Mathematical Foundations
of
Computer Science

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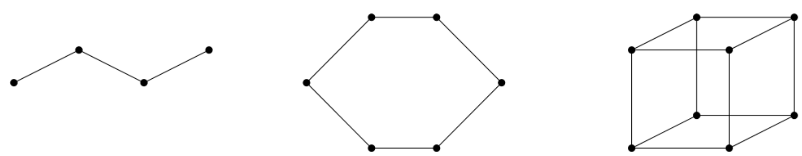
Group Mogicians

Exercise 5.1. For each of the graphs below, compute the number of automorphisms it has.



Justify your answer.

Answer The first graph's number of automorphisms is 2. Because only the symmetrical function and itself belong to the automorphisms.

The second graph's number of automorphisms is 12. The number of automorphisms include the graph itself(1), the symmetrical function(1), the function that a vertex point to the other vertex(5) and the symmertrical function(5).

The third graph's number of automorphisms is 48 by the formula of 5-2.

Exercise 5.2. Show that H_n has exactly $2^n \cdot n!$ automorphisms. Be careful: it is easy to construct $2^n \cdot n!$ different automorphisms. It is more difficult to show that there are no automorphisms other than those.

Proof. According to the definition of H_n, we can observe a layer structure that exists in H_n.
Pick a certain element $x_0 \in H_n$. Define $dis(x, y)$ as the number of different bits between x and y. Then L_i can be defined as the set made up of all the

elements x that $dis(x_0, x) = i$, $0 \le i \le n$.
Now there are some apparent observations.
First, $|H_n| = 2^n$. Second, $\forall x, y \in H_n$, only when $x \in L_i$, $y \in L_j$, and $|i - j| = 1$, there can possibly appear an edge between x and y.
Then we take a look at the automorphisms.
First, x_0 can be any element of H_n. So for x_0, there are 2^n different ways to choose.
Then, when x_0 is determined, there are n elements in L_1. And when it comes to construction of a automorphism, we can put them in any order. So there are $n!$ different ways to construct L_1.
Therefore, there are at least $2^n \cdot n!$ different automorphisms in all.
Now we represent automorphisms with function $F_(x)$. Here $F(x)$ is a function from any other graph to the graph which we structure with layers.
If there are more automorphisms, we can say that there exists two functions $F_1(x)$ and $F_2(x)$, and they go as follows.

$$\forall F_1(x_1) = F_2(x_2) \in L_0 \cup L_1\ , x_1 = x_2 . \exists F_1(x) \neq F_2(x). \tag{1}$$

There are two possible sub-situations.
The first is:

$$\exists F_1(x_1) \neq F_2(x_1),\ F_1(x_1) \in L_i,\ F_2(x_1) \in L_j,\ ,i \neq j. \tag{2}$$

Assume $F_1(x') = F_2(x') = x_0$. $F_1(x_1) \in L_i$ means $dis(x', x_i) = i$. Also $F_2(x_1) \in L_j$ means $dis(x', x_i) = j$. Since $i \neq j$, here comes a contradiction. So this sub-situation is impossible.
The second is:

$$\exists F_1(x_1) \neq F_2(x_1),\ F_1(x_1), F_2(x_1) \in L_i. \tag{3}$$

Assume k is the smallest number that meets:

$$\exists F_1(x_1) \neq F_2(x_1),\ F_1(x_1), F_2(x_1) \in L_k. \tag{4}$$

Then we know:

$$\forall i < k,\ if\ F_1(x) \in L_i,\ F_2(x) = F_1(x). \tag{5}$$

For x_1, there are k elements y that $F_1(y) = F_2(y) \in L_{k-1}$ and $dis(F_1(x_1), F_1(y)) = 1$, $dis(F_2(x_1), F_2(y)) = 1$.
Make these $F_1(y)$s form a set S_1, and the $F_2(y)$s form a set S_2. According to equation 5, $S_1 = S_2$.

Assume $F_1(x') = F_2(x') = x_0$. Make set Q_1 contains all the is that the ith bits of $F_1(x_1)$ and x_0 are different. Also define Q_2 for $F_2(x)$.
From equation 4, we see that $Q_1 \neq Q_2$.
With the same rule, we can define a set for every element y in S_1 and S_2. Name it as Q_y.
We can get some equations.

$$Q_1 = \bigcup_{y \in S_1} Q_y \tag{6}$$

$$Q_2 = \bigcup_{y \in S_2} Q_y \tag{7}$$

Since $S_1 = S_2$, we can see $Q_1 = Q_2$, which contradicts with equation 4.
Therefore, there exists no k that meets equation 4, this sub-situation is also impossible.
In conclusion, H_n has exactly $2^n \cdot n!$ automorphisms. $\square$

Exercise 5.3. Give an example of an asymmetric graph on six vertices.

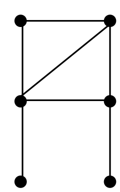


Figure 1: an asymmetric graph on six vertices

Exercise 5.4. Find an asymmetric tree.

