Mathematical Foundations of Computer Science

CS 499, Shanghai Jiao Tong University, Dominik Scheder

Group Name: **Mogicians**

1 Broken Chessboard and Jumping With Coins

1.1 Tiling a Damaged Checkerboard

Exercise 1.1. *Re-write the proof in your own way, using simple English sentences.*

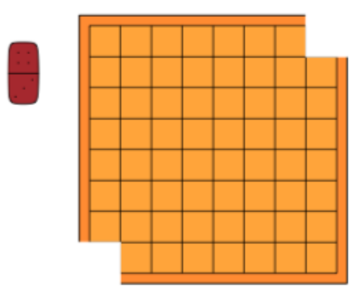

Figure 1: Damaged checkerboard

Proof. Color the checkerboard in black and white (Figure 2), then we can find that a single domino will always cover one black square and one white square. Since there are 30 white squares and 32 black squares in the damaged chessboard, it cannot be covered by dominoes (and will always leave 2 black squares).

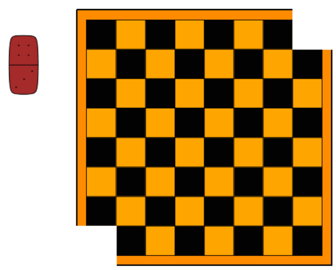

Figure 2: Colored checkerboard

□

Exercise 1.2. *Prove that the board below cannot be tiled. Try to find a short and simple argument.*

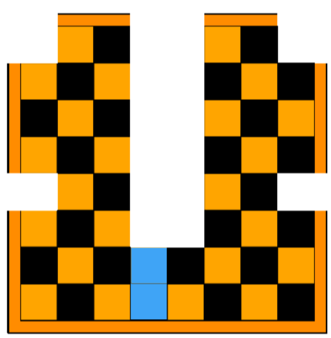

Figure 3: Colored checkerboard

Proof. First, focus on the two blue squares at the bottom. Obviously there are 5 ways to cover these squares(Figure 4). Each one will divide the board into two part.

However, none of these ways can divide the black and white squares evenly. Exercise 1.1 tells us that the board cannot be covered by dominoes if the amount of black squares and white squares are unequal. Thus this seriously damaged checkerboard can't be tiled.

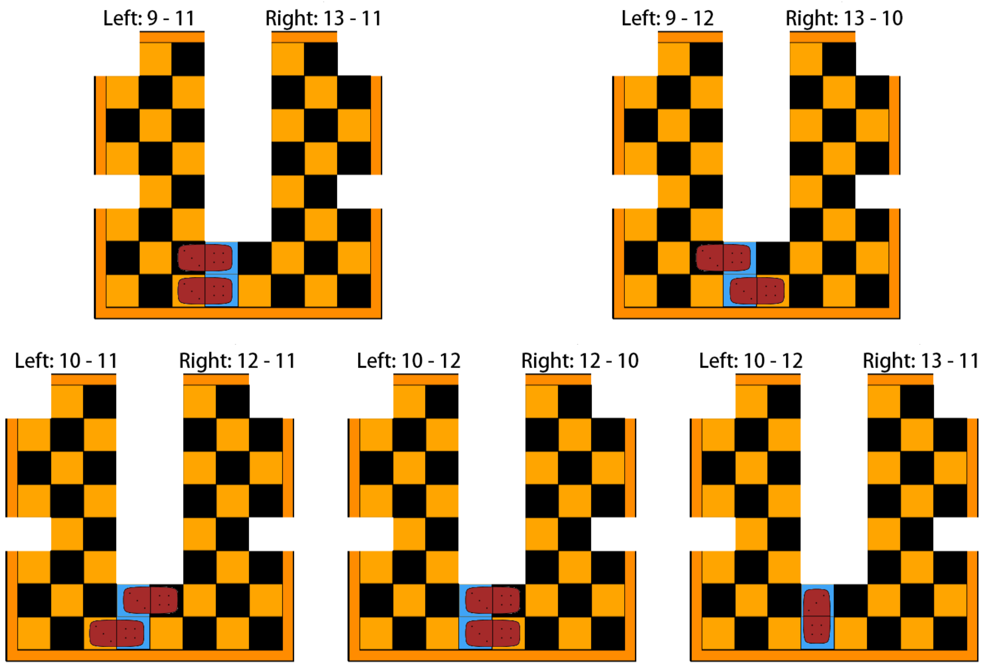

Figure 4: Five ways to cover the blue squares

□

1.2 Jumping with coins

Exercise 1.3. *You jump around with two coins. Show that you cannot increase the distance between the two coins.*

Figure 5: Jump around with two coins

Proof. According to the rules of the jumping, since there are only two coins, obviously the distance between the two coins won't change. □

Exercise 1.4. *You jump around with three coins. Show that you cannot start with an equilateral triangle and end up with a bigger equilateral triangle.Give a simple proof!*

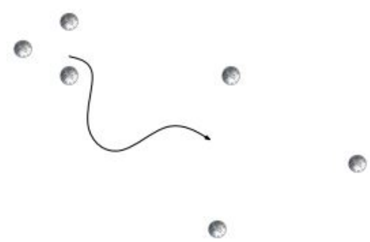

Figure 6: Jump around with three coins

Proof. Take a look at Figure 7, which indicates the general situation.

