COMP0009 Logic and Databases
Modal Logic

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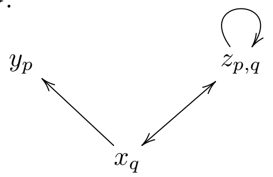
A modal propositional formula is either a proposition $p, q, r, \ldots$, a negated modal formula, a disjunction/conjunction/implication of two formulas, a box formula or a diamond formula. A Kripke frame $\mathcal{F} = (W, R)$ consists of a set W of worlds and a binary relation $R \subseteq W \times W$ of arrows. In other words, a Kripke frame is just a directed graph.

1. Consider the Kripke frame $\mathcal{F} = (\{x, y, z\}, \{(x, y), (x, z), (z, x), (z, z)\})$ and let v be the valuation $v(p) = \{y, z\}$, $v(q) = \{x, z\}$.



Are the following true?

(a) $\mathcal{F}, v, x \models \Box q$

(b) $\mathcal{F}, v, x \models (q \wedge \Box(q \to p))$

(c) $\mathcal{F}, v, x \models \Diamond\Box\bot$, where $\bot$ is any unsatisfiable formula, e.g. $(p \wedge \neg p)$.

Answer: First is not true, since there is an arrow $(x, y) \in R$ and $\mathcal{F}, v, y \not\models q$. Second is true, since $\mathcal{F}, v, x \models q$ and for any w where $(x, w) \in R$ either $w = y$ or $w = z$. In the first case we have $\mathcal{F}, v, y \not\models q$ and in the second $\mathcal{F}, v, z \models p$ either way $\mathcal{F}, v, w \models (p \to q)$, so $\mathcal{F}, v, x \models \Box(q \to p)$.

2. Write down a modal fromula that is true at a world w in a Kripke frame (regardless of the valuation) if and only if there are no outgoing edges from w (or no successors of w).
Answer: $\Box\bot$. If there are no successors of w then $\Box\bot$ holds vacuously at w, but it fails to hold if there are any outgoing edges.

3. Let $\mathcal{F}$ be one of the three Kripke Frames: (i) $\mathcal{Q} = (\mathbb{Q}, <)$, (ii) $\mathcal{N} = (\mathbb{N}, <)$, (iii) $\mathcal{N}^- = (\mathbb{N}, >)$, where $\mathbb{N}$ is the set of natural numbers, $\mathbb{Q}$ is the set of rational numbers and $<$ denotes strict inequality (reversed for $>$). For each modal formula below, state whether the formula is valid in the frame $\mathcal{F}$.

(a) $\Box p \to p$
Answer: Not valid in any of the 3 frames, since they are not reflexive. To see this, let v be the valuation where $v(p) = W \setminus \{w\}$ (p is false at w but true everywhere else, some world w). Then $\mathcal{F}, v, w \models \Box p$ but $\mathcal{F}, v, w \not\models p$, so $\mathcal{F} \not\models (\Box p \to p)$.

(b) $\Diamond(p \vee \neg p)$
Answer: Valid in $\mathcal{Q}$ and $\mathcal{N}$ since they have no final point, but not in $\mathcal{N}^-$ since $\mathcal{N}^-, 0 \not\models \Diamond\top$

(c) $\Diamond\Diamond p \to \Diamond p$
Answer: Valid in all three, since they are transitive

(d) $\Box\Box p \to \Box p$
Answer: Valid in $\mathcal{Q}$ since it is dense, but not in $\mathcal{N}$ or $\mathcal{N}^-$. E.g. let $v(p) = \mathbb{N} \setminus \{0, 1\}$. Then $\mathcal{N}, 0 \models \Box\Box p$ but $\mathcal{N}, 0 \not\models \Box p$ so $\mathcal{N}, 0 \not\models (\Box\Box p \to \Box p)$. For $\mathcal{N}^-$ let $v(p) = \{0\}$. Then $\mathcal{N}^-, 2 \models \Box\Box p$ but $\mathcal{N}^- \not\models \Box p$ so not valid in $\mathcal{N}^-$. For validity over $\mathcal{Q}$, suppose for contradiction that $\mathcal{Q} \not\models (\Box\Box p \to \Box p)$. Then there is a valuation v and a rational number q such that $\mathcal{Q}, v, q \models \Box\Box p$ but $\mathcal{Q}, v, q \models \neg\Box p$. From the latter, there is $r > q$ such that $\mathcal{Q}, v, r \not\models p$. But then, let $s = \frac{q+r}{2}$ (so $q < s < r$) and we have $\mathcal{Q}, v, s \not\models \Box p$, hence $\mathcal{Q}, v, q \not\models \Box\Box p$, contradicting the former assumption. Since $\mathcal{Q} \not\models (\Box\Box p \to \Box p)$ led to a contradiction, we deduce $\mathcal{Q} \models (\Box\Box p \to p)$.

