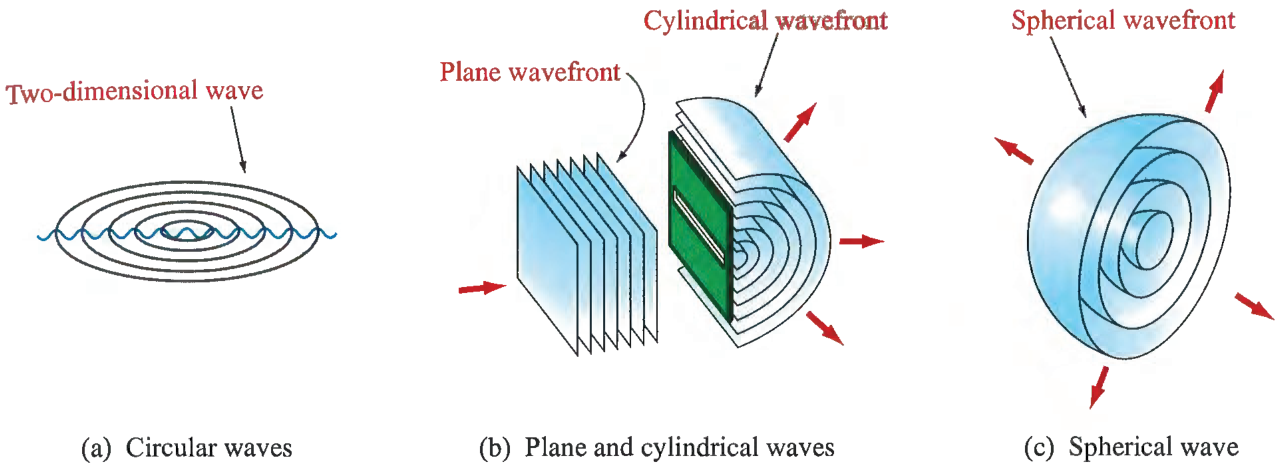

Figure 1-11: Examples of two-dimensional and three-dimensional waves: (a) circular waves on a pond, (b) a plane light wave exciting a cylindrical light wave through the use of a long narrow slit in an opaque screen, and (c) a sliced section of a spherical wave.

across a surface, like the ripples on a pond [Fig. 1-11(a)], and its disturbance can be described by two space variables. And by extension, a ***three-dimensional wave*** propagates through a volume and its disturbance may be a function of all three space variables. Three-dimensional waves may take on many different shapes; they include ***plane waves***, ***cylindrical waves***, and ***spherical waves***. A plane wave is characterized by a disturbance that at a given point in time has uniform properties across an infinite plane perpendicular to its direction of propagation [Fig. 1-11(b)]. Similarly, for cylindrical and spherical waves, the disturbances are uniform across cylindrical and spherical surfaces [Figs. 1-11(b) and (c)].

In the material that follows, we will examine some of the basic properties of waves by developing mathematical formulations that describe their functional dependence on time and space variables. To keep the presentation simple, we will limit our discussion to sinusoidally varying waves whose disturbances are functions of only one space variable, and we defer the discussion of more complicated waves to later chapters.

1-4.1 Sinusoidal Waves in a Lossless Medium

Regardless of the mechanism responsible for generating them, all linear waves can be described mathematically in common terms.

A medium is said to be ***lossless*** *if it does not attenuate the amplitude of the wave traveling within it or on its surface.*

By way of an example, let us consider a wave traveling on a lake surface, and let us assume for the time being that frictional forces can be ignored, thereby allowing a wave generated on the water surface to travel indefinitely with no loss in energy. If y denotes the height of the water surface relative to the mean height (undisturbed condition) and x denotes the distance of wave travel, the functional dependence of y on time t and the spatial coordinate x has the general form

$$\boxed{y(x,t) = A\cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \phi_0\right) \quad \text{(m)}, \qquad (1.17)}$$

where A is the ***amplitude*** of the wave, T is its ***time period***, λ is its ***spatial wavelength***, and ϕ_0 is a ***reference phase***. The quantity $y(x,t)$ can also be expressed in the form

$$y(x,t) = A\cos\phi(x,t) \quad \text{(m)}, \qquad (1.18)$$

where

$$\phi(x,t) = \left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \phi_0\right) \quad \text{(rad)}. \qquad (1.19)$$

