COMP0009 Exercises I. Logic Revision.

Robin Hirsch

October 14, 2021

EXERCISE 1 *For each of the following propositional formulas, find an equivalent formula written in* disjunctive normal form.

1. $((p \vee q) \wedge (\neg p \rightarrow \neg q))$
2. $\neg((p \rightarrow q) \rightarrow (q \rightarrow p))$
3. $(((p \vee q) \wedge (\neg q \vee \neg r)) \wedge (\neg p \vee r))$

EXERCISE 2 *Let L be a first order language to describe vertex colourings in graphs, with no constant symbols, no function symbols, two unary predicates R, B for red nodes and blue nodes respectively, one binary predicate E for the edge relation and one binary predicate symbol $=$ for equality between nodes of a graph. For each of the following statements about coloured graphs, write down an L-formula that expresses it.*

1. *there is an isolated node (not incident with any edge)*
2. *every node is coloured red or blue but not both*
3. *every blue node is adjacent to a red node*
4. *between any two nodes, there is a path from one to the other of length at most three.*
5. *the graph is reflexive, symmetric and transitive (look these up if you've forgotten).*

EXERCISE 3 *Let L be a first order language for arithmetic, with one constant symbol 1, one binary function symbols $+$ and two binary predicates $=, <$ (predicates written infix). Let $\mathbf{N}$ be the L-structure whose base is the set of natural numbers, and where all symbols are interpretted normally, i.e. 1 is interpretted as one, $+$ is interpretted as the binary function that adds its two arguments, $=$ is interpretted as equality and $<$ is interpretted as 'strictly less than', i.e. the set of all pairs (m, n) of natural numbers where m is less than n. Which of the following L-formulas is* valid *in $\mathbf{N}$.*

1. $\exists y(y = x)$
2. $\exists x \exists y(x + x = y)$
3. $\exists x(x + 1 = x)$
4. $\forall x \exists y(x = y + y \vee x = y + y + 1)$.
5. $\forall x \forall y \forall z((x < y \wedge y < z) \rightarrow x < z)$
6. $\forall x \forall y(x < y \rightarrow \exists z(x < z \wedge z < y))$.

Theory III, COMP0009, ????

Time allowed: 2.5 hours

Answer all three questions.

Marks for each part of each question are indicated in square brackets.

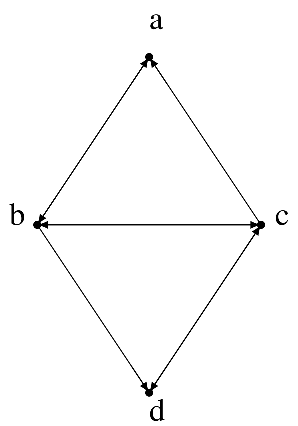
Calculators are NOT permitted.

1. a. Let $L = L(C, F, P)$ be a first-order language and let $\mathcal{M}$ be an L-structure. Let ϕ be a formula of this language. Prove that ϕ is valid in $\mathcal{M}$ if and only if $\neg\phi$ is not satisfiable in $\mathcal{M}$.

[Marks 8]

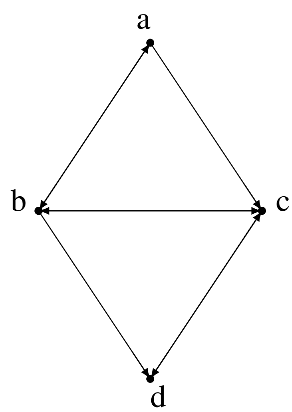
b. Let $C = F = \emptyset$ and $P = \{E^2, =\}$, where $=$ is the equality predicate, written infix. Each of the graphs below can be interpretted as an L-structure whose domain is the set of nodes of the graph and where $I(E^2) = \{(x, y) :$ there is an arrow from x to $y\}$. Which of these structures are models of the following formula? Give a brief reason for your answer.

$$\forall x \exists y (E^2(x, y) \land \forall z (E^2(y, z) \lor y = z))$$



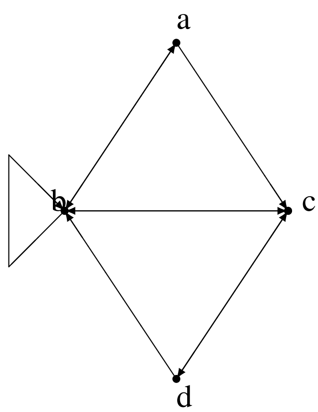
Graph 1
Edges:

