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Virtual Logic — Infinitesimals and Zero Numbers

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Introduction

This is column number 11 of our series on Virtual Logic. In this column we will discuss the mathematical subject of Newton's calculus using the notion of infinitesimals as an imaginary form of number. A form that allows us to reason to real answers and compute limits that would otherwise be ineluctable. I shall also discuss "zero numbers", a concept grounded in ordinary numbers that opens up the powers of zero so that 0,0x0,0x0x0,... are all distinct forms of the void. We shall see that these subjects are related to one another and that they are both related to the perennial topic of this column — paradox resolution and the relation of imagination/imaginary values to properties of a system as a whole.

First, I shall explain what sort of thing is an infinitesimal and how we will work with it. The idea of an infinitesimal was conceived by Newton and Leibniz in their invention of the calculus. The concept was that of a process (or calculation) whose results were getting ever-smaller so that if inspected, the value of the infinitesimal would be smaller than any named positive real number, and yet the infinitesimal is not zero! Newton called them "fluxions", tiny vanishing particles of flux, particles of the action that lies at the basis of the dynamic world. These vanishing particles are always in motion, elusive motion that avoids evaluation.

How can you imagine an infinitesimal? You can recognise yourself as a slow creature and think of the infinitesimal as a fantastically quick process. By the time you reach into your bag of numbers and pull one out for comparison, the value of the infintesimal has disappeared way below the value that you have chosen. In fact, you cannot take a reading of the value of the infinitesimal, it moves too fast to allow any reading to take place. And yet there it is vanishing away before your very eyes! Bishop Berkeley, in a fit of criticism, called the infinitesimals "ghosts of departed quantities" and wondered how Newton, Leibniz and their descendants could could muster the faith to believe in them.

In this article I shall present the idea of an infinitesimal as a kind of imaginary value related to ideal mathematical elements such as the square root of minus one or infinity.

In mathematics, almost every construction is "ideal" in the sense that it is an abstraction or idealization of experience. But some constructions, such as the "vanishing point" on the horizon where parallel lines meet, are particularly worth emphasizing as ideal or imaginary because they connote a place where we extend a given structure by adding new elements to it that inform that very structure. In self-observing systems of the human kind such an addition is the notion of a self. You will not find the self inside the system. Nor is it outside the system. It is not quite illusory either! The self is imaginary and because it is authentically imaginary, the self is real. In this way the study of infinitesimals is directly related to cybernetic concerns of emergence of values that relate to the structure of a system or a language as a whole.

Along with infinitesimals, we shall study the powers of zero in a system that says that zero zeros is not the same as zero! The concept of zero numbers is due independently to George Spencer-Brown [1] and to Frederick Joseph Staley [2]. Apparently, Spencer-Brown discovered the zero numbers in the course of extending Laws of Form to encompass ordinary arithmetic, while Joe Staley found them in contemplating the void as an underpinning of Mathematics. There is a curious affinity between infinitesimals and zero numbers, even though they are quite different. (Infinitesimals are not zero, but they adhere to zero. Powers of zero are not zero, but they are not at any distance from zero.)

The Paradox of Division by Zero

0NE/ZERO = (ZERO x ONE) / (ZERO x ZERO) = ZERO / (ZERO x ZERO)
(Since ZERO x ZERO = ZERO) = ZERO / ZERO = ONE
Hence ONE / ZERO = ONE
Hence ONE = ZERO x ONE = ZERO.
But we can not have ONE = ZERO!
So we say that you can not di vide by zero.

Here it is assumed that ZERO x ZERO = ZERO.

If you have the notion of subtraction then this fact is forced upon you, for 0= 1 + (-1) whence 0 x 0 = 0 x (1 + (-1)) = 0x1 + 0x(-1) = 0 + 0 =0. This calculation shows that 0x0=0 is a consequence of ax(b+c) = axb + axc (this is called the distibutivity of multiplication over addition), and the principle that 0xa=0 for any non-zero a.

