Foundations of Mathematics: Exam

Winter 2023/24

Originally held February 9, 2024. Slight alterations made later.

Please Read First

IMPORTANT: **Do not be intimidated by the number of questions.** The exam has a large number of questions so you can prioritize whatever you feel most comfortable with. You can get an A while solving only a subset of questions. Grades are determined based on points achieved in the overall exam.

Unless a question asks for an explanation or a proof, it is sufficient to provide calculations. When a question asks for an explanation or proof, keep it brief. Throughout, focus on calculation and formal reasoning, as we did on the blackboard in the lectures, and **keep prose to a minimum**. Space for your answers is provided with every question. If necessary, use the **blank pages at the end**.

Even when no explanation or intermediate steps are required, it's still fine to write them down as part of your solution, as long as you clearly mark the final answer. You can also use extra scratch paper (not handed in) for intermediate steps, but beware that copying results over to the exam sheets takes time.

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Matriculation Number: 7063772

Are you an Erasmus student (yes/no)? no

Points Achieved

Question	Topic	Max Points	Achieved
1.1	Linear Systems	40	
1.2	Independence	5	
1.3	Multiplication	2	
1.4	Rank	1	
1.5	Multiplication	5	
1.6	Rank	3	
1.7	Im, Ker	5	
2.1	Limits	8	
2.2	Derivatives	26	
2.3	Integrals	11	
2.4	Chain Rule	8	
2.5	Zero Jacobian	4	
2.6	Chain Rule	3	
2.7	Partial Deriv.	10	
2.8	Jacobians	9	
3.1	Neural Networks	9	
TOTAL			
Grade:			

149

1 Linear Algebra

1.1 Linear Systems of Equations (total: 40 points)

1.1.1 (total: 10 points)

Define

$$A := \begin{pmatrix} -2 & 7 & 5 \\ 2 & -5 & -7 \\ 1 & -3 & -3 \end{pmatrix}$$

Please compute the following:

1. (6 points) a **row reduced form** of the matrix

$$\begin{pmatrix} -2 & 7 & 5 & -17 \\ 2 & -5 & -7 & 23 \\ 1 & -3 & -3 & 10 \end{pmatrix}$$

   Explicitly show all row operations and the final result. Writing down the intermediate results is not strictly required, but can help us give partial credit for partially correct answers.

2. (1 points) the solution set of the equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -17 \\ 23 \\ 10 \end{pmatrix}$$

   Hint: Use the row reduced form computed above.

3. (1 points) the kernel of A, this is the solution set of the equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

   Hint: Use the row reduced from computed above.

4. (1 point) the rank of A

   Hint: Use the row reduced from computed above.

5. (1 points) does A have an inverse? No explanation needed.

YOUR ANSWER HERE

CONTINUE YOUR ANSWER HERE

1.1.2 (total: 10 points)

Define

$$A := \begin{pmatrix} -2 & 3 & 10 \\ 0 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix}$$

Please compute the following:

1. (6 points) a **row reduced form** of the matrix

$$\begin{pmatrix} -2 & 3 & 10 & -21 \\ 0 & 1 & 3 & -7 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

   Explicitly show all row operations and the final result. Writing down the intermediate results is not strictly required, but can help us give partial credit for partially correct answers.

2. (1 point) the solution set of the equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -21 \\ -7 \\ -1 \end{pmatrix}$$

3. (1 points) the kernel of A

4. (1 points) the rank of A

5. (1 points) does A have an inverse? No explanation needed.

YOUR ANSWER HERE

CONTINUE YOUR ANSWER HERE

1.1.3 (total: 10 points)

Define

$$A := \begin{pmatrix} -3 & -7 & 1 \\ 1 & 2 & 0 \\ -3 & -8 & 1 \end{pmatrix}$$

Please compute the following:

1. (6 points) a **row reduced form** of the matrix

$$\begin{pmatrix} -3 & -7 & 1 & 26 \\ 1 & 2 & 0 & -7 \\ -3 & -8 & 1 & 29 \end{pmatrix}$$

   Explicitly show all row operations and the final result. Writing down the intermediate results is not strictly required, but can help us give partial credit for partially correct answers.

2. (1 point) the solution set of the equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 26 \\ -7 \\ 29 \end{pmatrix}$$

3. (1 points) the kernel of A
4. (1 points) the rank of A
5. (1 points) does A have an inverse? No explanation needed.

