COMP0009 Exercises I. Logic Revision.

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EXERCISE 1 *For each of the following propositional formulas, find an equivalent formula written in* disjunctive normal form*.*

1. $((p \vee q) \wedge (\neg p \to \neg q))$
   Answer: p

2. $\neg((p \to q) \to (q \to p))$
   Answer: $(q \wedge \neg p)$

3. $(((p \vee q) \wedge (\neg q \vee \neg r)) \wedge (\neg p \vee r))$
   Answer: $(p \wedge \neg q \wedge r) \vee (q \wedge \neg r \wedge \neg p)$

EXERCISE 2 *Let L be a first order language to describe vertex colourings in graphs, with no constant symbols, no function symbols, two unary predicates R, B for red nodes and blue nodes respectively, one binary predicate E for the edge relation and one binary predicate symbol $=$ for equality between nodes of a graph. For each of the following statements about coloured graphs, write down an L-formula that expresses it.*

1. *there is an isolated node (not incident with any edge)*
   Answer: $\exists x \forall y(\neg E(x, y) \wedge \neg E(y, x))$

2. *every node is coloured red or blue but not both*
   Answer: $\forall x((R(x) \vee B(x)) \wedge \neg(R(x) \wedge B(x)))$

3. *every blue node is adjacent to a red node*
   Answer: $\forall x(B(x) \to \exists y((E(x, y) \vee E(y, x)) \wedge R(y)))$

4. *between any two nodes, there is a path from one to the other of length at most three.*
   Answer: $\forall x \forall y(x = y \vee E(x, y) \vee \exists z(E(x, z) \wedge E(z, y)) \vee \exists z \exists w(E(x, z) \wedge E(z, w) \wedge E(w, y)))$

5. *the graph is reflexive, symmetric and transitive (look these up if you've forgotten).*
   Answer: Reflexive $\forall x E(x, x)$, symmetric $\forall x \forall y(E(x, y) \to E(y, x))$ and transitive $\forall x \forall y \forall z((E(x, y) \wedge E(y, z)) \to E(x, z))$.

EXERCISE 3 *Let L be a first order language for arithmetic, with one constant symbol 1, one binary function symbols $+$ and two binary predicates $=, <$ (predicates written infix). Let $\mathbf{N}$ be the L-structure whose base is the set of natural numbers, and where all symbols are interpretted normally, i.e. 1 is interpretted as one, $+$ is interpretted as the binary function that adds its two arguments, $=$ is interpretted as equality and $<$ is interpretted as 'strictly less than', i.e. the set of all pairs (m, n) of natural numbers where m is less than n. Which of the following L-formulas is* valid *in $\mathbf{N}$.*

1. $\exists y(y = x)$
   Answer: yes

2. $\exists x \exists y(x + x = y)$
   Answer: yes

3. $\exists x(x+1=x)$
   Answer: no

4. $\forall x \exists y(x=y+y \vee x=y+y+1)$.
   Answer: yes, x is either even or odd

5. $\forall x \forall y \forall z((x<y \wedge y<z) \rightarrow x<z)$
   Answer: yes, $<$ is transitive

6. $\forall x \forall y(x<y \rightarrow \exists z(x<z \wedge z<y))$.
   Answer: No, not dense. e.g $3<4$ but there is no z between 3 and 4