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比特幣選擇權隱含風險中立機率密度

之平滑尾部導取

Extracting Smooth Tails of Option Implied Risk-neutral Densities

in the Bitcoin Market

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摘要

本研究旨在探討風險中立機率密度（Risk-neutral Density, RND）尾部配適方法對比特幣選擇權市場報酬率預測能力之影響，並提出創新的單點加斜率配適法，與 Birru & Figlewski (2012) 進行實證比較。

本研究以比特幣選擇權交易平台 Deribit 於 2021 年 1 月至 2024 年 4 月的交易數據為基礎，結合廣義柏拉圖分布（Generalized Pareto Distribution, GPD），建立完整的尾部延伸模型。透過比較距到期日 1 天與 7 天之選擇權樣本，檢驗 RND 統計特徵對現貨報酬率的預測能力。實證結果顯示，在計算效率方面，單點加斜率配適法較雙點配適法平均提升 10.95%。在日報酬率預測中，單點配適法建構之模型以偏態（Skewness）、中位數（Median）及前期報酬率為顯著預測變數（R-squared = 0.0130），表現優於雙點配適法（R-squared = 0.0098），此結果與 Conrad 等人 (2013) 及 Liu 與 Tsyvinski (2021) 關於偏態與動量效應的研究相呼應；在週報酬率預測方面，兩種方法表現接近，最佳預測變數組合為超額峰態（Excess Kurtosis）、中位數（Median）及加密貨幣恐懼與貪婪指數。超額峰態的預測力呼應 Amaya 等人 (2015) 研究結果，而恐懼與貪婪指數的顯著影響則與 He 等人 (2023) 的發現一致。樣本外預測結果顯示，無論在日報酬率或週報酬率預測中，兩種方法均產生正值的樣本外 R 平方，且單點配適法表現優於雙點配適法，顯示我們的預測模型具實質經濟價值 (Campbell & Thompson, 2008)。

本研究主要貢獻在於提出單點加斜率配適法，此方法提升 RND 尾部配適的效率與穩定性，更驗證了 RND 統計特徵在高波動性市場中的應用價值。

關鍵字：加密貨幣、比特幣、選擇權、隱含波動率、風險中立機率密度、廣義柏拉圖分布

Abstract

This study introduces an innovative approach to extracting smooth tails of risk-neutral density (RND) functions in Bitcoin options markets. We propose a novel one single point with one slope method that addresses limitations in Birru & Figlewski (2012), the kinks and computational inefficiencies.

Employing Deribit trading data (January 2021-April 2024) and integrating the Generalized Pareto Distribution (GPD), we demonstrate that our method enhances computational efficiency by 10.95% while ensuring smooth tail transitions without discontinuities. Empirical analysis reveals distinct predictive patterns across time horizons: for daily returns, our method identifies skewness, median, and lagged returns as significant predictors (R-squared = 0.0130), outperforming the conventional approach (R-squared = 0.0098); for weekly returns, excess kurtosis, median, and the Cryptocurrency Fear and Greed Index emerge as key predictors (Amaya et al., 2015; Conrad et al., 2013; M. He et al., 2023; Liu & Tsyvinski, 2021). Out-of-sample tests confirm the economic value of our approach with positive R-squared values across prediction horizons (Campbell & Thompson, 2008).

This research contributes both methodologically through enhanced RND estimation and empirically by revealing how cryptocurrency market risk characteristics vary across temporal dimensions, providing insights for risk management and investment strategies in highly volatile markets.

Keywords: Cryptocurrency, Bitcoin, Option, Implied Volatility, Risk-neutral Density, Generalized Pareto Distribution

Acknowledgement

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1. Introduction

1.1 Research Background and Motivation

Risk assessment has remained a core focus in the financial industry. In particular, with the rapid expansion of the cryptocurrency ecosystem, Bitcoin has witnessed explosive growth in its derivatives market (Akyildirim et al., 2020). According to The Block (*The Block*, 2025), Bitcoin futures and options trading volume exceeded $21 trillion in 2024, with options trading growing at 130% annually, far outpacing traditional derivatives markets. This reflects increasing demand for cryptocurrency risk management tools while providing researchers a unique perspective on price discovery in emerging markets (Zulfiqar & Gulzar, 2021).

Cryptocurrency markets differ significantly from traditional financial markets in several aspects such as 24/7 trading, extreme volatility, decentralized structure, and unique investors' composition. Bitcoin's historical annualized volatility frequently exceeds 100%, substantially higher than the 15-20% volatility of traditional stock indices (Liu & Tsyvinski, 2021). This extreme volatility makes risk management crucial while exhibiting specific challenges for interpreting option price information.

As a derivative financial instrument, options contain rich market information in their prices. Call options grant holders the right to purchase the underlying asset at a predetermined price at a specific future time while put options gives holders the right to sell. Their non-linear payoff structure reflects market expectations of future trends and volatility risk assessments (Hull, 2021). Option prices reveal market consensus on risk expectations and provide insights into market microstructure and investors' behavior (Bakshi et al., 2003). While the Black-Scholes model (1973) provides theoretical foundation for option pricing, the normal distribution assumption fails to capture the fattailed distribution and negative skewness commonly observed in financial markets, particularly in high-volatility cryptocurrency markets (Chordia et al., 2021).

In the option pricing theory, the Black-Scholes model (1973) assumes geometric Brownian motion and applies no-arbitrage principles. Nevertheless, observed option prices often deviate from theoretical values, with implied volatility exhibiting systematic differences across strike prices, forming "volatility smile" or "volatility skew" (Rubinstein, 1994). This indicates that market expectations differ from the log-normal distribution assumed by Black-Scholes, particularly in the tail regions.

Extracting the risk-neutral density (RND) from option prices captures the market's complete expectation of future price distributions. According to Breeden & Litzenberger (1978), the RND can be derived by taking the second derivative of option prices with respect to strike prices. To extract the RND, implied volatility is converted from a finite number of observed market option prices through the Black-Scholes model. To obtain the continuum of implied volatility curve, Hagan & West (2006) indicates that quadratic splines can avoid the risk of overfitting while maintaining curve smoothness to ensure model robustness and reliability, particularly in the case of high market volatility. Haslip & Kaishev (2014) shows that when dealing with complex derivative financial products such as lookback options, quadratic splines combined with Fourier transforms can provide efficient and accurate pricing results, achieving a good balance between computational efficiency and precision, though with the risk of discontinuous first derivatives. Bliss & Panigirtzoglou (2004), Figlewski (2008) and Monteiro et al (2008) demonstrate the interpolation of implied volatility using cubic spline functions. The cubic spline method has the advantages of high computational efficiency and relatively simple implementation, providing reasonable fitting results under most market conditions. Nevertheless, cubic splines only guarantee the continuity of the first derivative, while the second derivative may reveal discontinuous at certain nodes, resulting in non-smooth fitting when dealing with extreme volatility. Since cubic splines may lead to insufficiently smooth RNDs, quartic spline functions with a single knot have been extensively adopted for implied volatility curve fitting (Figlewski, 2008; Birru & Figlewski, 2012; Reinke, 2020). This approach ensures continuity of both first and second derivatives to construct smooth RND curves.

Due to limited market option prices, especially in deep out-of-the-money regions, RND extraction often lacks sufficient tail information. Previous research has developed two major streams of the RND tail-fitting methods: non-parametric and parametric approaches. Non-parametric methods make minimal assumptions about the underlying asset. Bondarenko (2000) proposes a non-parametric method for deriving the RND from option prices, finding daily RND changes correlate with index performance. Grith et al. (2012) explores kernel smoothing and spline functions in the RND estimation, highlighting their flexibility in capturing complex distributional features like skewness and multimodality. Monteiro & Santos (2022) addressed local constraint limitations in kernel-based estimation by imposing broader no-arbitrage constraints using the Heston model. Dong et al. (2024) introduced the Implied Willow Tree method, reconstructing complete risk-neutral processes directly from cross-maturity option data without preset parametric models.

Parametric methods commonly employ Extreme Value Theory (EVT) to extend RND tails. Figlewski (2008) pioneered using Generalized Extreme Value Distribution (GEV) to fit two points in each tail of the RND, requiring continuity conditions for density and cumulative density functions. Birru & Figlewski (2012) further replaced GEV with Generalized Pareto Distribution (GPD), discovering significant left skewness in the RND regardless of market volatility. Their findings confirmed that even during extreme market turbulence, the mean of RND remains close to the futures price, indicating effective no-arbitrage relationships. Other approaches include mixed distribution methods, which decompose the RND into core and tail components. Glatzer & Scheicher (2005) combined log-normal distributions for the core with GPD for the tails, demonstrating effectiveness in Eurozone bond markets. Markose & Alentorn (2011) parameterized the RND tails using Generalized Extreme Value Family to capture market expectations of extreme events. Monteiro et al. (2008) proposed density function extrapolation using cubic spline functions with non-negativity constraints and exponential functions for the tail extrapolation, though computationally simple but potentially limited in capturing complex tail features. Recent developments include novel parametric methods that do not rely on extreme value theory. Orosi (2015) established appropriate functional forms with parameter constraints to produce well-behaved risk-neutral density estimates. Uberti (2023) developed a semi-parametric estimation method combining parametric stability with non-parametric flexibility. Y. Li et al. (2024) introduced a Lognormal-Weibull mixture model offering improved performance when measuring skewness and analyzing multi-peak RNDs, demonstrating the continuous refinement in estimation methodology.

On top of implied volatility, RND contains vastly useful information in higher-order moments (i.e. skewness and excess kurtosis), which have been extensively used to predict asset returns. Bali & Murray (2013) and Conrad et al. (2013) found significant negative relationship between risk-neutral skewness and future stock returns. Similar findings appear in commodity futures (Fuertes et al., 2022), oil markets (Cortés et al., 2020), and foreign exchange (Chen et al., 2018). Recently, Böök et al. (2025) demonstrated that volatility, skewness, and kurtosis indicators derived from options data outperform historical return-based indicators in predicting stock risk premiums. While these relationships are established in traditional markets, research in cryptocurrency markets remains limited.

This study extends the framework of Birru & Figlewski (2012) to derive the entire RND because GPD is particularly adept at capturing the behavior of the extreme events. Nonetheless, we find several constraints in Birru & Figlewski (2012) when applied to cryptocurrency markets. First, Birru & Figlewski (2012) introduces discontinuities or kinks at the connection points. Also, it increases computational complexity, potentially affecting estimation stability.

To mitigate the limitations in Birru & Figlewski (2012), we propose a method which fits each tail of the RND using one single point with one slope to ensure the continuity without kinks for both the cumulative distribution function and density function. Our proposed approach imposes two essential conditions: (1) at the joining point, the cumulative probability of GPD must equal that of the empirical RND; and (2) the slope of the GPD density function must match the empirical RND's slope at this point. To further validate the effectiveness of our proposed method, this study examines the predictive power of the RND moments, such as skewness and excess kurtosis, for Bitcoin returns, in comparison with Birru & Figlewski (2012). Based on Bitcoin options trading data from the Deribit platform from January 2020 to April 2024, our proposed method significantly outperforms Birru & Figlewski (2012) in terms of daily returns and slightly surpasses in weekly returns. Further, the robustness tests using rolling window estimation (Campbell & Thompson, 2008) also confirmed our proposed method perform superior, in contrast with Birru & Figlewski (2012).

In the daily return prediction, skewness, median, and lagged returns are identified as significant predictors, with superior explanatory power (R-squared = 0.0130), compared to Birru & Figlewski (2012) (R-squared = 0.0098). The negative skewness coefficient aligns with Conrad et al. (2013), indicating higher pricing for negative risk. The significant predictive power of lagged returns supports Liu & Tsyvinski's (2021) argument about momentum effects in cryptocurrency markets. For weekly returns, both methods identify excess kurtosis, median, and the Crypto Fear and Greed Index as optimal predictors. Excess kurtosis exhibits a significant negative relationship with returns, consistent with Amaya et al. (2015), indicating that heightened tail risk leads to subsequent return reversals in longer horizons. The Crypto Fear and Greed Index shows a significant negative relationship with future returns, supporting M. He et al. (2023) that market sentiment serves as a contrarian indicator in cryptocurrency markets.

Our methodological innovation transcends computational efficiency by more precisely capturing extreme risk characteristics in the Bitcoin options market. Prior studies confirm RND higher-order moments' predictive power (Chang et al., 2013; Conrad et al., 2013), yet their applicability to cryptocurrency markets across varying time scales remains underexplored. Additionally, cryptocurrency investors exhibit distinctive behavioral patterns with heightened susceptibility to emotional factors (M. He et al., 2023). Integration of sentiment indicators into prediction models may reveal market dynamics beyond the explanatory scope of traditional financial theories.

This research make two main contributions to the literature as follows:

1. Methodological breakthrough: Our proposed one single point with one slope approach ensures no kinks in RND tail fitting through first derivative continuity conditions while reducing computational complexity compared to Birru & Figlewski (2012). Empirical results demonstrate superior smoothness in tail transitions and 10.95% faster computational efficiency, maintaining theoretical consistency with enhanced numerical stability.

2. Multi-time scale prediction framework: By comparing daily and weekly return

prediction models, this research reveals the differential predictive capabilities of RND characteristics across different time scales. We find that in daily return prediction, skewness (Conrad et al., 2013), median, and lagged returns (Liu & Tsyvinski, 2021) have significant predictive power; while in weekly return prediction, excess kurtosis (Amaya et al., 2015), median, and the Crypto Fear and Greed Index (M. He et al., 2023) become key predictive factors. These findings provide valuable insights into how predictive relationships in cryptocurrency markets vary across temporal dimensions.

As institutional investors continue to enter the cryptocurrency market, demand for professional risk management tools tends to grow substantially. Our proposed approach addresses this critical market need by offering a methodologically superior framework for RND estimation that eliminates kinks at connection points while enhancing computational efficiency. By establishing a more robust RND estimation framework, we advance the theoretical understanding of cryptocurrency market risk characteristics while providing practitioners with more reliable tools for risk assessment and derivatives pricing.

1.2 Thesis Structure

This research is divided into six chapters as follows:

"Chapter 1: Introduction": Explains the research background, motivation, and contributions.

"Chapter 2: Literature Review": Explores the theoretical development of risk-neutral density (RND), estimation methods, and related empirical research.

"Chapter 3: Data": Introduces research data sources and market overview.

"Chapter 4: Research Methodology": Details the research methods, including risk-neutral density calculation, tail fitting methods, and empirical regression model design.

"Chapter 5: Empirical Results": Analyzes regression results and compares performance differences between different fitting methods.

"Chapter 6: Conclusion": Summarizes the empirical results of this research and outlines directions for improvement in subsequent research.

2. Literature Review

2.1 Fundamental Theory of the Risk-neutral Density (RND)

The risk-neutral Density (RND) was first proposed by Breeden & Litzenberger (1978). They demonstrated that market expectations regarding the future price distribution of underlying assets could be extracted from option prices. This pioneering research opened a new direction for option pricing theory. Shimko (1993) further proposed a technique of converting market option prices into implied volatility space for interpolation. This method leverages the characteristic that implied volatility curves are smoother and more continuous than price data, helping to improve the accuracy of the RND estimation.

Christoffersen et al. (2013) conducted an in-depth exploration of the predictive power of option-implied information, discovering that risk-neutral skewness can effectively predict the direction and magnitude of future returns. Chang et al. (2013) approached from another angle, studying the relationship between risk-neutral kurtosis and market returns, indicating that higher risk-neutral kurtosis often presages greater future market volatility.

In extreme market scenarios, Birru & Figlewski (2012) conducted research on S&P 500 index options during the 2008 financial crisis, finding that the RND shapes undergo significant changes as market stress increases. To more accurately describe such extreme situations, they adopted the Generalized Pareto Distribution (GPD) to refine the RND tail estimation, a method that demonstrates excellent performance when handling extreme market conditions (Hosking & Wallis, 1987). Jackwerth (2020) further analyzed the risk-neutral probabilities of S&P 500 index options, discovering that markets often require a certain amount of time to fully reflect the impact of major events, providing important empirical evidence for understanding market information efficiency.

2.2 Methods for Deriving the Risk-neutral Density

Figlewski (2008) conducted in-depth research on U.S. market portfolios, proposing a comprehensive framework for risk-neutral density estimation, categorizing it into parametric and non-parametric methods, providing important references for practical applications. McNeil & Frey (2000) proposed an innovative method combining GARCH models with Extreme Value Theory (EVT) to estimate tail risk in financial time series. They particularly emphasized that the Generalized Pareto Distribution (GPD) has unique advantages in modeling extreme events in financial time series.

Orosi (2015) proposed a novel parametric method for estimating risk-neutral probability density functions. By setting appropriate functional forms and imposing constraints on model parameters, this method can produce risk-neutral density estimates with good properties. He et al. (2022) recommended using GPD to estimate the RND tails, a method particularly effective when dealing with extreme market conditions. Recently, Uberti (2023) proposed a new semi-parametric estimation method combining the stability of parametric methods with the flexibility of non-parametric methods, offering a new research direction for the RND estimation and demonstrating the continuous refinement trend in risk-neutral density estimation methodology.

Ammann & Feser (2019) conducted in-depth research on robust estimation methods for risk-neutral moments, proposing improved methods that effectively reduce estimation bias in situations of market noise, extreme price movements, or insufficient liquidity. Hayashi (2020) proposed an innovative method for analyzing risk-neutral probability density functions from volatility smiles, a method that can completely avoid approximation errors that might arise from numerical methods.

Recently, Dong et al. (2024) proposed the innovative Implied Willow Tree (IWT) method, addressing research gaps in reconstructing risk-neutral stochastic processes. Unlike traditional methods that only reconstruct risk-neutral probability density functions, this research directly reconstructs the complete risk-neutral process from cross-maturity market option price data without relying on any preset parametric models. Empirical evidence demonstrates the effectiveness of this method in pricing American options and Asian options, as well as its good capability in handling noise in the original data.

2.3 Moments of the RND

2.3.1 Higher-Order Moments

Higher-order moments of risk-neutral distributions, particularly skewness and kurtosis, play crucial roles in financial market research. Studies by Bali & Murray (2013) and Conrad et al. (2013) demonstrated that risk-neutral skewness exhibits a significant negative relationship with future stock returns, consistent with the theoretical expectation that investors prefer positive skewness. Empirical research by Kim & Park (2018) further confirmed that option-implied skewness is significantly negatively correlated with subsequent stock returns, a relationship that persists even after controlling for multiple firm characteristic variables.

Mei et al. (2017) investigated the impact of realized skewness and realized kurtosis on stock market volatility prediction. The out-of-sample prediction results indicate that both have significant negative effects on future volatility, with realized skewness outperforming realized kurtosis in medium to long-term prediction horizons, while showing limited effectiveness in short-term predictions.

Fuertes et al. (2022) found that risk-neutral skewness (RNSK) plays an important role in commodity futures price prediction. Strategies that involve buying futures with positive RNSK values and selling those with negative RNSK values generate significant excess returns, particularly when futures markets are in contango, providing new perspectives for asset allocation and risk management in commodity futures markets.

