Mathematical Foundations of Computer Science

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7 The Graph Score Theorem

- Homework assignment published on Monday, 2018-04-09.
- Submit questions and first solution by Sunday, 2018-04-15, 12:00 by email to dominik.scheder@gmail.com and the TAs.
- You will receive feedback by Wednesday, 2018-04-18.
- Submit your final solution by Sunday, 2018-04-22 to me and the two TAs.

Exercise 7.1. Describe, in simple sentences with a minimum of mathematical formalism, (1) the score of a graph, (2) what the graph score theorem is, (3) the idea of the graph score algorithm, (4) where the difficult part of its proof is. Imagine you have a friend who does not take this class, and think about how to answer the above questions to them.

1. the score of a graph: a sequence of numbers that describes the degree of each vertice.
2. the graph score theorem: Let $\mathbf{d} = (d_1, \dots, d_n)$ with $d_1 \leq \cdots \leq d_n$. Define $\mathbf{d}'$ by

$$\mathbf{d}'_i = \begin{cases} d_i - 1 & for \quad i = n - d_n, \ldots, n-1, n \\ d_i & for \quad i = 1, \ldots, n - d_n - 1 \end{cases}$$

Then there exist a graph with score $\mathbf{d}$ if and only if there exists a graph with score $\mathbf{d}'$. Furthermore, if $n = 1$, then there exists a graph with score d_1 if and only if $d_1 = 1$.

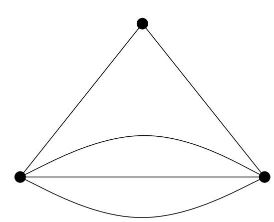
3. The graph score algorithm: a way to find out whether a group of numbers can be the graph scores of a graph.

4. Difficulties: The reverse implication is difficult, especially the proof of the claim that if there exist a graph G with the score $(d_1, d_2, \ldots, d_n)$, then there exist another graph G with the score $(d_1, d_2, \ldots, d_n)$, in which the n^{th} vertice has edges to the $(n - d_n)^{th}, (n - d_n - 1)^{th}, \ldots, (n-1)^{th}$ vertice.

7.1 Alternative Graphs

Now we will look at different notions of graphs. As defined in class and in the video lectures, a graph is a pair $G = (V, E)$ where V is a (usually finite) set, called the *vertices*, and $E \subseteq \binom{V}{2}$, called the set of *edges*.

Multigraphs. A *multigraph* is like a graph, but you can have several parallel edges between two vertices. You cannot, however, have self-loops. That is, there cannot be an edge from u to u itself. This is an example of a multigraph:



We can define degree and score for multigraphs, too. For example, this multigraph has score $(4, 4, 2)$. Obviously no graph can have this score.

Exercise 7.2. State a score theorem for multigraphs. **Remark.** This is actually simpler than for graphs.

Theorem 7.3 (Multigraph Score Theorem)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{N}_0^n$*. There is a multigraph with this score if and only if* $\forall 1 \le i \le n$, $a_i \ge 0$; $\sum_{i=1}^n a_i$ `is even;` $a_1 \le \sum_{i=2}^n a_i$.

Exercise 7.4. Prove your theorem.

Proof. 1. First we prove the sufficiency of the condition.

With a given score which satisfies the condition, we can use the algorithm below to construct a multigraph:

```
(a'_1, ..., a'_n) = (a_1, ..., a_n)
for i=1 to n
    if (a'_i == 0)
        continue
    for j=i to n
        if (a'_i ≥ a'_j)
            add a'_j edges between i and j
            a'_i -= a'_j
            a'_j = 0
        else
            add a'_i edges between i and j
            a'_j -= a'_i
            a'_i = 0
            break
while a'_i > 0
    find an edge (m,n) such that m, n < i
    remove edge (m,n)
    add edges (1,i),(i,k)
    a'_i -= 2
```

Note that the last while is executed only if $\forall j > i, a_j = 0$, which means a_i is the last positive number of the score. When entering the while block, a'_i must be even (because the total degree is even and it will be subtracted 2 whenever adds a new edge). So after the block, a'_i will be 0. And because a'_i is less than the total degree of the rest vertices, so we can always find an such edge in the while block.

After the construction, the remaining score of each vertex will be 0 and the degrees of the vertices in the graph equals to the original score.

2. Then we talk about the necessity of the condition.

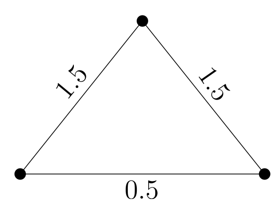
If there's a negative score, it can't be a multigraph score(because multigraph score cannot be negative.)

If the total number of degrees is odd, it can not be a graph score because of the handshaking theorem.