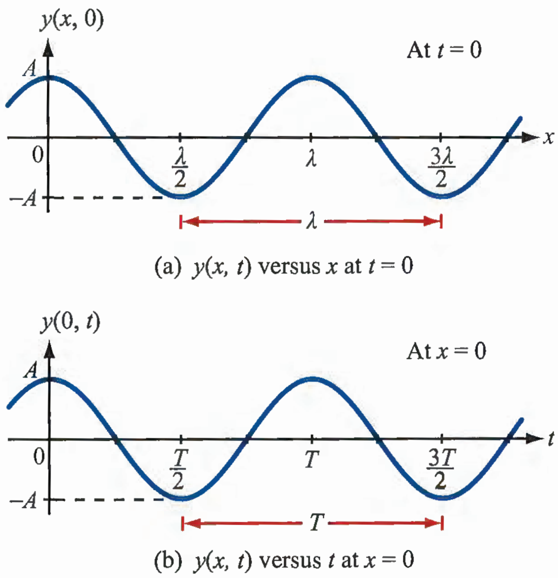

Figure 1-12: Plots of $y(x,t) = A\cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right)$ as a function of (a) x at $t = 0$ and (b) t at $x = 0$.

The angle $\phi(x,t)$ is called the ***phase*** of the wave, and it should not be confused with the reference phase ϕ_0, which is constant with respect to both time and space. Phase is measured by the same units as angles, that is, radians (rad) or degrees, with 2π radians $= 360°$.

Let us first analyze the simple case when $\phi_0 = 0$:

$$y(x,t) = A\cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right) \quad \text{(m)}. \tag{1.20}$$

The plots in Fig. 1-12 show the variation of $y(x,t)$ with x at $t = 0$ and with t at $x = 0$. The wave pattern repeats itself at a spatial period λ along x and at a temporal period T along t.

If we take time snapshots of the water surface, the height profile $y(x,t)$ would exhibit the sinusoidal patterns shown in Fig. 1-13. In all three profiles, which correspond to three different values of t, the spacing between peaks is equal to the wavelength λ, even though the patterns are shifted relative to one another because they correspond to different observation times. Because the pattern advances along the $+x$-direction at progressively increasing values of t, $y(x,t)$ is called a wave traveling in the $+x$-direction. If we track a given point on the wave, such as the peak P, and follow it in time, we can measure the ***phase velocity*** of the wave. At the peaks of the wave pattern, the phase $\phi(x,t)$ is equal to zero or multiples of 2π radians. Thus,

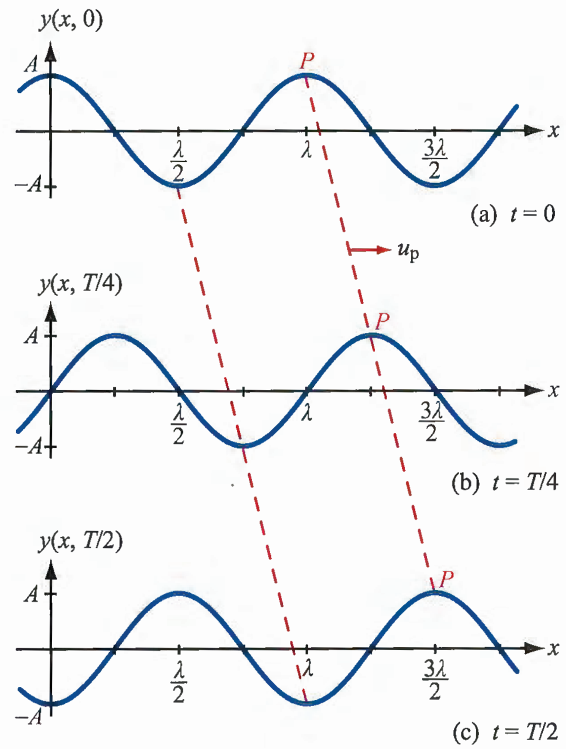

Figure 1-13: Plots of $y(x,t) = A\cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right)$ as a function of x at (a) $t = 0$, (b) $t = T/4$, and (c) $t = T/2$. Note that the wave moves in the $+x$-direction with a velocity $u_p = \lambda/T$.