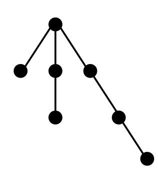


Figure 2: an asymmetric tree

Exercise 5.5. Show that there is no self-complementary graph on 999 vertices.

Proof. Since a self-complementary graph G and $\overline{G}$ has the same number of edges, for the graph $G+\overline{G} = \left(V, \binom{V}{2}\right)$, $|E| = \binom{V}{2}$ should be an even number. However, if $|V| = 999$, then $\binom{V}{2} = \frac{999\times998}{2}$, which is an odd number. So there is no self-complementary graph on 999 vertices. □

Exercise 5.6. Characterize the natural numbers n for which there is a self-complementary graph G on n vertices.

In order to make it possible to divide the graph into two isomorphic parts, the number of edges must be even: $\frac{n(n-1)}{2} = 2k, k \in \mathbb{N}$, so $n = 4k, 4k+1,\ k \in \mathbb{N}$(which is the necessary condition). We'll construct a self-conplementary subgraph and an isomorphism for each situation (to prove that this is also the sufficient condition).

1. $n = 4k$

First we number the verticles from 1 to n, and define a bijection from the integer addictive group modulo n to itself $f : \mathbb{N}_n \to \mathbb{N}_n,\ f(x) = x+1$(note that it's modulo-n addition).

Then we design a dividing strategy of K_n: we group the edges of K_n according to its "span", i.e. $S^{(i)} = \{(k, k+i) |\ k \in [n]\},\ i = 1, 2, \ldots, \frac{n}{2}$. Note that $\forall i < \frac{n}{2},\ |S^{(i)}| = n$, and $|S^{(\frac{n}{2})}| = \frac{n}{2}$. Furthermore, we divide each $S^{(i)}$ into two parts:

$$S^{(i)}_{even} = \{(k, k+i) |\ k \in [n], 2|k\}$$
$$S^{(i)}_{odd} = \{(k, k+i) |\ k \in [n], 2|k+1\}$$

Because the cardinality of each $S^{(i)}$ is even, so $\forall i,\ |S^{(i)}_{even}| = |S^{(i)}_{odd}|$. $f(x)$ is a bijection from $S^{(i)}_{even}$ to $S^{(i)}_{odd}$ and $\forall (s, s+i) \in S^{(i)}_{even},\ (f(s), f(t)) = (s+1, s+i+1) \in S^{(i)}_{odd}$ (Figures below are examples of n=4 and n=8, where blue/green lines represent odd/even edges).

So in conclusion,

$$E_1' \triangleq \bigcup_{i=1}^{\frac{n}{2}} S^{(i)}_{even}$$

$$E_2' \triangleq \bigcup_{i=1}^{\frac{n}{2}} S^{(i)}_{odd}$$

.(Actually if you rotate one of them a little bit in the figure, you'll find that the two subgraph are identical in shape.)

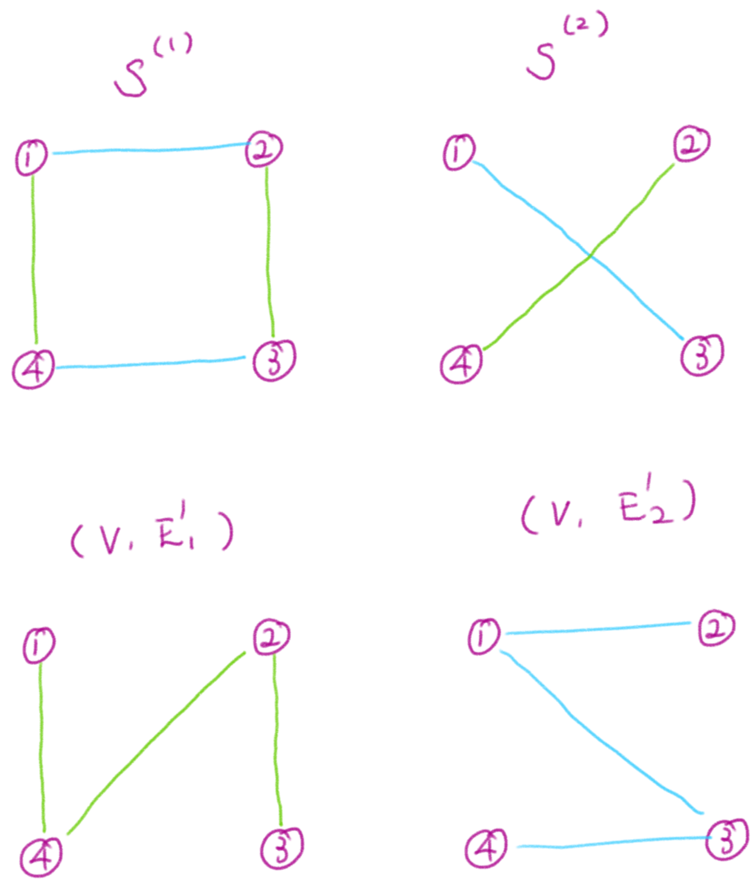


Figure 3: The Division of n=4

(V, E_1') and (V, E_2') are isomorphic and f is the isomorphism, while $(V, E_1') \cup (V, E_2') = G$, so they're both self-complementary.