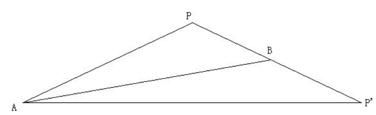

Figure 7: The general situation

Coin P jumps over coin B and reaches P'. Since the distance between P and B stays the same, the area of $\triangle ABP$ is the same as that of $\triangle ABP'$. In other words, when a coin jumps, the triangle's area stays the same.
Therefore, we cannot start with an equilateral triangle and end up with a bigger equilateral triangle. $\square$

You jump around with four coins which in the beginning form a square of side length 1.

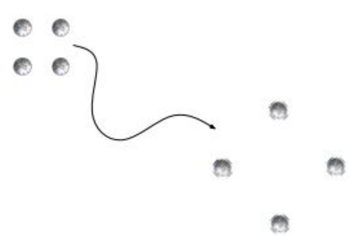

Figure 8: Jump around with four coins

Exercise 1.5. *Show that you cannot form a square of side length 2.*

Proof. Obviously, the jump is reversible. Therefore, to prove we cannot form a square of side length 2 from a square of side length 1, we will prove that we can't make a square smaller. Now we introduce grids and two axes that are

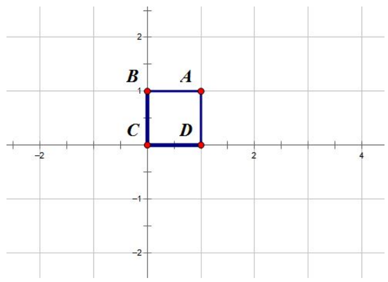

Figure 9: Introduce grid to the question

perpendicular to each other to the exercise. Make the square corners at grid intersection in the way shown in Figure 9. According to the axes, we can get the coordinates of the four vertices, which are obviously integers. And it's easy to find that whenever a jump happens, the coordinates are still integers. So the coins are always located on the intersections of the grid. Then we can see that our original square ABCD is the smallest square we can get through jumping.

Therefore, we can't make a square smaller. Due to the reversibility, it's also impossible to make a square bigger, let alone form a square of side length 2 from a square of side length 1. □

Exercise 1.6. *Show that you cannot achieve a position in which two coins are at the same position.*

Proof. Since the coins are put on the grid intersection, the location of each coin can be described according to the position relative to the coordinate origin (0, 0) of the axis. As is shown on Figure 9, the coordinate of the each coin is a binary integer pair $(x_i, y_i)(i \in \{A, B, C, D\})$, where the parity of each pair has four situations: (O, O), (E, E), (O, E) and (E, O) (O stands for *Odd* and E stands for *Even*). It is obvious that the parity situations of the four coins are different from each other and will never change during the whole jumping process. Therefore, we cannot achieve a position in which two coins are at the same place.

□

Exercise 1.7. *Show that you cannot form a larger square.*

Proof. The proof is exactly the same as that of Exercise 1.5. $\square$

2 Exclusion-Inclusion

2.1 Sets

Exercise 2.1. *Prove that* $|A \cup B| = |A| + |B| - |A \cap B|$*. Find ormulas for* $|A \cup B \cup C|$ *and* $|A \cup B \cup C \cup D|$*.*

Proof. 2.1.1

If $x \notin A \cup B$, then x is not counted at right.

If $x \in A$ and $x \notin B$, then $x \notin A \cap B$, x is counted once at right.

If $x \notin A$ and $x \in B$, then $x \notin A \cap B$, x is counted once at right.

If $x \in A$ and $x \in B$, then $x \in A \cap B$, x is added twice and subtracted once at right.

2.1.2

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Proof:

If $x \notin A$, $x \notin B$, $x \notin C$, then then x is not counted at right.

If $x \in A$, $x \notin B$, $x \notin C$, then x is counted once at right.

If $x \in A$, $x \in B$, $x \notin C$, then $x \in A \cap B$, x is added twice and subtracted once at right.

If $x \in A$, $x \in B$, $x \in C$, then $x \in A \cap B$, x is added four times and subtracted three times at right.

The rest of the situation can be proved similarly.

2.1.3

$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$

The proof is similar as previous formulas.

$\square$

Exercise 2.2. *Find a general formula for* $|A_1 \cup \cdots \cup A_n|$*.*

Proof.

$$|A_1 \cup \cdots \cup A_n| = \sum_{I \subseteq C} (-1)^{size(I)-1} |A_I|$$
$$C = \{1, 2, \cdots, n\} \qquad A_I = \bigcap_{i \in I} A_i$$

□

Exercise 2.3. *Justify the formula in the previous exercise in two ways.*

Proof. It is obvious that if $x \notin A_1 \cup \cdots \cup A_n$, than x will not be counted at right. What we want to prove is that each element in $A_1 \cup \cdots \cup A_n$ is counted once at right.

