NTT

NTT/INTT

- ✓ NTT is a variant of the Fast Fourier Transform (FFT) based on a finite field.
- ✓ It transforms polynomial multiplication into “pointwise multiplication," accelerating polynomial multiplication operations.
- ✓ Cooley-Tukey decomposition is used for NTT and Gentleman-Sande decomposition for INTT in a butterfly architecture.
- ✓ It adopts a Radix-2 Multi-path Delay Commutator (MDC) FFT, using 8 Butterfly Units (BU) to process data with N = 256.

NTT - Mathematical Derivation

- ✓ The Number Theoretic Transform (NTT) is defined as:

$$\hat{a}_j = \sum_{i=0}^{n-1} \psi^{2ij} a_i \mod q$$

Note:

$$\psi^{k+2n} = \psi^k$$
$$\psi^{k+n} = -\psi^k$$

- ✓ Using the Cooley-Tukey decomposition:

$$\hat{a}_j = \sum_{i=0}^{n/2-1} \psi^{4ij+2i} a_{2i} + \sum_{i=0}^{n/2-1} \psi^{4ij+2j+2i+1} a_{2i+1} \mod q$$

$$= \sum_{i=0}^{n/2-1} \psi^{4ij+2i} a_{2i} + \psi^{2j+1} \sum_{i=0}^{n/2-1} \psi^{4ij+2i} a_{2i+1} \mod q$$

- ✓ Define :

$$A_j = \sum_{i=0}^{n/2-1} \psi^{4ij+2i} a_{2i}, \quad B_j = \sum_{i=0}^{n/2-1} \psi^{4ij+2i} a_{2i+1}$$

- ✓ Thus, the transformed coefficients are:

$$\hat{a}_j = A_j + \psi^{2j+1} B_j \mod q$$
$$\hat{a}_{j+n/2} = A_j - \psi^{2j+1} B_j \mod q$$

INTT - Mathematical Derivation

- ✓ The Inverse Number Theoretic Transform (INTT) is given by:

$$\mathbf{a}_i = \sum_{j=0}^{n-1} \psi^{-(2i+1)j} \hat{a}_j \mod q$$

Note:

$$\psi^{k+2n} = \psi^k$$
$$\psi^{k+n} = -\psi^k$$

- ✓ Using the Gentleman-Sande decomposition:

$$\mathbf{a}_i = \left[\sum_{j=0}^{n/2-1} \psi^{-(2i+1)j} \hat{a}_j + \sum_{j=0}^{n/2-1} \psi^{-(2i+1)(j+n/2)} \hat{a}_{j+n/2} \right] \mod q$$

$$= \psi^{-i} \left[\sum_{j=0}^{n/2-1} \psi^{-2ij} \hat{a}_j + \sum_{j=0}^{n/2-1} \psi^{-2i(j+n/2)} \hat{a}_{j+n/2} \right] \mod q$$

- ✓ Define :

$$A_i = \sum_{j=0}^{n/2-1} \hat{a}_j \psi^{-4ij}, \quad B_i = \sum_{j=0}^{n/2-1} \hat{a}_{j+n/2} \psi^{-4ij}$$

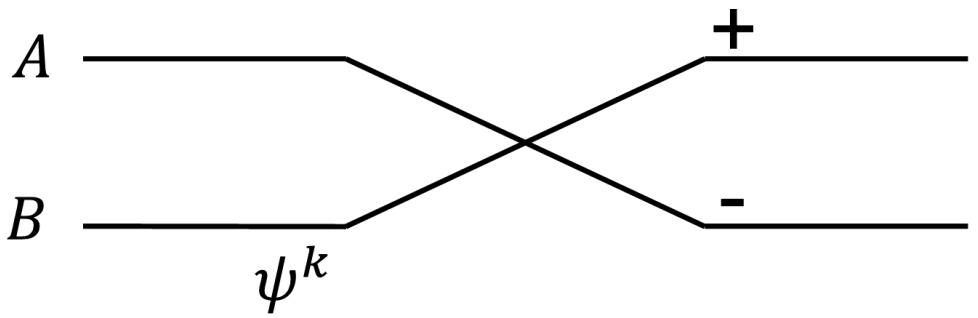
- ✓ Thus, the inverse transform coefficients are:

$$\mathbf{a}_{2i} = (A_i + B_i)\psi^{-2i} \mod q$$
$$\mathbf{a}_{2i+1} = (A_i - B_i)\psi^{-2i} \mod q$$