$\{(a, b),\ (b, a),$
$(c, a),\ (b, d),$
$(c, b),\ (b, c),$
$(c, d),\ (d, c)\}$



Graph 2
Edges:

$\{(a, b),\ (b, a),$
$(a, c),\ (c, b),$
$(b, c),\ (b, d),$
$(c, d),\ (d, c)\}$



Graph 3
Edges:

$\{(a, b),\ (b, a),$
$(a, c),\ (b, b),$
$(b, c),\ (c, b),$
$(d, b),\ (c, d),$
$(d, c)\}$

[Marks 9]

[Question 1 cont. on next page]

c. Now let $C = F = \emptyset$ and $P = \{=, N^1, Col^1, E^2, \rho^2\}$. The intension is that the domain D of a model $\mathcal{M} = (D, I)$ is partitioned into two parts, graph nodes and colours. If $N^1(x)$ holds it will mean that x designates a graph node and if $Col^1(x)$ holds then x designates a colour. As before, E^2 will be interpretted as the set of graph edges (so $E^2(x, y)$ will only hold if x and y are graph nodes and there is an arrow from x to y). $\rho^2(x, i)$ means that i designates a colour and the node designated x has that colour.

1. Write a first-order formula that means that every point in the model is either a graph node or a colour, but not both.
2. Write a first-order formula that means that the set of edges is irreflexive (you never have an edge from a graph node to itself) and symmetric (if there is an edge from one graph node to another then there is also a reverse edge).
3. Write down a first-order formula that means that every graph node has a unique colour.
4. Write down a first-order formula that means that two nodes connected by an edge must have different colours.
5. Write a formula that means that there are three colours and every node has one of these three colours.
6. Let ϕ be the conjunction of your answers to (1)–(5) above. Is ϕ satisfiable? If so, define a model. If not, give a proof.

[Marks 16]

[Marks Total = 33]

COMP0009 Logic and Databases
Exercises III: Propositional Tableaus.

Robin Hirsch

November 1, 2021

1. Prove the following by constructing closed tableaus.
   (a) $\vdash p \to p$
   (b) $\vdash p \vee \neg p$
   (c) $\vdash (p \to q) \to (p \to (q \vee r))$
   (d) $\vdash ((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r)$
   (e) $\vdash ((p \vee q) \to r) \to ((p \to r) \wedge (q \to r))$
   (f) $\vdash (p \to q) \to ((q \to r) \to (p \to r))$
2. What does it mean when we say that two propositional formulas are *equivalent* to each other?
3. Explain what is meant by *disjunctive normal form* (DNF).
4. Explain how a tableau can be used to convert a propositional formula into an equivalent, DNF formula.
5. For each of the following formulas construct a completed tableau and use your tableau to find an equivalent DNF formula.
   (a) $(p \to \neg q)$
   (b) $((p \vee \neg q) \wedge (p \to r))$.
   (c) $\neg((p \to q) \to (q \to p))$.
   (d) $\neg(((p \wedge q) \to r) \leftrightarrow ((p \to r) \vee (q \to r)))$.
6. For each of the following formulas build a complete tableau. Use the tableau to find a valuation satisfying the formula. Also use the tableau to find a DNF formula equivalent to the given formula.
   (a) $\neg((p \to q) \to (q \to p))$.
   (b) $((p \vee q \vee r) \wedge (\neg p \vee \neg q))$
   (c) $(p \to (p \wedge \neg p))$.
7. Suppose that *every* branch of a completed tableau for ϕ is open. Does it follow that ϕ is valid? If so, prove it. If not, give a counter-example.
8. You can use a tableau for ϕ to find a DNF equivalent to ϕ. A *CNF formula* (conjunctive normal form) is a conjunction of clauses each of which is a disjunction of literals. Can you think of a way of using a tableau to find a CNF formula equivalent to a given formula ϕ?

COMP0009 Logic & Databases
Exercises 4: Predicate Tableaus.

Robin Hirsch

November 13, 2021

In order to show that $A_1, A_2, \ldots, A_n \vdash B$ make a new tableau with $(A_1 \wedge A_2 \wedge \ldots \wedge A_n \wedge \neg B)$ at the root, keep expanding it until it is closed.
Binding priority: $\neg$ (binds tightest) then quantifiers, then $\wedge, \vee, \rightarrow$, so $\neg\exists x Px \rightarrow Qxx \wedge \neg Qyx$ means $((\neg(\exists x Px)) \rightarrow (Qxx \wedge \neg Qyx))$.

