Notes for Riemannian Geometry

Fourier

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1.1 Differentiable Manifolds

Differentiable Manifolds

A *differentiable manifold* of dimension n is a set M and a family of injective mappings $\mathbf{x}_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ of open sets U_α of $\mathbb{R}^n$ into M such that:

(1) $\bigcup_\alpha \mathbf{x}_\alpha(U_\alpha) = M$.

(2) for any pair α, β, with $\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) = W = \varnothing$, the sets $\mathbf{x}_\alpha^{-1}(W)$ and $\mathbf{x}_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the mappings $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$ are differentiable.

(3) The family $\{(U_\alpha, \mathbf{x}_\alpha)\}$ is maximal relative to the conditions (1) and (2).

The pair $(U_\alpha, \mathbf{x}_\alpha)$(or the mapping $\mathbf{x}_\alpha$) with $p \in \mathbf{x}_\alpha(U_\alpha)$ is called a *parametrization* 参数化 (or *system of coordinates* 坐标系) of M at p; $\mathbf{x}_\alpha(U_\alpha)$ is then called a *coordinate neighborhood* at p. A family $\{(U_\alpha, \mathbf{x}_\alpha)\}$ satisfying (1) and (2) is called a *differentiable structure* on M.

Open Sets

$A \in M$ is an *open set* in M if and only if $\mathbf{x}_\alpha^{-1}(A \cap \mathbf{x}_\alpha(U_\alpha))$ is an open set in $\mathbb{R}^n$ for all α.

Differentiable Mappings

$\varphi : M_1 \to M_2$ is *differentiable at* $p \in M_1$ if given a parametrization $\mathbf{y} : V \subset \mathbb{R}^m \to M_2$ at $\varphi(p)$ there exists a parametrization $\mathbf{x} : U \subset \mathbb{R}^n \to M_1$ at p such that $\varphi(\mathbf{x}(U)) \subset \mathbf{y}(V)$ and the mapping

$$\mathbf{y}^{-1} \circ \varphi \circ \mathbf{x} : U \subset \mathbb{R}^n \to \mathbb{R}^m \quad \text{(the expression of } \varphi \text{ in the parametrizations } \mathbf{x} \text{ and } \mathbf{y})$$

is differentiable at $\mathbf{x}^{-1}(p)$. φ is differentiable on an open set of M_1 if it is differentiable at all of the points of this open set.

Tangent Vector

Let M be a differentiable manifold. A differentiable function $\alpha : (-\varepsilon, \varepsilon) \to M$ is called a (differentiable) *curve* in M. Suppose that $\alpha(0) = p \in M$, and let $\mathcal{D}$ be the set of functions on M that are differentiable at p. The *tangent vector to the curve* α at $t = 0$ is a function $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ given by

$$\alpha'(0)f = \frac{\mathrm{d}(f \circ \alpha)}{\mathrm{d}t}\Big|_{t=0}, \quad f \in \mathcal{D}.$$

类比 f 在 $\alpha'(0)$ 方向的方向导数 $\boldsymbol{a} \cdot (\nabla f)$, $\alpha'(0) = (\boldsymbol{a} \cdot \nabla)$

If we choose a parametrization $\mathbf{x} : U \to M^n$ at $p = \mathbf{x}(0)$, we can express the function f and the curve α in this parametrization by

$$f \circ \mathbf{x}(q) = f(x_1, \dots, x_n), \quad q = (x_1, \dots, x_n) \subset U,$$

and

$$\mathbf{x}^{-1} \circ \alpha(t) = \big(x_1(t), \dots, x_n(t)\big),$$

respectively. Therefore, restricting f to α, we obtain

$$\begin{aligned}\alpha'(0)f &= \frac{\mathrm{d}(f \circ \alpha)}{\mathrm{d}t}\Big|_{t=0} = \frac{\mathrm{d}\big((f \circ \mathbf{x}) \circ (\mathbf{x}^{-1} \circ \alpha(t))\big)}{\mathrm{d}t}\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} f \circ \mathbf{x}\big(x_1(t), \dots, x_n(t)\big)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f\big(x_1(t), \dots, x_n(t)\big)\Big|_{t=0} \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big|_{t=0} x_i'(0) = \Big(\sum_{i=1}^n x_i'(0)\Big(\frac{\partial}{\partial x_i}\Big)_0\Big) f\end{aligned}$$