NTT/INTT - Butterfly diagram

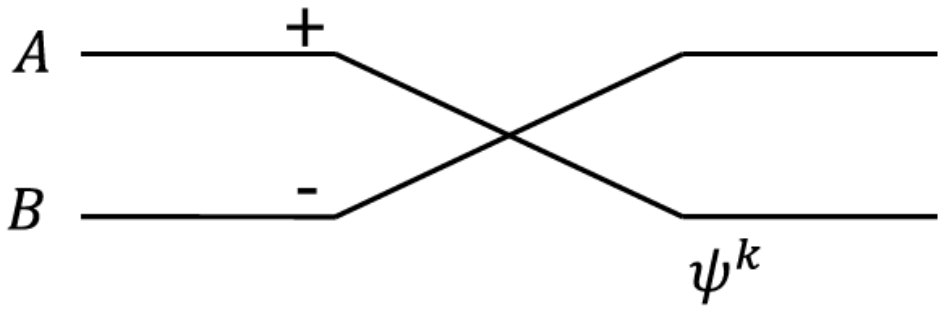
- ✓ Fundamental in FFT/NTT computations, is given by:

$$A = A + \psi^k B$$
$$B = A - \psi^k B$$

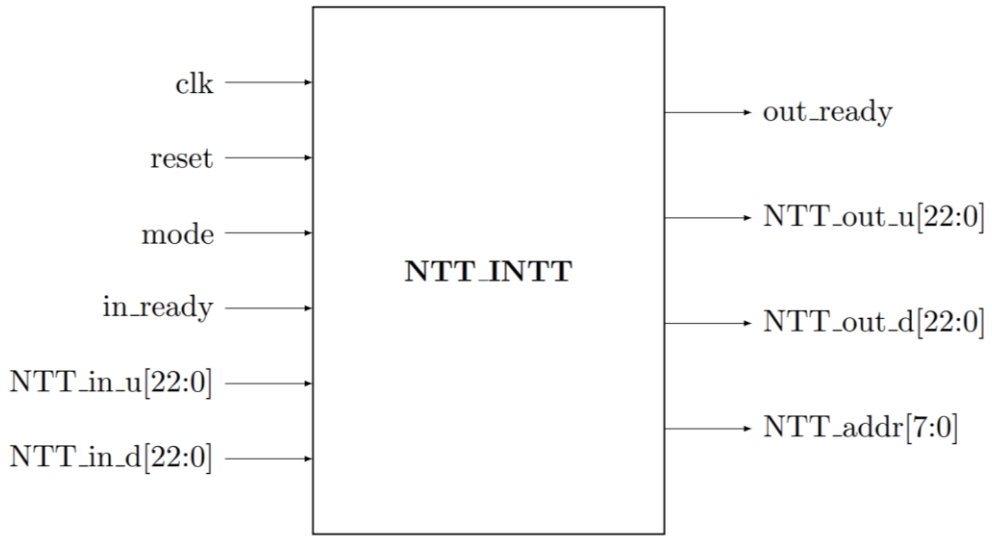


- ✓ Fundamental in FFT/NTT computations, is given by:

