Statistical Methods for Testing Cointegration

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Introduction to Cointegration

- **Definition**: Cointegration exists when two or more non-stationary time series (I(1)) share a stable long-term equilibrium relationship.
- **Mathematical Intuition**: For series $X_t, Y_t \sim I(1)$, if there exists β such that $Y_t - \beta X_t \sim I(0)$, they are cointegrated.
- **Application**: Critical for pairs trading, where price differences revert to a mean.

Engle-Granger Two-Step Method

- **Step 1**: Run OLS regression: $Y_t = \alpha + \beta X_t + \epsilon_t$.
- **Step 2**: Test if residual ϵ_t is stationary using ADF test:

$$\Delta\epsilon_t = \gamma\epsilon_{t-1} + \sum_{i=1}^{p} \phi_i \Delta\epsilon_{t-i} + u_t$$

- **Intuition**: If $\epsilon_t \sim I(0)$, β defines the cointegrating vector.
- **Pros**: Simple, suitable for two series.
- **Cons**: Sensitive to regression order, limited to single cointegration.

Johansen Test

- **Framework**: Based on VAR model for $Y_t = (Y_{1t}, \ldots, Y_{nt})'$:

$$\Delta Y_t = \Pi Y_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta Y_{t-i} + \epsilon_t$$

- **Test**: Estimate rank r of $\Pi = \alpha\beta'$ using Trace or Maximum Eigenvalue tests.
- **Trace Test Statistic**:

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^{n} \ln(1 - \hat{\lambda}_i)$$

- **Intuition**: r indicates the number of cointegrating relationships.
- **Pros**: Handles multiple series, robust to variable order.
- **Cons**: Computationally intensive, sensitive to lag selection.

Phillips-Perron and ADF Tests

- **Phillips-Perron (PP)**: Extends Engle-Granger by correcting for autocorrelation and heteroskedasticity in residuals using non-parametric methods.
- **Augmented Dickey-Fuller (ADF)**: Tests stationarity of residuals:

$$\Delta\epsilon_t = \alpha + \gamma\epsilon_{t-1} + \sum_{i=1}^{p} \phi_i \Delta\epsilon_{t-i} + u_t$$

- **Intuition**: If $\gamma < 0$ and significant, residuals are stationary, implying cointegration.
- **Pros**: PP is robust to heteroskedasticity; ADF is widely used.
- **Cons**: PP has lower power; ADF sensitive to lag choice.

KPSS Test

- **Approach**: Tests null hypothesis of stationarity (opposite to ADF).
- **Statistic**: Based on cumulative sum of residuals:

$$\text{KPSS} = \frac{1}{T^2} \sum_{t=1}^{T} S_t^2 / \hat{\sigma}^2, \quad S_t = \sum_{i=1}^{t} \epsilon_i$$

- **Intuition**: Complements ADF by testing if residuals are I(0).
- **Pros**: Enhances robustness when used with ADF.
- **Cons**: Sensitive to small samples.

ARDL Bounds Test and Others

- **ARDL Bounds Test**: Tests cointegration in ARDL model, flexible for I(0) or I(1) series.
- **CRDW**: Uses Durbin-Watson statistic to infer cointegration (less common).
- **Machine Learning**: PCA or neural networks to identify cointegration in high-dimensional data.
- **Intuition**: ARDL is robust to mixed integration orders; ML methods handle complex systems.
- **Pros**: ARDL is flexible; ML scales to large datasets.
- **Cons**: ARDL requires careful specification; ML lacks interpretability.

Comparison of Methods

- **Single vs. Multivariate**:
  - Engle-Granger, PP, ADF: Two series.
  - Johansen: Multiple series.
- **Flexibility**:
  - ARDL: Handles mixed I(0)/I(1) series.
  - Others assume I(1).
- **Robustness**:
  - KPSS + ADF: Complementary for residual stationarity.
  - Johansen: Robust to variable order.
- **Application**: Engle-Granger for pairs trading; Johansen for macroeconomic systems.

Takeaways

- Cointegration ensures long-term equilibrium for non-stationary series, key for pairs trading.
- **Engle-Granger**: Simple, two-series method but sensitive to regression order.
- **Johansen**: Robust for multiple series, ideal for complex systems.
- **PP/ADF/KPSS**: Complementary tests for residual stationarity.
- **ARDL**: Flexible for mixed integration orders.
- Choose method based on number of series, data properties, and computational resources.