COMP0009 Logic and Databases
Exercises 6: First order compactness. With answers.

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Let $\vdash$ be a proof system for first order logic (denoting either proof by an axiom system, by tableau or by some other method, sound and complete for first order validities.).

1. Let Γ be a set of first order sentences and let ϕ be a single sentence. Explain what $\Gamma \vdash \phi$ means.
   Answer: For axiom systems, there is a finite sequence $\phi_0, \phi_1, \ldots, \phi_k$ where each formula is either an instance of an axiom, an element of Γ or obtained from earlier elements of the sequence by a rule of inference, and $\phi_k = \phi$. For tableaus, there is a (finite) closed tableau with $\neg\phi$ at the root and the tableau is constructed by standard α, β, δ and γ expansions or by adding in nodes labelled by formulas in Γ.

2. Explain the notation $\Gamma \models \phi$.
   Answer: For every L-structure S if $S \models \Gamma$ (i.e. if all sentences in Γ are true in S) then $S \models \phi$.

3. Explain what it means when we say that a set Σ of L-formulas is *inconsistent*.
   Answer: It means $\Sigma \vdash \bot$

4. What does it mean when we say $\vdash$ is sound? What does it mean when we say $\vdash$ is strongly complete?
   Answer: Sound means $\Gamma \vdash \phi$ implies $\Gamma \models \phi$. It follows from soundness that inconsistent formulas do not have models. Strongly complete means $\Gamma \models \phi$ implies $\Gamma \vdash \phi$. It follows from strong completeness that all consistent sets of sentences have models

5. State the compactness theorem for first order logic.
   Answer: Σ is consistent if and only if every finite subset of Σ is consistent

Let $L = L(C, F, P)$ be a signature with constants $C = \{0, 1\}$, functions $F = \{+, \times\}$ and predicates $\{<, =\}$. Let $N = (\mathbb{N}, I)$ be the L-structure whose domain is the set of natural numbers, where $I(0) = 0$, $I(1) = 1$, where $I(+), I(\times)$ denote ordinary addition and multiplication of natural numbers (respectively) and where $I(<), I(=)$ denote ordinary less than or equal (respectively) on natural numbers.

6. Write down a closed term t in this language that denotes 7.
   Answer: e.g. $((1 + 1) \times ((1 + 1) + 1)) + 1$

7. Which elements of the domain $\mathbb{N}$ are named by closed terms in N?
   Answer: Every natural number is named by a closed term, n is named by $(((\ldots((1 + 1) + 1) + \ldots + 1) + 1$.

Let Σ be the set of all L-sentences true in N.

8. Write down an L-sentence in Σ.
Answer: e.g. $\forall x \forall y(x+y=y+x)$ *or* $\forall x(x<x+1)$ Write down an L sentence *not* in Σ.
Answer: e.g. $1=0$

Now let L^+ be the same signature as L, but also including one new constant symbol ω. Consider the infinite theory

$$\Sigma^+ = \Sigma \cup \{t < \omega : t \text{ is a closed } L\text{-term}\}$$

9. Let F be a finite subset of Σ^+. Prove that F is consistent. [Hint: find a *model* of F based on N, but with a suitable interpretation of ω.]
Answer: Since F is finite there is a 'largest' closed term t such that $t<\omega$ is in F, let $I(t)=n\in\mathbb{N}$. Now Let I' be same as standard interpretation I over $\mathbb{N}$ but let $I'(\omega)=n+1$. Then each sentence in Σ is true in $(\mathbb{N},I')$, since I' is same as standard interpretation on L-terms, and each formula $t<\omega$ in the finite set F is true in $(\mathbb{N},I')$ since $n<n+1$. Since F has a model, it must be consistent, by soundness.

10. Use the compactness theorem to prove that Σ^+ has a model $N^+=(\mathbb{N}^+,I^+)$.
Answer: Since every finite subset of Σ^+ is consistent (as shown in previous question) the compactness theorem tell us that Σ^+ is consistent. Strong completeness tells us that Σ^+ has a model.

11. If $m\in\mathbb{N}^+$ is named by a closed L-term t, then we say that m is a *standard* number, other elements of $\mathbb{N}^+$ are *non-standard*. Write down several closed L^+-terms denoting distinct, non-standard numbers.
Answer: E.g. $\omega, \omega+1, \ldots, \omega\times 2, \ldots, \omega\times\omega, \ldots$

12. Is $\forall x\forall y(x\times y=y\times x)$ true in N^+?
Answer: Yes. This sentence belongs to Σ and N^+ is a model of Σ.

13. Is there a number $m\in\mathbb{N}^+$ such that $\mathbb{N}^+\models m+1=\omega$? If so, is m interpretted as a standard number?
Answer: Yes. The sentence $\forall x(\neg x=0\to\exists y(y+1=x)$ is true in N hence it is included in Σ, hence N^+ is a model of it.

14. We say that m is a predecessor of n if $m+1=n$. Which elements of $\mathbb{N}^+$ have predecessors?
Answer: By previous part, all elements of $\mathbb{N}^+$ apart from zero have predecessors.

15. The principal of induction can be written as the second order formula $\forall P((P(0)\wedge\forall x(P(x)\to P(x+1)))\ \to\ \forall xP(x))$, where P is a unary predicate. Does the principal of induction hold in N^+?
Answer: No. Let P be a unary predicate interpreted as the set of standard numbers. Then $P(0)$ is true since 0 is standard and $P(n)\to P(n+1)$ is also true, since the successor of a standard number is standard. But $\forall xP(x)$ is not true, since $P(\omega)$ is false.