$$A = A + B$$
$$B = (A - B)\psi^{-k}$$



NTT_INTT - Module Pin Function Definition

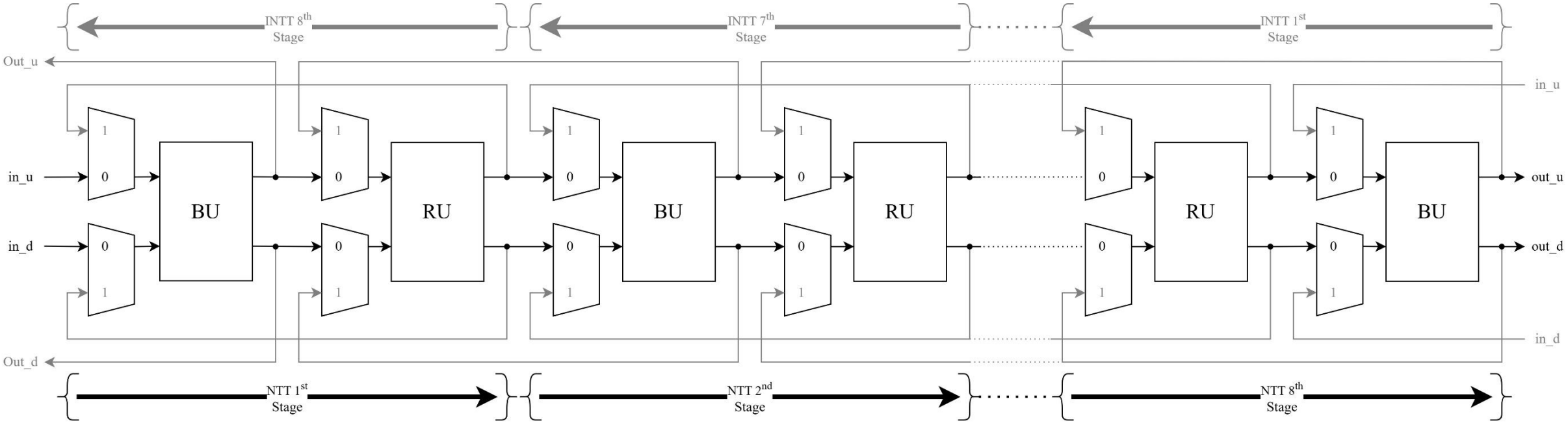


Pin name	I/O	Bit width	Function
clk	I	1	系統時脈
reset	I	1	系統同步重置信號
mode	I	1	0 : NTT mode / 1 : INTT mode
in_ready	I	1	由待轉換的MEM輸入資料準備好的指示信號
NTT_in_u	I	23	由待轉換的MEM輸入data
NTT_in_d	I	23	由待轉換的MEM輸入data
out_ready	O	1	轉換完成資料開始輸出的指示信號
NTT_out_u	O	23	轉換完成資料輸出data至指定MEM
NTT_out_d	O	23	轉換完成資料輸出data至指定MEM
NTT_addr	O	8	指定輸出的MEM位址

NTT_INTT - Module Task Assignment Table

Pin name	I/O	Bit length	Function
clk	I	1	系統時脈
reset	I	1	系統同步重置信號
mode	I	1	0 : NTT mode / 1 : INTT mode
in_ready	I	1	由待轉換的MEM輸入資料準備好的指示信號
NTT_in_u	I	23	由待轉換的MEM輸入data
NTT_in_d	I	23	由待轉換的MEM輸入data
out_ready	O	1	轉換完成資料開始輸出的指示信號
NTT_out_u	O	23	轉換完成資料輸出data至指定MEM
NTT_out_d	O	23	轉換完成資料輸出data至指定MEM
NTT_addr	O	8	指定輸出的MEM位址