In other words, the vector $\alpha'(0)$ can be expressed in the parametrization $\mathbf{x}$ by

$$\alpha'(0) = \sum_i x_i'(0)\Big(\frac{\partial}{\partial x_i}\Big)_0.$$

Differential of a Differentiable mapping

Let M_1^n and M_2^m be differentiable manifolds and let $\varphi : M_1 \to M_2$ be a differentiable mapping. For every $p \in M_1$ and for each $v \in T_pM_1$, choose a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to M_1$ with $\alpha(0) = p, \alpha'(0) = v$. Take $\beta = \varphi \circ \alpha$. The mapping $\mathrm{d}\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ given by $\mathrm{d}\varphi_p(v) = \beta'(0)$ is a linear mapping that does not depend on the choice of α.
$\mathrm{d}\varphi_p$ is called the *differential* of φ at p.

Proof. Let $\mathbf{x} : U \to M_1$ and $\mathbf{y} : V \to M_2$ be parametrizations at p and $\varphi(p)$, respectively. To express φ in these parametrizations, we can write

$$\mathbf{y}^{-1} \circ \varphi \circ \mathbf{x}(q) = \big(y_1(x_1, \dots, x_n), \dots, y_m(x_1, \dots, x_n)\big)$$

$$q = (x_1, \dots, x_n) \in U, \quad (y_1, \dots, y_m) \in V$$

On the other hand, expressing α in the parametrization $\mathbf{x}$, we obtain

$$\mathbf{x}^{-1} \circ \alpha(t) = \big(x_1(t), \dots, x_n(t)\big).$$

Therefore,

$$\mathbf{y}^{-1} \circ \beta(t) = \big(y_1(x_1(t), \dots, x_n(t)), \dots, y_m(x_1(t), \dots, x_n(t))\big)$$

It follows that the expression for $\beta'(0)$ with respect to the basis $\Big\{\Big(\frac{\partial}{\partial y_i}\Big)_0\Big\}$ of $T_{\varphi(p)}M_2$, associated to the parametrization $\mathbf{y}$, is given by

$$\beta'(0) = \Big(\sum_{i=1}^n \frac{\partial y_1}{\partial x_i} x_i'(0), \dots, \sum_{i=1}^n \frac{\partial y_m}{\partial x_i} x_i'(0)\Big).$$

The relation shows immediately that $\beta'(0)$ does not depend on the choice of α. In addition, it can be written as

$$\beta'(0) = \mathrm{d}\varphi_p(v) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} x_1'(0) \\ \vdots \\ x_n'(0) \end{bmatrix}; \quad \mathrm{d}\varphi_p = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

Therefore, $\mathrm{d}\varphi_p$ is a linear mapping of T_pM_1 into $T_{\varphi(p)}M_2$ whose matrix in the associated bases obtained from the parametrizations $\mathbf{x}$ and $\mathbf{y}$ is precisely the matrix $\Big(\frac{\partial y_i}{\partial x_j}\Big)$. □

Diffeomorphism 微分同胚

Let M_1 and M_2 be differentiable manifolds. A mapping $\varphi : M_1 \to M_2$ is a *diffeomorphism* if it is differentiable, bijective, and its inverse φ^{-1} is differentiable. φ is said to be a *local diffeomorphism* at $p \in M$ if there exist neighborhoods U of p and V of $\varphi(p)$ such that $\varphi : U \to V$ is a diffeomorphism.

Differentiable Mapping to Diffeomorphism

Let $\varphi : M_1^n \to M_2^n$ be a differentiable mapping and let $p \in M_1$ be such that $\mathrm{d}\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ is an isomorphism 同构. Then φ is a local diffeomorphism at p.

1.2 Immersions and embeddings; examples

Immersion 浸入, Embedding 嵌入 and Submanifold

Let M^m and N^n be differentiable manifolds. A differentiable mapping $\varphi : M \to N$ is said to be an *immersion* if $\mathrm{d}\varphi_p : T_pM \to T_{\varphi(p)}N$ is injective 内射，单射 for all $p \in M$.
If, in addition, φ is a homeomorphism 同胚 onto $\varphi(M) \subset N$, where $\varphi(M)$ has the subspace topology induced from N, we say that φ is an *embedding*.
If $M \subset N$ and the inclusion $i : M \subset N$ is an embedding, we say that M is a *submanifold* of N.