YOUR ANSWER HERE

CONTINUE YOUR ANSWER HERE

1.1.4 (total: 10 points)

$$A = \begin{pmatrix} -2 & 5 & 4 \\ 0 & -3 & 6 \\ 1 & -2 & -3 \end{pmatrix}$$

Please compute the following:

1. (6 points) a **row reduced form** of the matrix

$$\begin{pmatrix} -2 & 5 & 4 & -3 \\ 0 & -3 & 6 & -4 \\ 1 & -2 & -3 & -2 \end{pmatrix}$$

   Explicitly show all row operations and the final result. Writing down the intermediate results is not strictly required, but can help us give partial credit for partially correct answers.

2. (1 point) the solution set of the equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ -2 \end{pmatrix}$$

3. (1 points) the kernel of A
4. (1 points) the rank of A
5. (1 points) does A have an inverse? No explanation needed.

YOUR ANSWER HERE

CONTINUE YOUR ANSWER HERE

1.2 Linear Independence (5 points)

Consider the vectors:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

1. (2 points) Show that the set $\{v_1, v_2, v_3\}$ is not linearly independent.
2. (1 points) Show that the set $\{v_1, v_2\}$ is linearly independent.
3. (1 points) Find a vector v_4 such that the set $\{v_1, v_2, v_4\}$ is linearly independent. No explanation needed.
4. (1 points) Is the set $\{v_1, v_2, v_4\}$ a basis of $\mathbb{R}^3$? Briefly explain.

YOUR ANSWER HERE

1.3 Matrix Multiplication (2 points)

Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ -1 & -1 \end{pmatrix}$$

Considering the shapes of the matrices, which of the following matrix products exist? Just write "yes" or "no", no explanation needed.

1. AA (0.5 points) YOUR ANSWER HERE:
2. AB (0.5 points) YOUR ANSWER HERE:
3. BA (0.5 points) YOUR ANSWER HERE:
4. BB (0.5 points) YOUR ANSWER HERE:

1.4 Rank (1 points)

Consider a matrix $A \in \mathbb{R}^{5\times 1000}$, that is, a matrix with 5 rows and 1000 columns.

What is the largest rank that A could possibly have? No explanation needed.

Hint: The answer must be either 5 or 1000 – which one is right? Thinking about what happens after transforming A into row reduced form can help.

YOUR ANSWER HERE:

1.5 Matrix Multiplication (5 points)

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Find a matrix $B \in \mathbb{R}^{2\times 2}$ such that $AB \neq BA$. Provide B, AB, BA. No further explanation needed.

YOUR ANSWER HERE:

1.6 Rank and Addition (total: 3 points)

In class, we proved that $rank(AB) \leq rank(A), rank(B)$. Here, you will show that $rank(A+B)$ satisfies no such simple relationship.

Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Clearly, $rank(A) = 1$.

1. Find a matrix $B \in \mathbb{R}^{2\times 2}$ such that $rank(A) = 1$, $rank(B) = 1$, $rank(A+B) = 0$. (1 point)
2. Find a matrix $B \in \mathbb{R}^{2\times 2}$ such that $rank(A) = 1$, $rank(B) = 1$, $rank(A+B) = 1$. (1 point)
3. Find a matrix $B \in \mathbb{R}^{2\times 2}$ such that $rank(A) = 1$, $rank(B) = 1$, $rank(A+B) = 2$. (1 point)

In each case, give B and $A + B$. No further explanations are needed.
YOUR ANSWER HERE:

1.7 Images and Kernels (5 points)

Find some matrices $A, B \in \mathbb{R}^{2\times 2}$ such that

1. A, B both have some nonzero entries
2. $im(B) \subseteq ker(A)$

Explicitly provide A, B, $im(B)$ and $ker(A)$. No explanation needed.
YOUR ANSWER HERE:

2 Calculus

2.1 Computing Limits (total: 8 points)

Please compute the following limits:

$$\lim_{x\to 2} \left[(x-1)^2\right] \qquad \text{(2 points)} \tag{1}$$

$$\lim_{x\to 5} \frac{1}{x} \qquad \text{(2 points)} \tag{2}$$

$$\lim_{x\to 1} \frac{x^2-1}{x-1} \quad \textit{(Hint: Use } \cancel{\textit{binomial formula.)}} \qquad \text{(2 points)} \tag{3}$$

$$\lim_{x\to 1} \left[(-x)^{2024}\right] \qquad \text{(2 points)} \tag{4}$$

YOUR ANSWER HERE:

YOU ANSWER HERE:

2.2 Computing Derivatives (total: 26 points)

Please compute derivatives $f'(x)$ for the following functions $f(x)$. Besides the basic rules (including product rule and chain rule), you may use rules we derived in class (including derivatives of exp, log, x^n, etc.).