NTT_INTT – Block Diagram

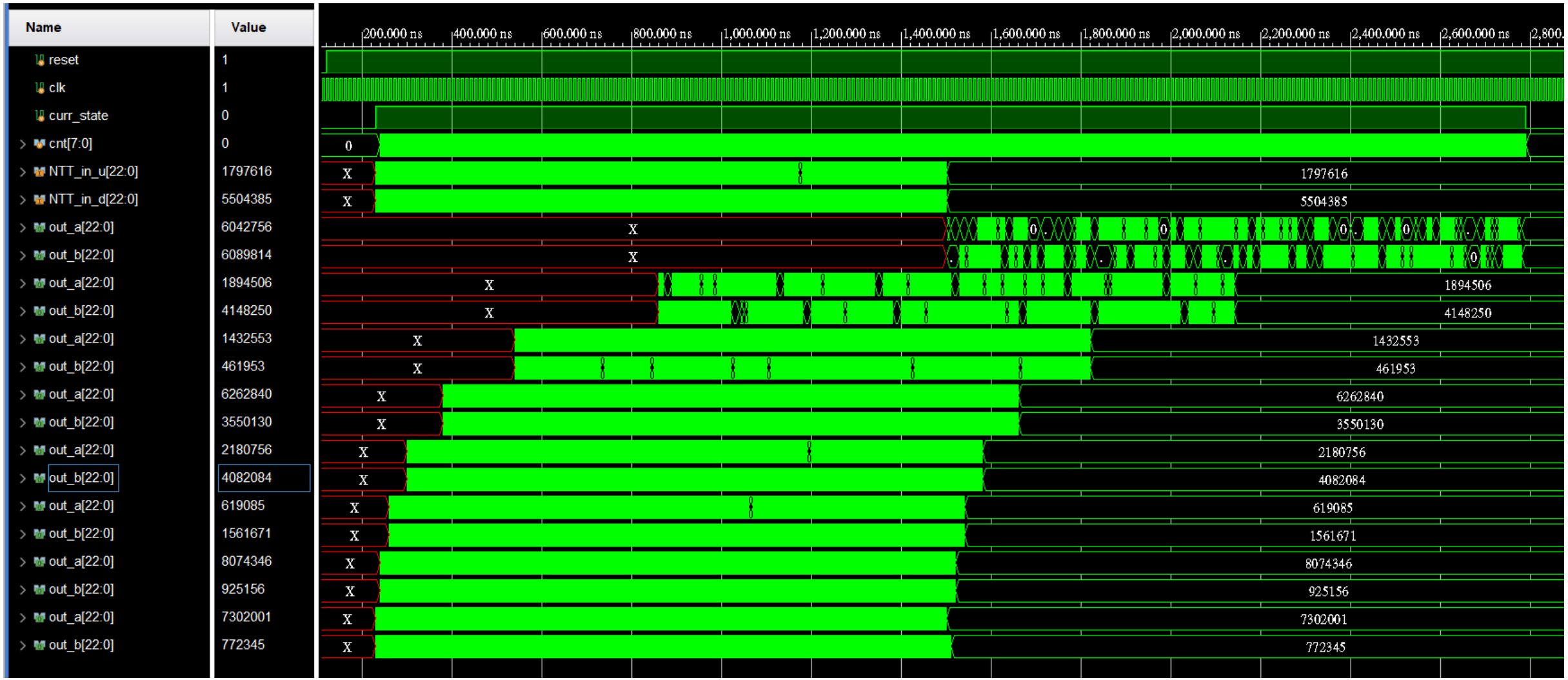


NTT – Timing Diagram

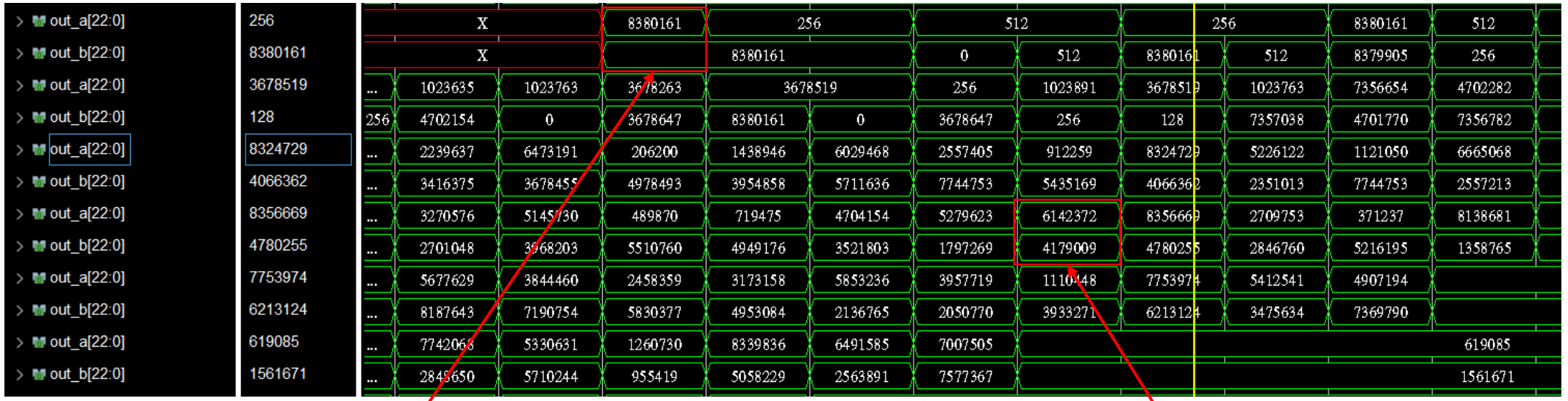
NTT - – Timing Diagram

5157967	8144313	5793821	5202208	1153275	2846100	3957586	3724921
1713881	8208803	3339230	2925445	2424097	7769886	614766	1691784
3162035	5357188	5215824	1368896	7067290	6755632	7118893	6953724
7669833	3774935	832905	2065634	2409715	899857	7254363	6880217
6641394	5558247	6436092	3879842	1575198	476033	3656372	1474475
4903989	1994723	4288883	7519296	4222356	5823871	2956723	193673
0	0	5223614	7662194	6608599	4509610	5031747	4300736
0	0	6917442	3547934	2905824	1225444	5940663	7276535
0	0	534788	7373421	1439604	627207	6967820	2051400
0	0	6170123	1867066	3095744	2358723	3793081	7466544
0	0	0	0	7483179	7841209	2168270	7466561
0	0	0	0	5524427	1571441	7132662	7833696
0	0	0	0	0	0	1797616	5504385
0	0	0	0	0	0	0	0

INTT – Timing Diagram



INTT – Timing Diagram



121	0	0	1023635	7356782	4258239	7992318	5864397	4518333
122	0	0	8380289	7356526	4010802	523968	607845	2328505
123	0	0	7356526	1023635	3940535	387971	3023220	4851779
124	0	0	1023763	256	8103694	7468350	2822564	2451780
125	0	0	1023635	4702154	2239637	3416375	3270576	2701048
126	0	0	1023763	0	6473191	3678455	5145730	3968203
127	8380161	8380161	3678263	3678647	206200	4978493	489870	5510760
128	256	8380161	3678519	8380161	1438946	3954858	719475	4949176
129	256	8380161	3678519	0	6029468	5711636	4704154	3521803
130	512	0	256	3678647	2557405	7744753	5279623	1797269
131	512	512	1023891	256	912259	5435169	6142372	4179009

Modular Reduction

- ✓ For a 46-bit value s, modular reduction is performed recursively by exploiting the relation :

$$2^{23} \equiv 2^{13} - 1 \; mod \; q$$

- ✓ The reduction ensures that the result falls within the interval $(-q, 2q)$ allowing for adjustments by adding q if negative or subtracting q if positive.
- ✓ After performing necessary additions or subtractions with q of the reduced result, the final output is determined by selecting the non-negative value.
- ✓ Each BU module contains one, and there are eight BU modules, requiring a total of eight modular reduction modules.

Modular Rduction - Mathematical Derivation

$$s[45:0] \equiv 2^{23}s[45:23] + s[22:0] \equiv 2^{13}s[45:23] - s[45:23] + s[22:0]$$