Prove the following, by constructing closed tableaux:

1. $\vdash (\exists x \forall y Cxy \rightarrow \forall x \exists y Cyx)$.
2. $\forall x(\exists y Fxy \rightarrow Gbx)$, $\forall x \forall y Fyx \vdash \exists x Gxx$.
3. $\forall x \forall y(\exists z Fyz \rightarrow Fxy)$, $Fab \vdash \forall y \exists x Fyx$.
4. $\forall x \forall y(Pxy \rightarrow \neg Pyx) \vdash \forall x \neg Pxx$.
5. $\forall x(Gx \rightarrow Hx)$, $\forall x(Hx \rightarrow Fx)$, $Ga \vdash \exists x(Gx \wedge Fx)$.
6. $\neg Cb \wedge \neg Cc$, $Ca \rightarrow \forall x Cx \vdash \neg Ca$.
7. $\exists x Fxb \rightarrow \forall x Gx$, $\forall x Fax \vdash \forall x(Hxc \rightarrow Gx)$.
8. $\forall x \forall y(\exists z Ayz \rightarrow Axy)$, $\neg Att \vdash \neg Ats$.
9. $\forall x \exists y(Ayx \wedge Cxy)$, $Awh \vdash \exists x Chx$.
10. $\forall x \exists y(Cx \rightarrow Py \wedge Axy) \vdash \forall x(Cx \rightarrow \exists y(Py \wedge Axy))$.
11. Ga, $\exists x(Gx \wedge Mx)$, $\forall x(Mx \rightarrow Fx) \vdash \exists x(Gx \wedge Fx)$.
12. $\forall x(Sx \wedge Fx \rightarrow Bx)$, $Sj \wedge Lj \vdash \forall x(Sx \wedge Lx \rightarrow Fx) \rightarrow Bj$.
13. $\forall x(Bxa \rightarrow Bxb) \vdash \forall x(\exists y(Cxy \wedge Bya) \rightarrow \exists z(Bzb \wedge Cxz))$.
14. $\forall x(Cx \rightarrow Fx) \vdash \forall x(\exists y(Txy \wedge Cy) \rightarrow \exists z(Txz \wedge Fz))$.
15. $\forall x(Bxh \rightarrow Bxw) \vdash \forall x(\exists y(Bxh \wedge Myx) \rightarrow \exists z(Bxw \wedge Mzx))$.
16. $\forall x(Rx \wedge Bx \rightarrow Cx)$, $\neg\exists x(Gx \wedge Cx) \vdash \forall x(Rx \wedge Bx \rightarrow \neg Gx)$.
17. $\forall x(Sx \wedge Mx \wedge \forall y(Gy \wedge My \rightarrow Dy) \rightarrow Cx)$, $\forall x(Gx \wedge Dx \wedge Mx \rightarrow \forall y(Gy \wedge My \rightarrow Dy)) \vdash$ $\forall x \forall y((Sx \wedge Mx) \wedge (Gy \wedge Dy \wedge My) \rightarrow Cx)$.
18. $\forall x(Gx \rightarrow Px \vee Rx)$, $\forall x(Fx \rightarrow Tx) \vdash (\forall x(Px \vee Rx \rightarrow Fx) \rightarrow \forall x(Gx \rightarrow Tx))$.
19. $\forall x(Cxj \wedge Cxm \rightarrow Bxm)$, $\exists x(Cxm \wedge \neg Bxm) \vdash \exists x(Cxm \wedge \neg Cxj)$.

Logic and Databases, COMP0009,

From a previous exam question.

Marks for each part of each question are indicated in square brackets.

Calculators are NOT permitted.

1. This question is about first-order tableaus.

   a. For each of these formulas state if the formula is an α-formula, β-formula, δ-formula, γ-formula, or a literal.

      1. $\neg P^2(x,y)$
      2. $\exists x(P^2(x,y) \vee P^2(y,x))$
      3. $\neg\exists x\forall y P'^2(x,y)$
      4. $\neg(\forall x P^2(x,x) \vee \neg\exists y P^2(x,y))$

   [8 marks]

   b. Explain how to expand a δ-formula in a tableau.

   [7 marks]

   c. Describe a good method of scheduling the expansion of nodes in a tableau. In particular, say which nodes should be expanded first and how you should schedule the expansion of γ nodes.