Hint: For several subquestions, you can use results from preceding subquestions. You can simply refer to your results from preceding subquestions.

$$f(x) = 5x - 1 \qquad \text{(1 points)} \tag{1}$$

$$f(x) = x^4 \qquad \text{(1 points)} \tag{2}$$

$$f(x) = x^{2023} \qquad \text{(1 points)} \tag{3}$$

$$f(x) = \exp(5x - 1) \qquad \text{(2 points)} \tag{4}$$

$$f(x) = \exp(x^4) \qquad \text{(2 points)} \tag{5}$$

$$f(x) = \frac{1}{x+1} \text{ (assuming } x \neq -1) \qquad \text{(2 points)} \tag{6}$$

$$f(x) = x^4 - x^2 + \frac{1}{x+1} \text{ (assuming } x \neq -1) \qquad \text{(2 points)} \tag{7}$$

$$f(x) = (5x - 1) \times (x - 2) \qquad \text{(2 points)} \tag{8}$$

$$f(x) = \frac{x-1}{x-2} \text{ (assuming } x \neq 2) \qquad \text{(2 points)} \tag{9}$$

$$f(x) = 1 + x^2 \qquad \text{(1 point)} \tag{10}$$

$$f(x) = \log(1 + x^2) \qquad \text{(2 points)} \tag{11}$$

$$f(x) = \frac{5x^2}{\log(1 + x^2)} \qquad \text{(3 points)} \tag{12}$$

$$f(x) = \exp\left(1 + \frac{5x^2}{\log(1 + x^2)}\right) \qquad \text{(3 points)} \tag{13}$$

$$f(x) = [\log(x)]^{10} \text{ (assuming } x > 0) \qquad \text{(2 points)} \tag{14}$$

YOUR ANSWER HERE:

CONTINUE YOUR ANSWER

2.3 Computing Antiderivatives (total: 11 points)

Please compute antiderivatives of the following functions. That is, for each f, find a function F such that $F'(x) = f(x)$. No further explanation needed.

Hint: For several subquestions, you can use results from preceding subquestions.

$$f(x) = x^3 \quad \text{(1 points)} \tag{1}$$

$$f(x) = (x+3) \quad \text{(1 points)} \tag{2}$$

$$f(x) = (x^3 + x^2 - x) \quad \text{(1 points)} \tag{3}$$

$$f(x) = \frac{1}{1+x} \quad \text{(2 points)} \tag{4}$$

$$f(x) = \frac{x^2}{7x} \quad \text{(2 points)} \tag{5}$$

$$f(x) = \exp(x) \quad \text{(2 points)} \tag{6}$$

$$f(x) = \exp(5x) \quad \text{(2 points)} \tag{7}$$

YOUR ANSWER HERE:

2.4 Applications of the Chain Rule (total: 8 points)

2.4.1 Inverse Functions (4 points)

Let f and g be functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x$. Use the Chain Rule to prove that

$$g'(x) = \frac{1}{f'(g(x))}$$

YOUR ANSWER HERE:

2.4.2 Inverse and Square (4 points)

Let f and g be functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^2$. Use the Chain Rule to prove that

$$g'(x) = \frac{2x}{f'(g(x))}$$

You may assume $x \neq 0$.

YOUR ANSWER HERE:

1 对两边求导

因为 $f(g(x)) = x$，我们对两边对 x 求导:

$$\frac{d}{dx}[f(g(x))] = \frac{d}{dx}[x]$$

使用 **链式法则**:

$$f'(g(x)) \cdot g'(x) = 1$$

2 代数变形

$$g'(x) = \frac{1}{f'(g(x))}$$

2.5 Where is the Jacobian Zero? (total: 4 points)

For the following function:

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 + x_1 - x_3$$

1. (3 points) Compute $Jac(f)$.
2. (1 point) Determine a point $x \in \mathbb{R}^3$ such that all entries of $Jac(f)$ at x are zero.

