Evaluating and Training HMMs

Computational Linguistics

Alexander Koller

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Outline

1. Language modeling: the forward algorithm.

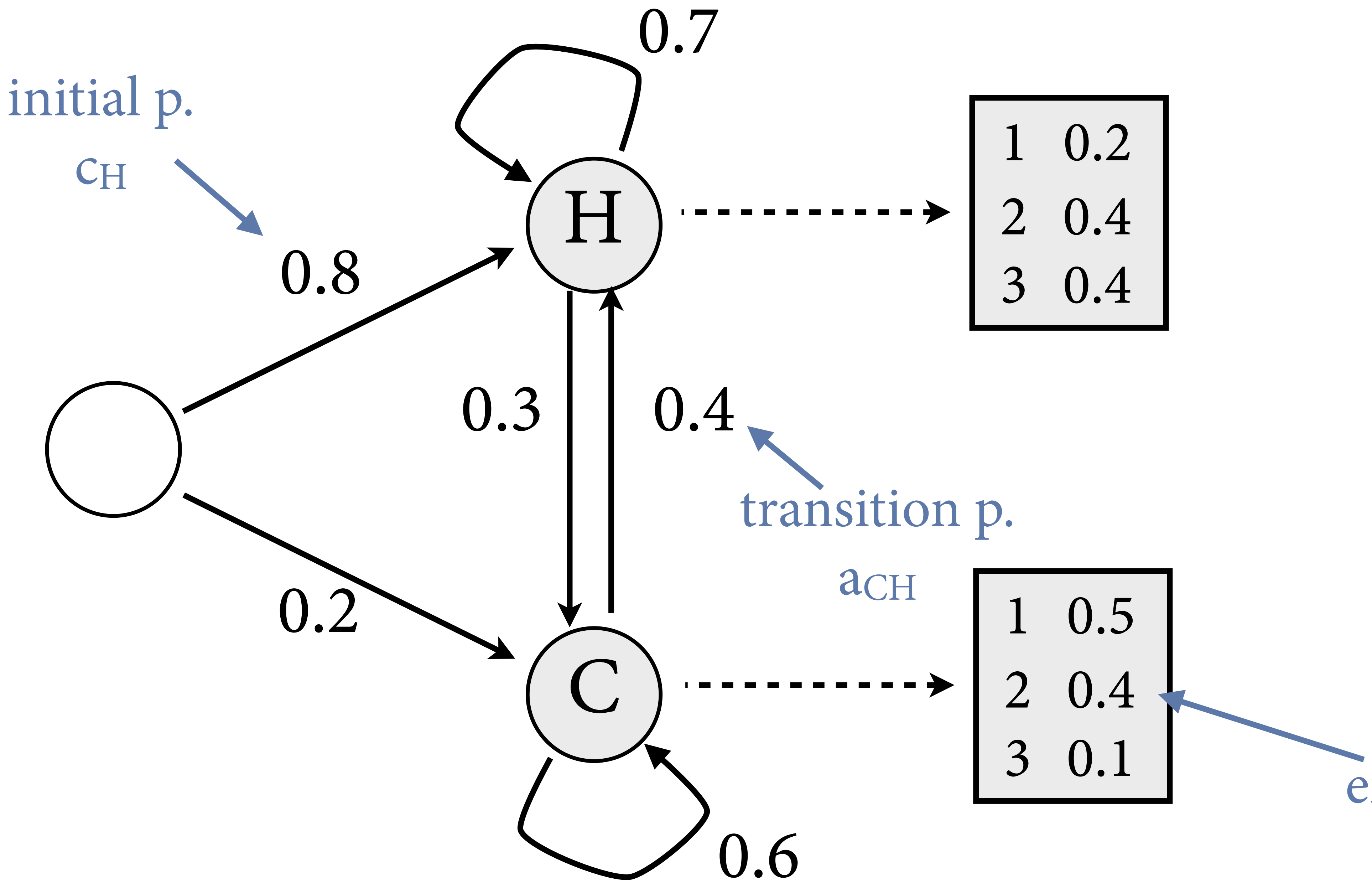
2. Supervised training: Maximum likelihood estimation.