It can be seen that if $\varphi : M^m \to N^n$ is an immersion, then $m \leqslant n$; the difference $n - m$ is called the *codimension* 余维数 of the immersion φ.

Examples

The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t^3, t^2)$ is a differentiable mapping but not an immersion because of the point $t = 0$.
$\mathrm{d}\alpha = (3t^2, 2t), \mathrm{d}\alpha(v) = (3t^2, 2t)v$. When $t = 0$, $\mathrm{d}\alpha = 0$, not an injective.

Immersion to Embedding

Let $\varphi : M_1^n \to M_2^m, n \leqslant m$, be an immersion of the differentiable manifold M_1 into the differentiable manifold M_2. For every point $p \in M_1$, there exists a neighborhood $V \subset M_1$ of p such that the restriction $\varphi|V \to M_2$ is an embedding.

Proof. This fact is a consequence of the inverse function theorem. Let $\mathbf{x}_1 : U_1 \subset \mathbb{R}^n \to M_1$ and $\mathbf{x}_2 : U_2 \subset \mathbb{R}^m \to M_2$ be a system of coordinates at p and at $\varphi(p)$, respectively, and let us denote by $(x_1, \dots, x_n)$ the coordinates of $\mathbb{R}^n$ and by $(y_1, \dots, y_m)$ the coordinates of $\mathbb{R}^m$. In these coordinates, the expression for φ, that is, the mapping $\tilde\varphi = \mathbf{x}_2^{-1} \circ \varphi \circ \mathbf{x}_1$, can be written

$$\tilde\varphi = \big(y_1(x_1, \dots, x_n), \dots, y_m(x_1, \dots, x_n)\big).$$

Let $q = \mathbf{x}_1^{-1}(p)$. Since φ is an immersion, we can suppose, renumbering the coordinates for both $\mathbb{R}^n$ and $\mathbb{R}^m$, if necessary, that

$$J\big(\tilde\varphi(q)\big) = \frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)}(q) = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} \neq 0.$$

To apply the inverse function theorem, we introduce the mapping $\phi = U_1 \times \mathbb{R}^{m-n} \to \mathbb{R}^m$ given by

$$\begin{aligned}&\phi(x_1, \dots, x_n, t_1, \dots, t_{m-n}) \\ &= \big(y_1(x_1, \dots, x_n), \dots, y_n(x_1, \dots, x_n), y_{n+1}(x_1, \dots, x_n) + t_1, \dots, y_m(x_1, \dots, x_n) + t_{m-n}\big)\end{aligned}$$

where $(t_1, \dots, t_{m-n}) \in \mathbb{R}^{m-n}$. It is easy to verify that ϕ restricted to U_1 coincides with $\tilde\varphi$ and that

$$\begin{aligned}\det\big(\mathrm{d}\phi(q)\big) &= \left|\begin{array}{ccc:ccc} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} & 0 & \cdots & 0 \\ \hdashline 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{array}\right| \\ &= \left|\begin{array}{c:c} \frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)}(q) & \mathbf{O}_{n\times(m-n)} \\ \hdashline \mathbf{O}_{(m-n)\times n} & \mathbf{E}_{m-n} \end{array}\right| = \frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)}(q) = J\big(\tilde\varphi(q)\big) \neq 0.\end{aligned}$$

It follows from the inverse function theorem, that there exist neighborhoods $W_1 \subset U_1 \times \mathbb{R}^{m-n}$ of q and $W_2 \subset \mathbb{R}^m$ of $\phi(q)$ such that the restriction $\phi|W_1$ is a diffeomorphism onto W_2. Let $\tilde V = W_1 \cap U_1$. Since $\phi|\tilde V = \tilde\varphi|\tilde V$ and $\mathbf{x}_i$ is a diffeomorphism, for $i = 1, 2$, we conclude that the restriction to $V = \mathbf{x}_1(\tilde V)$ of the mapping $\phi = \mathbf{x}_2 \circ \tilde\varphi \circ \mathbf{x}_1^{-1} : V \to \varphi(V) \subset M_2$ is a diffeomorphism, hence an embedding. □

1.3 Other examples of manifolds. Orientation

the Tangent Bundle 切丛

Let M^n be a differentiable manifold and let $TM = \big\{(p, v); p \in M, v \in T_pM\big\}$. We are going to provide the set TM with a differentiable structure (of dimension $2n$); with such a structure TM will be called the *tangent bundle* of M. This is the natural space to work with when treating questions that involve positions and velocities, as in the case of mechenics.

