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Virtual Logic — The MetaGame Paradox

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Introduction

This is column number 10. In this column we shall discuss a recent relative [1],[2] of the Russell paradox, the *Metagame Paradox.* This paradox is related to a set theoretic paradox about well-founded sets, the *Well-founded Set Paradox.* These two paradoxes are both related to the basic nature of any observing system that would include itself in its own observations. I give you these paradoxes and a comment on the nature of the observer. Judge them for yourself. Near the end of the column we show how, by turning the paradox around, there are new proofs of uncountability of certain infinities. These proofs have the remarkable quality that they use a bit of imaginary reasoning, only to have it vanish just as it appears! It is this role of the imagination that is central to our theme. We reason by an imaginary detour. And we obtain a real answer.

The culprit in the present case is the infinite descending form Omega:

$\Omega = <\Omega>$ whose presence has graced this column many times in the past as a Gremlin of Recursion and an Imaginary Logical Value. In the present context Omega is the exemplar of an infinite descent that is to be avoided. Omega then appears as the villain of the piece and can only appear as a possibility and a consummation devoutly to be avoided. It is, in fact, the avoidance of Omega that has made the proofs of uncountability that are found herein a secret subject known only to the few who would entertain a set that was not well-founded.

Such sets are quite well, thank you. They just fall into themselves forever.

The Metagame Paradox.

Metagame is a game of finite games. A *finite* game is a game that always ends after a finite amount of time (however long). Thus most board games and sports have this property. A chess game either ends by checkmate or by the no repetition rule, or even by a preset time limit. A game of golf ends after a certain number of holes have been played. Rules in Go prevent certain infinite repetitions (in a so-called "ko fight"). A boxing match has specific time limits.

If I say to you "Lets play Metagame!", then you get to say "Lets play *****." where ***** is the name of any finite game. Those two statements are the first two moves of any round of Metagame. After that I make the first play in ***** and we continue with the finite game ***** until it is over. The winner of that finite game is the winner of this particular round of Metagame.

Thus it might go like this.

Me: Lets play Metagame.
You: Lets play back gam mon.
We play back gam mon and you win.

Now we see that by its very definition, Metagame is a finite game. For indeed, every session of Metagame ends in a finite amount of time, the time it takes to play some specific finite game. Every session of Metagame consists in one play of some finite game.

Since Metagame is a finite game, it is a legal move for the second player to let ***** be Metagame itself! Then the following sequence of legal moves is possible:

Me: Lets play Metagame.
You: Lets play Metagame.
Me: Lets play Metagame.
You: Lets play Metagame.
Me: Lets play Metagame.
You: Lets play Metagame.

AdInfinitum.

Therefore Metagame is not finite.

We have shown that Metagame is finite and that the fact that it is finite implies that it is not finite. That is the Metagame Paradox.

Perhaps the reader would like to attempt his/her own resolution of this paradox before reading further. In any case, we shall not impose our opinions just yet. Instead, we describe another paradox.

The Paradox of the Well-Founded Sets

A set, as you know, is the mathematician's version of a collection, where the members of the collection are usually also sets. It is customary when writing a set to put commas between the members. Thus when I denote the set whose members are a,b and c, then I write: {a,b,c}. Lets recall a few sets, just for the sake of remembrance.

Recall that two sets are said to be equal if they have the same members. Thus any two empty sets are equal since they have exactly the same members, namely none! However {} is not equal to {{}} since the latter has a member, the empty set itself. In this way, distinct sets can be built up from nothing by the act of forming new sets from old. The oldest of all is the void, and from that issues the empty set. From the empty set issues the set whose member is the empty set, and we are on our way.