Cortés et al. (2020) examined the efficacy of using semi-parametric methods to derive implied risk-neutral density functions from West Texas Intermediate crude oil options. The results indicate that, compared to traditional log-normal distributions, log-SNP distributions more accurately capture the RND of oil prices, with skewness and kurtosis containing important information related to oil market expectations. When the market exhibits negative skewness, it indicates a higher probability of expected declines in the underlying price; high leptokurtosis suggests a higher probability of extreme price changes.

In recent research, Böök et al. (2025) proposed a new method for estimating volatility smiles, deriving robust conditional volatility, skewness, and kurtosis indicators from S&P 500 index short-term options data. These indicators demonstrate excellent performance in predicting U.S. equity risk premiums and market higher-order moments, outperforming equivalent indicators calculated from historical returns in both in-sample and out-of-sample tests.

2.3.2 Tail Risk and Extreme Value Theory

Regarding foundational research on extreme value theory, the threshold exceedance model proposed by Balkema & de Haan (1974) established an important foundation for subsequent research on extreme events in financial markets. Their research demonstrates that when samples exceed a sufficiently high threshold, their distribution converges to the Generalized Pareto Distribution, a finding with critical significance for estimating tail risks in financial markets.

Wang & Yen (2018) demonstrated that option-implied tail risk indicators constructed based on extreme value theory effectively predict future movements of underlying assets. The results reveal that tail risks implied by S&P 500 index and VIX options have significant predictive power for future returns, particularly pronounced during economic recessions. Chen et al. (2018) conducted an in-depth investigation into the relationship between crash risk and risk-neutral probability density functions, finding that higher-order moments of risk-neutral probability density functions possess significant explanatory power in predicting and explaining crash risk and risk premiums. Skewness exhibits a high positive correlation with risk premiums, while kurtosis shows a positive correlation with foreign exchange swap spreads, with higher kurtosis representing an increased probability of extreme market events.

Lehnert (2022) explored the relationship between risk-neutral skewness and hedge fund tail risk. The results indicate that short-selling behavior in index options markets leads to a negative relationship between risk-neutral market skewness and returns. This research challenges the traditional view of option-implied skewness as a downside risk indicator, providing new perspectives for understanding market risk preferences.

Conrad et al. (2013) further extended methods combining GARCH models with mixed normal distributions, particularly suitable for capturing asymmetric volatility characteristics in financial markets. Neumann & Skiadopoulos (2013) conducted in-depth research on higher-order risk-neutral moments of S&P 500 options, finding that market expectations regarding volatility, skewness, and kurtosis exhibit significant predictability, especially during periods of greater market volatility.

2.4 Empirical Applications in Financial Markets

2.4.1 Traditional Financial Markets

Mohrschladt & Schneider (2021) analyzed the differences in implied volatility between in-the-money and out-of-the-money options using high-frequency trading data, finding that in-the-money options contain important market information. Research by Li et al. (2024) indicates that risk-neutral skewness has significant predictive power for future stock returns, particularly pronounced during economic recessions, a finding consistent with the theoretical predictions of Cujean & Hasler (2017).

The latest research by Feng et al. (2024) revealed a strong association between market sentiment and option-implied volatility, especially during periods of higher market uncertainty. Köse et al. (2024) studied the influence of trader behavior on option price formation from a market microstructure perspective, finding that institutional investors' trading behavior significantly impacts the RND shape.

Regarding return prediction, Amaya et al. (2015) found that realized skewness has significant explanatory power for cross-sectional stock returns, with this predictive capability remaining significant after controlling for other risk factors. Bali & Zhou (2016) approached from the perspective of risk and uncertainty, finding a significant association between market uncertainty and expected returns, particularly during periods of higher macroeconomic uncertainty. The pioneering research by Campbell & Thompson (2008) proposed a framework for evaluating the economic value of predictive models and confirms that even seemingly minor predictive capabilities can yield considerable economic benefits.

Jondeau et al. (2020) investigated the predictive power of individual stock skewness for index futures returns. The results demonstrate that this indicator exhibits significant effectiveness in predicting S&P 500 index futures returns. This predictive relationship persists even after controlling for liquidity risk and economic cycle factors, and individual skewness also performs excellently in out-of-sample prediction of index futures returns.

2.4.2 Cryptocurrency Market

In the cryptocurrency market, Zulfiqar & Gulzar (2021) found that options trading provided by major cryptocurrency exchanges offers investors more diverse hedging instruments. Baur & Smales (2022) conducted in-depth research on trading behavior in the Bitcoin futures market, discovering that leveraged fund traders play a key role in the Bitcoin futures market, not only holding the largest positions but typically maintaining a net short position. These traders demonstrate an ability to accurately predict the largest market fluctuations.

López-Cabarcos et al. (2021) conducted an in-depth analysis of the relationship between investor sentiment and price volatility in the Bitcoin market, finding that social media sentiment indicators have significant predictive power for short-term price movements. This research emphasizes the importance of sentiment factors in cryptocurrency market pricing. Research by Chordia et al. (2021) indicated that the RND of the Bitcoin options market often exhibits significant left skewness and excess kurtosis. Akyildirim et al. (2020) further found that the cryptocurrency market exhibits higher volatility when investor fear sentiment rises, showing significant correlation with traditional market volatility indicators.

In research on the dynamic characteristics of cryptocurrency markets, Liu & Tsyvinski (2021) studied the risk and return characteristics of cryptocurrency markets, finding significant momentum effects. Li et al. (2021) further confirmed that the MAX momentum effect is particularly significant in cryptocurrency markets. Specifically, cryptocurrencies that have performed best in the past tend to maintain better performance in future periods, a phenomenon similar to the momentum effect in traditional financial markets but with greater intensity.

Liu & Chen (2024) found that cryptocurrencies with larger market capitalizations exhibit left-skewed characteristics, while those with smaller market capitalizations show right skewness, and asymmetric risk is negatively correlated with future returns, meaning cryptocurrencies with lower skewness typically generate higher returns. Liu et al. (2023) used machine learning methods to predict cryptocurrency returns, confirming that lagged returns possess strong predictive power.

3. Data

3.1 Data

This research collects historical trading data provided by Deribit exchange (*Deribit*, 2025), due to its leadership position in the global cryptocurrency options market. The data collection period covers from January 2020 to April 2024, including daily trading volume, closing prices, implied volatility, spot prices, futures prices and other trading information.

According to statistics from cryptocurrency information service provider The Block (*The Block*, 2025), among the three major trading platforms (Deribit, OKX, and Binance), Deribit not only possesses the largest trading volume, but its open interest also accounts for over 80% of the overall market share (as shown in Figure 3-1). This advantageous position mainly stems from its long operating history and robust market development strategy.

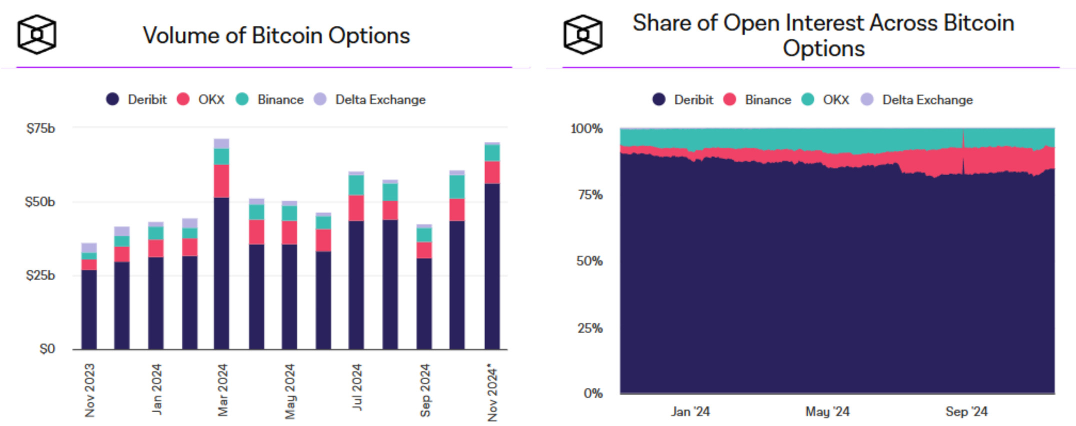

Figure 3-1: Statistics on Bitcoin Options Market Trading Volume and Open Interest
(Data Source: The Block Official Website)

Deribit was established in 2016 with headquarters in the Netherlands. Its name is a combination of "Derivatives" and "Bitcoin," making it the world's first professional exchange to launch cryptocurrency options products.

In terms of product design, Bitcoin European cash-settled options provided by Deribit adopt a 24/7 trading mechanism (*Deribit Options,* 2025), with expiration times uniformly set at 08:00 Coordinated Universal Time (UTC+0). Regarding expiration date selection, the exchange offers diversified product combinations, including: short-term contracts (1-day, 2-day, 3-day), medium-term contracts (1-week, 2-week, 3-week), month-end expiration long-term contracts (January, February, April, May, July, August, October, November), and quarterly expiration long-term contracts (March, June, September, December).

This diversified product design not only satisfies the trading needs of different investors but also helps enhance market liquidity and price discovery efficiency. Particularly after October 2020, Deribit added daily and weekly expiration products, significantly increasing market participation, with trading volume growing markedly as a result.

3.2 Overview of Bitcoin Options Trading Market

This study examines historical Bitcoin options data to investigate relationships between implied volatility and expected price movements through the Risk-neutral Density (RND) function analysis across varying market conditions.

Transaction counts (Figure 3-2) and trading volumes (Figure 3-3) have grown substantially since October 2020, coinciding with Deribit's product diversification strategy that introduced daily and weekly expiration products. Market activity surged again in late 2023, with call option volume reaching historic highs in February 2024 amid rising Bitcoin prices, reflecting market optimism. The increasing trading volume since H2 2023 indicates improved market liquidity and depth, enhancing price discovery efficiency and attracting institutional participation.

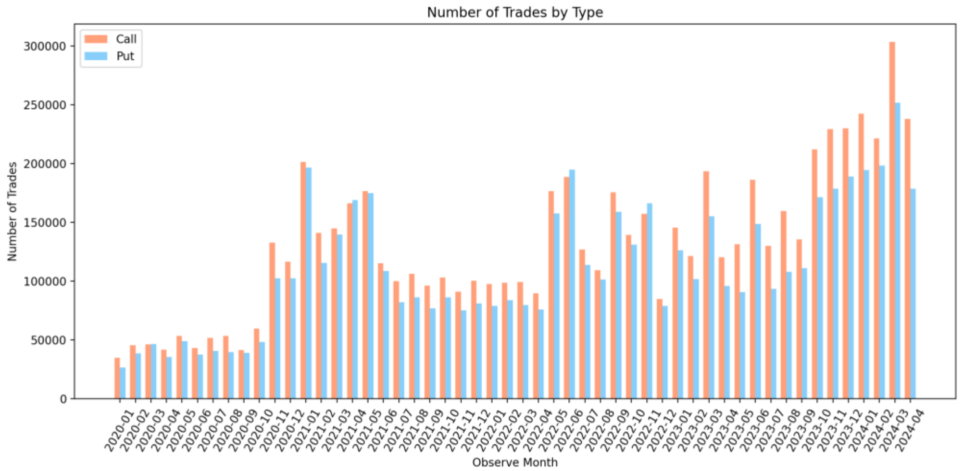

Figure 3-2: Monthly Transaction Counts of Bitcoin Call and Put Options from January 2020 to April 2024

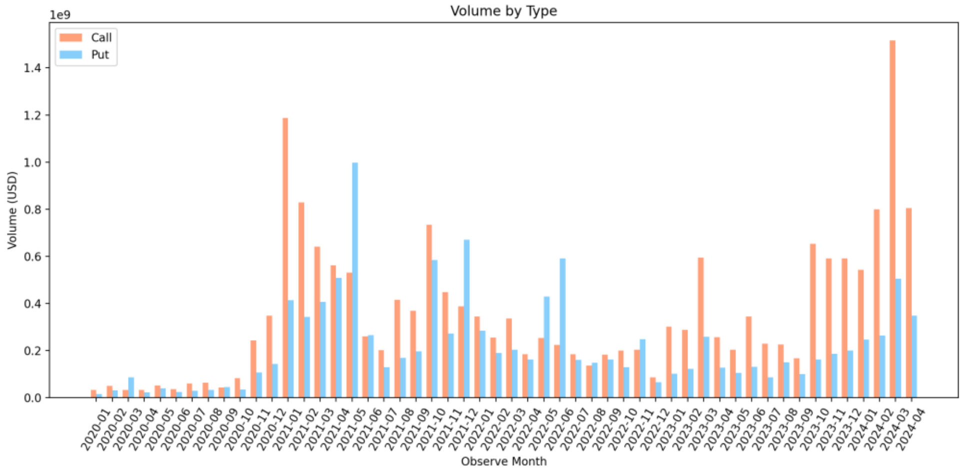

Figure 3-3: Monthly Trading Volume of Bitcoin Call and Put Options from January 2020 to April 2024

Trading activity heat maps reveal distinct patterns. Call options trading (Figure 3-4) concentrates around at-the-money positions (Moneyness ratio $\left(\frac{StrikePrice(K)}{SpotPrice(S)}\right)$ 0.9-1.1), with slightly out-of-the-money options (1.0-1.1) recording the highest volume ($2,072M), indicating investor preference for leveraged positions. Short-term call options (31-90 days) maintain substantial volume across various moneyness levels, while extremely short-term options (<14 days) show significant at-the-money activity, reflecting speculative demand. Long-term (>180 days) call options exhibit limited trading except for anomalous volume ($392M) in deep out-of-the-money positions (>2.0), likely reflecting institutional hedging strategies.

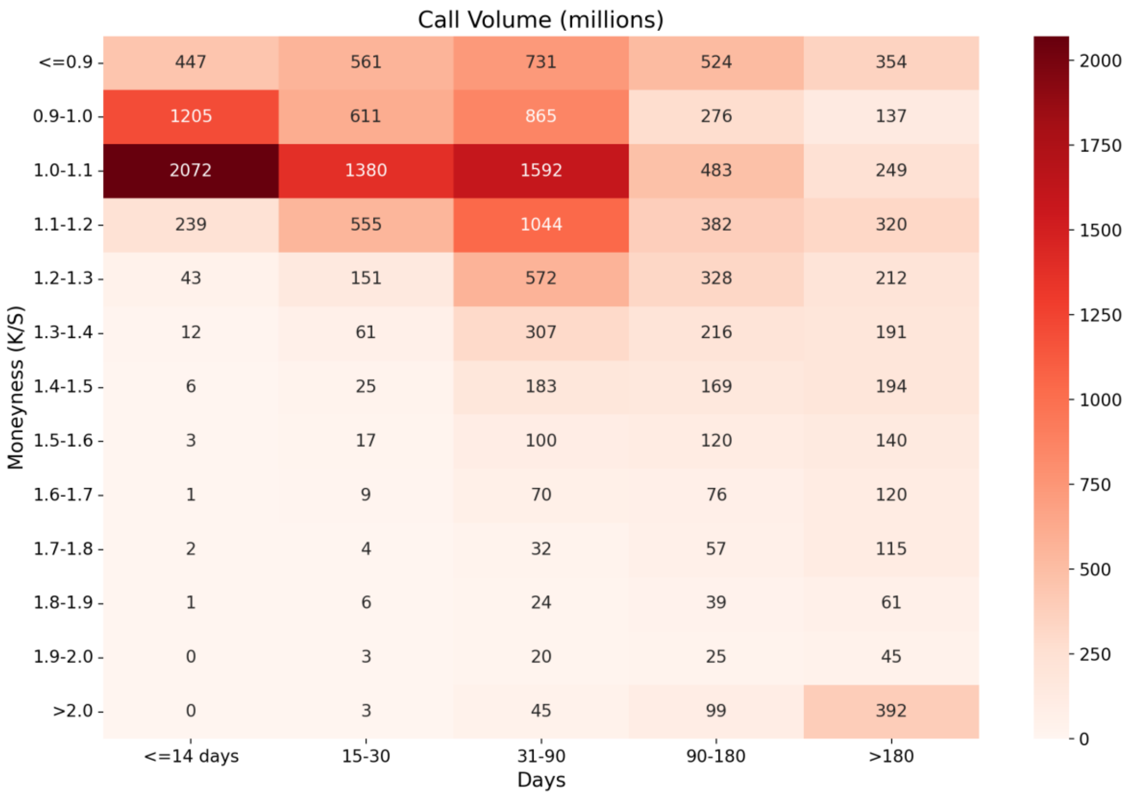

Figure 3-4: Heat Map of Total Bitcoin Call Options Trading Volume from January 2020 to April 2024

Put options (Figure 3-5) demonstrate different characteristics, with trading heavily concentrated near at-the-money (0.9-1.0) positions, peaking at \$1,676M. Deep out-of-the-money puts (≤0.7) show limited activity, suggesting minimal demand for protection against significant price declines. Short-term puts are most active near at-the-money, while medium-term puts (90-180 days) show substantial volume (\$821M) in deep in-the-money positions (>1.1). Long-term puts (>180 days) exhibit relatively uniform volume distribution across moneyness levels.

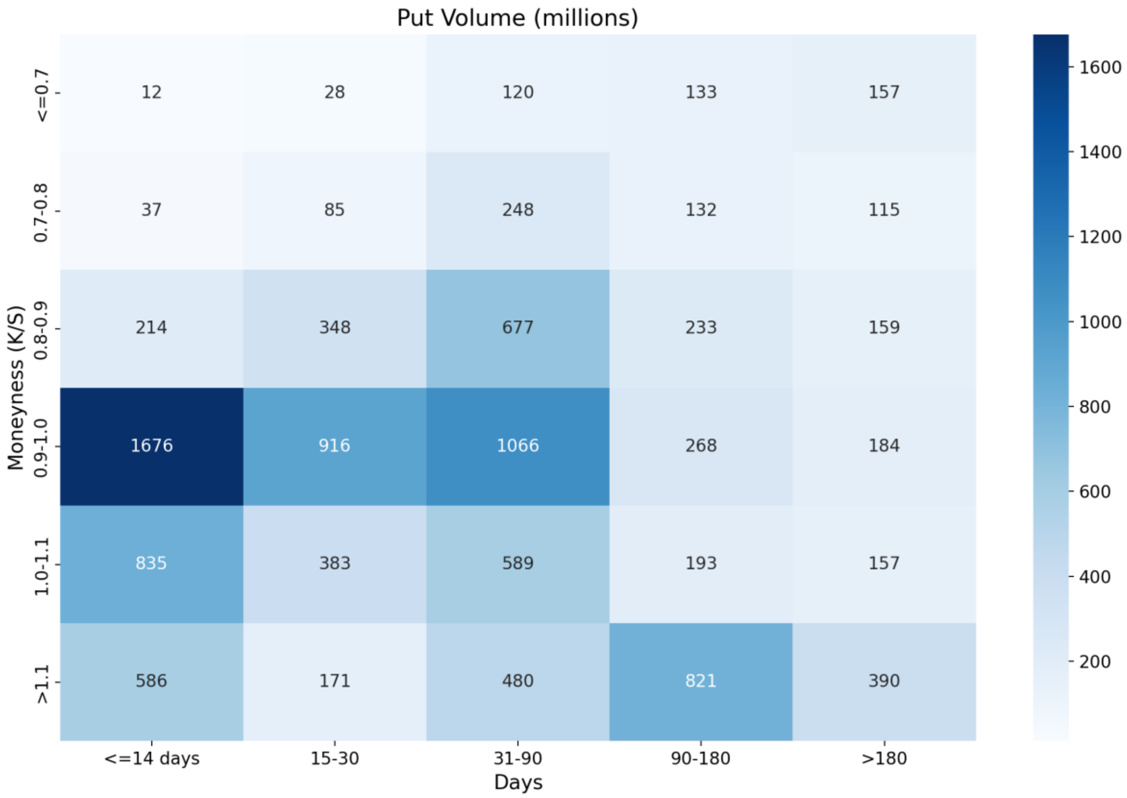

Figure 3-5: Heat Map of Total Bitcoin Put Options Trading Volume from January 2020 to April 2024

The market demonstrates considerable depth with trading concentrated around at-the-money positions and preference for short-term strategies. Higher call option volumes relative to puts reflect bullish market sentiment, while anomalous trading patterns in specific segments likely represent specialized institutional strategies. Despite its significant growth, the market remains characterized by predominantly short-term speculative trading behavior.