Proof. Let $\big\{(U_\alpha, \mathbf{x}_\alpha)\big\}$ be a maximal differentiable structure on M. Denote by $(x_1^\alpha, \dots, x_n^\alpha)$ the coordinates of U_α and by $\Big\{\frac{\partial}{\partial x_1^\alpha}, \dots, \frac{\partial}{\partial x_n^\alpha}\Big\}$ the associated bases to the tangent spaces of $\mathbf{x}_\alpha(U_\alpha)$. For every α, define

$$\mathbf{y}_\alpha : U_\alpha \times \mathbb{R}^n \to TM,$$

by

$$\mathbf{y}_\alpha(x_1^\alpha, \dots, x_n^\alpha, u_1, \dots, u_n) = \Big(\mathbf{x}_\alpha(x_1^\alpha, \dots, x_n^\alpha), \sum_{i=1}^n u_i \frac{\partial}{\partial x_i^\alpha}\Big), \quad (u_1, \dots, u_n) \in \mathbb{R}^n.$$

We are going to show that $\big\{(U_\alpha \times \mathbb{R}^n, \mathbf{y}_\alpha)\big\}$ is a differentiable structure on TM. Since $\bigcup_\alpha \mathbf{x}_\alpha(U_\alpha) = M$ and $(\mathrm{d}\mathbf{x}_\alpha)_q(\mathbb{R}^n) = T_{\mathbf{x}_\alpha(q)}M$, $q \in U_\alpha$, we have that

$$\bigcup_\alpha \mathbf{y}_\alpha(U_\alpha \times \mathbb{R}^n) = TM,$$

which verifies condition (1). Now let

$$(p, v) \in \mathbf{y}_\alpha(U_\alpha \times \mathbb{R}^n) \cap \mathbf{y}_\beta(U_\beta \times \mathbb{R}^n).$$

Then

$$(p, v) = \big(\mathbf{x}_\alpha(q_\alpha), \mathrm{d}\mathbf{x}_\alpha(v_\alpha)\big) = \big(\mathbf{x}_\beta(q_\beta), \mathrm{d}\mathbf{x}_\beta(v_\beta)\big),$$

where $q_\alpha \in U_\alpha$, $q_\beta \in U_\beta$, $v_\alpha, v_\beta \in \mathbb{R}^n$. Therefore,

$$\mathbf{y}_\beta^{-1} \circ \mathbf{y}_\alpha(q_\alpha, v_\alpha) = \mathbf{y}_\beta^{-1}\big(\mathbf{x}_\alpha(q_\alpha), \mathrm{d}\mathbf{x}_\alpha(v_\alpha)\big) = \big((\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha)(q_\alpha), \mathrm{d}(\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha)(v_\alpha)\big).$$

Since $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$ is differentiable, $\mathrm{d}(\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha)$ is as well. It follows that $\mathbf{y}_\beta \circ \mathbf{y}_\alpha$ is differentiable, which follows condition (2) and completes the example. □

Regular Surfaces in $\mathbb{R}^n$

The natural generalization of the notion of a regular surface in $\mathbb{R}^3$ is the idea of a surface of dimension k in $\mathbb{R}^n$, $k \leqslant n$. A subset $M^k \subset \mathbb{R}^n$ is a *regular surface of dimension* k if for every $p \in M$ there exists a neighborhood V of p in $\mathbb{R}^n$ and a mapping $\mathbf{x} : U \subset \mathbb{R}^k \to M \cap V$ of an open set $U \subset \mathbb{R}^k$ onto $M \cap V$ such that:

(1) $\mathbf{x}$ is a differentiable homeomorphism.

(2) $\mathrm{d}\mathbf{x}_q : \mathbb{R}^k \to \mathbb{R}^n$ is injective for all $q \in U$.