   [8 marks]

   d. For each of these formulas construct a tableau with the formula at the root and state whether the formula is satisfiable or not.

      1. $(\forall x\forall y(P^2(x,y) \to \neg P^2(y,x)) \wedge \exists x P^2(x,x))$
      2. $(\exists x Q^1(x) \wedge \forall x\exists y P^2(x,y))$.
      3. $\exists x\forall y P^2(x,y) \wedge \neg\forall x\exists y P^2(y,x)$.

   [10 marks]

[Total=33 marks]

COMP0009 Logic and Databases
Exercises 6: First order compactness.

Robin Hirsch

November 30, 2020

Let $\vdash$ be a proof system for first order logic (denoting either proof by an axiom system, by tableau or by some other method, sound and complete for first order validities.).

1. Let Γ be a set of first order sentences and let ϕ be a single sentence. Explain what $\Gamma \vdash \phi$ means.
2. Explain the notation $\Gamma \models \phi$.
3. Explain what it means when we say that a set Σ of L-formulas is *inconsistent*.
4. What does it mean when we say $\vdash$ is sound? What does it mean when we say $\vdash$ is strongly complete?
5. State the compactness theorem for first order logic.

Let $L = L(C, F, P)$ be a signature with constants $C = \{0, 1\}$, functions $F = \{+, \times\}$ and predicates $\{<, =\}$. Let $N = (\mathbb{N}, I)$ be the L-structure whose domain is the set of natural numbers, where $I(0) = 0,\ I(1) = 1$, where $I(+), I(\times)$ denote ordinary addition and multiplication of natural numbers (respectively) and where $I(<), I(=)$ denote ordinary less than or equal (respectively) on natural numbers.

6. Write down a closed term t in this language that denotes 7.
7. Which elements of the domain $\mathbb{N}$ are named by closed terms in N?

Let Σ be the set of all L-sentences true in N.

8. Write down an L-sentence in Σ. Write down an L sentence *not* in Σ.

Now let L^+ be the same signature as L, but also including one new constant symbol ω. Consider the infinite theory

$$\Sigma^+ = \Sigma \cup \{t < \omega : t \text{ is a closed } L\text{-term}\}$$

9. Let F be a finite subset of Σ^+. Prove that F is consistent. [Hint: find a *model* of F based on N, but with a suitable interpretation of ω.]
10. Use the compactness theorem to prove that Σ^+ has a model $N^+ = (\mathbb{N}^+, I^+)$.
11. If $m \in \mathbb{N}^+$ is named by a closed L-term t, then we say that m is a *standard* number, other elements of $\mathbb{N}^+$ are *non-standard*. Write down several closed L^+-terms denoting distinct, non-standard numbers.

12. Is $\forall x \forall y (x \times y = y \times x)$ true in N^{+}?

13. Is there a number $m \in \mathbb{N}^{+}$ such that $\mathbb{N}^{+} \models m + 1 = \omega$? If so, is m interpretted as a standard number?

14. We say that m is a predecessor of n if $m + 1 = n$. Which elements of $\mathbb{N}^{+}$ have predecessors?

15. The principal of induction can be written as the second order formula $\forall P((P(0) \wedge \forall x(P(x) \rightarrow P(x+1))) \rightarrow \forall x P(x))$, where P is a unary predicate. Does the principal of induction hold in N^{+}?

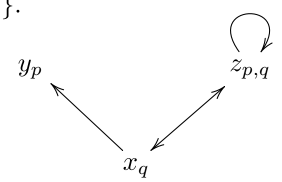
COMP0009 Logic and Databases
Modal Logic

Robin Hirsch

December 2, 2019

A modal propositional formula is either a proposition $p, q, r, \ldots$, a negated modal formula, a disjunction/conjunction/implication of two formulas, a box formula or a diamond formula. A Kripke frame $\mathcal{F} = (W, R)$ consists of a set W of worlds and a binary relation $R \subseteq W \times W$ of arrows. In other words, a Kripke frame is just a directed graph.

1. Consider the Kripke frame $\mathcal{F} = (\{x, y, z\}, \{(x, y), (x, z), (z, x), (z, z)\})$ and let v be the valuation $v(p) = \{y, z\},\ v(q) = \{x, z\}$.



   Are the following true?

   (a) $\mathcal{F}, v, x \models \Box q$

   (b) $\mathcal{F}, v, x \models (q \land \Box(q \to p))$

   (c) $\mathcal{F}, v, x \models \Diamond\Box\bot$, where $\bot$ is any unsatisfiable formula, e.g. $(p \land \neg p)$.