4. Research Methodology

4.1 Deriving the Risk-neutral Density (RND)

In the following text, symbols C, P, S, K, and T represent as follows: C is call option price, P is put option price, S is underlying asset current price, K is strike price, r is risk-free rate, T is days to option expiration. This research also uses $f(K)$ to represent the Risk-neutral Density Function (RND) and $F(K) = \int_{-\infty}^{K} f(z)dz$ to represent the Risk-neutral Distribution Function.

The call option price is the expected payoff before its expiration day T, discounted back to the present value. Under risk-neutral conditions, this expected price can be calculated based on risk-neutral probability and discounted using the risk-free rate, as follows:

$$C = \int_{K}^{\infty} e^{-rT} (S_T - K) f(S_T) dS_T \tag{1}$$

Next, by taking the first partial derivative of call option price with respect to strike price, we can derive the risk-neutral distribution function $F(K)$, as follows:

$$\begin{aligned}
\frac{\partial C}{\partial K} &= \frac{\partial}{\partial K}\left[\int_{K}^{\infty} e^{-rT} (S_T - K) f(S_T) dS_T\right] \\
&= e^{-rT}\left[-(K - K)f(K) + \int_{K}^{\infty} -f(S_T) dS_T\right] \\
&= -e^{-rT}\int_{K}^{\infty} f(S_T) dS_T \\
&= -e^{-rT}[1 - F(K)]
\end{aligned}$$

Rearranging terms, we obtain the risk-neutral distribution function $F(K)$:

$$F(K) = e^{rT}\frac{\partial C}{\partial K} + 1 \tag{2}$$

Then, by taking another partial derivative of equation (2) with respect to strike price, we can derive the RND at strike price:

$$f(K) = \frac{\partial}{\partial K}\left[e^{rT}\frac{\partial C}{\partial K} + 1\right] = e^{rT}\frac{\partial^2 C}{\partial K^2} \tag{3}$$

In actual options trading markets, since strike prices are in discrete form, we can use observed option prices and apply Finite Difference Methods (FDM) to obtain approximate solutions to equations (2) and (3). Assuming that at time T to expiration, there are N options with different strike prices in the market, where K_1 represents the lowest strike price and K_n represents the highest strike price. We uses three options with strike prices K_{n-1}, K_n and K_{n+1} to calculate the approximate value centered at K_n, as follows:

$$F(K_n) \approx e^{rT}\left[\frac{C_{n+1} - C_{n-1}}{X_{n+1} - X_{n-1}}\right] + 1 \tag{4}$$

$$f(K_n) \approx e^{rT}\frac{C_{n+1} - 2C_n + C_{n-1}}{(\Delta X)^2} \tag{5}$$

Equations (1) to (5) explain how to theoretically derive the RND between strike prices K_2 and K_{n-1} from a set of call option prices. Similar derivation methods can also be applied to extract the RND from put option prices. For put options, the equivalent expressions corresponding to equations (2) to (5) are as follows:

$$F(K) = e^{rT}\frac{\partial P}{\partial K} \tag{6}$$

$$f(K) = e^{rT} \frac{\partial^2 P}{\partial K^2} \tag{7}$$

$$F(K_n) \approx e^{rT} \left[\frac{P_{n+1} - P_{n-1}}{X_{n+1} - X_{n-1}} \right] \tag{8}$$

$$f(K_n) \approx e^{rT} \frac{P_{n+1} - 2P_n + P_{n-1}}{(\Delta X)^2} \tag{9}$$

In this research, ΔX is a fixed constant value used to construct artificially spaced option prices to fill gaps between discrete strike prices in the market. This approach addresses the problem of sparse or uneven trading data and ensures consistent spacing between strike prices, facilitating numerical calculations through finite difference methods and improving the accuracy of estimation.

4.2 Deriving the RND Using Bitcoin Options

The method introduced in the previous section assumes the existence of a set of option prices that perfectly conform to theoretical pricing relationships (Equation (1)). Nevertheless, when applied to option prices traded in actual markets, several important issues and challenges arise. First, market imperfections in observed option prices must be carefully addressed; otherwise, the derived the RND may exhibit unacceptable characteristics, such as negative values in certain regions. Second, appropriate methods must be found to complete the tails of the RND outside the range from K_2 to K_{n-1}. This section introduces methods proposed in this research and reviewed literature for extracting the RND from market option prices, and explains the techniques we adopt here.

4.2.1 Applying the Black-Scholes Model

In traditional financial markets, option pricing models (such as the Black-Scholes model) typically use risk-free interest rates as parameters, which are generally represented by the yields of low-risk assets like government bonds. Nevertheless, in cryptocurrency markets like Bitcoin, the applicability of risk-free interest rates is limited and therefore not widely used. This is because the Bitcoin market lacks a unified risk-free asset. Due to the decentralized nature of cryptocurrency markets, there are no widely accepted risk-free assets like government bonds, making it difficult to determine a single universal risk-free rate, thus limiting its applicability in this market.

Additionally, Bitcoin price volatility is far higher than traditional assets. This highly volatile characteristic has a more significant impact on option prices than risk-free interest rates, causing traders to focus more on changes in implied volatility rather than risk-free rates. Furthermore, in cryptocurrency markets, the interest rate environment may be influenced by exchange rules and market supply and demand, not necessarily related to traditional risk-free rates, making traditional interest rate indicators difficult to reflect the actual situation in cryptocurrency markets. Moreover, the cost of holding Bitcoin differs from the cost of holding traditional currencies or assets, including security aspects and technological risks, which are difficult to quantify through risk-free interest rates, further limiting the applicability of risk-free rates in Bitcoin option pricing.

This research uses Bitcoin option trading prices from the Deribit exchange, which has adopted a more suitable model (priced in Bitcoin) to compute option prices, adapting to the characteristics of the cryptocurrency market and meeting trading market needs. To meet research requirements, this study observes the traditional Black-Scholes model (Equation (10)) and compares it with the calculation formula provided by Deribit exchange (Equation (11)). It can be seen that multiplying the Deribit exchange quote $C_{Deribit}$ by the Bitcoin spot price S_0 yields the Bitcoin option price denominated in US dollars.

$$C_{BS} = S_0 \times N(d_1) - Ke^{-rT} \times N(d_2)$$

$$\Rightarrow C_{BS} = S_0 \times \left[N(d_1) - \frac{Ke^{-rT}}{S_0} \times N(d_2) \right] \ and\ F = S_0 e^{rT}$$

$$\Rightarrow \frac{C_{BS}}{S_0} = N(d_1) - \frac{K}{F} \times N(d_2) \tag{10}$$

$$C_{Deribit} = N(d_1) - \frac{K}{F} \times N(d_2) \tag{11}$$

$$where\ d_1 = \frac{\ln\left(\frac{F}{K}\right) + \left(\frac{\sigma^2}{2}\right) \times T}{\sigma \times \sqrt{T}}$$

$$d_2 = d_1 - \sigma \times \sqrt{T}$$

Where C_{BS} is the Black-Scholes call option price (denominated in USD), $C_{Deribit}$ is the Deribit exchange Bitcoin call option price (denominated in Bitcoin), S_0 is the Bitcoin spot price, $N(x)$ is the cumulative distribution function of the normal distribution, K is the strike price, r is the risk-free interest rate, T is the option's time to expiration, F is the Bitcoin futures price, ln is the natural logarithm, σ is the annualized standard deviation.

4.2.2 Calculating Bitcoin Option Implied Volatility

In the Bitcoin options market, traders are predominantly risk-seeking and tend to operate with out-of-the-money (OTM) options, primarily due to their lower cost, high leverage effect, and particularly strong sensitivity to volatility. For buyers, given the extremely high volatility of the Bitcoin market itself, these options are highly attractive to speculators and high-risk-preferring investors. Despite the higher probability of these options expiring worthless, traders are still willing to take on such risks. For sellers, since

Bitcoin price volatility is significantly higher than traditional financial markets, the premium levels for OTM options are usually higher, further enhancing the incentives for seller participation, making OTM options a core tool for many sellers to create stable cash flow. In summary, OTM options have better trading volume and liquidity, and their prices can more efficiently reflect market sentiment. This research uses OTM option trading data to compute implied volatility; nevertheless, to avoid anomalies caused by unreasonable trades in deeply out-of-the-money areas, options data with strike prices below $10 is excluded.

Shimko (1993) proposed converting market option prices to implied volatility (IV) before interpolation, as implied volatility curves are typically smoother and more continuous than price data, making them suitable for interpolation and smoothing processes. The interpolated curves are then converted back to call option prices to extract the RND. This method does not rely on option prices conforming to Black-Scholes model assumptions, but rather uses the Black-Scholes formula merely as a calculation tool to convert data into a form more suitable for smoothing.

Figlewski (2008) proposed a method aimed at resolving abnormal fluctuations in implied volatility data near at-the-money options, particularly the jump phenomenon in call and put option prices near the at-the-money point. Such jumps may lead to non-smooth implied volatility curves, thereby affecting the stability of the RND calculation. Following this method, if a strike price is between 0.9 and 1.1 times the futures price, the average of call and put implied volatilities is taken as the data point. The formula is as follows:

$$IV_{mix}(K) = 0.5 \times IV_{call}(K) + 0.5 \times IV_{put}(K)$$
$$K \in [0.9 \times F, 1.1 \times F]$$

Figure 4-1 presents the relationship between Bitcoin option implied volatility and strike price, showing that the market has a higher preference for OTM options. At the same time, significant fluctuations in implied volatility can be observed near the at-the-money point. To reduce the dramatic volatility of implied volatility at the at-the-money position, this research takes the average value of call and put implied volatilities for data within the range of 0.9 to 1.1 times the futures price as the basis for subsequent smoothing. As shown in Figure 4-2, the green marked points represent the averaged implied volatility, which effectively reduces the fluctuation amplitude of call and put implied volatility near the at-the-money point. For data outside this range, the implied volatility data of OTM options are directly adopted. After the above processing, the final constructed implied volatility data are shown in Figure 4-3.

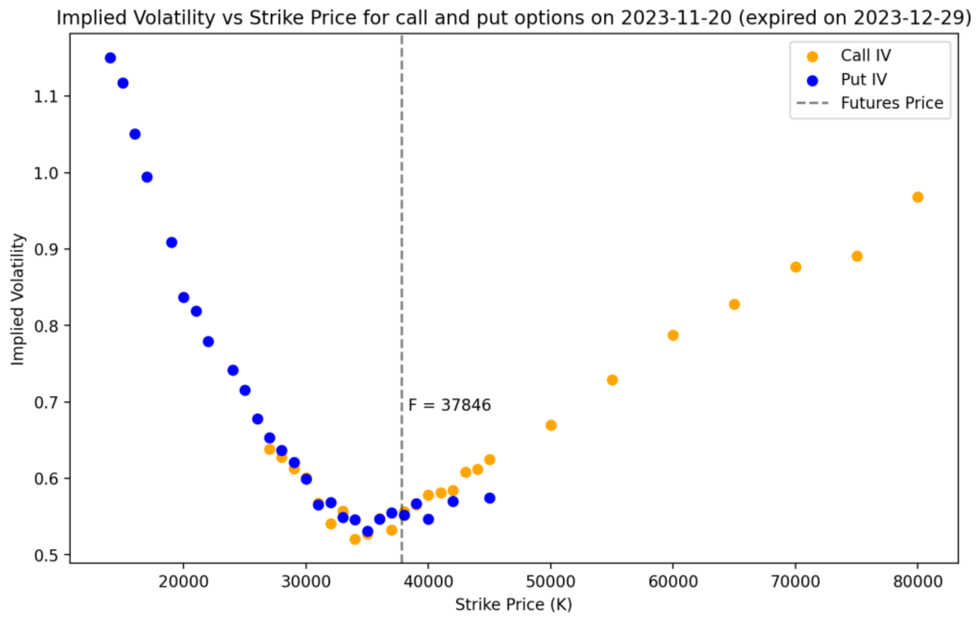

Figure 4-1: Bitcoin Option Implied Volatility Distribution on November 20, 2023 (Expiring December 29, 2023)

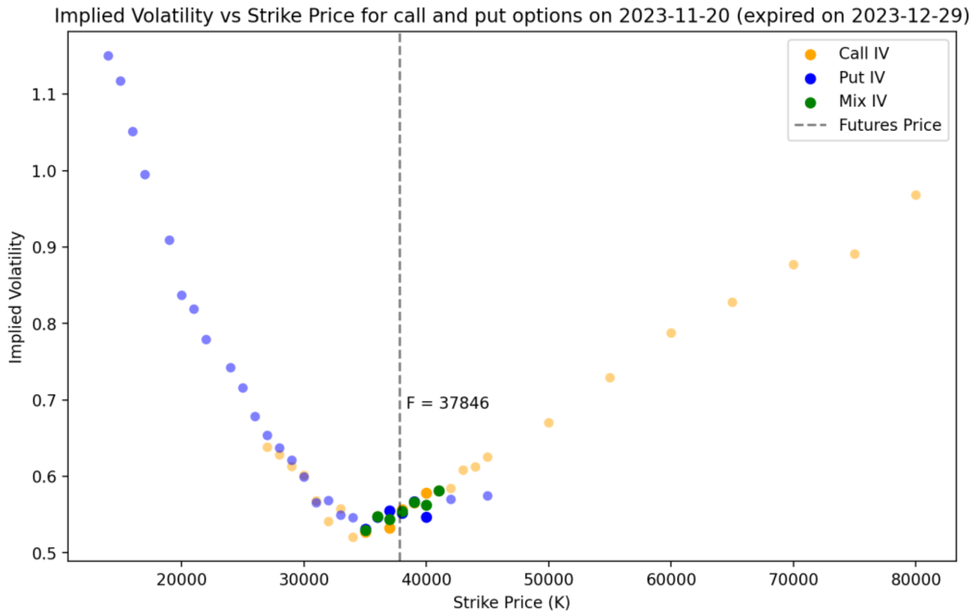

Figure 4-2: Bitcoin Option Implied Volatility Distribution on November 20, 2023 (Expiring December 29, 2023)

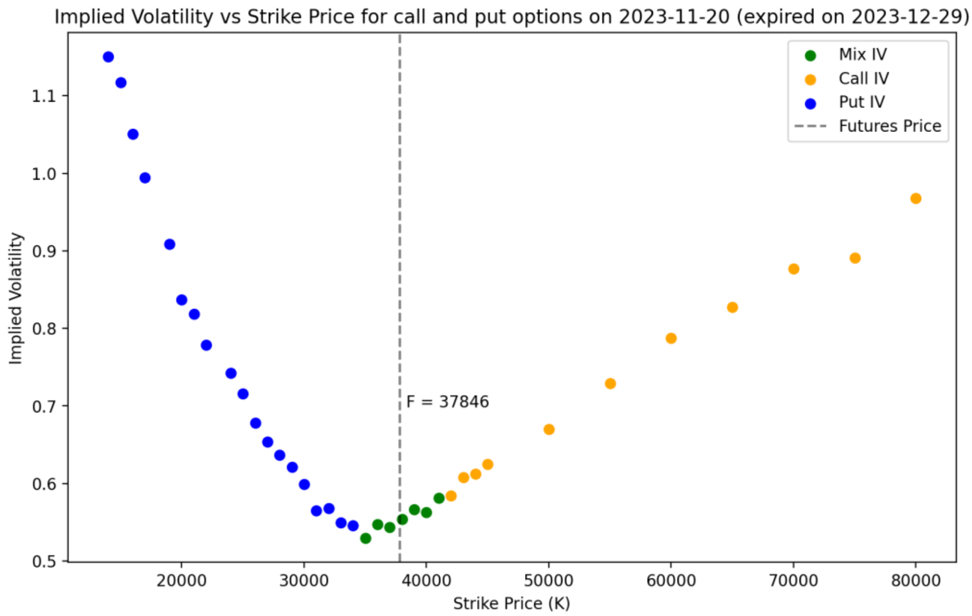

Figure 4-3: Bitcoin Option Implied Volatility Distribution on November 20, 2023 (Expiring December 29, 2023)

4.2.3 Fitting Bitcoin Option Implied Volatility Curve

To more precisely fit the processed implied volatility data, this research adopts a 4th-order spline function with a single knot for curve fitting. The knot is placed at the futures price, a design that allows greater flexibility at this key position while maintaining the overall continuity of the curve. Using a 4th-order spline function ensures that the fitted curve has third-order continuous differentiability (C^3 continuity), effectively capturing subtle changes in the implied volatility curve while avoiding over-fitting problems.

The mathematical representation of the 4th-order spline function is as follows:

$$S(x) = \begin{cases} \sum_{i=0}^{4} a_i \, (x - x_0)^i, & x < k \\ \sum_{i=0}^{4} b_i \, (x - x_0)^i, & x \geq k \end{cases}$$

Where k is the knot position (i.e., the futures price), x_0 is the reference point, and a_i and b_i are coefficients to be determined. At the knot, the function must satisfy the following continuity conditions:

$$\begin{cases} \sum_{i=0}^{4} a_i (k - x_0)^i = \sum_{i=0}^{4} b_i (k - x_0)^i \\ \sum_{i=1}^{4} i \, a_i (k - x_0)^{i-1} = \sum_{i=1}^{4} i \, b_i (k - x_0)^{i-1} \\ \sum_{i=2}^{4} i(i-1) a_i (k - x_0)^{i-2} = \sum_{i=2}^{4} i(i-1) b_i (k - x_0)^{i-2} \\ \sum_{i=3}^{4} i(i-1)(i-2) a_i (k - x_0)^{i-3} = \sum_{i=3}^{4} i(i-1)(i-2) b_i (k - x_0)^{i-3} \end{cases}$$

These conditions ensure the continuity of the function value and its first, second, and third derivatives at the knot.

Setting the sole knot at the futures price has important economic significance, as this position typically corresponds to at-the-money options. This knot placement divides the curve into two segments, corresponding to areas above and below the futures price, allowing the fitted curve to more accurately reflect volatility characteristics near the at-the-money point. This segmented fitting method is particularly suitable for handling the asymmetric features that may appear in option implied volatility before and after the at-the-money position.