2. $n = 4k + 1$

For each $G = (V, E)$ such that $n = 4k$, we can divide V into two parts: $V_1 = \{v \mid 2|deg(v)\}$ and $V_2 = \{v \mid 2|(deg(v) + 1)\}$. Then each $v \in V_1$ has an odd degree in $\overline{G}$, and each $v \in V_2$ has an even degree in $\overline{G}$.

If $n = 4k + 1$, we can construct a self-complementary graph $G = (V, E)$ on $4k$ vertices $v_1, v_2, \cdots, v_{4k}$, and divide V into V_1 and V_2 by the above method. Then if $v \in V_1$, $f(v) \in V_2$, and if $v \in V_2$, $f(v) \in V_1$. It is obvious that $G' = (V', E')$ such that $V' = V \cup \{v_{4k+1}\}, E' = E \cup \{(u, v_{4k+1}) \mid u \in V_1\}$

is a self-complementary graph and $f(v_{4k+1}) = v_{4k+1}$, because $(u, v_{4k+1}) \in E' \Leftrightarrow u \in V_1 \Leftrightarrow f(u) \in V_2 \Leftrightarrow (f(u), f(v_{4k+1})) \in E''$ (Suppose that overline(G')=(V",E")).

Above all, There is a self-complementary graph on n vertices if and only if $n = 4k$ or $n = 4k + 1$, such that $k \in \mathbb{N}$.

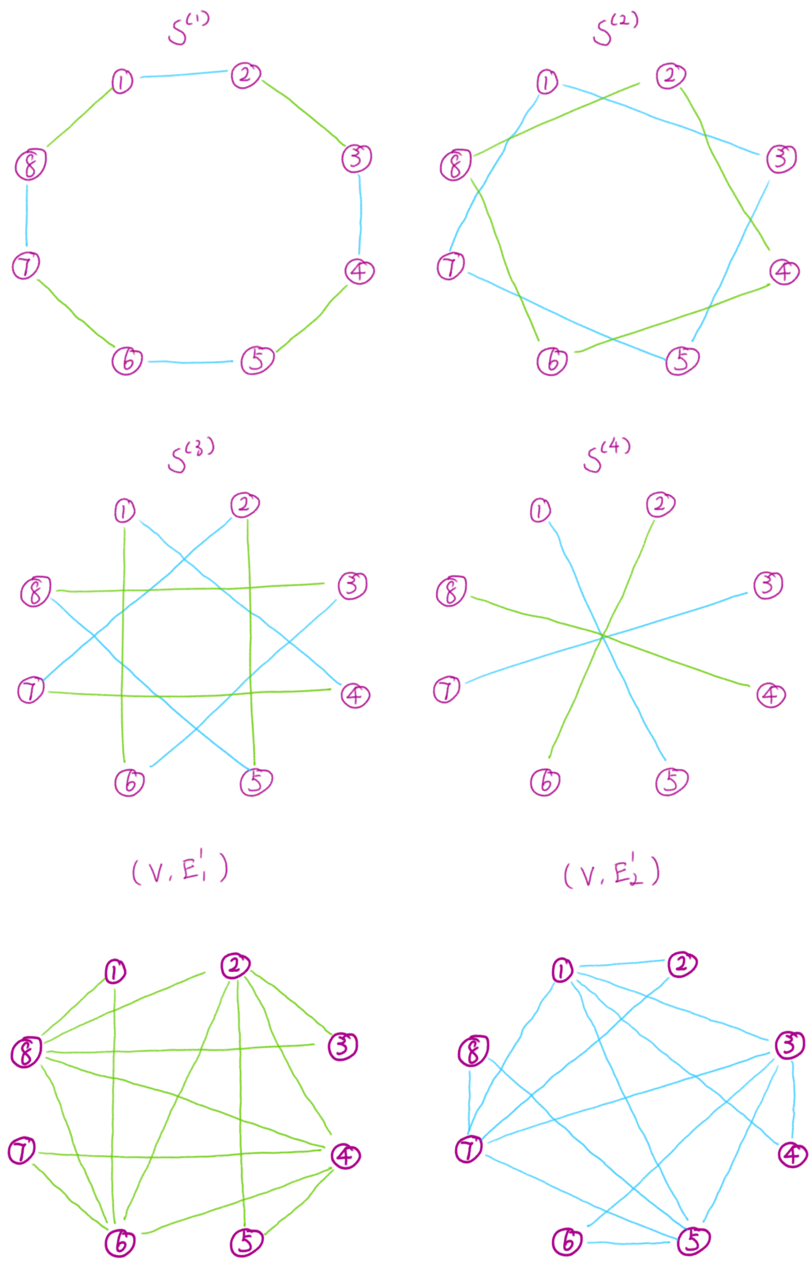


Figure 4: The Division of n=8