (a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13} \cdots, a_{n+1\,n-1}, a_{nn}, a_{n-1\,n+1}, \cdots)$. It is clear that $x \to r$.

In this way,we can proof that each element in $\mathbb{R}^{\mathbb{N}}$ can be mapped to a unique element r in $\{0,1\}^{\mathbb{N}}$, and the converse is also true. So $\mathbb{R}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$. Combine $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$, we get the conclusion $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$. $\square$

Next, let us view $\{0,1\}^{\mathbb{N}}$ as a partial ordering: given two elements $\mathbf{a},\mathbf{b} \in \{0,1\}^{\mathbb{N}}$, that is, sequences $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$, we define $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for all $i \in \mathbb{N}$. Clearly, $(0,0,\dots)$ is the minimum element in this ordering and $(1,1,\dots)$ the maximum.

Exercise 2.12. Give a countably infinite chain in $\{0,1\}^{\mathbb{N}}$. Remember that a set A is countably infinite if $A \cong \mathbb{N}$.

Answer Construct a map between $\mathbb{N}$ and $\{0,1\}^{\mathbb{N}}$:

$$f(n) = (\underbrace{1,1,\dots,1}_{\text{k ones}}, 0, 0 \dots)$$

We can find that $\forall i < j,\ f(i) < f(j)$.(Equal at the first i positions and after the j^{th} position, less at the i^{th} to j^{th}). So $A = \{f(1),\ f(2),\ \dots\}$ is an infinite chain.

Because f is a bijection between $\mathbb{N}$ and A, so A is countable.

Exercise 2.13. Find a countably infinite antichain in $\{0,1\}^{\mathbb{N}}$.

Answer Construct a map between $\mathbb{N}$ and $\{0,1\}^{\mathbb{N}}$:

$$f(n) = (0, 0, \dots, 0, \underbrace{1}_{k^{th}\text{ position}}, 0, 0 \dots)$$

We can find that $\forall i \neq j,\ f(i)$ and $f(j)$ are incomparable.(Greater at the i^{th} position and less at the j^{th}). So $A = \{f(1),\ f(2),\ \dots\}$ is an infinite antichain.

Similarly, because f is a bijection between $\mathbb{N}$ and A, so A is countable.

Exercise 2.14. Find an uncountable antichain in $\{0,1\}^{\mathbb{N}}$. That is, an antichain A with $A \cong \mathbb{R}$.

Answer First, we find a function F(x) whose definition domain is $\{0,1\}^{\mathbb{N}}$. To describe the function, we make $x_{i,j}$ denote the ith and jth bits of infinite bit sequence x. Then F(x) can be described as follows.

$$F(x)_{2i-1,2i} = \begin{cases} 01 & x_i = 0, i \geq 1 \\ 10 & x_i = 1, i \geq 1 \end{cases} \tag{2}$$

Let set A be the value domain of F(x). Since the function above is a bijection obviously, we can say $A \cong \{0,1\}^{\mathbb{N}}$. And according to the definition, all the elements of A share the same number of 1s, which means they can't compare with each other. In other words, A is an uncountable antichain.

****Exercise 2.15.** *Find an uncountable chain in $\{0,1\}^{\mathbb{N}}$. That is, an antichain A with $A \cong \mathbb{R}$.*

Answer First, we construct a function G(x) whose definition domain is $\{0,1\}^{\mathbb{N}}$.
To describe the function, we make $x_{(i,j)}$ denote the ith to the jth bits of infinite bit sequence x. Define ToTen(y) is a function to transfer a binary number y to decimal. Then G(x) can be described as follows.

$$G(x)_{(2^i,2^{i+1}-1)} = \underbrace{1,1,1,1,1....1,1,1}_{ToTen(x_{(1,i)})\ 1s}\underbrace{0,0,0,0,0....0,0,0}_{(2^i - ToTen(x_{(1,i)}))\ 0s} \tag{3}$$

Let set A be the value domain of G(x).
Obviously, G(x) is a bijective function from $\{0,1\}^{\mathbb{N}}$ to A. Therefore, $A \cong \{0,1\}^{\mathbb{N}} \cong \mathbb{R}$. In other words, A is uncountable.
Given two bit sequences a and b, define $a \preceq b$ if $a \leq b$ as binary numbers. For $(\{0,1\}^{\mathbb{N}}, \preceq)$, $\{0,1\}^{\mathbb{N}}$ is obviously a chain.
Given two elements $x1, x2 \in \{0,1\}^{\mathbb{N}}$. If $x1 \preceq x2$, $\forall i \in \mathbb{N}_0^n, x1_{(1,i)} \preceq x2_{(1,i)}$.
According to the definition of G(x), if $x1_{(1,i)} \preceq x2_{(1,i)}$, $G(x1)_{(2^i,2^{i+1}-1)} \leq G(x2)_{(2^i,2^{i+1}-1)}$. This means when $x1 \preceq x2$, $G(x1) \leq G(x2)$.
So $\forall x1, x2 \in \{0,1\}^{\mathbb{N}}$, because x1 and x2 are comparable for $\preceq$, G(x1) and G(x2) are comparable for $\leq$. Due to the fact that G(x) is a bijective function, A is a chain for $\leq$.
In conclusion, A is an uncountable chain.

3 Questions

3.1

We have almost no idea on exercise 2.15,

3.2

(A generalization of exercise 2.1 2.3)

Let E_4 be the set of all equivalence relations on $\{1, 2, \ldots, n\}$. Note that E_4 is ordered by set inclusion, i.e.,

$$(E_4, \{(R_1, R_2) \in E_4 \times E_4 | R_1 \subset R_2\}) \tag{4}$$

is a partial ordering.

Q1 What's the size of the largest chain?

Conjecture n

***Q2** What's the size of the largest anti-chain?