YOUR ANSWER HERE:

2.6 Application of the Chain Rule (3 points)

Consider any function f whose derivative is well-defined, and define g by

$$g(x) = f(exp(x))$$

Compute $g'(x)$ in terms of $f'(x)$.

YOUR ANSWER HERE:

2.7 Partial Derivatives (total: 10 points)

Compute the partial derivatives of the following functions.

1. $f(x, y) = 2x^2y$ (2 points)
2. $f(x, y) = -x + 2y$ (2 points)
3. $f(x, y) = \frac{\exp(10x)}{\log(y)}$ $(y > 0)$ (2 points)
4. $f(x, y) = \log(1 + \exp(-x^4y^2))$ (4 points)

YOUR ANSWER HERE:

YOUR ANSWER CONTINUED:

2.8 Jacobians and Partial Derivatives (total: 9 points)

2.8.1 (6 points)

Let

$$f(x_1, x_2) = \begin{pmatrix} x_1 x_2 \\ \exp(x_1) x_2 \\ x_1^2 \end{pmatrix}$$

Provide $Jac(f) \in \mathbb{R}^{3\times 2}$. No explanation needed.

YOUR ANSWER HERE:

2.8.2 (3 points)

Let

$$f(x_1, x_2) = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 x_2 \\ \exp(x_1) x_2 \\ x_1^2 \end{pmatrix}$$

Compute $Jac(f) \in \mathbb{R}^{2\times 2}$ using the result from the previous question. Explicitly write it as a 2×2-matrix.

YOUR ANSWER HERE:

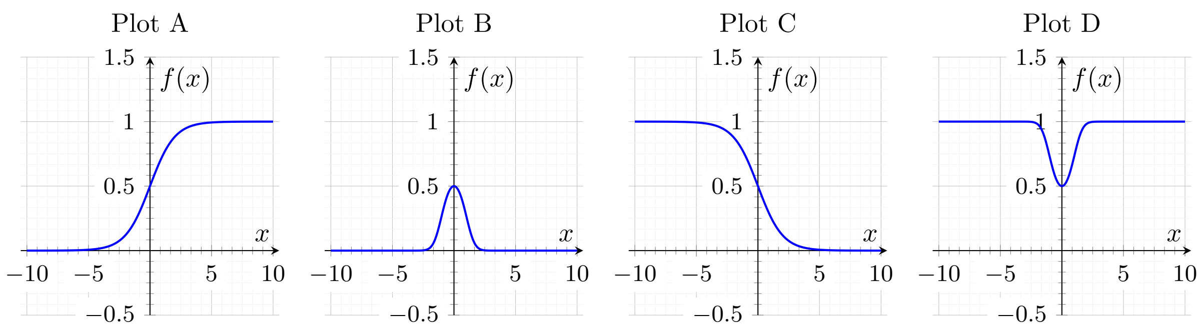
3 Neural Networks

3.1 Derivative of Logistic Regression (total: 9 points)

Here, you will compute the derivatives for a simple machine learning algorithm (simpler than the ones we covered in class): logistic regression. Consider

$$f(x) = \frac{1}{1 + \exp(-x)}$$

1. (1 point) Which of the following plots represents f? No explanation needed.



2. (3 points) Compute $f'(x)$.
3. (3 points) Let $v \in \mathbb{R}^n$. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, ..., x_n) = \sum_{i=1}^{n} v_i x_i$$

   Compute $Jac(h)$.

   Hint: Recall that $\sum_{i=1}^{n} v_i x_i = v_1 x_1 + \cdots + v_n x_n$.

4. (2 points) Let $v \in \mathbb{R}^n$. Define $s : \mathbb{R}^n \to \mathbb{R}$ by

$$s(x_1, ..., x_n) = f(\sum_{i=1}^{n} v_i x_i)$$

   where f is as defined above. Using the results from the previous questions, compute $\frac{\partial s}{\partial x_i}$ for $i = 1, \ldots, n$.

YOUR ANSWER HERE:

CONTINUE YOUR ANSWER HERE:

YOUR ANSWER HERE:

YOUR ANSWER HERE: