

Ljung-Box Test in ARIMA Modeling

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Introduction

- ▶ Model diagnostics are essential in time series analysis.
- ▶ Ljung-Box test examines whether residuals are independently distributed.
- ▶ A good ARIMA model should have white noise residuals.

Ljung-Box Test Definition

Test Statistic:

$$Q = n(n+2) \sum_{k=1}^h \frac{\hat{\rho}_k^2}{n-k} \quad (1)$$

where:

- ▶ n is the sample size,
- ▶ h is the number of lags considered,
- ▶ $\hat{\rho}_k$ is the autocorrelation at lag k .

Hypothesis Testing:

- ▶ H_0 : The residuals are white noise (no autocorrelation).
- ▶ H_1 : The residuals exhibit autocorrelation.

Intuition Behind the Ljung-Box Test

- ▶ Checks whether residuals from a fitted model are independently distributed.
- ▶ If residuals show autocorrelation, the model may be misspecified.
- ▶ A high Q value suggests significant autocorrelation.
- ▶ Q follows a χ^2 distribution with h degrees of freedom.

Choosing the Number of Lags h

- ▶ A common choice: $h \approx \log(n)$.
- ▶ Too few lags may miss hidden autocorrelation.
- ▶ Too many lags can lead to over-rejection of H_0 .

Example Calculation

Given:

- ▶ $n = 100$ observations,
- ▶ Autocorrelations at first three lags: $\hat{\rho}_1 = 0.1$, $\hat{\rho}_2 = 0.08$, $\hat{\rho}_3 = 0.05$,
- ▶ Consider $h = 3$.

Compute Q :

$$\begin{aligned} Q &= 100(102) \left(\frac{0.1^2}{100-1} + \frac{0.08^2}{100-2} + \frac{0.05^2}{100-3} \right) \\ &= 100(102) \left(\frac{0.01}{99} + \frac{0.0064}{98} + \frac{0.0025}{97} \right) \\ &= (\text{final computed value}) \end{aligned}$$

Interpreting the Results

- ▶ Compare Q to the χ_h^2 critical value.
- ▶ If $Q > \chi_h^2$, reject H_0 (residuals are not white noise).
- ▶ If $Q < \chi_h^2$, residuals show no significant autocorrelation.
- ▶ p-value interpretation:
 - ▶ $p < 0.05 \rightarrow$ significant autocorrelation.
 - ▶ $p > 0.05 \rightarrow$ residuals resemble white noise.

Summary

- ▶ The Ljung-Box test helps verify model adequacy.
- ▶ A well-fitted ARIMA model should have white noise residuals.
- ▶ Choosing the right number of lags is critical.
- ▶ Always complement statistical tests with visual diagnostics.