Inverse Image of a Regular Value

Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable mapping of an open set U of $\mathbb{R}^n$. A point $p \in U$ is defined to be a *critical point* of F if the differential $\mathrm{d}F_p : \mathbb{R}^n \to \mathbb{R}^m$ is not surjective 满射. The image $F(p)$ of a critical point is called a *critical value* of F. A point $a \in \mathbb{R}^m$ that is not a critical value is said to be a *regular value* of F. Note that any point $a \notin F(U)$ is trivially a regular value of F and that if there exists a regular value of F in $\mathbb{R}^m$, then $n \geqslant m$.
Now let $a \in F(U)$ be a regular value of F. We are going to show that the *inverse image* $F^{-1}(a) \subset \mathbb{R}^n$ is a regular surface of dimension $n - m = k$. $F^{-1}(a)$ is then a submanifold of $\mathbb{R}^n$.

Orientation

Let M be a differentiable manifold. We say that M is *orientable* if M admits a differentiable structure $\big\{(U_\alpha, \mathbf{x}_\alpha)\big\}$ such that:

- for every pair α, β, with $\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) = W \neq \varnothing$, the differential of the change of coordinates $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$ has positive determinant.

In the opposite case, we say that M is *non-orientable*. If M is orientable, a choice of a differentiable structure satisfying $(\cdot)$ is called an *orientation* of M. M is said to be *oriented*. Two differentiable structures that satisfy $(\cdot)$ *determine the same orientation* if their union again satisfies $(\cdot)$.
If M is orientable and connected, there exist exactly two distinct orientations on M.
Now let M_1 and M_2 be differentiable manifolds and let $\varphi : M_1 \to M_2$ be a diffeomorphism. It is easy to verify that M_1 is orientable if and only if M_2 is orientable. If, additionally, M_1 and M_2 are connected and oriented, φ induces an orientation on M_2 which may or may not coincide with the initial orientation of M_2. In the first case, we say that φ *perverses the orientation* and in the second case, that φ *reverses the orientation*.

Theorem

If M can be covered by two coordinate neighborhoods V_1 and V_2 in such a way that the intersection $V_1 \cap V_2$ is connected, then M is orientable. Indeed, since the determinant of the differential of the coordinate change $\neq 0$, it does not change sign in $V_1 \cap V_2$; if it is negative at a single point, it suffices to change the sign of one of the coordinates to make it positive at that point, hence on $V_1 \cap V_2$.

Examples

待补充

1.4 Vector Fields; brackets. Topology of Manifolds

Vector Fields

A *vector field* X on a differentiable manifold M is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_p(M)$. In terms of mappings, X is a mapping of M into the tangent bundle TM. The field is *differentiable* if the mapping $X : M \to TM$ is differentiable.

Considering a parametrization $\mathbf{x} : U \subset \mathbb{R}^n \to M$ we can write

$$X(p) = \sum_{i=1}^n a_i(p) \frac{\partial}{\partial x_i}, \tag{1}$$

where each $a_i : U \to \mathbb{R}$ is a function on U and $\Big\{\frac{\partial}{\partial x_i}\Big\}$ is the basis associated to $\mathbf{x}$, $i = 1, \dots, n$. It is clear that X is differentiable if and only if the functions a_i are differentiable for some (and, therefore, for any) parametrization.
Occasionally, it is convenient to use the idea suggested above and think of a vector field as a mapping $X : \mathcal{D} \to \mathcal{F}$ from the set $\mathcal{D}$ of differentiable functions on M to the set $\mathcal{F}$ of functions on M, defined in the following way

$$(Xf)(p) = \sum_i a_i(p) \frac{\partial f}{\partial x_i}(p), \tag{2}$$

where f denotes, by abuse of notation, the expression of f in the parametrization $\mathbf{x}$. Indeed, this idea of a vector as a directional derivative was precisely what was used to define the notion of tangent vector. It is easy to check that the function Xf does not depend on the choice of parametrization $\mathbf{x}$. In this context, it is immediate that X is differentiable if and only if $X : \mathcal{D} \to \mathcal{D}$, that is, $Xf \in \mathcal{D}$ for all $f \in \mathcal{D}$.
Observe that if $\varphi : M \to M$ is a diffeomorphism, $v \in T_pM$ and f is a differentiable function in a neighborhood of $\varphi(p)$, we have

$$\big(\mathrm{d}\varphi(v)f\big)\varphi(p) = v(f \circ \varphi)(p)$$

Indeed, let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a differentiable curve with $\alpha'(0) = v, \alpha(0) = p$. Then

$$\big(\mathrm{d}\varphi(v)f\big)\varphi(p) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi \circ \alpha)\Big|_{t=0} = v(f \circ \varphi)(p).$$

2 Affine Connections 仿射联络; Riemannian Connections