At the implementation level, this research uses the LSQUnivariateSpline method from Python's SciPy package for curve fitting. This method employs the least squares approach for parameter estimation, effectively handling non-uniformly distributed data points and achieving segmented fitting through specified internal knots. Through the combination of the least squares method and knot placement, LSQUnivariateSpline provides a flexible and stable mathematical tool capable of fitting smooth and accurate implied volatility curves in non-uniformly distributed data (as shown in Figure 4-4), providing a solid foundation for subsequent risk-neutral density function (RND) calculations. The least squares method formula is as follows:

$$\min_{\{a_i\},\{b_i\}} \sum_{j=1}^{n} [y_j - S(x_j)]^2$$

Where y_j is the actual observed value, $S(x_j)$ is the observed value of the spline function at x_j, $\{a_i\},\{b_i\}$ is the set of parameters to be estimated for the spline function, and n is the total number of data points.

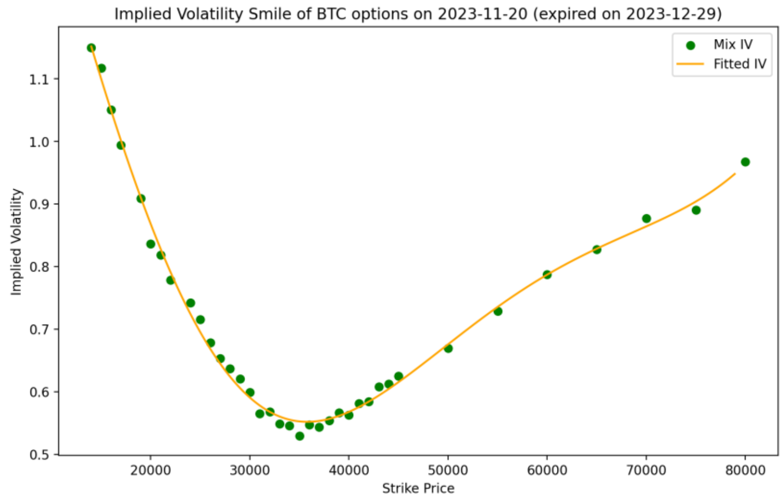

Figure 4-4: Bitcoin Option Implied Volatility Fitted Curve on November 20, 2023 (Expiring December 29, 2023)

4.2.4 Extracting the RND from Bitcoin Options

After completing the fitting of the implied volatility curve, the calculation of the risk-neutral probability density follows. First, this research uses the fitted implied volatility curve, combined with the pricing model adopted by the Deribit exchange (Formula (11)), to calculate theoretical call option prices at different strike prices, as shown in Figure 4-5.

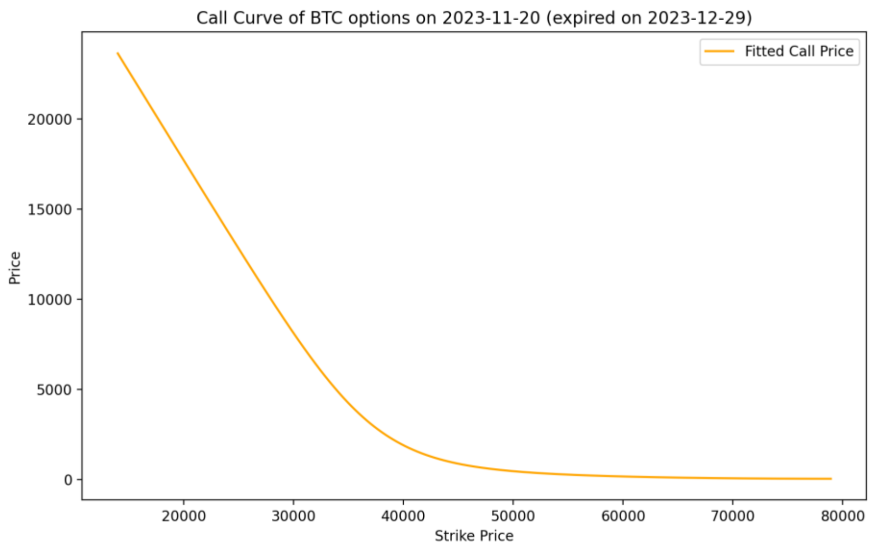

Figure 4-5: Bitcoin Option Theoretical Call Option Prices on November 20, 2023 (Expiring December 29, 2023)

After obtaining the theoretical call option prices, this research employs the finite difference method (central difference method) for discrete data differentiation to derive the risk-neutral probability density. Compared to forward or backward differences, the central difference method can effectively reduce truncation errors (Formulas (4) and (5)). To ensure the stability and accuracy of numerical calculations, this research adopts an equidistant partitioning approach in setting the price spacing, with ΔX set to 0.1. Choosing a smaller price spacing not only provides more refined density estimation, but the equidistant partitioning approach also helps improve the stability of numerical differentiation calculations.

After completing the risk-neutral probability density calculation, to ensure the reliability and reasonableness of the data, this research focuses on the cumulative distribution function (CDF) and right cumulative probability for data validation. In terms of data integrity validation, this research first ensures that both cumulative distribution function values and right cumulative probability values do not contain missing values, with observations containing missing values excluded from the analysis scope. Furthermore, to maintain theoretical consistency of calculation, this research further restricts these two probability values to be strictly between 0 and 1, while excluding boundary values equal to 0 or 1, to avoid extreme cases affecting subsequent analysis.

$$\begin{cases} F(K)\ exists\ (non-missing) \\ F_R(K)\ exists\ (non-missing) \\ 0 < F(K) < 1 \\ 0 < F_R(K) < 1 \end{cases}$$

Where $F(K)$ is the cumulative distribution function, $F_R(K)$ is the right cumulative probability, and $F_R(K) = 1 - F(K)$ 。

The validation criteria are set based on three aspects of consideration. First, from the perspective of theoretical consistency, these criteria ensure that the estimation conforms to the basic properties of cumulative distribution functions in probability theory, while also satisfying the basic requirements of probability density functions. Second, in terms of numerical stability, this filtering approach can avoid computational problems in subsequent analyses due to extreme or abnormal values, effectively improving the reliability of the overall estimation. Finally, from a practical application perspective, removing abnormal values that might lead to misinterpretation better ensures that results accurately reflect market participants' true price expectations.

The derive risk-neutral probability density function and its cumulative distribution function are shown in Figure 4-6 and Bitcoin Option Risk-neutral Probability Cumulative Distribution Function on November 20, 2023 (Expiring December 29, 2023) Figure 4-7:

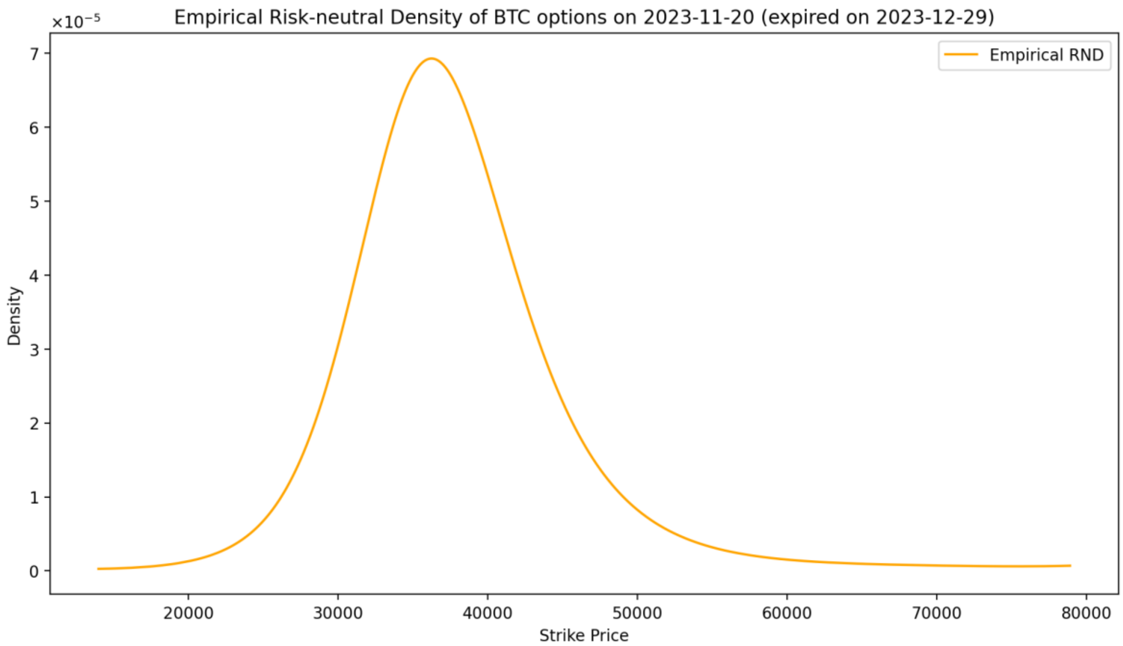

Figure 4-6: Bitcoin Option Risk-neutral Probability Density Function on November 20, 2023 (Expiring December 29, 2023)

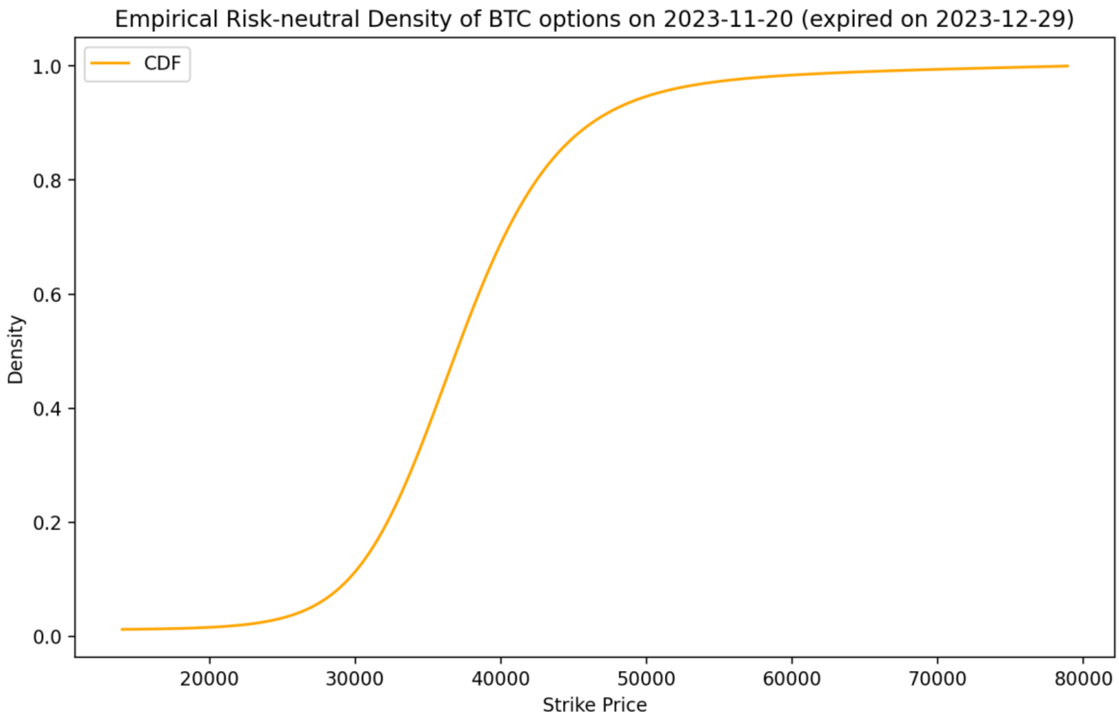

Figure 4-7: Bitcoin Option Risk-neutral Probability Cumulative Distribution Function on November 20, 2023 (Expiring December 29, 2023)

4.3 Fitting the Tails of the RND

4.3.1 Two-Point Fitting Method with GEV

The risk-neutral probability density (RND) extracted from market option prices can only cover the range of effective trading strike prices. To fully describe market expectations, the tails of the RND need to be extended. Figlewski (2008) proposed using the Generalized Extreme Value Distribution (GEV) to fit the tails of the RND. This method requires setting three conditions for each tail to ensure the continuity of tail fitting:

$$\text{Right tail conditions: } \begin{cases} F_{GEVR}\big(K(\alpha_{1R})\big) = \alpha_{1R} \\ f_{GEVR}\big(K(\alpha_{1R})\big) = f_{EMP}\big(K(\alpha_{1R})\big) \\ f_{GEVR}\big(K(\alpha_{2R})\big) = f_{EMP}\big(K(\alpha_{2R})\big) \end{cases}$$

$$\text{Left tail conditions: } \begin{cases} F_{GEVL}\big(-K(\alpha_{1L})\big) = 1 - \alpha_{1L} \\ f_{GEVR}\big(-K(\alpha_{1L})\big) = f_{EMP}\big(K(\alpha_{1L})\big) \\ f_{GEVR}\big(-K(\alpha_{2L})\big) = f_{EMP}\big(K(\alpha_{2L})\big) \end{cases}$$

Where F_{GEVR} and f_{GEVR} are the cumulative distribution function and probability density function of the right tail GEV, respectively, F_{GEVL} is the cumulative distribution function of the left tail GEV, f_{EMP} is the RND function derived in this research, and $K(\alpha)$ is the strike price corresponding to the αquantile of the Empirical RND. The fitting conditions from Figlewski (2008) can be summarized as:

(1) At the first joining point, the cumulative probability of the GEV tail must equal the cumulative probability of the RND

(2) At the first joining point, the GEV density function value must equal the density function value of the RND

(3) At the second joining point, the GEV density function value must equal the density function value of the RND

4.3.2 GPD Distribution Theory

This research adopts the Generalized Pareto Distribution (GPD) for tail fitting. The central position of GPD in extreme value theory stems from the groundbreaking research by Balkema & Haan (1974) , who proved that when observations exceed a sufficiently high threshold, their excess distribution asymptotically converges to GPD. This property makes GPD particularly suitable for describing tail events. The choice of GPD over GEV is primarily based on its parameter structure advantage: GPD only requires setting two parameters: scale parameter (σ) and shape parameter (ξ), which is more concise compared to GEV's three parameters: location parameter (μ), scale parameter (σ), and shape parameter (ξ). This concise parameter structure not only enhances computational efficiency but also reduces the risk of over-fitting. Additionally, research by Hosking & Wallis (1987) further verified the threshold stability property of GPD, making it more advantageous in practical applications. According to extreme value theory, GPD and GEV share the same shape parameter in the tail, meaning that both distributions have the same asymptotic properties when describing extreme events. Nevertheless, GPD is more intuitive in practical applications, especially in the field of financial risk management, as shown in the research by McNeil & Frey (2000). Birru & Figlewski (2012) further confirmed that the performance of GPD and GEV in tail fitting is quite similar, and both outperform the lognormal distribution.

The mathematical expression of the GPD cumulative distribution function (CDF) is as follows:

$$F_{GPD}(x;\sigma,\xi) = \begin{cases} 1-(1+\xi\frac{x}{\sigma})^{-\frac{1}{\xi}}, & \xi \neq 0 \\ 1-exp\,(-\frac{x}{\sigma}), & \xi = 0 \end{cases}$$

The mathematical expression of the GPD probability density function (PDF) is as follows:

$$f_{GPD}(x;\sigma,\xi) = \begin{cases} \frac{1}{\sigma}(1+\xi\frac{x}{\sigma})^{-\frac{1}{\xi}-1}, & \xi \neq 0 \\ \frac{1}{\sigma}exp\,(-\frac{x}{\sigma}), & \xi = 0 \end{cases}$$

Where $\sigma > 0$ is the scale parameter, used to control the degree of dispersion of the distribution, with larger σ values indicating greater variability in the data. The shape parameter ξ determines the type and tail characteristics of the distribution:

(1) When $\xi > 0$: The distribution is a Pareto Distribution with heavy-tailed characteristics; the distribution has infinite support, with domain $[0,\infty)$; the tail decays more slowly

(2) When $\xi = 0$: The distribution degenerates to an Exponential Distribution with a fixed decay rate; it is the simplest continuous memoryless distribution; the tail

decays at a moderate speed

(3) When $\xi < 0$: The distribution belongs to the Beta Family with finite support characteristics; the distribution function is only defined on the interval $\left[0, -\frac{\sigma}{\xi}\right)$; it is less commonly used in financial market applications because asset returns typically do not have a clear upper limit

4.3.3 Two-Point Fitting Method with GPD

Birru & Figlewski (2012) proposed a two points fitting method for GPD tails, using two joining points to compare the density function values of GPD and the RND. The tail fitting conditions are set as follows:

(1) At the first joining point, the GPD density function value must equal the RND density function value

(2) At the second joining point, the GPD density function value must equal the RND density function value

The mathematical expressions are as follows:

Right tail conditions:

$$\begin{cases} f_{GPD}(K(\alpha_{1R})) = f_{EMP}(K(\alpha_{1R})) & (PDF\ condition) \\ f_{GPD}(K(\alpha_{2R})) = f_{EMP}(K(\alpha_{2R})) & (PDF\ condition) \end{cases}$$

Left tail conditions:

$$\begin{cases} f_{GPD}(K(\alpha_{1L})) = f_{EMP}(K(\alpha_{1L})) & (PDF\ condition) \\ f_{GPD}(K(\alpha_{2L})) = f_{EMP}(K(\alpha_{2L})) & (PDF\ condition) \end{cases}$$

The scale parameter and shape parameter of GPD are solved by minimizing the following objective functions:

Right tail parameter minimization objective function:

$$\min_{\sigma,\xi}\{[f_{GPD}(K(\alpha_{1R}))-f_{EMP}(K(\alpha_{1R}))]^2+[f_{GPD}(K(\alpha_{2R}))-f_{EMP}(K(\alpha_{2R}))]^2\}$$

Left tail parameter minimization objective function:

$$\min_{\sigma,\xi}\{[f_{GPD}(K(\alpha_{1L}))-f_{EMP}(K(\alpha_{1L}))]^2+[f_{GPD}(K(\alpha_{2L}))-f_{EMP}(K(\alpha_{2L}))]^2\}$$

Where α_{1L} is preset to 0.05; α_{2L} is preset to 0.02; α_{1R} is preset to 0.95; α_{2R} is preset to 0.98. The empirical RND curve and its CDF completed using the two points fitting method are shown in Figure 4-8 and Figure 4-9.