$$\equiv 2^{23}s[45:33] + 2^{13}s[32:23] - s[45:23] + z$$

$$\equiv 2^{13}\,(s[45:33] + s[32:23]) - (s[45:33] + s[45:23]) + z$$

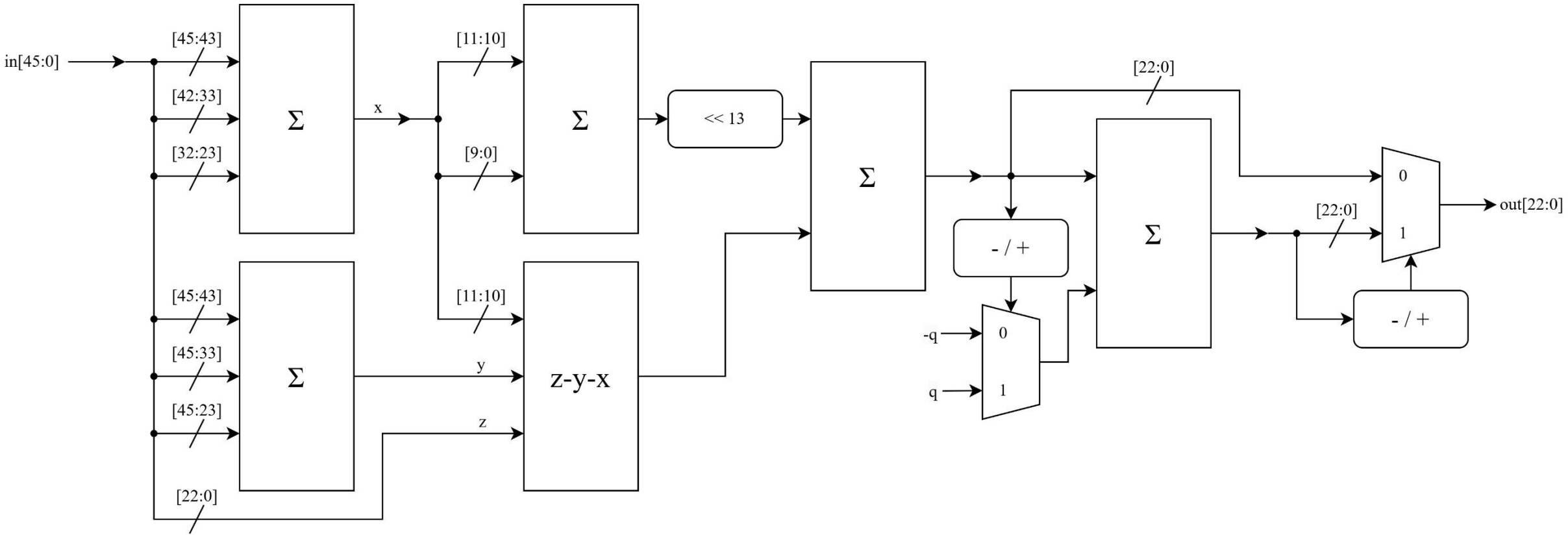
$$\equiv 2^{23}s[45:43] + 2^{13}\,(s[42:33] + s[32:23]) - (s[45:33] + s[45:23]) + z$$

$$\equiv 2^{13}\,(s[45:43] + s[42:33] + s[32:23]) - (s[45:43] + s[45:33] + s[45:23]) + z$$

$$\equiv 2^{13}x - y + z \equiv 2^{23}x[11:10] + 2^{13}x[9:0] - y + z$$

$$\equiv 2^{13}\,(x[11:10] + x[9:0]) - (y + x[11:10]) + z \pmod q$$

Modular Rduction – Block Diagram



Modular Reduction – Timing Diagram

Name	Value	10.000 ns	15.000 ns	20.000 ns	25.000 ns	30.000 ns	35.000 ns	40.000 ns	45.000 ns	50.000 ns	55.000 ns
in[45:0]	98765432	0		70368744177663		123456789		8380416		987654321	
out[22:0]	7145532	0		49144		6130951		8380416		7145532	
in[45:0]	00003ade	000000000000		3fffffffffff		0000075bcd15		0000007fe000		00003ade68b1	
out[22:0]	6d083c	000000		00bff8		5d8d07		7fe000		6d083c	
x[11:0]	075	000		805		00e		000		075	
y[23:0]	000075	000000		802005		00000e		000000		000075	
z[22:0]	5e68b1	000000		7fffff		5bcd15		7fe000		5e68b1	
d[11:0]	075	000		007		00e		000		075	
e[26:0]	05e683c	0000000		7ffdff8		05bcd07		07fe000		05e683c	
f[23:0]	6d083c	000000		00bff8		5d8d07		7fe000		6d083c	
adjust[23:0]	801fff	801fff									
g[23:0]	ed283b	801fff		80dff7		ddad06		ffffff		ed283b	