Suppose that x belongs to k A_i $(k \geqslant 1)$, then x is added $C_k^{size(I)}$ times when size(I) is an odd number and subtracted $C_k^{size(I)}$ times when size(I) is an even number. The total of them is

$$T = C_k^1 - C_k^2 + C_k^3 - \cdots + (-1)^{i-1} \cdot C_k^i + \cdots + (-1)^{k-1} \cdot C_k^k$$
$$\because (1+x)^k = C_k^0 + C_k^1 \cdot x + \cdots + C_k^k \cdot x^k$$
$$\therefore T = C_k^0 - (1-1)^k = 1$$
$$QED$$

□

3 Feasible Intersection Patterns

Exercise 3.1. *Find sets A_1, A_2, A_3, A_4 such that all pairwise intersections have size 3 and all three-wise intersections have size 1. Formally,*
1.$|A_i \cap A_j| = 3$ *for all* $\{i, j\} \in \binom{[4]}{2}$,
2.$|A_i \cap A_j \cap A_k| = 1$ *for all* $\{i, j, k\} \in \binom{[4]}{3}$.

Proof. For example, we can find sets:
$A_1 = \{1, 2, 3, 5, 6, 7\}$
$A_2 = \{1, 2, 4, 5, 8, 9\}$
$A_3 = \{1, 3, 4, 6, 8, 10\}$
$A_4 = \{2, 3, 4, 7, 9, 10\}$. □

Exercise 3.2. *Show that if we insist that $|A_i| = 5$ for all i, then the task from the above exercise cannot be solved.*
In the spirit of the previous questions, let us call a sequence $(a_1, a_2, \cdot, a_n) \in \mathbb{N}$ feasible if there are sets $A_1, \cdot, A_n$such that all k-wise intersections have size a_k. That is, $|A_i| = a_1$ for all i, $|A_i \cap A_j| = a_2$ for all $i \neq j$ and so on. The previous exercise would thus state that $(5,3,1,0)$ is not feasible, but $(6,3,1,0)$ is, as one solution of Exercise 3.1 shows.

Proof. Through the Principle of inclusion-exclusion:$|A_1 \cup A_2 \cup \cdots A_m| = \sum_{1 \leq i \leq m} |A_i| - \sum_{1 \leq i < j \leq m} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq m} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{m-1} |A_1 \cap A_2 \cap \cdots \cap A_m|$, we can list these equations:
$|A_1 \cup A_2 \cup A_3 \cup A_4| = \sum_{1 \leq i \leq 4} |A_i| - \sum_{1 \leq i < j \leq 4} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq 4} |A_i \cap A_j \cap A_k| = 4 \times 5 - 6 \times 3 + 1 \times 4 = 6$
$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 2 \times 5 - 3 = 7$
It has already seen that $|A_1 \cup A_2 \cup A_3 \cup A_4| < |A_1 \cup A_2|$, but it is impossible.
So we have proved that if $|A_i| = 5$ for all i,then the task from the above exercise cannot be solved. □

Exercise 3.3. *Suppose I give you a sequence $(a_1, \cdots, a_n)$. Find a way to determine whether such a sequence is feasible or not.*

Proof. Define that $A_I := \left(\bigcap_{i \in I} A_i\right)$, $I \subseteq \{1, 2, 3, \cdots n\}$, define $i = |I|$, so $|A_I|$ is equal to a_i.
Define that $B_I := \left(\bigcap_{i \in I} A_i\right) \setminus \left(\bigcup_{j \notin I} A_j\right)$.Define b_i that $|B_I|$ is equal to $|b_i|$.
Define a set J which is the subset of I, we can easily list that: $|A_I| = |B_J|$
Because B_J consist of A_I, we can get a equation:
$b_i = a_i - \sum_{j=i+1}^{n} C_{n-i}^{j-i} b_j (i < n)$
$b_i = a_n (i = n)$
We can list these conditions to judge whether the sequence $(a_1, \cdots, a_n)$ is feasible or not:
1.a_i need to be non-negative integers.
2.a_i need to be decreasing as i increasing.
3.b_i need to be non-negative integers.
If the sequence $(a_1, \cdots, a_n)$ meet these conditions, the sequence is feasible. □

4 Our Questions

After working on the proof for the above questions, we also come up with some more questions.
Our first question is about the coins. We have proven that it's impossible to make a equilateral or a square bigger through jumping without changing its shape. We can't help thinking about whether the regularity is true to other regular polygons. So our question is can we make a general regular polygon bigger through jumping without changing its shape?
Our second question is related to Exercise 3.3. As we can see, the situation that this exercise describes is quite simplified. In the exercise, the size of all the A_Is is preset to the same when Is are of the same size. So our question is that if we are given random numbers as the size of A_I, how the way we use in Exercise 3.3 might change?