If there's a vertex whose degree is greater than the sum of the rest, then the "demand" of this vertex exceeds the degrees that the rest can "provide" (even if we connect this vertex to all other vertices we can't meet its need)

So the three conditions are necessary. $\square$

Weighted graphs. A weighted graph is a graph in which every edge e has a non-negative weight w_e. In such a graph the *weighted degree* of a vertex u is $\mathrm{wdeg}(u) = \sum_{\{u,v\}\in E} w_{\{u,v\}}$.



This is an example of a weighted graph, which has score $(3, 2, 2)$. Obviously no graph and no multigraph can have this score.

Exercise 7.5. State a score theorem for weighted graphs. That is, state something like

Theorem 7.6 (Weighted Graph Score Theorem)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{R}_0^n$. *There is a weighted graph with this score if and only if*

$$a_1 \leq \sum_{i=2}^{n} a_i$$

.

Remark. This is actually even simpler.

Exercise 7.7. Prove your theorem.

Proof. 1. First we prove the sufficiency of the condition.

With a given score which satisfies the condition, we can use the algorithm below to construct a weighted graph:

```
(a'_1, ..., a'_n) = (a_1, ..., a_n)
for i=1 to n
    if (a'_i == 0)
        continue
    for j=i to n
        if (a'_i ≥ a'_j)
            add an edge weighting a'_j between i and j
            a'_i -= a'_j
            a'_j = 0
        else
            add an edge weighting a'_i between i and j
            a'_j -= a'_i
            a'_i = 0
            break
    while a'_i > 0
        find an edge (1,j) such that j < i
        w=weight(1,j)
        if w > a'_i
            w -= a'_i/2
            add edges (1,j),(1,k) both weighting a'_i/2
            a'_i = 0
            break
        else
            w=0
            add edges (1,j),(1,k) both weighting w/2
            a'_i -= w
```

Note that the last while is executed only if $\forall j > i, a_j = 0$, which means a_i is the last positive number of the score. After the block, a'_i will be 0. And because a'_i is less than the total degree of the rest vertices, so we can always find a such edge in the while block.

After the construction, the remaining score of each vertex will be 0 and the degrees of the vertices in the graph equals to the original score.

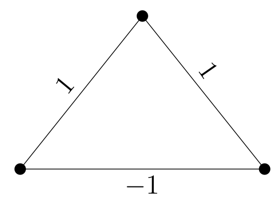
2. Then we talk about the necessity of the condition.

If there's a negative score, it can't be a weighted graph score(because weighted graph score cannot be negative.)

If there's a vertex whose degree is greater than the sum of the rest, then the "demand" of this vertex exceeds the degrees that the rest can "provide" (even if we connect this vertex to all other vertices we can't meet its need).

So the two conditions are necessary. $\square$

Allowing negative edge weights. Suppose now we allow negative edge weights, like here:



This "graph with real edge weights" has score $(2, 0, 0)$. This score is impossible for graphs, multigraphs, and weighted graphs with non-negative edge weights.

Exercise 7.8. State a score theorem for weighted graphs when we allow negative edge weights. That is, state a theorem like

Theorem 7.9 (Score Theorem for Graphs with Real Edge Weights)**.** *Let $(a_1, \ldots, a_n) \in \mathbb{R}^n$. There is a graph with real edge weights with this score if and only if: $n = 1$ and $a_1 = 0$, or $n = 2$ and $a_1 = a_2$, or $n \geq 3$.*

Exercise 7.10. Prove your theorem.

Proof. If $n = 1$, then a_1 should be 0 since there is no selfloop. If $n = 2$, then $a_1 = a_2$ since each edge connects vertex 1 and vertex 2.

If $n = 3$, for an arbitrary score $(a_1, \ldots, a_n)$, we can use the algorithm below to construct a real-weighted graph:

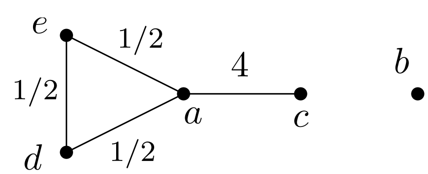
for i=1 to n
  arbitrarily select two verticles j, k different from i
  add edges (i, j) and (i, k) with weight $\frac{1}{2}a_i$
  add an edge (j, k) with weight $-\frac{1}{2}a_i$
  $a_i = 0$

Since the operation above can make $a_i = 0$, and doesn't change a_j and a_k, the program will end up with all a_i to be 0. From a given $(a_1, \ldots, a_n)$, we can easily build a real-weighted graph using the edges refered in the algorithm. $\square$

Exercise 7.11. For each student ID $(a_1, \ldots, a_n)$ in your group, check whether this is (1) a graph score, (2) a multigraph score, (3) a weighted graph score, or (4) the score of a graph with real edge weights.