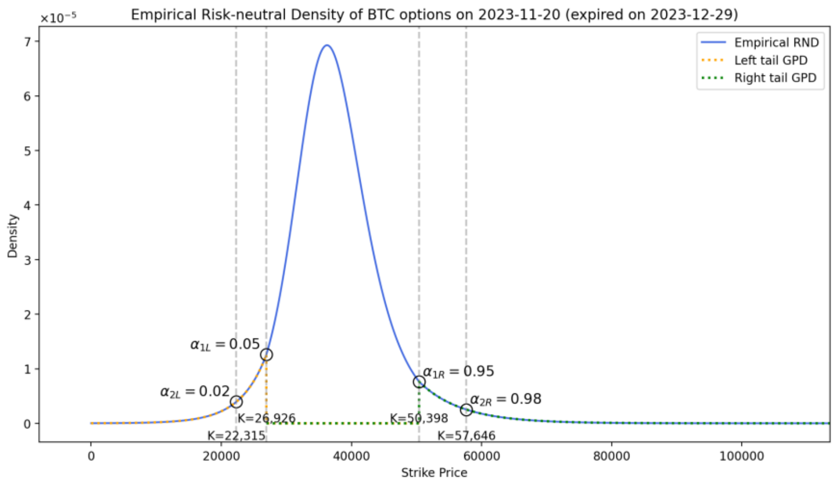

Figure 4-8: Bitcoin Option Empirical RND and GPD Tail Fitting on November 20, 2023 (Two Points Fitting Method)

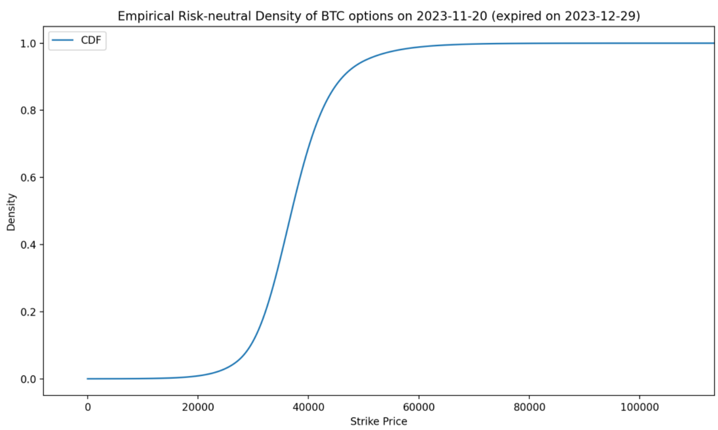

Figure 4-9: CDF of Bitcoin Option Empirical RND on November 20, 2023 (Two Points Fitting Method)

4.3.4 One Single Point with one Slope Fitting Method with GPD

This research proposes using one single point with one slope fitting method with GPD. Our proposed method not only considers the fitting of CDF values but also adds continuity conditions for the slope of the density function. The main advantage of this method is that it can simultaneously ensure the continuity and smoothness of the density function while simplifying the fitting process and improving computational efficiency. The tail fitting conditions set in this research are as follows:

(1) At the joining point, the cumulative probability of GPD must equal the cumulative probability of the RND.

(2) At the joining point, the slope of the GPD density function must equal the slope of the RND density function.

The mathematical expressions are as follows:

Right tail conditions:

$$\begin{cases} F_{GPD}(K(\alpha_{1R})) = \alpha_{1R} & (CDF\ condition) \\ \dfrac{f_{GPD}(K(\alpha_{1R}) + \Delta x) - f_{GPD}(K(\alpha_{1R}))}{\Delta x} = \dfrac{f_{EMP}(K(\alpha_{1R}) + \Delta x) - f_{EMP}(K(\alpha_{1R}))}{\Delta x} & (Slope\ condition) \end{cases}$$

Left tail conditions:

$$\begin{cases} F_{GPD}(-K(\alpha_{1L})) = \alpha_{1L} & (CDF\ condition) \\ \dfrac{f_{GPD}(-K(\alpha_{1L}) + \Delta x) - f_{GPD}(-K(\alpha_{1L}))}{\Delta x} = \dfrac{f_{EMP}(K(\alpha_{1L}) + \Delta x) - f_{EMP}(K(\alpha_{1L}))}{\Delta x} & (Slope\ condition) \end{cases}$$

The scale parameter and shape parameter of GPD are solved by minimizing the following objective functions:

Right tail parameter minimization objective function:

$$\min_{\sigma,\xi}\left\{\left[F_{GPD}\big(K(\alpha_{1R})\big)-\alpha_{1R}\right]^2 + \left[\frac{f_{GPD}(K(\alpha_{1R})+\Delta x)-f_{GPD}(K(\alpha_{1R}))}{\Delta x} - \frac{f_{EMP}(K(\alpha_{1R})+\Delta x)-f_{EMP}(K(\alpha_{1R}))}{\Delta x}\right]^2\right\}$$

Left tail parameter minimization objective function:

$$\min_{\sigma,\xi}\left\{\left[F_{GPD}(-K(\alpha_{1L}))-\alpha_{1L}\right]^2 + \left[\frac{f_{GPD}(-K(\alpha_{1L})+\Delta x)-f_{GPD}(-K(\alpha_{1L}))}{\Delta x} - \frac{f_{EMP}(K(\alpha_{1L})+\Delta x)-f_{EMP}(K(\alpha_{1L}))}{\Delta x}\right]^2\right\}$$

Where Δx is a fixed constant value used to construct artificially spaced option prices, preset to 0.1; α_{1L} is preset to 0.05; α_{1R} is preset to 0.95.

This research performs fitting separately for the left and right tails. For the left tail, we select the point with a cumulative probability of 5% as the joining point; for the right tail, we select the point with a cumulative probability of 95% as the joining point. This design ensures the smoothness of tail fitting while maintaining the continuity of the overall distribution. To minimize fitting errors, this research adopts the least squares method for parameter estimation, solving for the scale parameter and shape parameter of GPD through numerical optimization methods. The completed empirical RND curve and its CDF are shown in Figures 4-10 and 4-11.

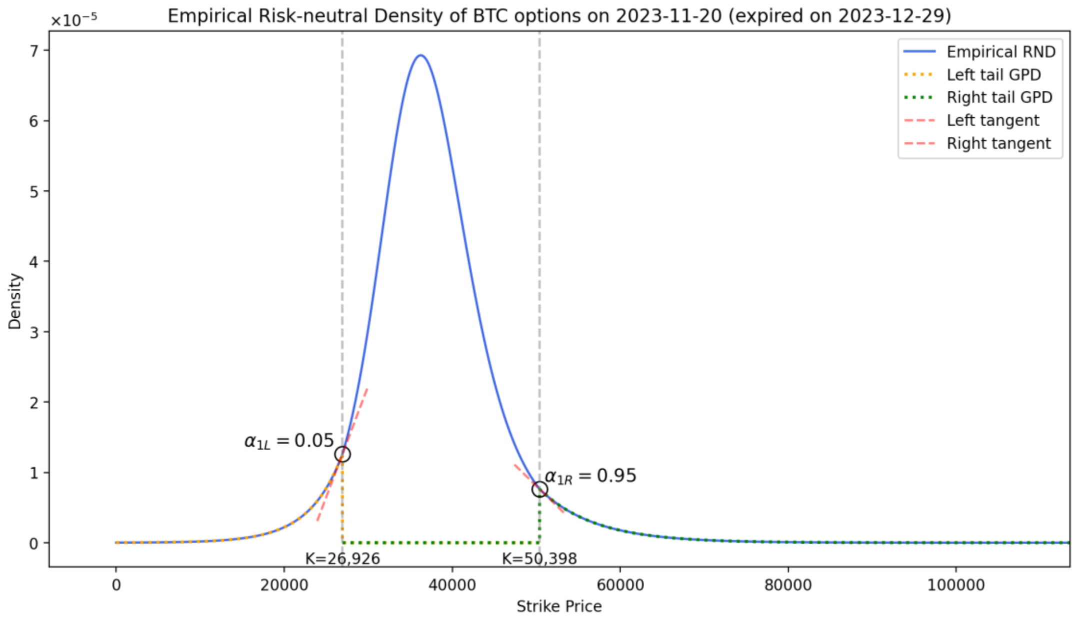

Figure 4-10: Bitcoin Option Empirical RND and GPD Tail Fitting on November 20, 2023 (Single Point Fitting Method)

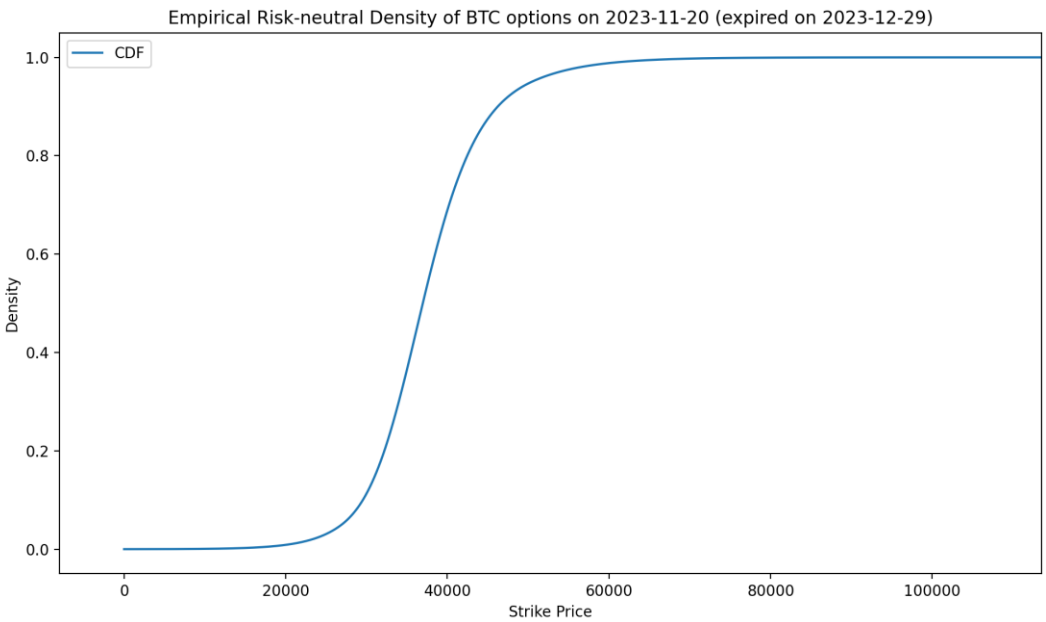

Figure 4-11: CDF of Bitcoin Option Empirical RND on November 20, 2023 (Single Point Fitting Method)

4.4 Deriving the Moments of the RND

This section introduces in detail the moments of the risk-neutral probability density (RND), including moments of various orders and their derived statistics, to analyze market participants' expectations regarding future Bitcoin price movements.

4.4.1 Definition and Implications of Moments

Moments are important statistical measures for describing the characteristics of probability distributions and can be divided into raw moments and central moments. For discrete strike prices, the nth-order raw moment is defined as:

$$m'_n = E[K^n] = \sum_{i=1}^{N} K_i^n f(K_i)\,\Delta K$$

And the nth-order central moment is defined as:

$$m_n = E[(K - \bar{K})^n] = \sum_{i=1}^{N} (K_i - \bar{K})^n f(K_i)\,\Delta K$$

Where $f(K_i)$ is the risk−neutral probability density function (RND); ΔK is the price interval, set to 0.1; $\bar{K}$ is the expected value, i.e., the first−order raw moment m'_1; N is the number of observations.

4.4.2 Moments of the RND

Recent research shows that option-implied moments not only effectively describe market expectations but also possess significant predictive power. Chang et al. (2013) found that risk-neutral skewness can effectively predict stock returns, especially during periods of high market volatility; while Neumann & Skiadopoulos (2013) pointed out that changes in risk-neutral kurtosis often lead market trends, providing important signals for investment decisions. This research computes the following four main moments:

1. Mean

The mean is the first-order raw moment, reflecting the overall market expectation for the future price level of the underlying asset. Bali & Murray (2013) pointed out that under risk-neutral pricing theory, the mean of the RND should equal the forward price discounted by the risk-free rate, providing an important arbitrage constraint. The formula is as follows:

$$\overline{K} = m_1' = E[K] = \sum_{i=1}^{N} K_i f(K_i)\,\Delta K$$

2. Standard Deviation

The standard deviation is the square root of the second-order central moment, measuring the degree of price dispersion. Christoffersen et al. (2013) found that option-implied standard deviation has stronger predictive power than historical volatility, especially in emerging markets. Standard deviation reflects market expectations for price volatility, with higher values indicating greater uncertainty among market participants regarding future price movements. The formula is as follows:

$$\sigma = \sqrt{m_2} = \sqrt{E[(K-\overline{K})^2]} = \sqrt{\sum_{i=1}^{N} (K_i-\overline{K})^2 f(K_i)\,\Delta K}$$

3. Skewness

Skewness is the standardized third-order central moment, describing the asymmetry of the distribution:

$$\text{Skewness} = \frac{m_3}{\sigma^3} = \frac{E[(K-\overline{K})^3]}{\sigma^3} = \frac{\sum_{i=1}^{N}(K_i-\overline{K})^3 f(K_i)\,\Delta K}{\sigma^3}$$

The skewness coefficient has important implications in financial markets. When positive skewness is observed, it indicates that the price distribution has a longer right tail, implying that market participants expect a higher probability of significant upward movements, reflecting overall optimistic market sentiment. Conversely, negative skewness indicates that the price distribution has a longer left tail, representing a greater perceived downside risk in the market, usually reflecting higher hedging demand among market participants. Research by Conrad et al. (2013) shows that risk-neutral skewness not only reflects market sentiment but also contains investors' expectations for extreme events. Li et al. (2024) shows that the dynamic changes in the skewness coefficient can often serve as a leading indicator of market sentiment shifts, with its changing trends providing important reference value for investment decisions.

4. Excess Kurtosis

Excess Kurtosis is the standardized fourth-order central moment minus 3 (the kurtosis value of the normal distribution), used to describe the tail characteristics of the distribution. It is calculated as:

$$\text{Excess Kurtosis} = \frac{m_4}{\sigma^4} - 3 = \frac{E[(K - \bar{K})^4]}{\sigma^4} - 3 = \frac{\sum_{i=1}^{N}(K_i - \bar{K})^4 f(K_i)\,\Delta K}{\sigma^4} - 3$$

In practical applications, excess kurtosis is an important indicator for assessing extreme market risk. Amaya et al. (2015) pointed out that excess kurtosis can effectively capture extreme market risk, with its predictive power being particularly significant during financial crises. When positive excess kurtosis is observed, it indicates that the distribution has more pronounced fat-tail characteristics compared to the normal distribution, meaning that the probability of extreme events occurring is higher than expected under a normal distribution.

4.4.3 Market Implications of the Moments

The moments of the RND not only provide a mathematical description of the distribution but also contain rich market information. From the perspective of price expectations, the mean of the distribution reflects the consensus expectation of the market for future price levels. If it deviates significantly from the actual market price, it may suggest potential arbitrage opportunities. Such price deviations often provide profit opportunities for arbitrageurs while also contributing to the self-adjusting mechanism of market prices.

In terms of risk assessment, standard deviation, as a traditional indicator of volatility risk, combined with the extreme risk information provided by excess kurtosis, allows investors to assess market risk more comprehensively. Especially during periods of increased market volatility, changes in these indicators often provide timely risk warning signals, helping investors adjust their portfolios in a timely manner to respond to market changes. Market sentiment monitoring can be achieved through the changing trends in skewness, reflecting shifts in market sentiment and providing a quantitative basis for dynamic adjustment of investment strategies.

This research uses regression analysis to empirically test the predictive power of the RND characteristics for spot returns. By analyzing the correlation between these moments and future market trends, we can more deeply evaluate their application value in the Bitcoin market, providing a more solid quantitative foundation for investment decisions.

4.5 Regression Analysis

4.5.1 Theoretical Background for Regression Models

To explore the predictive power of Bitcoin option-implied risk-neutral probability density (RND) for underlying asset price movements, this research adopts multiple regression analysis for testing. The dependent variable (Y) is set as the logarithmic return of Bitcoin, while the independent variables (X) gradually incorporate moments of the RND such as Mean, Standard Deviation, Skewness, Excess Kurtosis, Median, as well as market sentiment indicators like the Cryptocurrency Fear and Greed Index and the Chicago Board Options Exchange Volatility Index (VIX), along with historical returns from the previous 1 to 4 periods, to construct the most explanatory predictive model.

Bali & Zhou (2016) demonstrate that moments of the Risk-neutral Density (RND) functions, particularly skewness and kurtosis, effectively predict cross-sectional asset return variations, reflecting market participants' risk preferences and containing crucial pricing information. Amaya et al. (2015) further establish that the RND excess kurtosis specifically excels in predicting extreme market risk, a finding particularly relevant for high-volatility markets such as cryptocurrencies.

López-Cabarcos et al. (2021) examine relationships between Bitcoin volatility, stock market performance, and investor sentiment, indicating that during market stability periods, VIX returns and investor sentiment significantly impact Bitcoin volatility. Akyildirim et al. (2020), utilizing high-frequency data, identify a time-varying positive correlation between cryptocurrencies and market panic indicators (VIX, VSTOXX), which intensifies during periods of financial market stress. M. He et al. (2023) employ the daily updated Cryptocurrency Fear and Greed Index as a predictor, demonstrating significant in-sample and out-of-sample predictive power for individual cryptocurrency and market index returns over one-day to one-week horizons.

Liu & Tsyvinski (2021) identify significant momentum effects in cryptocurrency markets, with Bitcoin's current period returns substantially predicting returns for the subsequent 1-6 days. Y. Li et al. (2021) document a positive MAX momentum effect, where cryptocurrencies exhibiting higher extreme daily returns tend to generate superior future returns. Liu et al. (2023), applying machine learning methodologies to cryptocurrency return prediction, determine that previous 1-day returns possess the strongest predictive power, exceeding the combined effect of all other variables.

4.5.2 Regression Model Specification

Campbell & Thompson (2008) introduced the concept of "economic significance threshold," showing that by gradually introducing predictive variables and setting strict statistical significance standards, one can effectively distinguish variables with substantial predictive power. In Gu et al. (2020), a "staged variable introduction framework" was proposed specifically for high-dimensional data, which can alleviate overfitting problems compared to models that introduce all variables at once.

This research adopts a multi-level regression analysis, gradually expanding from univariate to four-variable models, to systematically explore the predictive power of various risk-neutral probability density characteristics for future returns. The following details the design of regression models at each level:

1. Univariate Regression Model

The univariate regression model is mainly used to test the explanatory power of individual variables for future returns. Its basic form is:

$$R_t = \beta_0 + \beta_1 Variable_{i,t-1} + \varepsilon_t, i \in \{1, 2, \dots, 11\}$$

Where $R_t = ln(\frac{Close_t}{Close_{t-1}})$ is the return of Bitcoin price in the next period (T). If the option sample expires in 7 days, then the return is calculated using the closing price from the current day to the closing price 7 days later (expiration date).

$Variable_i$ is sequentially replaced with the following variables for univariate regression analysis: Mean, Standard Deviation (Std), Skewness, Excess Kurtosis, Median, Cryptocurrency Fear and Greed Index, Chicago Board Options Exchange Volatility Index (VIX), T-1 Return, T-2 Return, T-3 Return, T-4 Return.

2. Bivariate Regression Model

Considering the importance of Skewness in option pricing theory, this research designs bivariate models with Skewness as a fixed factor. The model is set as follows:

$$R_t = \beta_0 + \beta_1 Skewness_{t-1} + \beta_2 Variable_{i,t-1} + \varepsilon_t, i \in \{1, 2, \ldots, 10\}$$

$Variable_i$ is sequentially replaced with the following variables, paired with Skewness for bivariate regression analysis: Mean, Standard Deviation (Std), Excess Kurtosis, Median, Cryptocurrency Fear and Greed Index, Chicago Board Options Exchange Volatility Index (VIX), T-1 Return, T-2 Return, T-3 Return, T-4 Return.

