Theory III, COMP0009, ????

Time allowed: 2.5 hours

Answer all three questions.

Marks for each part of each question are indicated in square brackets.

Calculators are NOT permitted.

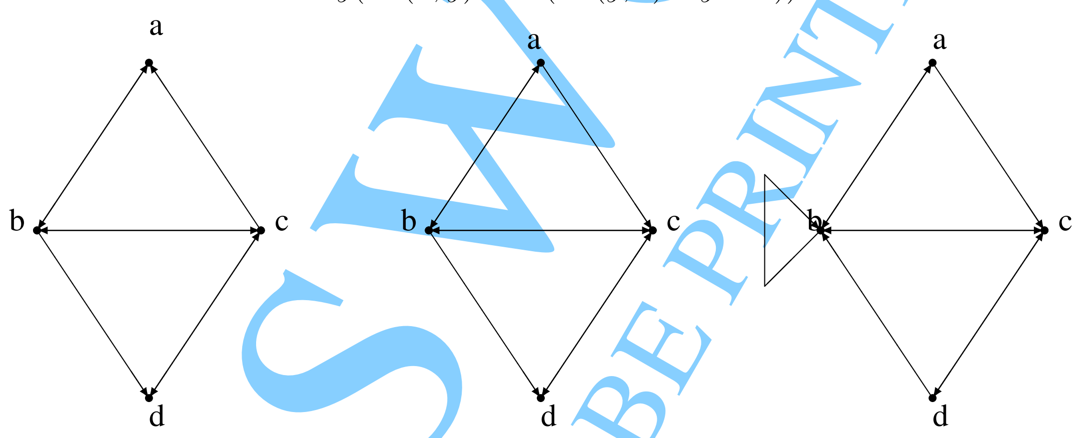
1. a. Let $L = L(C, F, P)$ be a first-order language and let $\mathcal{M}$ be an L-structure. Let ϕ be a formula of this language. Prove that ϕ is valid in $\mathcal{M}$ if and only if $\neg\phi$ is not satisfiable in $\mathcal{M}$.

ϕ is valid over $\mathcal{M} = (D, I)$ iff for all variable assignments $A : vars \to D$ we have $\mathcal{M}, A \models \phi$ iff there is no var. assig. A with $\mathcal{M}, A \models \neg\phi$ iff $\neg\phi$ is not satisfiable in $\mathcal{M}$.

[Marks 8]

b. Let $C = F = \emptyset$ and $P = \{E^2, =\}$, where $=$ is the equality predicate, written infix. Each of the graphs below can be interpretted as an L-structure whose domain is the set of nodes of the graph and where $I(E^2) = \{(x, y) :$ there is an arrow from x to $y\}$. Which of these structures are models of the following formula? Give a brief reason for your answer.

$$\forall x \exists y (E^2(x, y) \wedge \forall z (E^2(y, z) \vee y = z))$$



Graph 1	Graph 2	Graph 3
Edges:	Edges:	Edges:
$\{(a,b),\ (b,a),$	$\{(a,b),\ (b,a),$	$\{(a,b),\ (b,a),$
$(c,a),\ (b,d),$	$(a,c),\ (c,b),$	$(a,c),\ (b,b),$
$(c,b),\ (b,c),$	$(b,c),\ (b,d),$	$(b,c),\ (c,b),$
$(c,d),\ (d,c)\}$	$(c,d),\ (d,c)\}$	$(d,b),\ (c,d),$
		$(d,c)\}$

1 = yes (for each x use either $y \mapsto b$ or $y \mapsto c$). 2=no ($x \mapsto d$ falsifies formula). 3=no ($x \mapsto d$ falsifies the formula since y would have to map to b or to c, but neither of these has edges to all other nodes).

[Marks 9]

[Question 1 cont. on next page]

c. Now let $C = F = \emptyset$ and $P = \{=, N^1, Col^1, E^2, \rho^2\}$. The intension is that the domain D of a model $\mathcal{M} = (D, I)$ is partitioned into two parts, graph nodes and colours. If $N^1(x)$ holds it will mean that x designates a graph node and if $Col^1(x)$ holds then x designates a colour. As before, E^2 will be interpretted as the set of graph edges (so $E^2(x, y)$ will only hold if x and y are graph nodes and there is an arrow from x to y). $\rho^2(x, i)$ means that i designates a colour and the node designated x has that colour.

1. Write a first-order formula that means that every point in the model is either a graph node or a colour, but not both.
   $\forall x((N^1(x) \vee Col^1(x)) \wedge \neg(N^1(x) \wedge Col^1(x)))$

2. Write a first-order formula that means that the set of edges is irreflexive (you never have an edge from a graph node to itself) and symmetric (if there is an edge from one graph node to another then there is also a reverse edge).
   $\forall x \forall y(\neg E^2(x, x) \wedge (E^2(x, y) \rightarrow E^2(y, x)))$

3. Write down a first-order formula that means that every graph node has a unique colour.
   $\forall x(N^1(x) \rightarrow \exists i(Col^1(i) \wedge \rho^2(x, i) \wedge \forall j(Col^1(j) \wedge \neg(i = j) \rightarrow \neg\rho^2(x, j))))$

4. Write down a first-order formula that means that two nodes connected by an edge must have different colours.
   Either $\neg\exists x \exists y \exists j(E^2(x, y) \wedge \rho^2(x, i) \wedge \rho^2(y, i))$ or $\forall x \forall y \forall i \forall j((E^2(x, y) \wedge \rho^2(x, i) \wedge \rho^2(y, j)) \rightarrow \neg(i = j))$

5. Write a formula that means that there are three colours and every node has one of these three colours.
   $\exists i \exists j \exists k \forall x(\rho^2(x, i) \vee \rho^2(x, j) \vee \rho^2(x, k))$.

6. Let ϕ be the conjunction of your answers to (1)–(5) above. Is ϕ satisfiable? If so, define a model. If not, give a proof.
   Yes. Take any model with less than 3 graph nodes, three colours, and give each graph node a unique distinct colour.

[Marks 16]

[Marks Total = 33]