Notes for Functional Anaysis

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1 Normed and Banach spaces

1.1 Vector spaces

1.1.1 Definition for vector spaces

(1) For all $x_1, x_2, x_3 \in X$, $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$.

(2) There exists an element, denoted by $\mathbf{0}$ (called the *zero vector*) such that for all $x \in X$, $x + \mathbf{0} = x = \mathbf{0} + x$.

(3) For every $x \in X$, there exists an element, denoted by $-x$, such that $x + (-x) = (-x) + x = \mathbf{0}$.

(4) For all x_1, x_2 in X, $x_1 + x_2 = x_2 + x_1$.

(5) For all $x \in X$, $1 \cdot x = x$.

(6) For all $x \in X$ and all $\alpha, \beta \in \mathbb{K}$, $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$.

(7) For all $x \in X$ and all $\alpha, \beta \in \mathbb{K}$, $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$.

(8) For all $x_1, x_2 \in X$ and all $\alpha \in \mathbb{K}$, $\alpha \cdot (x_1 + x_2) = \alpha \cdot x_1 + \alpha \cdot x_2$.

1.1.2 Some examples

- $\mathbb{R}^d = \mathbb{R} \times \ldots \times \mathbb{R}$ (d times): 常见的矢量，有限维.
- $C[a,b]$: all continuous functions $\mathbf{x} : [a,b] \to \mathbb{K}$
  连续函数集，可看成一个无限维向量，每一个 $\mathrm{d}x$ 分别对应一个维度.
- $C^1[a,b] = \{\mathbf{x} : [a,b] \to \mathbb{R} | \, \mathbf{x}$ is continuously differentiable on $[a,b]\}$
  $$C^1[a,b] \subset C[a,b].$$
- $\ell^p := \left\{ (a_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$, $p \in [1, \infty)$ 收敛的数列，无限维.
- ℓ^∞: ℓ^p for any $p \in [1, \infty)$, $\ell^p \subset \ell^\infty$.
  c_{00}: all sequences that are eventually 0
  c_0: all sequences that converge to 0
  c: all suquences that are convergent
  $c_{00} \subset c_0 \subset c \subset \ell^\infty$.
- $L^p[a,b]$: " = "$\left\{\mathbf{x} : [a,b] \to \mathbb{R} \;\middle|\; \underbrace{\int_a^b}_{\text{Lebesgue}} |\mathbf{x}(t)|^p \, \mathrm{d}t < \infty \right\}$, $p \in [1, \infty)$
  积分收敛的函数，但不一定连续.
  Each element in $L^p[a,b]$ is not a functions $\mathbf{x}$, but rather an equivalence class $[\mathbf{x}]$ of functions, where
  $$[\mathbf{x}] = \left\{\mathbf{y} : [a,b] \to \mathbb{R} \;\middle|\; \underbrace{\int_a^b}_{\text{Lebesgue}} |\mathbf{x}(t) - \mathbf{y}(t)|^p \, \mathrm{d}t = 0 \right\}(\text{不等的部分零测}).$$
  If $\mathbf{x}, \mathbf{y} \in C[a,b]$, then $\mathbf{x} = \mathbf{y}$.

1.1.3 Convex sets

X is a vector space, $C \subset X$, for all $x, y \in C$, and all $\alpha \in [0,1]$, $(1-\alpha)x + \alpha y \in C$

1.2 Normed spaces

1.2.1 Definition for normed spaces

A *norm* on X is a function $\|\cdot\| : X \to [0, +\infty)$ such that:

(1) (Positive definiteness)
For all $x \in X$, $\|x\| \geqslant 0$. If $x \in X$ and $\|x\| = 0$, then $x = \mathbf{0}$.

(2) For all $\alpha \in \mathbb{K}(\mathbb{R}$ or $\mathbb{C})$ and for all $x \in X$, $\|\alpha x\| = \alpha\|x\|$.

(3) (Triangle inequity) For all $x, y \in X$, $\|x + y\| \leqslant \|x\| + \|y\|$.

A *normed space* is a vector space X equipped with a norm.

1.2.2 Some examples

- $\mathbb{R}^d, \|\cdot\|_p$:
  $$\|\mathbf{x}\|_p := (|x_1|^p + \ldots + |x_d|^p)^{\frac{1}{p}}, \mathbf{x} \in \mathbb{R}^d$$
  texicab norm: $\|\cdot\|_1$-norm
  Euclidean norm: $\|\cdot\|_2$-norm
  $\max\{|x_1|, \ldots, |x_d|\}$: $\|\cdot\|_\infty$-norm
- $C[a,b], \|\cdot\|_p$:
  $$\|\mathbf{x}\|_1 := \int_a^b |\mathbf{x}(t)| \, \mathrm{d}t,$$
  $$\|\mathbf{x}\|_2 := \sqrt{\int_a^b |\mathbf{x}(t)|^2 \, \mathrm{d}t},$$
  $$\|\mathbf{x}\|_\infty = \sup_{t\in[a,b]} |\mathbf{x}(t)| = \max_{t\in[a,b]} |\mathbf{x}(t)|,$$
  $$\langle \mathbf{x}, \mathbf{y} \rangle := \sqrt{\int_a^b \mathbf{x}(t)\big(\mathbf{y}(t)^*\big) \, \mathrm{d}t} \implies \|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$
- $C^1[a,b], \|\mathbf{x}\|_{1,\infty}$
  $$\|\mathbf{x}\|_{1,\infty} := \|\mathbf{x}\|_\infty + \|\mathbf{x}'\|_\infty$$
- $\ell^p, \|\cdot\|_p$
  $$\left\|(a_n)_{n\in\mathbb{N}}\right\|_p := \Big(\sum_{n=1}^{\infty} (a_n)_{n\in\mathbb{N}}\Big)^{\frac{1}{p}},\ (a_n)_{n\in\mathbb{N}} \in \ell^p.$$
  When $p = \infty$, $\left\|(a_n)_{n\in\mathbb{N}}\right\|_\infty = \sup_{n\in\mathbb{N}} |a_n|$, $(a_n)_{n\in\mathbb{N}} \in \ell^\infty$
- $L^p, \|\cdot\|_p$
  $$\|\mathbf{x}\|_p := \Big(\int_a^b |\mathbf{x}(t)| \, \mathrm{d}t\Big)^{\frac{1}{p}}$$