3. Three-Variable Regression Model

The three-variable model further incorporates Excess Kurtosis as a fixed factor, forming:

$$R_t = \beta_0 + \beta_1 Skewness_{t-1} + \beta_2 ExcessKurtosis_{t-1} + \beta_3 Variable_{i,t-1} + \varepsilon_t, i \in \{1, 2, \ldots, 9\}$$

$Variable_i$ is sequentially replaced with the following variables, paired with Skewness and Excess Kurtosis for three-variable regression analysis: Mean, Standard Deviation (Std), Median, Cryptocurrency Fear and Greed Index, Chicago Board Options Exchange Volatility Index (VIX), T-1 Return, T-2 Return, T-3 Return, T-4 Return.

4. Four-Variable Regression Model

Building on the three-variable model, the four-variable model adds the Standard Deviation (Std) variable. The complete model is as follows:

$$R_t = \beta_0 + \beta_1 Skewness_{t-1} + \beta_2 ExcessKurtosis_{t-1} + \beta_3 Std_{t-1} + \beta_4 Variable_{i,t-1} + \varepsilon_t,$$

$i \in 1,2,\dots,8$

$Variable_i$ is sequentially replaced with the following variables, paired with Skewness, Excess Kurtosis, and Standard Deviation for four-variable regression analysis: Mean, Median, Cryptocurrency Fear and Greed Index, Chicago Board Options Exchange Volatility Index (VIX), T-1 Return, T-2 Return, T-3 Return, T-4 Return.

4.5.3 Validation of Model Effectiveness

Our proposed method of single point compared to Birru & Figlewski (2012) used in existing literature has the advantage of computational efficiency in practical applications. To objectively validate the effectiveness of this method, this research adopts three quantitative indicators for evaluation: Mean Squared Error (MSE), Out-of-sample R-squared (R^2_{OS}) (Campbell & Thompson, 2008; Welch & Goyal, 2008), and computational efficiency, to comprehensively compare the accuracy and practicality of predictive models constructed by the two methods.

1. Mean Squared Error (MSE) Calculation

Mean Squared Error is a commonly used indicator for evaluating the accuracy of predictive models (Orosi, 2015). Its calculation formula is as follows:

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

Where y_i is the actual observed value (i.e., the actual Bitcoin return), $\hat{y}_i$ is the model′s predicted value, and n is the sample size. A smaller MSE value indicates smaller prediction errors and higher prediction accuracy.

2. Out-of-sample R-squared (R_{OS}^2) Calculation

Referring to the research methods of Campbell & Thompson (2008) and Welch & Goyal (2008), this research adopts out-of-sample R-squared (R_{OS}^2) to evaluate the predictive power of the model. R_{OS}^2 measures the improvement in prediction of the predictive model relative to the historical average benchmark model. Its calculation formula is as follows:

$$R_{OS}^2 = 1 - \frac{\sum_{t=s_0+1}^{T}(R_{t+1} - \hat{R}_{t+1})^2}{\sum_{t=s_0+1}^{T}(R_{t+1} - \bar{R}_{t+1})^2}$$

Where R_{t+1} is the actual return, $\hat{R}_{t+1}$ is the predicted value from the predictive model, $\bar{R}_{t+1}$ is the historical average return (benchmark model), s_0 is the initial in-sample period length, and T is the total sample size. When R_{OS}^2 is greater than zero, it indicates that the predictive model outperforms the historical average benchmark model; the larger the R_{OS}^2 value, the more significant the improvement in prediction.

This research adopts a rolling window approach for out-of-sample prediction, with the initial in-sample period set to 80% of the total sample size, and gradually advancing forward for prediction. By comparing the R_{OS}^2 values of one single point with one slope method and Birru & Figlewski (2012), the difference in out-of-sample predictive power between the two methods can be objectively evaluated.

3. Computational Efficiency Comparison

In addition to prediction accuracy, this research also values the practical applicability of the method, especially its computational efficiency when processing large amounts of data. To objectively evaluate the computational performance of the two methods, this research selects ten representative option expiration dates, calculates the 7-day risk-neutral probability density for each expiration date, and uses both one single point with one slope method and Birru & Figlewski (2012) to fit the tails with the Generalized Pareto Distribution (GPD). To ensure the reliability and stability of the results, this research performs ten repeated computations for each method, records their execution times, and computes the average, thereby comprehensively evaluating the differences in computational efficiency between the two methods in practical applications.

5. Empirical Results

This chapter presents empirical comparisons between our proposed one single point with one slope method and Birru & Figlewski (2012) for RND tail fitting. We analyze fitting characteristics, computational efficiency, and predictive performance across different option expiration horizons.

We focus on Bitcoin options with 1-day and 7-day expiration periods due to the cryptocurrency market's unique characteristics. Bitcoin's 24-hour trading environment, substantial liquidity (tens of billions in daily volume), and efficient price discovery mechanism make it ideal for studying RND characteristics. Additionally, the market's rapid information processing and technology-oriented trader base enable short-term products to effectively capture real-time risk assessments.

Section 5.1 evaluates fitting performance and computational efficiency of both methods. Sections 5.2 and 5.3 analyze the predictive capabilities for options with 1-day and 7-day expiration periods, respectively, examining how different fitting methodologies impact return prediction across timeframes.

5.1 Analysis of Fitting Effects

5.1.1 Comparison between One Single Point with One Slope Method and Birru & Figlewski (2012)

To further compare the fitting effects of our proposed method and Birru & Figlewski (2012), this research selects samples from observation dates July 10, 2022 (expiration date July 11, 2022) and September 27, 2023 (expiration date September 28, 2023) for analysis. From the results, one single point with one slope method demonstrates better stability, with smoother and more continuous fitting curves, indicating that this method has stronger adaptability when handling different market scenarios.

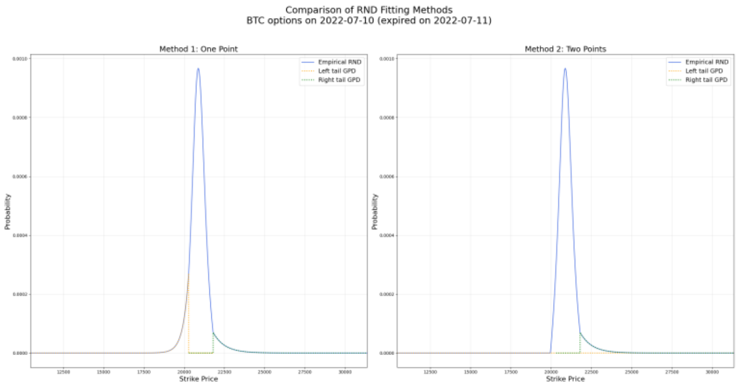

Figure 5-1: Comparison of Bitcoin Option GPD Tail Fitting on July 10, 2022

(Left: Our proposed method; Right: Birru & Figlewski (2012))

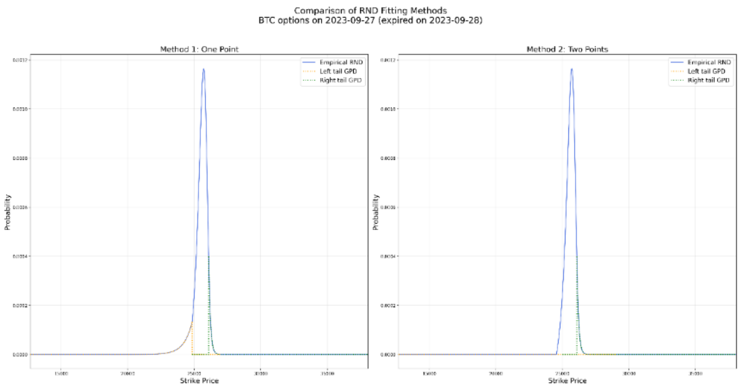

Figure 5-2: Comparison of Bitcoin Option GPD Tail Fitting on September 27, 2023

(Left: Our proposed method; Right: Birru & Figlewski (2012))

Examination of the one single point with one slope method (left panels) reveals robust RND curve fitting across both samples, maintaining continuity at joining points. Notably, the method produces gradually decreasing tails that conform to probability density function properties, demonstrating superior robustness in handling extreme values.

Conversely, while Birru & Figlewski (2012) method (right panels) generally produces reasonable fitting curves, it occasionally exhibits fitting failures or discontinuities under certain market conditions. This limitation stems from the simultaneous continuity requirements at two joining points, which becomes problematic during extreme market fluctuations or highly skewed price distributions where parameter combinations satisfying dual conditions are difficult to identify.

From an implementation perspective, the one single point with one slope method offers enhanced computational efficiency and reduced fitting failures—advantages particularly valuable in large sample analyses. The method's more stable fitting results facilitate subsequent moment calculations and analyses. Based on these comparative advantages, we conclude that the one single point with one slope method provides superior practical value for constructing Bitcoin options' risk-neutral probability density functions relative to Birru & Figlewski (2012).

5.1.2 Comparison of Computational Efficiency

We evaluated computational efficiency by testing both methods across 10 option expiration dates (September-December 2023) using identical hardware configurations. Each test generated 10 weekly RNDs with GPD tail fitting, as presented in Table 5-1.

Our proposed one single point with one slope method demonstrated superior performance, averaging 309.86 seconds execution time (range: 297.58-313.52 seconds) compared to Birru & Figlewski's (2012) 347.95 seconds (range: 346.49-349.10 seconds)—representing a 10.95% reduction in computation time.

This efficiency advantage stems from fundamental algorithmic differences. While Birru & Figlewski (2012) requires simultaneous satisfaction of continuity conditions at two joining points, introducing additional optimization constraints, our method addresses only a single joining point, enabling more direct optimization with faster parameter convergence. Our approach also exhibits greater execution time stability (lower standard deviation), indicating more consistent computational performance—a critical advantage for large-scale empirical analyses.

The one single point with one slope method offers significant practical benefits for applications requiring large sample processing or real-time analysis, with important implications for automated trading strategies and risk monitoring systems.

Table 5-1: Comparison of Computational Efficiency for Bitcoin Option GPD Tail Fitting
(Left: Our proposed method; Right: Birru & Figlewski (2012))

Execution Time:	2025/2/5 00:48
Execution Conditions:	Each time generates 10 weekly return RNDs with GPD distribution tail fitting.
Option Expiration Dates:	2023/9/22, 2023/9/29, 2023/10/13, 2023/10/20, 2023/10/27, 2023/11/10, 2023/11/17, 2023/11/24, 2023/12/15, 2023/12/22

Our proposed method		Birru & Figlewski (2012)	
1st Execution Time (sec)	297.58	1st Execution Time (sec)	346.49
2nd Execution Time (sec)	300.23	2nd Execution Time (sec)	346.95
3rd Execution Time (sec)	313.52	3rd Execution Time (sec)	347.78
4th Execution Time (sec)	313.51	4th Execution Time (sec)	347.68
5th Execution Time (sec)	312.25	5th Execution Time (sec)	347.81
6th Execution Time (sec)	311.37	6th Execution Time (sec)	347.94
7th Execution Time (sec)	313.52	7th Execution Time (sec)	348.39
8th Execution Time (sec)	311.86	8th Execution Time (sec)	349.10
9th Execution Time (sec)	312.36	9th Execution Time (sec)	348.59
10th Execution Time	312.42	10th Execution Time	348.80
Shortest Execution Time (sec)	297.58	Shortest Execution Time (sec)	346.49
Longest Execution Time (sec)	313.52	Longest Execution Time (sec)	349.10
Average Execution Time (sec)	309.86	Average Execution Time (sec)	347.95

5.2 Regression Analysis with 1 Day to Expiration

5.2.1 Fitting Tails with GPDs Based on One Single Point with One Slope Method

This section uses option products expiring daily from January 10, 2021, to April 30, 2024, deriving the RND from the observation date one day before expiration, and constructs complete the RND functions using one single point with one slope method. Moments such as mean, standard deviation, skewness, and kurtosis are then calculated as explanatory variables. Using the next period's Bitcoin spot return as the explained variable, multi-level regression analysis is conducted to observe whether the RND has predictive effects. The descriptive statistics of the variables are shown in Table 5-2, with a total of 832 samples. Observing Skewness and Excess Kurtosis, it can be found that the RND functions constructed using our proposed method have outliers in skewness and excess kurtosis, which in turn affect the mean and standard deviation of these variables.

Table 5-2: Descriptive Statistics of the RND Moments and Bitcoin Returns for Products with 1 Day to Expiration (Our proposed method)

	Count	Mean	Std	Min	25%	Median	75%	Max
T Return (Y)	832	-0.0002	0.0336	-0.1670	-0.0155	-0.0003	0.0156	0.1353
Mean	832	35891.0208	14144.2251	0.0000	25047.4826	34274.1105	46035.6010	72614.4638
Std	832	1784.3452	1422.5234	0.0000	821.3941	1463.6223	2291.7552	12224.7221
Skewness	832	0.4054	7.2140	-12.5062	-0.7712	0.2451	1.0921	193.7010
Excess Kurtosis	832	56.6637	1392.5124	-140.8251	-0.2814	1.6822	4.2421	40084.7494
Median	832	36972.0948	14057.6155	7860.0000	25800.8750	35687.3500	46933.4000	72597.4000
Fear and Greed Index	832	46.9892	22.4770	6.0000	26.0000	49.0000	68.2500	95.0000
VIX	832	20.0917	5.2866	12.0700	16.1875	19.2250	23.0300	37.2100
T-1 Return	832	-0.0001	0.0337	-0.1670	-0.0155	-0.0003	0.0156	0.1353
T-2 Return	832	-0.0001	0.0337	-0.1670	-0.0153	-0.0003	0.0161	0.1353
T-3 Return	832	-0.0001	0.0337	-0.1670	-0.0153	-0.0003	0.0160	0.1353
T-4 Return	832	-0.0003	0.0337	-0.1670	-0.0157	-0.0004	0.0156	0.1353

Following the regression analysis set in Section 4.5, the univariate regression results are shown in Table 5-3. It can be observed that Mean, Skewness, Median, and T-4 Return are significant, mainly concentrated in the RND distribution characteristics. The predictive ability of market sentiment indicators such as the Cryptocurrency Fear and Greed Index and Volatility Index (VIX) is extremely low, indicating that technical indicators may be more valuable than market sentiment indicators.

Table 5-3: Univariate Regression Results for Products with 1 Day to Expiration (Our proposed method)

	Coefficient	p value	Significance	R-squared
Mean	-0.0866	0.0124	**	0.0075
Std	-0.0247	0.4761		0.0006
Skewness	0.0614	0.0770	*	0.0038
Excess Kurtosis	0.0515	0.1379		0.0027
Median	-0.0707	0.0416	**	0.0050
Fear and Greed Index	-0.0023	0.9462		0.0000
VIX	-0.0010	0.9777		0.0000
T-1 Return	-0.0402	0.2471		0.0016
T-2 Return	0.0276	0.4261		0.0008
T-3 Return	0.0093	0.7893		0.0001
T-4 Return	0.0617	0.0752	*	0.0038

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

Table 5-4 presents bivariate regression results with Skewness as a fixed variable. The addition of Median and T-4 Return variables provides stable predictive capability with increased explanatory power (higher R-squared values), suggesting that RND median and short-term momentum effects contain significant predictive information for Bitcoin returns. Three-variable and four-variable regression analyses are documented in Appendix Table 1 and Appendix Table 2.

Table 5-4: Bivariate Regression Results for Products with 1 Day to Expiration (Our proposed method)

	Coef	p value	Sig	Skewness_Coef	Skewness_p	Skewness_Sig	R-squared
Mean	-0.0796	0.0229	**	0.0502	0.1510		0.0100
Std	-0.0230	0.5064		0.0607	0.0803	*	0.0043
Excess Kurtosis	-0.0631	0.5565		0.1211	0.2595		0.0042
Median	-0.0730	0.0352	**	0.0640	0.0647	*	0.0091
Fear and Greed Index	-0.0063	0.8551		0.0618	0.0758	*	0.0038
VIX	0.0009	0.9792		0.0614	0.0771	*	0.0038
T-1 Return	-0.0347	0.3194		0.0581	0.0956	*	0.0050
T-2 Return	0.0298	0.3906		0.0624	0.0724	*	0.0046
T-3 Return	0.0100	0.7722		0.0615	0.0765	*	0.0039
T-4 Return	0.0606	0.0802	*	0.0602	0.0821	*	0.0074

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

After multi-level regression analysis, this section finally adopts a regression model with three variables: Skewness, Median, and T-4 Return. The model results are shown in Table 5-5, indicating that Skewness and T-4 Return have positive effects on Bitcoin spot return prediction, while Median has a negative effect; the Mean Squared Error (MSE) shows the high volatility of Bitcoin. This section also uses this model as a basis to attempt to add a fourth variable to find a more explanatory model. Nevertheless, none of the added variables are statistically significant (Appendix Table 3), indicating that this model has already demonstrated relatively stable predictive ability.

Table 5-5: Regression Results for Products with 1 Day to Expiration (Our proposed method)

	Coefficient	p value	Significance	R-squared	MSE
Skewness	0.0629	0.0690	*	0.0130	0.9858
Median	-0.0746	0.0311	**		
T-4 Return	0.0626	0.0705	*		

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

5.2.2 Fitting Tails with GPDs Based on Birru & Figlewski (2012)

This section uses option products expiring daily from January 10, 2021, to April 30, 2024, deriving the RND from the observation date one day before expiration, and constructs complete the RND functions using Birru & Figlewski (2012). Moments such as mean, standard deviation, skewness, and kurtosis are then calculated as explanatory variables. Using the next period's Bitcoin spot return as the explained variable, multi-level regression analysis is conducted to observe whether the RND has predictive effects. The descriptive statistics of the variables are shown in Table 5-6, with a total of 831 samples, one less than our proposed method. This is because Birru & Figlewski (2012) encountered fitting problems, indicating that one single point with one slope method is more stable. Observing Skewness and Excess Kurtosis, it can be found that the statistical characteristic data calculated using Birru & Figlewski (2012) is less affected by outliers.

Table 5-6: Descriptive Statistics of the RND Moments and Bitcoin Returns for Products with 1 Day to Expiration (Birru & Figlewski (2012))

	Count	Mean	Std	Min	25%	Median	75%	Max
T Return (Y)	831	-0.0003	0.0336	-0.1670	-0.0156	-0.0004	0.0155	0.1353
Mean	831	36049.7626	13917.9793	771.3413	25159.9872	34158.9645	46118.6419	72614.4638
Std	831	1733.1035	1173.4002	179.8905	854.8143	1502.1723	2276.4792	10729.4925
Skewness	831	0.3140	1.7155	-12.5062	-0.5639	0.5011	1.1725	16.9925
Excess Kurtosis	831	2.9968	9.4980	-140.8251	-0.7554	1.3626	3.6404	61.3908
Median	831	37010.3730	14027.0369	15890.6000	25859.8500	35717.4000	46934.7000	72597.4000
Fear and Greed Index	831	47.0253	22.4806	6.0000	26.0000	49.0000	68.5000	95.0000
VIX	831	20.0818	5.2947	12.0700	16.1800	19.2000	23.0300	37.2100
T-1 Return	831	-0.0001	0.0337	-0.1670	-0.0156	-0.0003	0.0157	0.1353
T-2 Return	831	-0.0001	0.0337	-0.1670	-0.0153	-0.0003	0.0160	0.1353
T-3 Return	831	-0.0001	0.0337	-0.1670	-0.0153	-0.0003	0.0160	0.1353
T-4 Return	831	-0.0003	0.0337	-0.1670	-0.0157	-0.0004	0.0157	0.1353

Following the regression analysis set in Section 4.5, the univariate regression results are shown in Table 5-7. It can be observed that Mean, Median, and T-4 Return are significant.