Here is the empty set, a set with no members at all and indicated by empty brackets:

0= {}

Here is the set whose member is the empty set:

1={{}}

Here is the set whose members are the empty set and the set whose member is the empty set:

2={ {}, {{}} }

I have called these three sets by the names of the first three numbers because 0 has zero members, 1 has one member and 2 has two members. Note that 1 = {0} and 2 = {0,1}. We can continue with

3={0, 1, 2} = { {}, {{}}, { {}, {{}} } }
4={0,1,2,3} = { {}, {{}}, { {}, {{}} } ,{ {}, {{}}, { {}, {{}} } } }

and so on. In this way, starting from nothing, we build zero and all the positive integers.

Another sequence of sets that we can build are the sets

[0]={}
[1] ={{}}
[2] = {{{}}}
[3] = {{{{}}}}
[4] = {{{{{{}}}}}}

and so on, where [N] is a finite nest of N+1 brackets.

Each of these sets has only one member, but there is a nesting of membership with [0] a member of [1], [1] a member of [2], [2] a member of [3], [3] a member of [4] and so on. The set [N] has members of members in a chain that is N deep. If we go to infinity and form an infinite nest of brackets we get a set

M = {{{{{{{ ... }}}}}}}

This set has the property that M is a member of itself!

M = { M }

The self-referential, or reentering, set M has *an infinite descending chain of membership*. It is a little hall of mirrors. You walk inside, crossing the set boundary and you find M itself. You walk inside this M and you find M again. This never stops. It is like walking down a staircase that never ends.

A set is said to be *well-founded* if it has no infinite descending chains of membership. Each integer set N={0,1,2, ..., N-1} is well-founded and each "integer descent" set [N] is also well-founded, but the set M is not well-founded.

(It is quite well, thank you, just not well-founded). In fact any set that has itself as a member is not well-founded.

There are sets that are not members of themselves that are nevertheless not well-founded. Consider the sets A and B.

A's only mem ber is B and B's only mem ber is A
A = {B}
B = {A}

By definition, A and B are distinct from one another, so that neither set is a member of itself, but since each is a member of the other, neither is well-founded. Together they form a miniature society.

Now consider *the set WF of all well-founded sets.* Clearly WF is itself well-founded. If we wish to look at a membership chain in WF, we start with WF and choose a member S of WF. This member S is well-founded and so any descending chain of members constructed from S will stop in a finite number of steps.

Since WF is well-founded and WF is the set of all well-founded sets, we have proved that WF is a member of itself. But no set that is a member of itself can be well-founded! *This is the Well-Founded Set Paradox.*

I am sure that the reader can see clearly that the Well-Founded Set Paradox is very similar in structure to the Metagame Paradox. A well-founded set is like a finite game where a move in the game is to choose a member of the set and successive moves choose members of members. Well-foundedness assures us that these choices end after a finite number of steps. Putting WF into itself makes it possible to choose WF from its members, and then to choose WF again from the members of that WF, and to keep going forever. Yet before we put WF inside itself, WF was indeed well-founded!

The Observer

Looking outward, every observation that I make has a stop. I can look to the walls of my room. I can look beyond to the lake through my window. I can look at the waves on the lake. I can choose a pair of binoculars and look at the froth on the waves. At any given epoch there is a limit on this penetration of observations. The world as I find it is all such chains of observation, each ending in a void of perception after a finite number of steps. So my world is a finite world. And that world itself can become the subject of my observation. That subject I call "myself" since it indicates all of my world, and my observation of that world. But then upon observing myself, I find that I can observe myself within myself in an infinite telescope of observations. It would seem that my world is infinite after all!