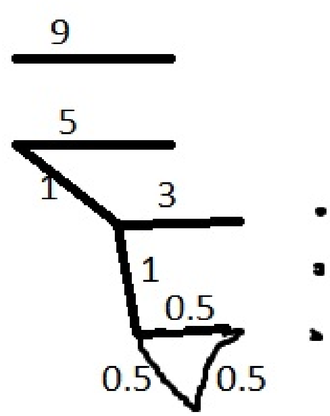
Whenever the answer is *yes*, show the graph, when it is *no*, give a short argument why.

Example Solution. My work ID is 50411. This is a weighted graph score, as shown by this picture:

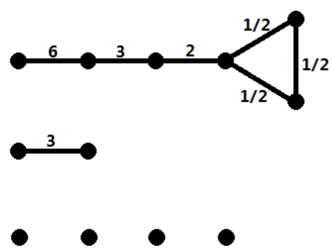


This settles (3). It is not a multigraph score, because BLABLABLA. I won't give more details, as it might give too many hints about Exercise 7.2. Alright, this settles (2). Note that I do *not* need to answer (4), as this is already answered by (3). Neither do I need to answer (1), as a "no" for (2) implies a "no" for (1).

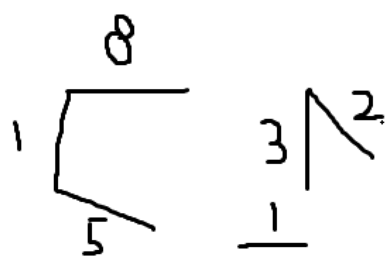
Answer My student ID is 516030910259. The corresponding $(a_1, \ldots, a_n)$ is $(9, 9, 6, 5, 5, 3, 2, 1, 1, 0, 0, 0)$. It' neither a graph score nor a multigraph score because the sum of digits is odd. It's a weighted graph score, also a negative weighted score, as shown by this picture:



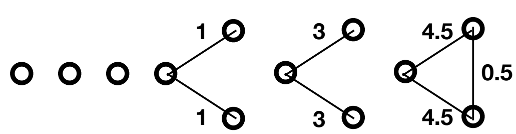
My student ID is 516030910303. The score is $(9, 6, 5, 3, 3, 3, 1, 1, 0, 0, 0, 0)$. The sum is an odd number, so it is a weighted graph score, which is as follows:



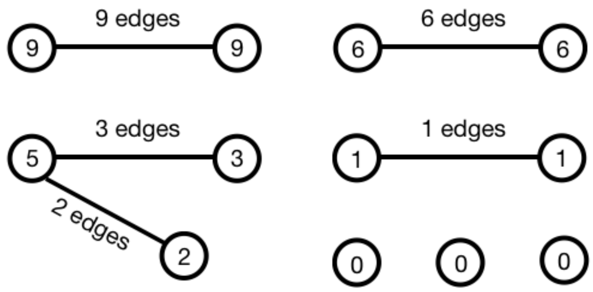
My student ID is 516030910258. The score is $(9, 8, 6, 5, 5, 3, 2, 1, 1, 0, 0, 0)$. The sum is an even number, and it will not be a graph score because the number of $a_i (a_i \neq 0)$ is less than 9+1. It's a multigraph score, also a weighted graph because multigraph is a special situation of weighted graph, also a nagative weighted graph, as shown by this picture:



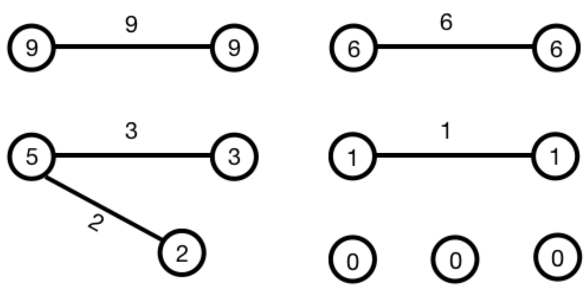
My student ID is 515030910632. The score is $(9, 6, 5, 5, 3, 3, 2, 1, 1, 0, 0, 0)$. The score is an odd number, so it is a weighted graph score, which is as follows: a weighted graph score, which is as follows:



Xu Yue: My student ID is 516030910269. The score is $(9, 9, 6, 6, 5, 3, 2, 1, 1, 0, 0, 0)$. (1) It cannot be a graph because the verticle 1 require 9 other verticles whose degree is greater than 0, while there're only 8. (2) The sum is an even number and a_1 is less than the sum of the rest, so there's a mutigraph with this score because of theorem 7.3, and it's shown below.



(3) a_1 is less than the sum of the rest, so there's a mutigraph with this score because of theorem 7.6, and it's shown below.



(4) Because every weighted graph is a real-weighted graph, so my result of (4) is identical to (3)