(e) $\Box\bot \vee \Diamond\Box\bot$.
Answer: Not valid in $\mathcal{N}$ or $\mathcal{Q}$ but valid in $\mathcal{N}^-$. Reason: the formula $\Box\bot$ can only hold at a point in a Kripke frame with no successors at all (then it holds vacuously). In $\mathcal{Q}$ or $\mathcal{N}$ every point has a successor, so $\Box\bot \vee \Diamond\Box\bot$ is not even satisfiable, certainly not valid. However, in $\mathcal{N}^-$ the point 0 has no successor, therefore $\mathcal{N}^-, v, 0 \models \Box\bot$ (vacuously), regardless of the valuation v. Every natural number n is either 0 or strictly bigger than 0, in the former case we have $\mathcal{N}^-, v, 0 \models \Box\bot$ and in the latter case we have $\mathcal{N}^-, v, n \models \Diamond\Box\bot$. So $\mathcal{N}^- \models (\Box\bot \vee \Diamond\Box\bot)$.

4. (a) Suppose $\mathcal{F} \models \Diamond\top$ for some Kripke frame $\mathcal{F}$ where $\top$ is any valid formuula, e.g. $p \vee \neg p$. What does this tell you about $\mathcal{F}$? What if $\mathcal{F}$ is transitive and irreflexive?
Answer: Every world has an outgoing edge, there are no 'end worlds'. If $\mathcal{F}$ is irreflexive and transitive there must be infinitely many worlds.

(b) Find a Kripke frame $\mathcal{F}$ such that $\mathcal{F} \models (\Diamond\Box p \to \Box\Diamond p)$. Find a Kripke frame $\mathcal{G}$ such that $\mathcal{G} \not\models (\Diamond\Box p \to \Box\Diamond p)$. Can you describe the frame property that this formula defines?
Answer: Convergence: if $(x, y), (x, z) \in R$ then there is w such that $(y, w), (z, w) \in R$. If a frame is not convergent there are x, y, z with $(x, y), (x, z) \in R$ but there is no w with $(y, w) \in R$ and $(z, w) \in R$. Consider the valuation v where $v(p) = \{y' \in W : (y, y') \in R\}$. Then $\mathcal{F}, v, y \models \Box p$, so $\mathcal{F}, v, x \models \Diamond\Box p$. But $\mathcal{F}, v, z \not\models \Diamond p$ (as there is no w with $(y, w), (z, w) \in R$) so $\mathcal{F}, v, x \not\models \Diamond\Box p \to \Box\Diamond p$, hence $\mathcal{F} \not\models \Diamond\Box p \to \Box\Diamond p$. Conversely, suppose $\mathcal{F}$ has the convergence property: if $(x, y), (x, z) \in R$ then there is $w \in W$ where $(y, w), (z, w) \in R$. Let v be any valuation, let $x \in W$ be any world, and suppose $\mathcal{F}, v, x \models \Diamond\Box p$. So there is $y \in W$ where $(x, y) \in R$ and $\mathcal{F}, v, y \models \Box p$. For any $z \in W$ if $(x, z) \in R$ then we know that there is $w \in R$ with $(y, w) \in R$ and $(z, w) \in R$. Since $(y, w) \in R$ and $\mathcal{F}, v, y \models \Box p$ we know that $\mathcal{F}, v, w \models p$. Hence $\mathcal{F}, v, z \models \Diamond p$. This shows that $\mathcal{F}, v, x \models \Box\Diamond p$. Hence $\mathcal{F} \models \Diamond\Box p \to \Box\Diamond p$. This convergence property is called the Church-Rosser *property.*