The Well-Founded Forms

In this article I have not discussed how to resolve these paradoxes. In a sense the best compliment you can pay to a paradox is not to “solve” it, but to use its form of reason to make progress in a non-paradoxical context. This is the virtual logic of the paradox. In our case lets consider the collection of *forms*, where a *finite form* is a well-formed (sic) expression in parentheses with exactly one full outer parenthesis such as

<> or
<<>> or
<<><>>

Since any form has the appearance <ABC...> where A, B and C are forms, I shall call the outer parentheses the *outer boundary* of the form. A form has *members*, namely the forms that can be obtained by crossing inward from the outer boundary. Thus the members of the form <ABC> are A, B and C. The members of <<><>> are two copies of <>. (In forms we allow multiple copies of forms.) We can go to infinite forms by allowing a form to have infinitely many members. For example <AAAA...> is an infinite form whose members are infinitely many copies of A. We have the definition of well-founded forms just as for sets. A form is well-founded if it has only finite descending chains of membership. Some infinite forms are well-founded. For example < <> <> <> <> ...> is well-founded. Of course

$$\Omega = <<<<<<<...>>>>>>> = <\Omega>$$

is not well-founded since it is a member of itself.

Now suppose that you could make a list of all the well-founded forms. Say this list is A_1, A_2, A_3, A_4 and so on. Then you can form

$$B= <A_1A_2A_3A_4...>$$

and it follows that B is a well-founded form.

Furthermore, B cannot be on the list, for if it were on the list then B would be a member of itself and hence not well-founded. We have proved that there is no way to make a list (even though infinite) of all the well-founded forms. The existence of any list makes it possible to always construct a well-founded form that is not on the list.

The collection of well-founded forms is uncountably infinite.

For a summary of this argument see Figure 1.

Discussion

In this essay we have seen how the MetaGame paradox and the Well-Founded Sets Paradox operate both to produce contradiction and to produce new forms of

