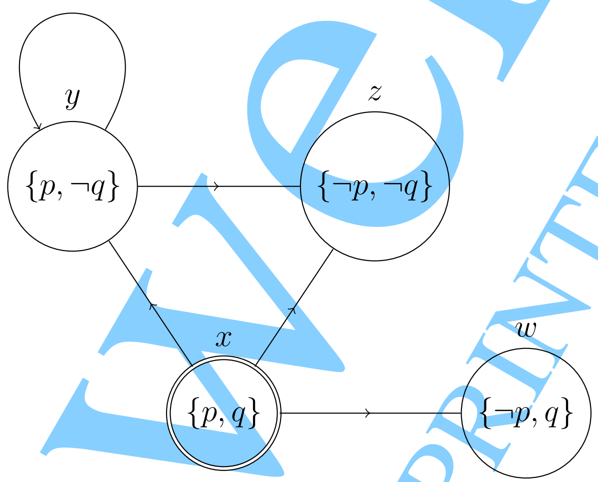
Marks for each part of each question are indicated in square brackets.

Calculators are NOT permitted.

1. a. Consider a Kripke frame with worlds $V = \{x, y, z, w\}$ and edges $E = \{(x, y), (x, z), (x, w), (y, y), (y, z)\}$. Let v be the propositional valuation $v(p) = \{x, y\},\ v(q) = \{x, w\}$.



Which of the following are true?

1. $(V, E), v, x \models \Box p$
2. $(V, E), v, x \models \Diamond p$
3. $(V, E), v, x \models \Diamond(p \wedge q)$
4. $(V, E), v, x \models \Diamond\Box\bot$
5. $(V, E), v, x \models \Diamond(p) \wedge \Box(p \to \Box\neg q)$.

No, yes, no, yes, yes

b. Let (V, E) be the Kripke frame above. Which of the following hold?

1. $(V, E) \models (\Box p \to p)$
2. $(V, E) \models (\Box p \to \Box\Box p)$
3. $(V, E) \models (\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q))$.

No, cos not reflexive. Yes, it is transitive. Yes, this is valid over all frames.

[Question 1 cont. over page]

c. For each formula below use a tableau to find a Kripke model of the formula.

1. $\Diamond p \wedge \Box(p \to \Diamond p)$
2. $\Diamond p \wedge \Box(p \to \Diamond\neg p) \wedge \Box(p \vee \Diamond p)$.

Also, use tableaus to find *transitive* Kripke models for both formulas.

For the first formula, let $V = \{x, y, z\},\ E = \{(x, y), (y, z)\},\ v(p) = \{y, z\}$, formula is true at x.

For the second formula let $V = \{x, y, z\},\ E = \{(x, y), (y, z)\}$ and $v(p) = \{y\}$, formula is true at x.

For transitive frames, in the first case $V = \{x_0, x_1, x_2, \ldots\},\ E = \{(x_i, x_j) : 0 \le i < j\}$ and $v(p) = \{x_i : i \ge 1\}$. In the second case, same frame but $v(p) = \{x_{2i+1} : i \ge 0\}$ (alternates between p and $\neg p$).

d. A frame (V, E) is *dense* if $(v, w) \in E$ implies there is $u \in V$ such that $(v, u) \in E$ and $(u, w) \in E$. Write down a modal formula ϕ such that for all frames (V, E) we have $(V, E) \models \phi$ if and only if (V, E) is dense. Prove that your formula defines the class of dense frames.

$(\Box\Box p \to \Box p)$ (or $(\Diamond p \to \Diamond\Diamond p)$). Suppose (V, E) is not dense, then there is an edge $(x, y) \in E$ but no node z where $(x, z), (z, y) \in E$. Let v be valuation $v(p) = V \setminus \{y\}$. p does not hold at y. There is an edge (x, y) therefore $(V, E), v, x \not\models \Box p$. But for any node z if there is an edge (x, z) then there is no edge (z, y) therefore $(V, E), v, x \models \Box\Box p$. Therefore the implication is not valid in (V, E). Conversely, suppose (V, E) is dense. Let v be any valuation and let $x \in V$ be any world. Suppose $(V, E), v, x \models \Box\Box p$. For any $y \in V$ where $(x, y) \in E$ we know by density that there is z where $(x, z), (z, y) \in E$, so $(V, E), v, y \models p$. This shows that $(V, E), v, x \models \Box\Box p$ implies $(V, E), v, x \models \Box p$, as required. You might find it easier to use the equivalent $\Diamond$ form of the density axiom.