3. Unsupervised training: The Expectation Maximization algorithm.

4. EM for HMMs: the forward-backward algorithm.

5. EM example and remarks.

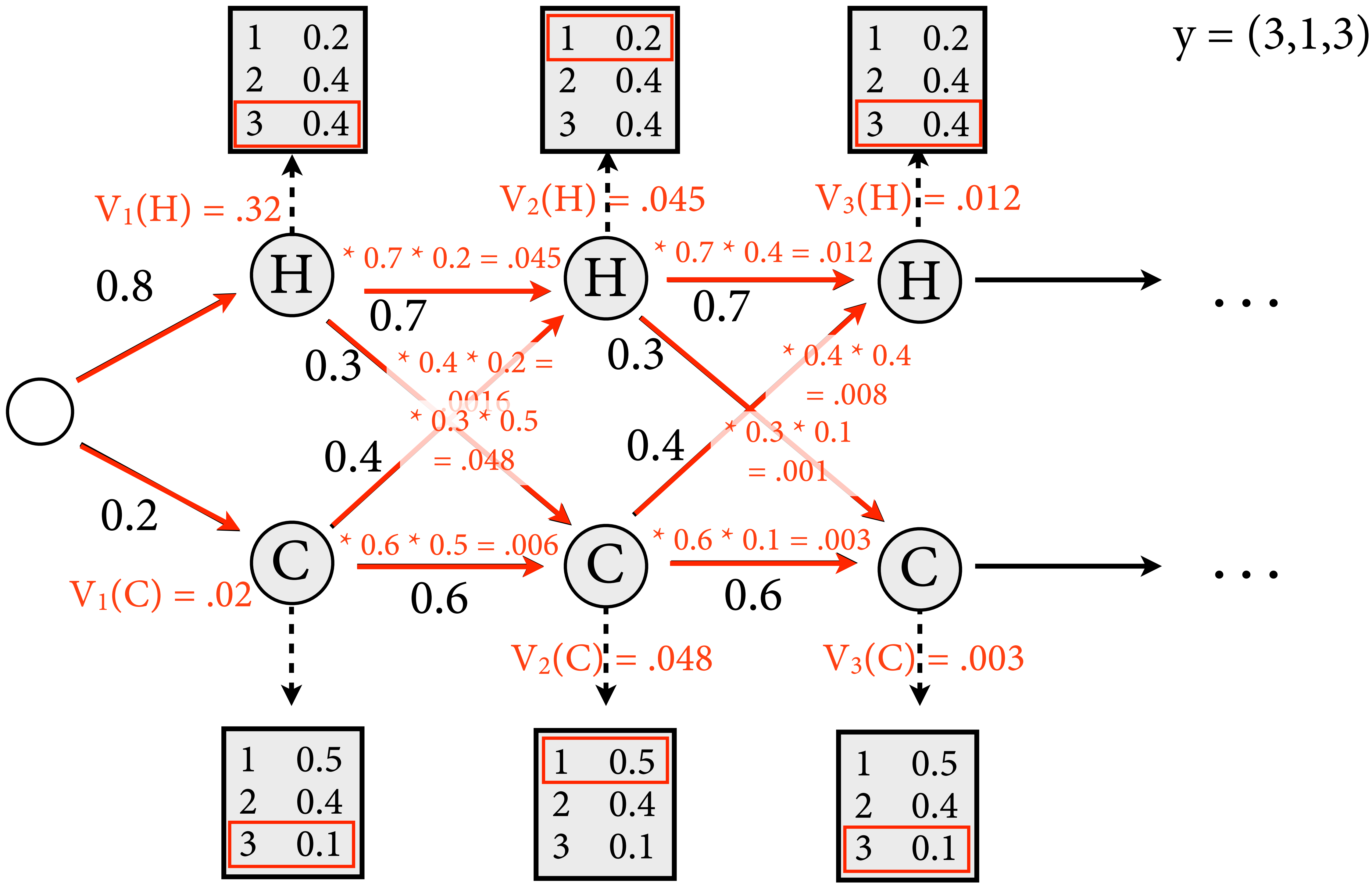
Example: Eisner's Ice Cream



States represent weather on a given day: Hot, Cold

Outputs represent number of ice creams Jason eats that day