We see then that we could attempt to let go of 0x0=0 in a context of only positive or zero numbers since in that context there is no proof that 0 x 0 = 0 in the form shown above. In other words, if we work in positive arithmetic then zero is not seen as a combination of two integers. Any time you add two positive integers you get another positive integer. The positive integers including zero is called the "natural numbers" by mathematicians. Up until the middle fifteen hundreds the concept of negative numbers was barely heard of. Even today, most people do not have a full confortability with negative numbers. (After all who wants to have a negative bank balance?!) Elementary counting happens first without subtraction and without the concept of an absence as a something. This will be the context of the zero numbers that we shall discuss.

Here is Joe Staley [2] on the concept of the zero numbers:

According to arithmetic 00=0, zero times zero is just zero. On the other hand, zero times anything is just zero number of that thing, none of them; so zero times zero says no zeros, none, not one of them. But zero is one zero, not none.

It would have been possible to resolve this contradiction by several arguments. We might say that 00=0 fits into a consistent mathematical system, so there. Or we might argue about the difference between zero as an object of multiplication, zero times as a verb, and '0' as a symbol, and so on. It was possible, however, to take another path, the path of envoidment, of accepting 00 not equal to 0 and then developing the consequences of a further discrimination (00 < 0), the path taken in the following.

The result of this process of envoidment is to produce a substantialextension of the known numbers. We develop here an arithmetic of numbers smaller than zero but larger than any negative number. This number set has a rich and resonant set of strata, an architecture of sudden ellipses, and a new foundation for the natural numbers, wherein '1' is found as the mutual limits of ascending an descending sequences of zero numbers.

We repeat:

Zero times any thing
Is just zero number of that thing,
None of them;
So zero
times zero says
No zeros,
None,
Not one of them.

This is a neccessary thought in order to enter the realm of an observing system, a system that can reflect on its own use of language.

Infinitesimals

If we replace ZERO by an infinitesimal number Q that is not equal to zero, then we do not have Q x Q = Q and so if we were to write

Q / Q x Q

this would just be the same as 1/Q.

Of course since Q, Q x Q , Q x Q x Q , ... give smaller and smaller infinitesimals, it follows that their reciprocals 1/Q, 1/ Q x Q , 1/ Q x Q x Q, .. will be larger and larger infinities. We can live with this! Adding such numbers will enlarger and enrich the number system and there will not be any problems related to the division by zero. Infinitesimals are not zero.

We could go into this domain but I am going to postpone it and enter the next section where we work with the simplest possible sort of inifinitesimal, one whose square is zero. Dear reader, please bear with me. This next turn will reward your attention.

Square Zero Infinitesimals

Now I want to talk about an even stranger infinitesimal. I will denote it by Z. The square of Z is zero!

$$Z \times Z = 0.$$

You can think that Z is so very small that when you multiply Z by itself it just vanishes. (After all a very small number times a very small number is a much smaller number, so Z is just the limiting case of this phenomemnon.) But Z itself is distinct from zero, and Z/2 is smaller than Z, Z/4 is smaller than Z/2 and so on. You cannot divide by Z.

The infinitesimal Z is perfectly suited to doing calculus. Now why should we dwell on this matter of the calculus? The answer is that it speaks to the fundamental nature of the moment. For calculus is mathematics devised to capture in quantitative terms the dynamics of transition from moment to moment to moment. Newton found the calculus in his attempt to fathom gravity and the motions of the moon, the planets, the stars and the objects of our everyday space. The transition from moment to moment is mediated in this model by an infinitesimal and so the next moment of time T+Z is infinitely close to the present moment T. So close that they are glued together, and yet they are distinct.

In calculus we are concerned with measuring how a function of a variable t changes when the t value changes by an infinitesimal amount. A function f(t) is just a rule that assigns to each value of t a new value f(t). For example, f(t) could be the velocity of a bird flying by your window at the time t.

We are concerned with the difference f(t+Z) — f(t), the amount that f(t) changes in an infinitesimal amount of time. We make the assumption that an infinitesimal change in t will result in an infinitesimal change in f(t). Thus we write

$$f(t+Z) - f(t) = f'(t)Z$$

and call f′(t) the derivative of f at t. We say that a function is differentiable at x if there is a well-defined new function f′(t) that satisfies this equation for any infinitesimal Z.

A simple example of differentiablility is illustrated by the function f(t)= t^2= tt. Here we have f(t+Z)-f(t) = (t+Z)(t+Z)-tt = tt + Zt +tZ +ZZ - tt = tZ+Zt =2tZ. Note that ZZ=0.

Thus

$$f(t+Z) - f(t) = 2tZ.$$

By our definition, f′(t)=2tZ in this example.

We have used this calculation involving the ideal infinitesimal Z to reason to the very real answer that an (accelerating) bird whose position is given by the square of the time will have velocity equal to twice the value of that time. This is innocent enough for the bird at time zero, but I would not want to be in the way of that same bird after one minute. If you replace the bird by a safe dropped from the tenth story of your apartment building, then the mathematics is essentally the same, and we all know that it is a bad idea to be under that safe. The calculus lets you find the exact values related to your safety from accelerating birds and falling safes! It also allows one to find out about the orbits of the planets and the behaviour of electrons, radio waves and all manner of varying things.

The whole structure of the calculus can be based on this simple system of infinitesimals. Eventually this will be the way calculus is taught in the schools.

To see how efficient this is, the reader already familiar with the calculus will be astonished at the simplicity of the proof of the chain rule:

Let $F(x) = f(g(x)$.
Then $F(x+Z)= f(g(x+Z))= f(g(x)+g'(x)Z) =f(g(x)) + f'(g(x))g'(x)Z$
Thus $F'(x) = f'(g(x))g'(x)$.

This is the so-called chain rule, a rule for differentiation that is often left unproved in calculus texts due to the complexity of the usual derivation.

We have, in this section, gone very quickly into the basics of the calculus.

The real point of this section is the structure of the infinitesimal change. By allowing an imaginary number Z with the property that Z x Z = 0, we illuminate the structure of the moment and the movement of time. It remains for the reader and the author to continue this meditation in both mathematical and cybernetic directions.

Now we are going to look more carefully at the previously mentioned concept of zero numbers. Zero numbers arise naturally in the context of Spencer-Brown's Laws of Form [1]. Laws of Form has been a constant topic in this column because it is a cogent mathematical expression of the idea that "a universe comes into being in the making of a distinction". That universe is the universe of an observing system. The universe itself (in the largest sense) is one turn of the world into a that that is seen and a that that is seen. Those thats (in a plethora of reference) are different from each other, and yet they are the same. The difference between the universe that sees and the universe that is seen is infinitesimal. Or perhaps it is a power of zero!

Zero Numbers

Recall that zero numbers are distinct powers of zero. Here is how zero numbers arise for Spencer-Brown. He begins by letting <> (the mark in his notation) denote the number one. Then <><> denotes two and <><><> denotes three and so on with a row of n marks denoting the number n: n = <><><>...<>, 0=, 1=<>, 2=<><>, 3=<><><>, 4=<><><><>, 5=<><><><><> ad infinitum.

It turns out that 0^n = 0x0x0x0x...x0 = < <><><>...<> >. That is, 0^n is denoted by a bracket around a row of n marks. To see this we need to do a little arithmetic in the style of Spencer-Brown.

In the Spencer-Brown arithmetic (henceforward referred to as SB arithmetic) it is given that <<A>> = A for any A, <<A><B>>C = <<AC><BC>> for any A,B,C and AB = BA for any A and B. The reasons for these rules will become quickly apparent. Addition is defined by A+B = AB. Multiplication is defined by AxB = <<A><B>>.

Watch the results of 2x3: 2x3 = <<2><3>> = << <><> > <3> > = < < <<3>> <<3>> > > = <<3>><<3>> = 3 3 = <><><> <><><> = <><><><><><> =6.

It should now be clear how Crossing: <<A>>=A and Transfer: <<A><B>>C=<<AC><BC>> interact to create the structure of multiplication in SB.

Now look at 0x0 = <<0><0>>. Since 0 = "nothing", we conclude that 0x0 = <<><>>. This justifies our interpretation of 0^n as < <> <> <> ... <>>. Zero numbers live naturally in the SB arithmetic.

Now consider <3>A for any number A: <3>A = < <><><> >A = <<A><A><A>> = AxAxA = A^3. By using Transfer, we have found that <n>A = A^n for any numbers n and A. Exponentiation is a natural consequence of the interaction of zero numbers and ordinary numbers!

Zero numbers are quite analogous to infinitesimals. But they are a realm unto themselves. For the product of a zero number and any ordinary number is simply zero. And the sum of a zero number and any ordinary number is simply that number. Thus it is with some joy that we can point to the exponential relation

0^n.A=A^n

as a genuine form of communication between the void-realm of the zero numbers and the solid realm of ordinary number reality.

Now Together

We end this essay with an injunction to the reader that she think on both infinitesimals and the zero numbers and feel the presence of this realm of number prior to number, prior to geometry. A realm that has a delicate beginning and powerful possibilities for calculation and for articulation of the subtlety of the moment, an articulation of the subtlety of the relation between the observer and the observed who are the same and yet not the same.

I am sure that all that we have said seems tantalizing to some readers (as it does to this author). For one begins to feel that here in the presence of infinitesimals and zero numbers one is at the gateway to genuinely new insights into the the relationship of mathematics and reality. Surely the infinitesimals have something to tell us about the very small! They should tell us about the Planck scale in quantum physics, about ultimate units of space and of time. In asking these questions we enter into a construction site. Some aspects are well worked out, like all the known uses of the calculus. Other aspects take a new flavor, and some are just beyond the reach of saying. Let me finish with a couple of examples.

It is well known to mathematicians and physicists that infinitesimals in different spatial directions should not commute with one another! In particular it is well-known that if dx and dy represent infinitesimals pointing in perpendicular spatial directions then we should have the rule dxdy = -dydx. Now don't ask why! (Unless you want to take courses for five years.) But from the point of view of square zero infinitesimals this is not a mystery. After all we are given that dxdx = 0 and dydy=0. And surely (dx +dy) should also be a square zero infintesimal since it just points in the sum of the spatial directions of dx and dy. But this means that 0 = (dx +dy)(dx +dy) = dxdx + dxdy +dydx + dydy = dxdy + dydx. So 0= dxdy + dy dx. This is the same thing as saying that dxdy = -dydx.

Down there in the tiniest parts of the world processes do not commute with one another. No way around this, and this non-commuting is where quantum mechanics starts. We can start with the idea that space is non-commutative in the small,understanding that this comes from the gluing of moment to moment in the realm of the observer, and begin to articulate a quantum story of the world. It is work in progress. See References [3] through [8]. Even in these references I had not quite understood the importance of infinitesimals. The invitation is out. Think on these matters. Let the author of this essay know your thoughts. There is a lot to come.

V. PostScript

For more about the Spencer-Brown Arithmetic, the reader can consult [1],[9] and [13],[14],[15]. For an extraordinary construction of all numbers great and small (infinite and infinitesimal) read the amazing book *On Numbers and Games* by John Horton Conway [10]. The book by Henle and Kleinberg [11] is an excellent introduction to modern theories of infinitesimals based on the work of the logician Abraham Robinson.

Finally one of the earliest American proponents of infinitesimals was the philoshopher/ mathematician Charles Saunders Peirce. We let him have the last word: ([12] Vol. 4, p. 355)

> ... the conception of continuity involves no contradiction and cannot be dispensed with. From this discussion flows the irresistible consequence that infinitesimals exist wherever there is continuity. The logic of the differential calculus is set forth from this point of view. The doctrine of lim its is not de nied. .... But it is shown to be an un nec es sar ily round about way.

Infinitesimals exist wherever there is continuity. Let us be direct from now on.

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