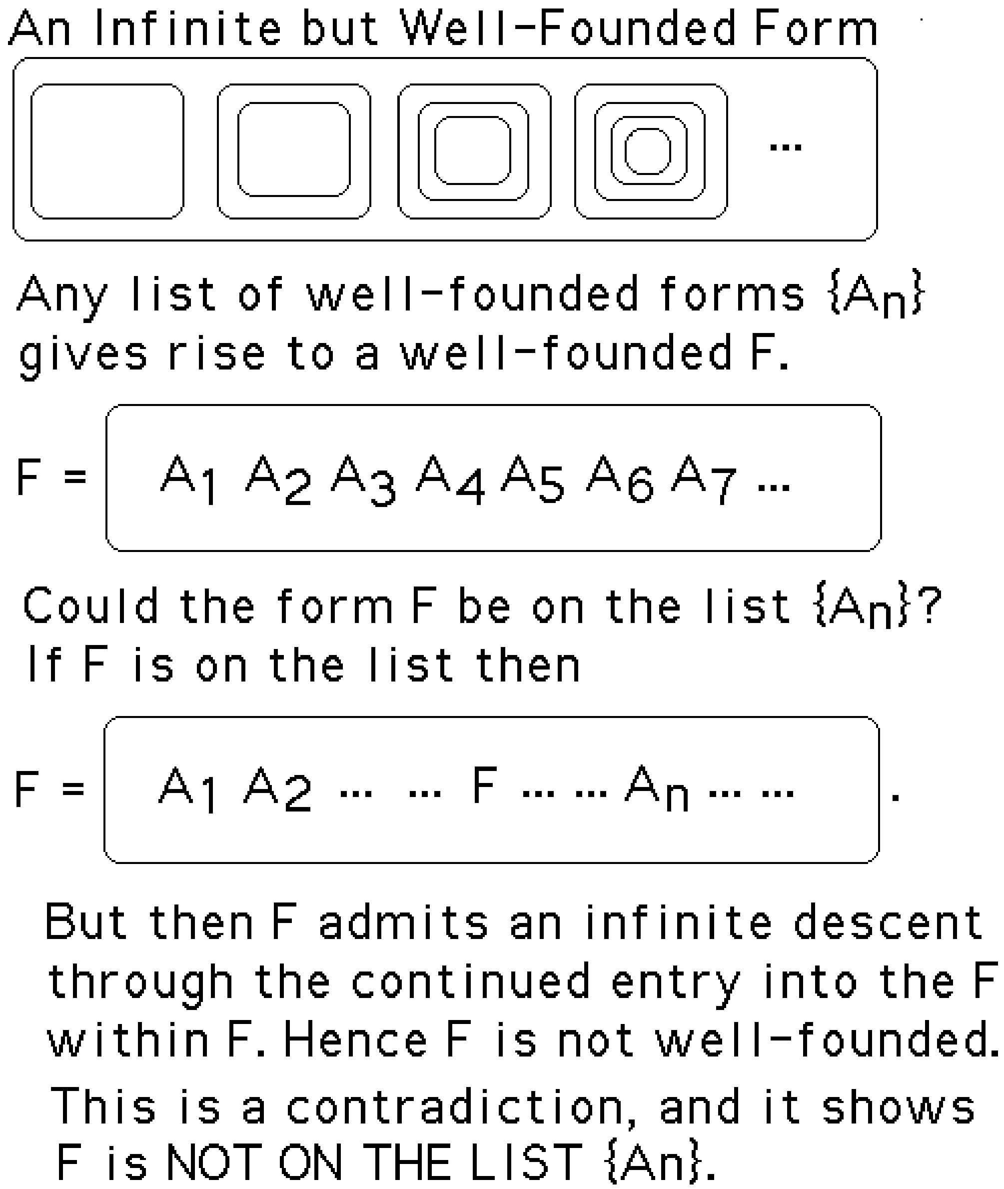

Figure 1 - Any List of Well-Founded Forms is Incomplete

reasoning. The level of contradiction in these paradoxes remains a puzzle. We seem to feel that the concept of a finite game is a valid one and that it should not be neccessary to legislate the list of present finite games before we define an entity like MetaGame. Yet if we do not perform such legislation, then MetaGame cannot be finite. The alternative is to declare that MetaGame is not a game! But a game is a certain kind of human interaction, and I have played MetaGame. Feels like a game to me.

Gail Reed suggests the way out that some games are just finite *and* infinite. What do you think?

In this essay we have seen how the MetaGame paradox leads to a particular way that certain infinite collections are uncountable. The surprising turn of this argument is that given a list of (well-founded) forms, the entire list can itself be used to make a new well-founded form that is not on the list. The new form cannot be on the given list exactly because its membership in the list would entail the infinite descent that is prohibited for well-founded forms. A remarkable feature of this argument is that if applies uniformly to finite and to infinite lists.

For example, suppose that {A,B,C} is a list of well-founded forms. Then F = <ABC> is also a well-founded form but if (for example) F =A, then F=<FBC> is not well-founded. Hence F is distinct from A, B and C.

The most extreme version of this is to consider a single well-founded form A. Thus F=<A> must be distinct from A because if A=F, then F=<F> and then F would not be well-founded.

The essential and curious feature of these proofs is the evanescent appearance of the Ω = <Ω>, a quintessential self-referential non-well-founded form. The form Ω appears momentarily as a villain that would spoil the finite descent character of the players in the list. Even though Ω appears in the proof only to be eliminated she is nevertheless an essential ingredient in the argument. This argument uses the imaginary form Ω (imaginary with respect to the class of well-founded forms) to reason to a real answer.

In our cognition and in the action of a self-observing system the imaginary value takes the linguistic form of reference to the self. From the point of view of the finite, the self is an illusion, a word that fills in a singularity in the system's picture of the world. But from the meta-viewpoint there is an extension to realms where selves are as real as the imagination of the observer. Domains arise in which the imaginary is real. The real and the finite is understood to be understood only in this wider domain.

References

1. Raymond Smullyan, 5000 B.C. and Other Philosophical Fantasies, St. Martin's Press, New York (1983), pp. 40-41.
2. William S. Zwicker, Playing Games with Games: The Hypergame Paradox, American Mathematical Monthly, (1987), Vol. 94, pp. 507-514.
3. Jorge Luis Borges, Labyrinths — Selected Stores and Other Writings, ed. by Yates and Irby, New Directions Pub. Co. (1962).

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