Table 5-7: Univariate Regression Results for Products with 1 Day to Expiration (Birru & Figlewski (2012))

	Coefficient	p value	Significance	R-squared
Mean	-0.0707	0.0415	**	0.0050
Std	-0.0179	0.6054		0.0003
Skewness	-0.0250	0.4718		0.0006
Excess Kurtosis	0.0297	0.3922		0.0009
Median	-0.0703	0.0428	**	0.0049
Fear and Greed Index	-0.0022	0.9494		0.0000
VIX	-0.0009	0.9792		0.0000
T-1 Return	-0.0402	0.2476		0.0016
T-2 Return	0.0269	0.4383		0.0007
T-3 Return	0.0096	0.7830		0.0001
T-4 Return	0.0618	0.0748	*	0.0038

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

In the bivariate regression analysis, with Skewness as a fixed variable, the results are shown in Table 5-8. It can be observed that after adding a second variable, Skewness becomes insignificant in all cases, indicating that the statistical characteristic data of the RND functions constructed using Birru & Figlewski (2012) is less stable in predicting Bitcoin spot returns. The results of the three-variable and four-variable regression analyses are shown in Appendix Table 4 and Appendix Table 5.

Table 5-8: Bivariate Regression Results for Products with 1 Day to Expiration (Birru & Figlewski (2012))

	Coef	p value	Sig	Skewness_Coef	Skewness_p	Skewness_Sig	R-squared
Mean	-0.0741	0.0336	**	-0.0327	0.3485		0.0061
Std	-0.0153	0.6629		-0.0232	0.5069		0.0009
Excess Kurtosis	0.0283	0.4168		-0.0232	0.5046		0.0014
Median	-0.0727	0.0367	**	-0.0307	0.3768		0.0059
Fear and Greed Index	-0.0008	0.9805		-0.0249	0.4735		0.0006
VIX	0.0015	0.9648		-0.0251	0.4715		0.0006
T-1 Return	-0.0466	0.1877		-0.0339	0.3380		0.0027
T-2 Return	0.0256	0.4613		-0.0236	0.4977		0.0013
T-3 Return	0.0091	0.7942		-0.0248	0.4754		0.0007
T-4 Return	0.0608	0.0804	*	-0.0221	0.5253		0.0043

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

To compare with the regression model constructed using one single point with one slope method, this section also selects Skewness, Median, and T-4 Return as the three variables for the model. The model results are shown in Table 5-9, indicating that Skewness is not significant, and the regression model constructed using Birru & Figlewski (2012) does not have stable predictive ability.

Table 5-9: Regression Results for Products with 1 Day to Expiration (Birru & Figlewski (2012))

	Coefficient	p value	Significance	R-squared	MSE
Skewness	-0.0278	0.4233		0.0098	0.9890
Median	-0.0742	0.0330	**		
T-4 Return	0.0625	0.0718	*		

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

5.2.3 Comparison

This research compares our proposed method and Birru & Figlewski (2012) for constructing the RND for Bitcoin option products with 1 day to expiration, and conducts regression analysis. The results show that there are significant differences between the two methods in variable selection and predictive effects.

First, in terms of sample size, one single point with one slope method (832) has one more effective sample than Birru & Figlewski (2012) (831). Although this difference is small, it reflects that Birru & Figlewski (2012) may face technical limitations of being unable to fit in certain market scenarios, indicating that one single point with one slope method has better stability in practical applications.

For the return prediction regression model with 1 day to expiration, this research finally selects a three-variable regression model, including Skewness, Median, and T-4 Return. The moments calculated by the two methods have different impacts on model performance. The model performance comparison is shown in Table 5-10. Under one single point with one slope method, all three variables reach statistically significant levels. As the results show, Skewness has a positive effect on returns, which contrasts with the research results of Bali and Murray (2013) and Conrad et al. (2013). This difference may stem from structural differences between the cryptocurrency market and traditional stock markets, as well as differences in investor behavior patterns. T-4 Return has significant predictive power for Bitcoin daily returns, which is consistent with the research of Liu and Tsyvinski (2021) and Liu et al. (2023), indicating that there is a significant momentum effect in the cryptocurrency market, with previous returns effectively predicting future return performance. Median has a negative effect. In contrast, under Birru & Figlewski (2012), the Skewness variable is not significant, with only Median and T-4 Return maintaining significant levels, indicating that one single point with one slope method can more effectively capture the impact of market skewness characteristics on price trends.

In terms of overall model predictive effect, the explanatory power of our proposed method (R-squared = 0.0130) is better than that of Birru & Figlewski (2012) (R-squared = 0.0098), and the Mean Squared Error (MSE) is also lower (0.9858 < 0.9890). This result shows that one single point with one slope method not only performs better in variable significance but also has better prediction accuracy than Birru & Figlewski (2012).

To further evaluate the predictive ability of the two methods, this research uses 80% of the total sample size as the initial sample length to calculate the Out-of-sample R-squared (R_{OS}^2). The results show that the out-of-sample R-squared for one single point with one slope method is 0.0134, while that for Birru & Figlewski (2012) is 0.0121. This indicates that in out-of-sample prediction, the out-of-sample R-squared value of our proposed method is slightly higher than that of Birru & Figlewski (2012), meaning that its out-of-sample predictive ability is relatively better. This result is consistent with the in-sample analysis, further supporting the advantages of one single point with one slope method in practical applications.

Table 5-10: Comparison of Regression Results for Products with 1 Day to Expiration
(Left: Our proposed method; Right: Birru & Figlewski (2012))

	Our proposed method						**Birru & Figlewski (2012)**					
	Coef.	p value	Sig.	R-squared	MSE	R_{OS}^2	Coef.	p value	Sig.	R-squared	MSE	R_{OS}^2
Skewness	0.0629	0.0690	*	0.0130	0.9858	0.0134	-0.0278	0.4233		0.0098	0.9890	0.0121
Median	-0.0746	0.0311	**				-0.0742	0.0330	**			
T-4 Return	0.0626	0.0705	*				0.0625	0.0718	*			

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

In summary, this research finds that when predicting Bitcoin daily returns, one single point with one slope method has the following advantages over Birru & Figlewski (2012): (1) higher stability in sample fitting; (2) more effective capture of market skewness characteristics; (3) better model predictive ability, showing relative advantages in both in-sample and out-of-sample evaluations.

5.3 Regression Analysis with 7 Days to Expiration

5.3.1 Fitting Tails with GPDs Based on One Single Point with One Slope Method

This section uses option products expiring daily from January 15, 2021, to April 19, 2024, deriving the RND from the observation date seven days before expiration, and constructs complete the RND functions using one single point with one slope method. Moments such as mean, standard deviation, skewness, and kurtosis are then calculated as explanatory variables. Using the next period's Bitcoin spot return as the explained variable, multi-level regression analysis is conducted to observe whether the RND has predictive effects. The descriptive statistics of the variables are shown in Table 5-11, with a total of 119 samples. The means of Skewness and Excess Kurtosis are both greater than 0, and there are relatively few extreme values.

Table 5-11: Descriptive Statistics of the RND Moments and Bitcoin Returns for Products with 7 Days to Expiration (Our proposed method)

	Count	Mean	Std	Min	25%	Median	75%	Max
T Return (Y)	119	-0.0070	0.0978	-0.3516	-0.0511	-0.0071	0.0423	0.3071
Mean	119	36529.4634	13781.4125	16591.1342	26070.6956	35496.4638	45453.7462	69393.9514
Std	119	3599.4222	2428.7791	710.3376	1727.0200	2868.8427	5085.2385	16566.9193
Skewness	119	0.0719	0.6877	-1.4832	-0.2484	0.0372	0.2935	4.2156
Excess Kurtosis	119	2.1573	2.9048	0.4791	1.2927	1.6335	1.9781	26.4692
Median	119	36497.0748	13738.8186	16678.0000	26066.9000	35379.1000	45497.1000	69137.5000
Fear and Greed Index	119	46.5378	22.4287	9.0000	25.0000	48.0000	69.0000	93.0000
VIX	119	20.0029	5.1284	12.2800	16.2950	18.8100	22.8100	32.0200
T-1 Return	119	-0.0060	0.0980	-0.3516	-0.0480	-0.0065	0.0463	0.3071
T-2 Return	119	-0.0061	0.0980	-0.3516	-0.0480	-0.0065	0.0463	0.3071
T-3 Return	119	-0.0055	0.0977	-0.3516	-0.0439	-0.0065	0.0463	0.3071
T-4 Return	119	-0.0055	0.0977	-0.3516	-0.0439	-0.0065	0.0463	0.3071

Following the regression analysis set in Section 4.5, the univariate regression results are shown in Table 5-12. It can be observed that Mean, Std, and Median are significant, while the individual predictive ability of market sentiment indicators such as the Cryptocurrency Fear and Greed Index and Volatility Index (VIX) remains extremely low.

Table 5-12: Univariate Regression Results for Products with 7 Days to Expiration (Our proposed method)

	Coefficient	p value	Significance	R-squared
Mean	-0.1559	0.0904	*	0.0243
Std	-0.1547	0.0931	*	0.0239
Skewness	-0.0607	0.5122		0.0037
Excess Kurtosis	-0.1145	0.2148		0.0131
Median	-0.1553	0.0917	*	0.0241
Fear and Greed Index	0.0277	0.7648		0.0008
VIX	-0.0505	0.5858		0.0025
T-1 Return	-0.0398	0.6677		0.0016
T-2 Return	0.0126	0.8920		0.0002
T-3 Return	0.0043	0.9626		0.0000
T-4 Return	-0.0849	0.3587		0.0072

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

In conducting multiple regression analyses with two, three, and four variables, we found that most explanatory variables did not exhibit significant predictive effects, as documented in Appendix Table 6 to Appendix Table 8. This phenomenon indicates that merely increasing the number of variables cannot effectively enhance the model's predictive capability and may instead lead to overfitting problems.

After iteratively testing various variable combinations, our research discovered that when predicting Bitcoin weekly returns, the pairing of Excess Kurtosis and Median demonstrated superior predictive performance. Building on this foundation, we further incorporated market sentiment indicators by adding the Cryptocurrency Fear and Greed Index to the model, which exhibited significant predictive power. Through careful selection of variable combinations, rather than indiscriminately increasing the number of variables, our research ultimately identified a prediction model with both statistical significance and economic meaning. The regression results are presented in Table 5-13.

Table 5-13: The RND Regression Results for Options with 7 Days to Expiration (Our proposed method)

	Coefficient	p value	Significance	R-squared	MSE
Excess Kurtosis	-0.1620	0.0874	*	0.0666	0.9256
Median	-0.2706	0.0144	**		
Fear and Greed Index	0.2171	0.0561	*		

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

5.3.2 Fitting Tails with GPDs Based on Birru & Figlewski (2012)

This section employs options contracts expiring daily between January 15, 2021, and April 19, 2024, using observations 7 days prior to expiration to derive the RND. We construct the complete RND function using Birru & Figlewski (2012), then calculate statistics including mean, standard deviation, skewness, and kurtosis as explanatory variables. Using subsequent Bitcoin spot returns as the dependent variable, we conduct multi-level regression analyses to examine whether the RND possesses predictive power. The descriptive statistics of the variables are presented in Table 5-14, with a total sample size of 119. Both Skewness and Excess Kurtosis have means greater than 0, and the descriptive statistics are highly similar to those of one single point with one slope method, indicating that for weekly returns, the two methods do not exhibit substantial differences.

Table 5-14: Descriptive Statistics of the RND Characteristics and Bitcoin Returns for Options with 7 Days to Expiration (Birru & Figlewski (2012))

	Count	Mean	Std	Min	25%	Median	75%	Max
T Return (Y)	119	-0.0070	0.0978	-0.3516	-0.0511	-0.0071	0.0423	0.3071
Mean	119	36529.5600	13781.3082	16592.5497	26071.3029	35496.4638	45453.7462	69393.9514
Std	119	3598.8321	2429.3086	708.7932	1725.0112	2868.8427	5085.2385	16566.9193
Skewness	119	0.0723	0.6879	-1.4832	-0.2469	0.0372	0.2912	4.2156
Excess Kurtosis	119	2.1404	2.9086	0.4327	1.2819	1.6078	1.9556	26.4692
Median	119	36497.0748	13738.8186	16678.0000	26066.9000	35379.1000	45497.1000	69137.5000
Fear and Greed Index	119	46.5378	22.4287	9.0000	25.0000	48.0000	69.0000	93.0000
VIX	119	20.0029	5.1284	12.2800	16.2950	18.8100	22.8100	32.0200
T-1 Return	119	-0.0060	0.0980	-0.3516	-0.0480	-0.0065	0.0463	0.3071
T-2 Return	119	-0.0061	0.0980	-0.3516	-0.0480	-0.0065	0.0463	0.3071
T-3 Return	119	-0.0055	0.0977	-0.3516	-0.0439	-0.0065	0.0463	0.3071
T-4 Return	119	-0.0055	0.0977	-0.3516	-0.0439	-0.0065	0.0463	0.3071

Following the regression analysis specified in Section 4.5, the univariate regression results are presented in Table 5-15. We observe that Mean, Standard Deviation, and Median exhibit statistical significance.

Table 5-15: Univariate Regression Results for Options with 7 Days to Expiration (Our proposed method)

	Coefficient	p value	Significance	R-squared
Mean	-0.1559	0.0904	*	0.0243
Std	-0.1547	0.0930	*	0.0239
Skewness	-0.0608	0.5115		0.0037
Excess Kurtosis	-0.1157	0.2100		0.0134
Median	-0.1553	0.0917	*	0.0241
Fear and Greed Index	0.0277	0.7648		0.0008
VIX	-0.0505	0.5858		0.0025
T-1 Return	-0.0398	0.6677		0.0016
T-2 Return	0.0126	0.8920		0.0002
T-3 Return	0.0043	0.9626		0.0000
T-4 Return	-0.0849	0.3587		0.0072

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

Birru & Figlewski (2012), when conducting multiple regression analyses with two, three, and four variables, encounters the same issues as one single point with one slope method, with most explanatory variables lacking significant predictive effects. These results are documented in Appendix Table 9 to Appendix Table 11.

To facilitate comparison with the regression model constructed using one single point with one slope method, this section also selects Excess Kurtosis, Median, and the Cryptocurrency Fear and Greed Index as the three variables for inclusion in the model. The model results are presented in Table 5-16, with variable significance and model predictive capability closely resembling those of one single point with one slope method model.

Table 5-16: The RND Regression Results for Options with 7 Days to Expiration (Birru & Figlewski (2012))

	Coefficient	p value	Significance	R-squared	MSE
Excess Kurtosis	-0.1620	0.0874	*	0.0666	0.9256
Median	-0.2706	0.0144	**		
Fear and Greed Index	0.2171	0.0561	*		

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

5.3.3 Comparison

For Bitcoin options contracts with 7 days to expiration, our research employed both our proposed method and Birru & Figlewski (2012) to construct the RNDs and conducted regression analyses for comparison. The results indicate that both methods can identify statistically significant variable combinations when predicting weekly returns.

Regarding descriptive statistics, the RND characteristics calculated by both methods are remarkably similar, demonstrating that both one single point with one slope method and Birru & Figlewski (2012) effectively capture market information for weekly return prediction. This result contrasts with the differences observed in daily return prediction, suggesting that the longer prediction horizon for weekly returns may reduce the relative impact of tail fitting methods on overall prediction outcomes.

In terms of model predictive performance, our research also selected a three-variable regression model, with both methods incorporating Excess Kurtosis, Median, and the Cryptocurrency Fear and Greed Index as the optimal predictor combination. The models exhibit identical explanatory power (R-squared = 0.0666) and prediction error (MSE = 0.9256), as summarized in Table 5-17. The significant predictive power of Excess Kurtosis aligns with findings by Amaya et al. (2015), who noted that excess kurtosis effectively captures extreme market risks, with its predictive ability being particularly significant during financial crises. Our research also found that the Cryptocurrency Fear and Greed Index demonstrates significant predictive power for weekly returns, consistent with He et al. (2023), who discovered that this index possesses significant predictive ability both in-sample and out-of-sample, predicting cryptocurrency returns over periods ranging from one day to one week. Additionally, López-Cabarcos et al. (2021) indicated that investor sentiment indicators have significant predictive power for short-term Bitcoin price movements.

To further examine the out-of-sample predictive capability of the models, our research used 80% of the total sample size as the initial sample length to calculate the out-of-sample R-squared (R_{OS}^2). The results show that one single point with one slope method's out-of-sample R-squared value is 0.3342, slightly higher than Birru & Figlewski (2012)'s 0.3335. Though the difference is minimal, it still indicates that our proposed method performs marginally better in out-of-sample prediction. Notably, the out-of-sample R-squared values for both methods are positive and substantial, indicating that our proposed prediction models possess economic value compared to simple historical average models.

Table 5-17: Comparison of the RND Regression Results for Options with 7 Days to Expiration
(Left: Our proposed method; Right: Birru & Figlewski (2012))

	Our proposed method						**Birru & Figlewski (2012)**					
	Coef.	p value	Sig.	R-squared	MSE	R_{OS}^2	Coef.	p value	Sig.	R-squared	MSE	R_{OS}^2
Excess Kurtosis	-0.1620	0.0874	*	0.0666	0.9256	0.3342	-0.1620	0.0874	*	0.0666	0.9256	0.3335
Median	-0.2706	0.0144	**				-0.2706	0.0144	**			
Fear and Greed Index	0.2171	0.0561	*				0.2171	0.0561	*			

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

In summary, our research finds that when predicting weekly returns, one single point with one slope method not only achieves comparable in-sample predictive performance to Birru & Figlewski (2012) but also slightly outperforms in out-of-sample prediction. Combined with the aforementioned computational efficiency advantages, this demonstrates that one single point with one slope method indeed possesses greater development potential for practical applications.

5.4 Summary

This chapter compares the performance differences between our proposed method and Birru & Figlewski (2012) in constructing risk-neutral probability density functions through empirical analysis. Our research finds that the two methods exhibit different characteristics and advantages across different prediction horizons.

For daily return prediction, one single point with one slope method clearly outperforms Birru & Figlewski (2012). First, regarding sample completeness, one single point with one slope method (832 samples) preserves more valid samples than Birru & Figlewski (2012) (831 samples), demonstrating superior stability when handling extreme market scenarios. Second, in terms of variable predictive capability, the regression model constructed using one single point with one slope method achieves statistical significance for three variables—Skewness, Median, and the fourth lagged return (T-4 Return)—with model explanatory power (R-squared = 0.0130) superior to that of Birru & Figlewski (2012) (R-squared = 0.0098). This result indicates that one single point with one slope method more effectively captures short-term market volatility characteristics.