2. Write down a modal fromula that is true at a world w in a Kripke frame (regardless of the valuation) if and only if there are no outgoing edges from w (or no successors of w).

3. Let $\mathcal{F}$ be one of the three Kripke Frames: (i) $\mathcal{Q} = (\mathbb{Q}, <)$, (ii) $\mathcal{N} = (\mathbb{N}, <)$, (iii) $\mathcal{N}^- = (\mathbb{N}, >)$, where $\mathbb{N}$ is the set of natural numbers, $\mathbb{Q}$ is the set of rational numbers and $<$ denotes strict inequality (reversed for $>$). For each modal formula below, state whether the formula is valid in the frame $\mathcal{F}$.

   (a) $\Box p \to p$

   (b) $\Diamond(p \lor \neg p)$

   (c) $\Diamond\Diamond p \to \Diamond p$

   (d) $\Box\Box p \to \Box p$

   (e) $\Box\bot \lor \Diamond\Box\bot$.

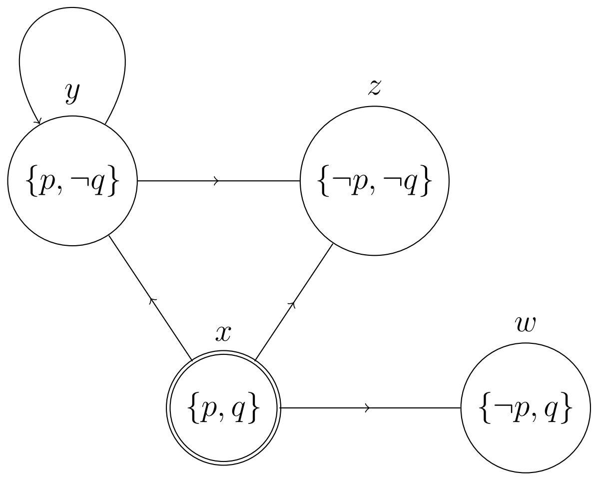
4. (a) Suppose $\mathcal{F} \models \Diamond\top$ for some Kripke frame $\mathcal{F}$ where $\top$ is any valid formuula, e.g. $p \lor \neg p$. What does this tell you about $\mathcal{F}$? What if $\mathcal{F}$ is transitive and irreflexive?

   (b) Find a Kripke frame $\mathcal{F}$ such that $\mathcal{F} \models (\Diamond\Box p \to \Box\Diamond p)$. Find a Kripke frame $\mathcal{G}$ such that $\mathcal{G} \not\models (\Diamond\Box p \to \Box\Diamond p)$. Can you describe the frame property that this formula defines?

Marks for each part of each question are indicated in square brackets.

Calculators are NOT permitted.

1. a. Consider a Kripke frame with worlds $V = \{x, y, z, w\}$ and edges $E = \{(x, y), (x, z), (x, w), (y, y), (y, z)\}$. Let v be the propositional valuation $v(p) = \{x, y\},\ v(q) = \{x, w\}$.



Which of the following are true?

1. $(V, E), v, x \models \Box p$
2. $(V, E), v, x \models \Diamond p$
3. $(V, E), v, x \models \Diamond(p \wedge q)$
4. $(V, E), v, x \models \Diamond\Box\bot$
5. $(V, E), v, x \models \Diamond(p) \wedge \Box(p \to \Box\neg q)$.

b. Let (V, E) be the Kripke frame above. Which of the following hold?

1. $(V, E) \models (\Box p \to p)$
2. $(V, E) \models (\Box p \to \Box\Box p)$
3. $(V, E) \models (\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q))$.

[Question 1 cont. over page]

c. For each formula below use a tableau to find a Kripke model of the formula.

1. $\Diamond p \wedge \Box(p \to \Diamond p)$
2. $\Diamond p \wedge \Box(p \to \Diamond\neg p) \wedge \Box(p \vee \Diamond p)$.

Also, use tableaus to find *transitive* Kripke models for both formulas.

d. A frame (V, E) is *dense* if $(v, w) \in E$ implies there is $u \in V$ such that $(v, u) \in E$ and $(u, w) \in E$. Write down a modal formula ϕ such that for all frames (V, E) we have $(V, E) \models \phi$ if and only if (V, E) is dense. Prove that your formula defines the class of dense frames.