$$\begin{aligned}\|\mathbf{x}\|_\infty &:= \operatorname{essup} |\mathbf{x}(t)| \,(\text{essential supremum, 本质上确界}) \\ &:= \inf\big\{M : |\mathbf{x}(t)| \leqslant M \text{ for almost all } t \in [a,b]\big\}\end{aligned}$$

1.3 Topology of normed spaces

1.3.1 Open ball

Let $(X, \|\cdot\|)$ be a normed space, $x \in X$, and $r > 0$.
The *open ball* $B(x,r)$ with *center* x and *radius* r is difined by
$$B(x,r) = \{y \in X : \|x - y\| < r\}.$$

1.3.2 Open set

Let $(X, \|\cdot\|)$ be a normed space. A set $U \subset X$ is said to be *open* if for every $x \in U$, there exists an $r > 0$ such that $B(x,r) \subset U$.

- X and $\varnothing$ are also open.
- Any union or finite intersection of open sets is open.

1.3.3 Closed set

Let $(X, \|\cdot\|)$ be a normed space. A set F is *closed* if its complement $X \backslash F$ is open.

- X and $\varnothing$ are also closed.

1.3.4 Dense set

A subset D of a normed space $(X, \|\cdot\|)$ is said to be *dense* in X if for all $x \in X$ and all $\epsilon > 0$, there exists a $y \in D$ such that $\|x - y\| < \epsilon$.
That is, if we take any $x \in X$ and consider any ball $B(x, \epsilon)$ centered at X, it contains a point from D.

- $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ are dense in $\mathbb{R}$.
- c_{00} is dense in ℓ^2.

1.3.5 Separable spaces

A normed space X is called separable if it has a countable dense set, that is, there exists a set $D := \{x_1, x_2, x_3, \ldots\}$ in X such that for every $r > 0$ and every $x \in X$, there exists an $x_n \in D$ such that $\|x_n - x\| < r$.

- $\mathbb{R}$ is separable, since we can simply take $D = \mathbb{Q}$.
- ℓ^p is separable for all $1 \leqslant p < \infty$.
- ℓ^∞ is not separable.

1.3.6 Topology

If we consider the collection $\mathcal{O}$ of all open sets in a normed space $(X, \|\cdot\|)$, we notice that it has the three properties:

(1) $\varnothing, X \in \mathcal{O}$

(2) If $U_i \in \mathcal{O}$ for all $i \in I$, then $\bigcup_{i\in I} U_i \in \mathcal{O}$

(3) If $U_1, \ldots, U_n$ is a finite collection of sets from $\mathcal{O}$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{O}$.

Any collection $\mathcal{O}$ of subsets of X that satisfy properties above is called a *topology* on X and $X, \mathcal{O}$ is called a *topology space.*

1.4 Sequences in a normed space; Banach spaces

1.4.1 Convergent sequence

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and let $L \in X$. The sequence $(x_n)_{n\in\mathbb{N}}$ is said to be *convergent* (in X) with *limit* L if
$$\forall\epsilon > 0, \exists N \in \mathbb{N}, \text{such that } \forall n > N, \|x_n - L\| < \epsilon$$
$$\lim_{n\to\infty} \|x_n - L\| = 0.$$

- A convergent sequence has a unique limit.
- Consider the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ converging to $\mathbf{0}$ in the normed space $(C[0,1], \|\cdot\|_\infty)$, where $\mathbf{x}_n = \dfrac{\sin(2\pi nt)}{n}$.

1.4.2 Cauchy sequence

A sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in a normed space $(X, \|\cdot\|)$ is called a *Cauchy sequence* if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$, such that for all $m, n \in \mathbb{N}$ satisfying $m, n > \mathbb{N}$, $\|x_m - x_n\| < \epsilon$.

- Every convergent sequence is *Cauchy.*

1.4.3 Banach space

Banach space or *complete normed space*: a normed space with which the set Cauchy sequence = convergent sequence.

- **Banach space**: $(\mathbb{R}, |\cdot|)(\mathbb{C}, |\cdot|)(\ell^p, \|\cdot\|_p)(C[a,b], \|\cdot\|_\infty)\big(L^2[a,b], \|\cdot\|_2\big)$...
- *none*-**Banach space**: $(\mathbb{Q}, |\cdot|)(C[a,b], \|\cdot\|_2)$...

1.4.4 Corollaries

- Every Cauchy sequence in a normed space is bounded.
- Every real sequence has a *momotone* subsequence.
- momotone + bounded $\implies$ convergent(单调有界必收敛).
- (Bolzano-Weierstrass Theorem) Every bounded real sequence has a convergent subsequence.
- Every real Cauchy sequence in $\mathbb{R}$ is convergent.
- Finite-dimensional normed spaces are Banach.