BU

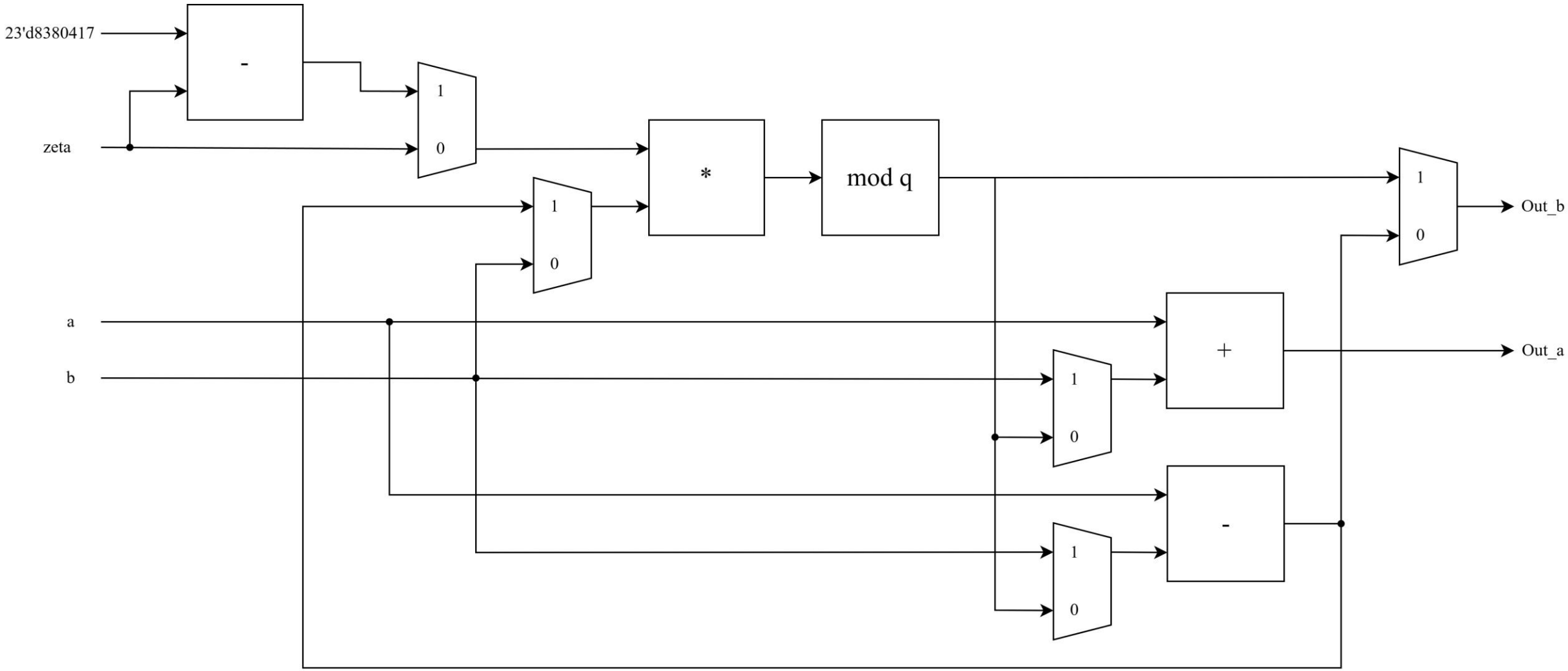
- ✓ The Butterfly Unit is formed based on the symmetry and parity properties in NTT and INTT computations.
- ✓ For a pair of input numbers a and b, together with a corresponding twiddle factor w, the butterfly operation in a finite field (modulo p) proceeds as follows:

$$x=(a+w\times b) \bmod q$$

$$y=(a-w\times b) \bmod q$$

- ✓ The structures of NTT and INTT are similar, but the twiddle factors used in INTT are the modular inverses of those in NTT. A normalization factor must also be applied at the end.
- ✓ Eight BU modules are used in the NTT/INTT of the thesis.

BU – Block Diagram



RU

- ✓ To match the butterfly structure, the output of each stage's BU is reordered accordingly.
- ✓ There are a total of 7 RUi, where $1 \leq i \leq 7$, in our implemented NTT and INTT.
- ✓ In NTT/INTT, the MEM depth of each RUi Stage is different:

$$\text{NTT} : \text{MEM\_Depth\_i} = 2 \times ((8 - i) - 1)$$

$$\text{INTT} : \text{MEM\_Depth\_i} = 2 \times (i - 1)$$

RUi – Block Diagram