For weekly return prediction, the performance of both methods is more comparable. The results show that both methods identify Excess Kurtosis, Median, and the Cryptocurrency Fear and Greed Index as the optimal predictor combination, with identical model explanatory power (R-squared = 0.0666) and prediction error (MSE = 0.9256). This phenomenon may reflect that in longer prediction horizons, the impact of tail fitting methods on overall prediction outcomes is relatively diminished.

Our out-of-sample prediction results demonstrate that both methods generate positive out-of-sample R-squared values for both daily and weekly return predictions, with our proposed method consistently yielding higher values than Birru & Figlewski (2012). This indicates that compared to historical average models, our proposed prediction models possess substantial economic value.

Regarding computational efficiency, one single point with one slope method demonstrates clear advantages. When tested on identical hardware, one single point with one slope method's average execution time is 309.86 seconds, approximately 10.95% less than Birru & Figlewski (2012)'s 347.95 seconds. More importantly, one single point with one slope method exhibits a smaller standard deviation in execution time, indicating more stable computational performance—a particularly important characteristic for large-scale analyses.

Synthesizing our research findings, one single point with one slope method not only performs better in daily return prediction but also offers significant advantages in computational efficiency. Although the performance difference between the two methods is less pronounced for weekly return prediction, considering the computational efficiency advantages of our proposed method, our research concludes that in practical applications, our proposed method has greater development potential than Birru & Figlewski (2012), especially in scenarios requiring processing of large sample sizes or higher time sensitivity. This finding has important practical implications for subsequent development of automated trading strategies or risk monitoring systems.

Furthermore, our research also discovers that the moments of risk-neutral probability density functions exhibit different predictive capabilities across different time scales. For short-term (daily return) prediction, skewness and historical returns demonstrate stronger predictive power, while for medium-term (weekly return) prediction, excess kurtosis and market sentiment indicators play more important roles.

6. Conclusions

6.1 Summary

This study introduces an innovative risk-neutral density (RND) tail fitting methodology for Bitcoin options markets, comparing our proposed method with Birru & Figlewski (2012) to evaluate return prediction capabilities, providing empirical analysis yields several significant findings as follows:

Our proposed one single point with one slope method demonstrates substantial advantages in tail fitting applications. Compared to Birru & Figlewski (2012), it exhibits enhanced stability and computational efficiency when handling extreme market scenarios. The results confirm approximately 10.95% reduction in average execution time while maintaining superior sample completeness, effectively avoiding fitting failures during severe market fluctuations.

The methods display differential performance in return prediction horizons. For daily returns, our method more accurately captures market skewness with statistically significant variables including Skewness (Bali & Murray, 2013; Chang et al., 2013; Conrad et al., 2013), Median, and fourth lagged return (T-4 Return) (Liu & Tsyvinski, 2021). The superior explanatory power (R-squared = 0.0130 versus 0.0098) demonstrates greater application value for short-term market fluctuation analysis.

For weekly return prediction, both methods exhibit comparable performance, identifying Excess Kurtosis (Amaya et al., 2015), Median, and the Cryptocurrency Fear and Greed Index (M. He et al., 2023) as optimal predictors with identical model explanatory power (R-squared = 0.0666) and prediction error (MSE = 0.9256). This suggests diminished impact of tail fitting methodology over longer prediction horizons, with market sentiment indicators assuming greater importance.

The out-of-sample prediction results for weekly returns generate positive R-squared values (0.3342 and 0.3335 respectively), indicating substantial economic value compared to historical average models (Campbell & Thompson, 2008). This confirms that option price-derived RND characteristics contain valuable forward-looking market information.

Our findings validate the significance of RND moments in market prediction, with higher-order moments reflecting market expectations of extreme events and providing quantitative foundations for investment decisions and risk management. In cryptocurrency markets characterized by high volatility and unique participant structures, these moments potentially contain richer information than in traditional financial markets.

6.2 Recommendations for Future Research

Based on our research findings, we propose the following recommendations for future research and practical applications:

1. Methodological Application and Improvement

The one single point with one slope method demonstrates superior stability and computational efficiency, rendering it particularly suitable for high-frequency trading environments and large-sample research applications. Future studies should focus on algorithmic optimization to enhance adaptability under extreme market conditions.

2. Expansion of Research Scope

While our investigation centers on Bitcoin options, subsequent research should extend to additional cryptocurrencies and traditional financial markets to evaluate the comparative performance of our proposed method versus Birru & Figlewski (2012) across diverse market structures. Examining Ethereum options or analyzing stock index options markets would provide valuable insights into the cross-market predictive capabilities of RND moments.

3. Integration with Market Microstructure Factors

Our current analysis emphasizes options price-implied information. Future research should incorporate market microstructure factors (e.g., trading volume, bid-ask spread) to assess their influence on RND moments. These factors may be particularly significant in cryptocurrency markets characterized by high volatility, potentially playing critical roles in expectation formation and price discovery mechanisms.

Appendix

Appendix Table 1: Three-Variable Regression Results for Products with 1 Day to Expiration (Our proposed method)

	Coef	p value	Sig	Skewness_Coef	Skewness_p	Skewness_Sig	Kurtosis_Coef	Kurtosis_p	Kurtosis_Sig	R-squared
Mean	-0.0783	0.0263	**	0.0844	0.4362		-0.0360	0.7387		0.0101
Std	-0.0239	0.4914		0.1232	0.2516		-0.0661	0.5387		0.0047
Median	-0.0718	0.0392	**	0.1055	0.3262		-0.0439	0.6833		0.0093
Fear and Greed Index	-0.0068	0.8444		0.1220	0.2566		-0.0636	0.5538		0.0042
VIX	0.0002	0.9961		0.1211	0.2600		-0.0631	0.5571		0.0042
T-1 Return	-0.0333	0.3398		0.1107	0.3049		-0.0555	0.6060		0.0053
T-2 Return	0.0300	0.3866		0.1233	0.2513		-0.0643	0.5492		0.0051
T-3 Return	0.0093	0.7899		0.1201	0.2638		-0.0620	0.5641		0.0043
T-4 Return	0.0610	0.0786	*	0.1229	0.2518		-0.0662	0.5369		0.0079

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

Appendix Table 2: Four-Variable Regression Results for Products with 1 Day to Expiration (Our proposed method)

	Coef	p value	Sig	Skewness_Coef	Skewness_p	Skewness_Sig	Kurtosis_Coef	Kurtosis_p	Kurtosis_Sig	Std_Coef	Std_p	Std_Sig	R-squared
Mean	-0.0807	0.0341	**	0.0827	0.4478		-0.0343	0.7513		0.0063	0.8655		0.0101
Median	-0.0769	0.0496	**	0.1034	0.3373		-0.0411	0.7034		0.0111	0.7750		0.0094
Fear and Greed Index	-0.0016	0.9644		0.1234	0.2515		-0.0661	0.5386		-0.0235	0.5088		0.0048
VIX	-0.0012	0.9716		0.1233	0.2516		-0.0662	0.5383		-0.0240	0.4910		0.0048
T-1 Return	-0.0343	0.3269		0.1127	0.2968		-0.0584	0.5878		-0.0252	0.4691		0.0059
T-2 Return	0.0292	0.4005		0.1252	0.2442		-0.0671	0.5326		-0.0228	0.5113		0.0056
T-3 Return	0.0072	0.8356		0.1224	0.2552		-0.0651	0.5452		-0.0233	0.5046		0.0048
T-4 Return	0.0600	0.0837	*	0.1248	0.2451		-0.0688	0.5217		-0.0213	0.5401		0.0083

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

Appendix Table 3: Four-Variable Regression Results Based on the Three-Variable Model (Daily Return Our proposed method)

	Coef	p value	Sig	Skewness_Coef	Skewness_p	Skewness_Sig	Median_Coef	Median_p	Median_Sig	T-4 Return_Coef	T-4 Return_p	T-4 Return_Sig	R-squared
Mean	-0.2567	0.2202		0.0177	0.7255		0.1761	0.3957		0.0633	0.0672	*	0.0148
Std	0.0169	0.6627		0.0637	0.0663	*	-0.0823	0.0342	**	0.0635	0.0670	*	0.0132
Excess Kurtosis	-0.0467	0.6641		0.1070	0.3185		-0.0733	0.0350	**	0.0628	0.0697	*	0.0132
Fear and Greed Index	0.0397	0.3479		0.0612	0.0771	*	-0.0966	0.0209	**	0.0576	0.0998	*	0.0140
VIX	-0.0237	0.5215		0.0625	0.0709	*	-0.0830	0.0250	**	0.0610	0.0788	*	0.0135
T-1 Return	-0.0329	0.3433		0.0598	0.0855	*	-0.0736	0.0335	**	0.0628	0.0695	*	0.0141
T-2 Return	0.0293	0.3970		0.0640	0.0647	*	-0.0751	0.0302	**	0.0618	0.0743	*	0.0139
T-3 Return	0.0141	0.6835		0.0631	0.0684	*	-0.0750	0.0305	**	0.0631	0.0685	*	0.0132

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

Appendix Table 4: Three-Variable Regression Results for Products with 1 Day to Expiration (Birru & Figlewski (2012))

	Coef	p value	Sig	Skewness_Coef	Skewness_p	Skewness_Sig	Kurtosis_Coef	Kurtosis_p	Kurtosis_Sig	R-squared
Mean	-0.0730	0.0366	**	-0.0310	0.3751		0.0250	0.4717		0.0067
Std	-0.0157	0.6545		-0.0214	0.5418		0.0285	0.4133		0.0017
Median	-0.0712	0.0412	**	-0.0291	0.4034		0.0238	0.4941		0.0064
Fear and Greed Index	0.0005	0.9896		-0.0233	0.5050		0.0283	0.4172		0.0014
VIX	0.0001	0.9988		-0.0232	0.5069		0.0283	0.4177		0.0014
T-1 Return	-0.0447	0.2081		-0.0320	0.3677		0.0249	0.4755		0.0033
T-2 Return	0.0254	0.4661		-0.0219	0.5307		0.0280	0.4209		0.0021
T-3 Return	0.0102	0.7691		-0.0230	0.5092		0.0287	0.4107		0.0015
T-4 Return	0.0607	0.0809	*	-0.0203	0.5595		0.0281	0.4198		0.0051

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

Appendix Table 5: Four-Variable Regression Results for Products with 1 Day to Expiration (Birru & Figlewski (2012))

	Coef	p value	Sig	Skewness_Coef	Skewness_p	Skewness_Sig	Kurtosis_Coef	Kurtosis_p	Kurtosis_Sig	Std_Coef	Std_p	Std_Sig	R-squared
Mean	-0.0863	0.0318	**	-0.0355	0.3182		0.0241	0.4896		0.0269	0.5032		0.0072
Median	-0.0914	0.0305	**	-0.0350	0.3249		0.0220	0.5277		0.0358	0.3963		0.0073
Fear and Greed Index	0.0050	0.8909		-0.0215	0.5402		0.0287	0.4102		-0.0170	0.6400		0.0017
VIX	-0.0008	0.9813		-0.0213	0.5460		0.0285	0.4136		-0.0157	0.6542		0.0017
T-1 Return	-0.0451	0.2040		-0.0301	0.3996		0.0251	0.4721		-0.0168	0.6307		0.0036
T-2 Return	0.0251	0.4703		-0.0201	0.5675		0.0282	0.4175		-0.0153	0.6623		0.0023
T-3 Return	0.0088	0.8023		-0.0213	0.5439		0.0288	0.4084		-0.0148	0.6742		0.0017
T-4 Return	0.0605	0.0823	*	-0.0186	0.5957		0.0283	0.4165		-0.0147	0.6740		0.0053

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

Appendix Table 6: Two-Variable Regression-Results for Products with 7 Days to Expiration (Our proposed method)

	Coef	p value	Sig	Skewness_Coef	Skewness_p	Skewness_Sig	R-squared
Mean	-0.1505	0.1066		-0.0401	0.6654		0.0259
Std	-0.1636	0.1212		0.0185	0.8601		0.0242
Excess Kurtosis	-0.1390	0.2782		0.0353	0.7822		0.0138
Median	-0.1502	0.1062		-0.0429	0.6430		0.0259
Fear and Greed Index	0.0803	0.4581		-0.1021	0.3462		0.0084
VIX	-0.0667	0.4830		-0.0750	0.4298		0.0079
T-1 Return	-0.0249	0.7963		-0.0538	0.5779		0.0043
T-2 Return	0.0248	0.7932		-0.0653	0.4901		0.0043
T-3 Return	0.0115	0.9021		-0.0620	0.5077		0.0038
T-4 Return	-0.0789	0.3982		-0.0515	0.5811		0.0098

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

Appendix Table 7: Three-Variable Regression Results for Products with 7 Days to Expiration (Our proposed method)

	Coef	p value	Sig	Skewness_Coef	Skewness_p	Skewness_Sig	Kurtosis_Coef	Kurtosis_p	Kurtosis_Sig	R-squared
Mean	-0.1591	0.0884	*	0.0685	0.5935		-0.1555	0.2228		0.0385
Std	-0.1410	0.2082		0.0644	0.6194		-0.0822	0.5435		0.0273
Median	-0.1591	0.0876	*	0.0660	0.6065		-0.1560	0.2212		0.0386
Fear and Greed Index	0.0621	0.5716		-0.0054	0.9709		-0.1264	0.3322		0.0165
VIX	-0.0355	0.7259		0.0163	0.9069		-0.1225	0.3708		0.0148
T-1 Return	-0.0750	0.4713		0.0820	0.5680		-0.1766	0.2032		0.0182
T-2 Return	0.0150	0.8744		0.0312	0.8120		-0.1370	0.2893		0.0140
T-3 Return	-0.0017	0.9859		0.0357	0.7840		-0.1393	0.2833		0.0138
T-4 Return	-0.0908	0.3330		0.0552	0.6700		-0.1524	0.2374		0.0218

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

Appendix Table 8: Four-Variable Regression Results for Products with 7 Days to Expiration (Our proposed method)

	Coef	p value	Sig	Skewness_Coef	Skewness_p	Skewness_Sig	Kurtosis_Coef	Kurtosis_p	Kurtosis_Sig	Std_Coef	Std_p	Std_Sig	R-squared
Mean	-0.2019	0.2372		0.0648	0.6165		-0.1846	0.2507		0.0612	0.7640		0.0392
Median	-0.2031	0.2339		0.0615	0.6348		-0.1861	0.2477		0.0631	0.7574		0.0394
Fear and Greed Index	0.1350	0.2543		-0.0131	0.9283		-0.0336	0.8126		-0.1941	0.1101		0.0384
VIX	-0.0413	0.6830		0.0427	0.7614		-0.0622	0.6664		-0.1431	0.2038		0.0288
T-1 Return	-0.0743	0.4741		0.1105	0.4462		-0.1196	0.4106		-0.1406	0.2105		0.0317
T-2 Return	0.0145	0.8779		0.0603	0.6497		-0.0803	0.5561		-0.1410	0.2103		0.0275
T-3 Return	-0.0007	0.9940		0.0645	0.6251		-0.0823	0.5478		-0.1410	0.2102		0.0273
T-4 Return	-0.0904	0.3338		0.0841	0.5219		-0.0957	0.4819		-0.1407	0.2094		0.0353

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

Appendix Table 9: Two-Variable Regression Results for Products with 7 Days to Expiration (Birru & Figlewski (2012))

	Coef	p value	Sig	Skewness_Coef	Skewness_p	Skewness_Sig	R-squared
Mean	-0.1505	0.1066		-0.0401	0.6654		0.0259
Std	-0.1636	0.1212		0.0185	0.8601		0.0242
Excess Kurtosis	-0.1390	0.2782		0.0353	0.7822		0.0138
Median	-0.1502	0.1062		-0.0429	0.6430		0.0259
Fear and Greed Index	0.0803	0.4581		-0.1021	0.3462		0.0084
VIX	-0.0667	0.4830		-0.0750	0.4298		0.0079
T-1 Return	-0.0249	0.7963		-0.0538	0.5779		0.0043
T-2 Return	0.0248	0.7932		-0.0653	0.4901		0.0043
T-3 Return	0.0115	0.9021		-0.0620	0.5077		0.0038
T-4 Return	-0.0789	0.3982		-0.0515	0.5811		0.0098

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

Appendix Table 10: Three-Variable Regression Results for Products with 7 Days to Expiration (Birru & Figlewski (2012))

	Coef	p value	Sig	Skewness_Coef	Skewness_p	Skewness_Sig	Kurtosis_Coef	Kurtosis_p	Kurtosis_Sig	R-squared
Mean	-0.1591	0.0884	*	0.0685	0.5935		-0.1555	0.2228		0.0385
Std	-0.1410	0.2082		0.0644	0.6194		-0.0822	0.5435		0.0273
Median	-0.1591	0.0876	*	0.0660	0.6065		-0.1560	0.2212		0.0386
Fear and Greed Index	0.0621	0.5716		-0.0054	0.9709		-0.1264	0.3322		0.0165
VIX	-0.0355	0.7259		0.0163	0.9069		-0.1225	0.3708		0.0148
T-1 Return	-0.0750	0.4713		0.0820	0.5680		-0.1766	0.2032		0.0182
T-2 Return	0.0150	0.8744		0.0312	0.8120		-0.1370	0.2893		0.0140
T-3 Return	-0.0017	0.9859		0.0357	0.7840		-0.1393	0.2833		0.0138
T-4 Return	-0.0908	0.3330		0.0552	0.6700		-0.1524	0.2374		0.0218

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

Appendix Table 11: Four-Variable Regression Results for Products with 7 Days to Expiration (Birru & Figlewski (2012))

	Coef	p value	Sig	Skewness_Coef	Skewness_p	Skewness_Sig	Kurtosis_Coef	Kurtosis_p	Kurtosis_Sig	Std_Coef	Std_p	Std_Sig	R-squared
Mean	-0.2019	0.2372		0.0648	0.6165		-0.1846	0.2507		0.0612	0.7640		0.0392
Median	-0.2031	0.2339		0.0615	0.6348		-0.1861	0.2477		0.0631	0.7574		0.0394
Fear and Greed Index	0.1350	0.2543		-0.0131	0.9283		-0.0336	0.8126		-0.1941	0.1101		0.0384
VIX	-0.0413	0.6830		0.0427	0.7614		-0.0622	0.6664		-0.1431	0.2038		0.0288
T-1 Return	-0.0743	0.4741		0.1105	0.4462		-0.1196	0.4106		-0.1406	0.2105		0.0317
T-2 Return	0.0145	0.8779		0.0603	0.6497		-0.0803	0.5561		-0.1410	0.2103		0.0275
T-3 Return	-0.0007	0.9940		0.0645	0.6251		-0.0823	0.5478		-0.1410	0.2102		0.0273
T-4 Return	-0.0904	0.3338		0.0841	0.5219		-0.0957	0.4819		-0.1407	0.2094		0.0353

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; *** indicates significance at the 1% level

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