$$\phi(x,t) = \frac{2\pi t}{T} - \frac{2\pi x}{\lambda} = 2n\pi, \quad n = 0, 1, 2, \ldots \tag{1.21}$$

Had we chosen any other fixed height of the wave, say y_0, and monitored its movement as a function of t and x, this again would have been equivalent to setting the phase $\phi(x,t)$ constant such that

$$y(x,t) = y_0 = A\cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right), \tag{1.22}$$

or

$$\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} = \cos^{-1}\left(\frac{y_0}{A}\right) = \text{constant.} \quad (1.23)$$

The apparent velocity of that fixed height is obtained by taking the time derivative of Eq. (1.23),

$$\frac{2\pi}{T} - \frac{2\pi}{\lambda}\frac{dx}{dt} = 0, \quad (1.24)$$

which gives the ***phase velocity*** u_p as

$$\boxed{u_\text{p} = \frac{dx}{dt} = \frac{\lambda}{T} \quad \text{(m/s)}.} \quad (1.25)$$

The phase velocity, also called the ***propagation velocity***, is ***the velocity of the wave pattern*** as it moves across the water surface. The water itself mostly moves up and down; when the wave moves from one point to another, the water does not move physically along with it.

The ***frequency*** of a sinusoidal wave, f, is the reciprocal of its time period T:

$$\boxed{f = \frac{1}{T} \quad \text{(Hz)}.} \quad (1.26)$$

Combining the preceding two equations yields

$$\boxed{u_\text{p} = f\lambda \quad \text{(m/s)}.} \quad (1.27)$$

The wave frequency f, which is measured in cycles per second, has been assigned the unit (Hz), named in honor of the German physicist Heinrich Hertz (1857–1894), who pioneered the development of radio waves.

Using Eq. (1.26), Eq. (1.20) can be rewritten in a more compact form as

$$y(x,t) = A\cos\left(2\pi f t - \frac{2\pi}{\lambda}x\right) = A\cos(\omega t - \beta x), \quad (1.28)$$

where ω is the ***angular velocity*** of the wave and β is its ***phase constant*** (or ***wavenumber***), defined as

$$\boxed{\begin{aligned} \omega &= 2\pi f \quad \text{(rad/s)}, && (1.29a)\\ \beta &= \frac{2\pi}{\lambda} \quad \text{(rad/m)}. && (1.29b)\end{aligned}}$$

In terms of these two quantities,

$$u_\text{p} = f\lambda = \frac{\omega}{\beta}. \quad (1.30)$$

So far, we have examined the behavior of a wave traveling in the $+x$-direction. To describe a wave traveling in the $-x$-direction, we reverse the sign of x in Eq. (1.28):

$$y(x,t) = A\cos(\omega t + \beta x). \quad (1.31)$$

The direction of wave propagation is easily determined by inspecting the signs of the t and x terms in the expression for the phase $\phi(x,t)$ given by Eq. (1.19): if one of the signs is positive and the other is negative, then the wave is traveling in the positive x-direction, and if both signs are positive or both are negative, then the wave is traveling in the negative x-direction. The constant phase reference ϕ_0 has no influence on either the speed or the direction of wave propagation.

We now examine the role of the phase reference ϕ_0 given previously in Eq. (1.17). If ϕ_0 is not zero, then Eq. (1.28) should be written as

$$y(x,t) = A\cos(\omega t - \beta x + \phi_0). \quad (1.32)$$

A plot of $y(x,t)$ as a function of x at a specified t or as a function of t at a specified x will be shifted in space or time, respectively, relative to a plot with $\phi_0 = 0$ by an amount proportional to ϕ_0. This is illustrated by the plots shown in Fig. 1-14. We observe that when ϕ_0 is positive, $y(t)$ reaches its peak value, or any other specified value, sooner than when $\phi_0 = 0$. Thus, the wave with $\phi_0 = \pi/4$ is said to ***lead*** the wave with $\phi_0 = 0$ by a ***phase lead*** of $\pi/4$; and similarly, the wave with $\phi_0 = -\pi/4$ is said to ***lag*** the wave with $\phi_0 = 0$ by a ***phase lag*** of $\pi/4$. A wave function with a negative ϕ_0 takes longer to reach a given value of $y(t)$, such as its peak, than the zero-phase reference function.

When its value is positive, ϕ_0 signifies a phase lead in time, and when it is negative, it signifies a phase lag.

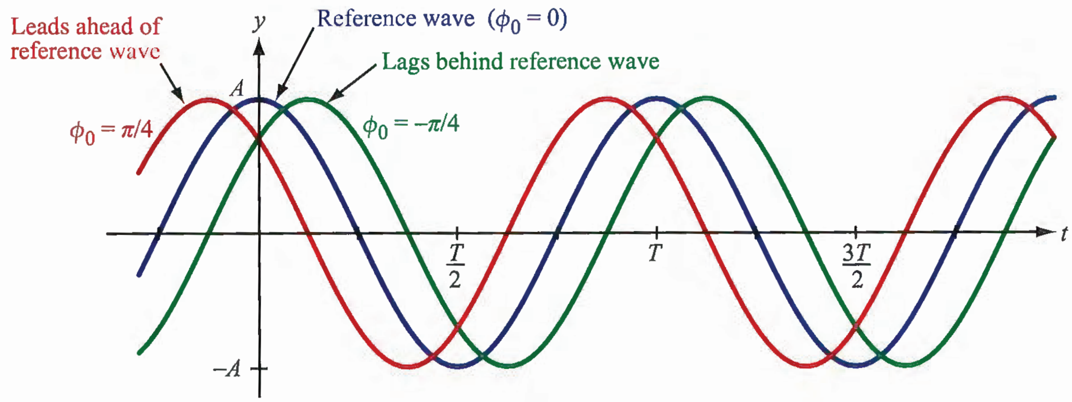

Figure 1-14: Plots of $y(0, t) = A\cos[(2\pi t/T) + \phi_0]$ for three different values of the reference phase ϕ_0.

Exercise 1-1: Consider the red wave shown in Fig. E1.1. What is the wave's (a) amplitude, (b) wavelength, and (c) frequency, given that its phase velocity is 6 m/s?

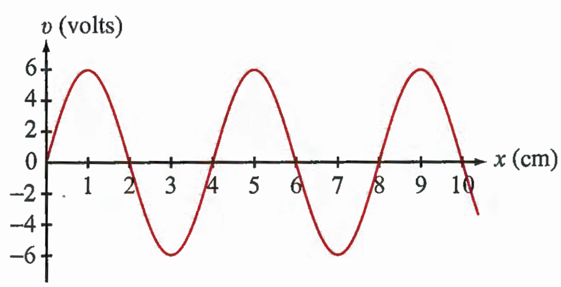

Figure E1.1

Answer: (a) $A = 6$ V, (b) $\lambda = 4$ cm, (c) $f = 150$ Hz.

Exercise 1-2: The wave shown in red in Fig. E1.2 is given by $\upsilon = 5\cos 2\pi t/8$. Of the following four equations:

(1) $\upsilon = 5\cos(2\pi t/8 - \pi/4)$,

(2) $\upsilon = 5\cos(2\pi t/8 + \pi/4)$,

(3) $\upsilon = -5\cos(2\pi t/8 - \pi/4)$,

(4) $\upsilon = 5\sin 2\pi t/8$,

(a) which equation applies to the green wave? (b) which equation applies to the blue wave?

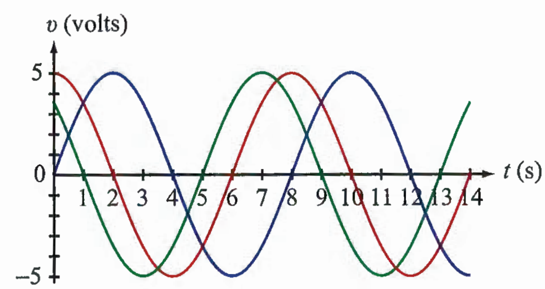

Figure E1.2

Answer: (a) #2, (b) #4.

Exercise 1-3: The electric field of a traveling electromagnetic wave is given by

$$E(z, t) = 10\cos(\pi \times 10^7 t + \pi z/15 + \pi/6) \qquad \text{(V/m)}.$$

Determine (a) the direction of wave propagation, (b) the wave frequency f, (c) its wavelength λ, and (d) its phase velocity u_p.

Answer: (a) $-z$-direction, (b) $f = 5$ MHz, (c) $\lambda = 30$ m, (d) $u_p = 1.5 \times 10^8$ m/s.