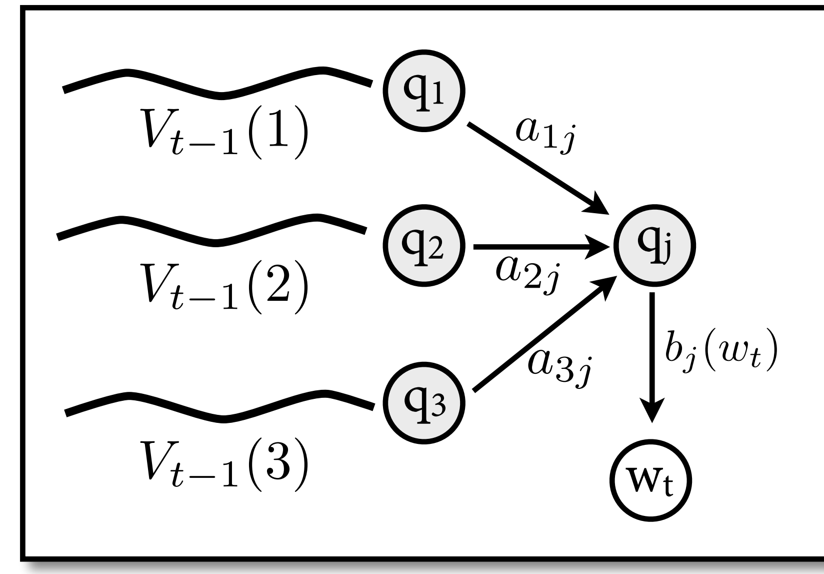
Best partial runs



$$V_t(j) = \max_{x_1,\ldots,x_{t-1}} P(y_1,\ldots,y_t,x_1,\ldots,x_{t-1},X_t=q_j)$$

The Viterbi Algorithm

$$V_t(j) = \max_{x_1,\ldots,x_{t-1}} P(y_1,\ldots,y_t,x_1,\ldots,x_{t-1},X_t = q_j)$$



- Base case, t = 1:

$$V_1(j) = P(y_1, X_1 = q_j) = b_j(y_1) \cdot c_j$$

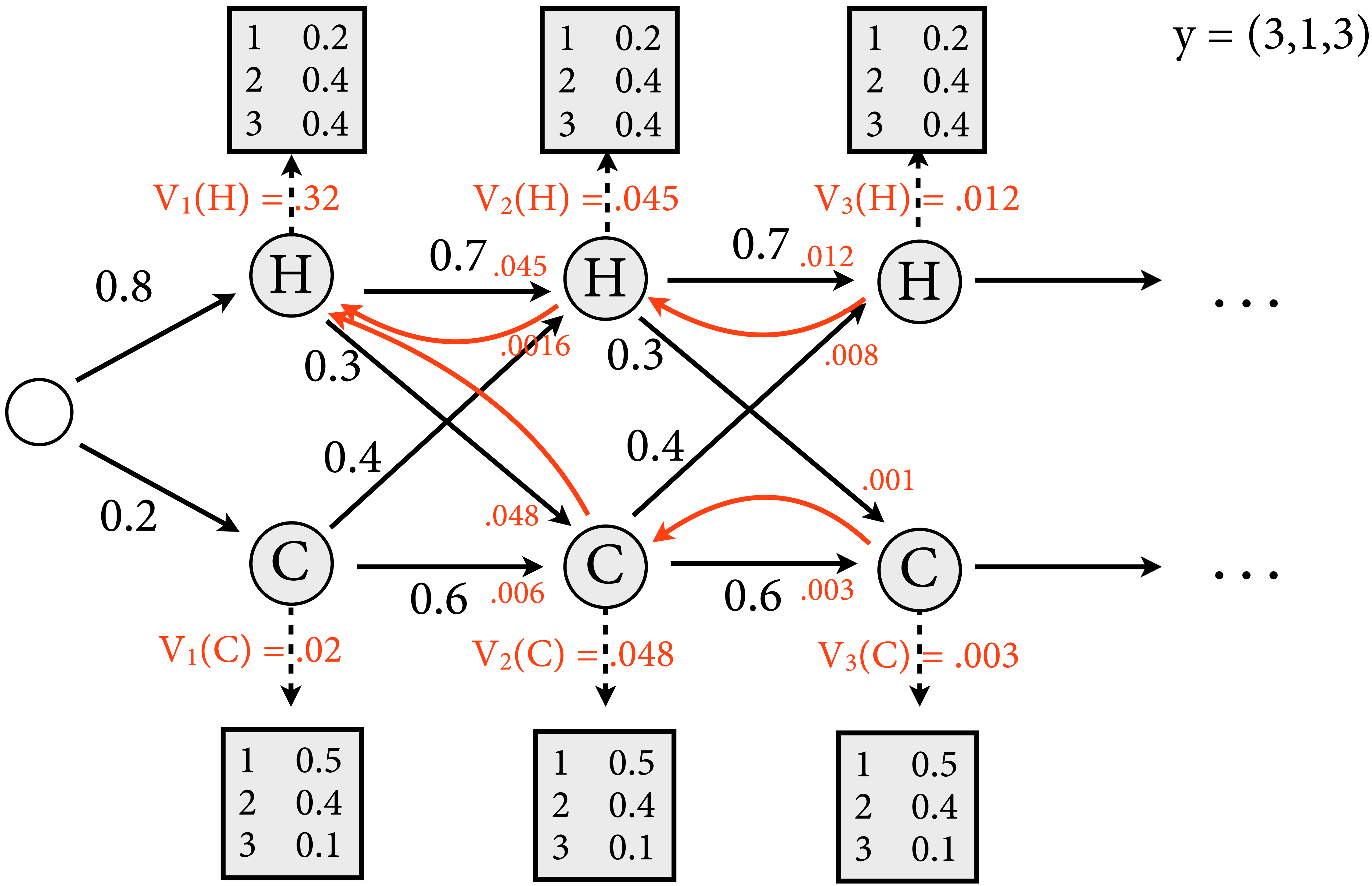
- Inductive case, for t = 2, …, T:

$$V_t(j) = \max_{i=1}^{N} V_{t-1}(i) \cdot a_{ij} \cdot b_j(y_t)$$

- Once we have calculated all V values, we can easily calculate prob of best path:

$$\max_{x_1,\ldots,x_T} P(x_1,y_1,\ldots,x_T,y_T) = \max_{q\in Q} V_T(q)$$

Viterbi Algorithm: Example



$$V_t(j) = \max_{x_1,\ldots,x_{t-1}} P(y_1,\ldots,y_t,x_1,\ldots,x_{t-1},X_t=q_j)$$

$$V_t(j) = \max_{i=1}^{N} V_{t-1}(i)\cdot a_{ij}\cdot b_j(y_t)$$

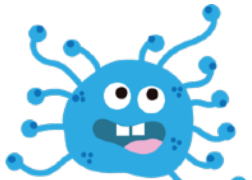
Backpointers

- In the end, we need to reconstruct the sequence of states $x_1, \ldots, x_T$ with max probability.

- For each t, j: remember the value of i for which the maximum was achieved in *backpointer* $bp_t(j)$.

$$V_t(j) = \max_{i=1}^{N} V_{t-1}(i) \cdot a_{ij} \cdot b_j(y_t)$$

- Then just follow backpointers from right to left.



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Question 2: Language modeling

- Given an HMM and a string $y_1, \ldots, y_T$, what is the likelihood $P(y_1 \ldots y_T)$?

- We can compute $P(y_1 \ldots y_T)$ efficiently with the *forward algorithm.*

DT	NN	VBD	NNS	IN	DT	NN	
The	representative	put	chairs	on	the	table.	p_1
DT	JJ	NN	VBZ	IN	DT	NN	
The	representative	put	chairs	on	the	table.	p_2

$P(\text{sentence}) = p_1 + p_2$

Question 2: Likelihood, P(y)

- How likely is it that Jason Eisner ate 3 ice creams on day 1, 1 ice cream on day 2, 3 ice creams on day 3?

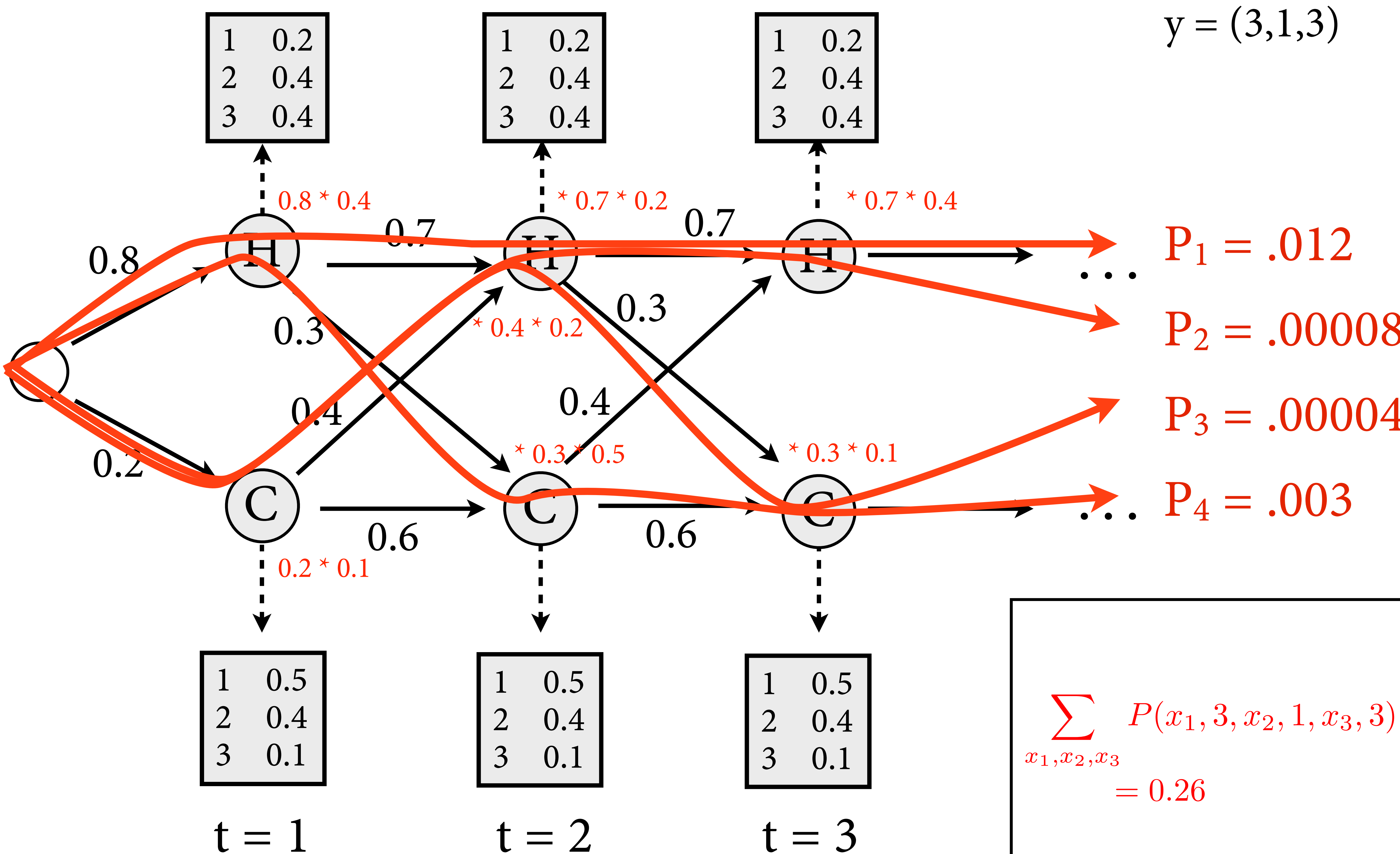
- Want to compute: P(y = 3, 1, 3).

- Same problem as with max:
  - Output 3, 1, 3 can be emitted by many different state sequences.
  - Obtain by marginalization:

$$P(3,1,3) = \sum_{x_1,x_2,x_3 \in Q} P(x_1,3,x_2,1,x_3,3)$$

  - Naive computation is far too slow.

Ice cream trellis



$$\sum_{x_1,x_2,x_3} P(x_1,3,x_2,1,x_3,3)$$
$$= 0.26$$

The Forward Algorithm

- Key idea: *Forward probability* $\alpha_t(j)$ that HMM outputs $y_1, \ldots, y_t$ and then ends in $X_t = q_j$.

$$\begin{aligned} \alpha_t(j) &= P(y_1, \ldots, y_t, X_t = q_j) \\ &= \sum_{x_1, \ldots, x_{t-1}} P(y_1, \ldots, y_t, X_1 = x_1, \ldots, X_{t-1} = x_{t-1}, X_t = q_j) \end{aligned}$$

- From this, can compute easily

$$P(y_1, \ldots, y_T) = \sum_{q \in Q} \alpha_T(q)$$

The Forward Algorithm

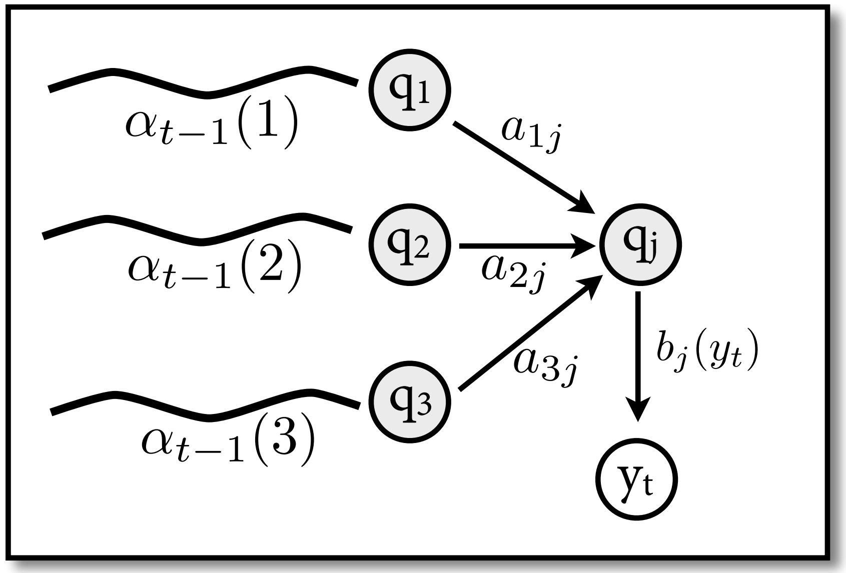
$$\alpha_t(j) = P(y_1, \ldots, y_t, X_t = q_j)$$

- Base case, t = 1:

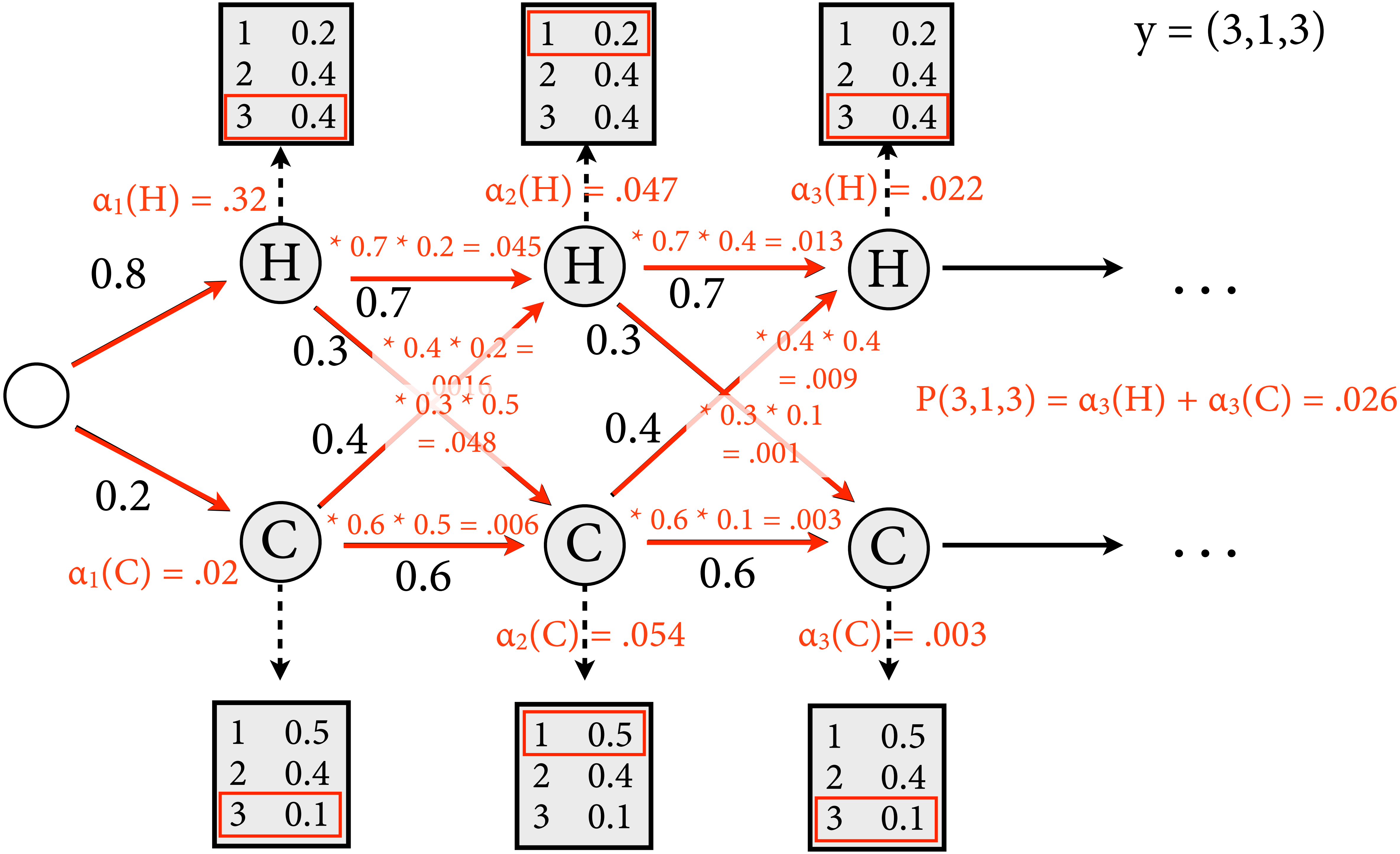
$$\alpha_1(j) = P(y_1, X_1 = q_j) = b_j(y_1) \cdot c_j$$

- Inductive case, compute for t = 2, …, T:

$$\begin{aligned}
\alpha_t(j) &= P(y_1, \ldots, y_t, X_t = q_j) \\
&= \sum_{i=1}^{N} P(y_1, \ldots, y_{t-1}, X_{t-1} = q_i) \cdot P(X_t = q_j \mid X_{t-1} = q_i) \cdot P(y_t \mid X_t = q_j) \\
&= \sum_{i=1}^{N} \alpha_{t-1}(i) \cdot a_{ij} \cdot b_j(y_t)
\end{aligned}$$



P(3,1,3) with Forward



$$\alpha_t(j) = P(y_1, \ldots, y_t, X_t = q_j)$$

$$\alpha_1(j) = b_j(y_1) \cdot a_{0j}$$

$$\alpha_t(j) = \sum_{i=1}^{N} \alpha_{t-1}(i) \cdot a_{ij} \cdot b_j(y_t)$$

Runtime

- Forward and Viterbi have the same runtime, dominated by inductive step:

$$V_t(j) = \max_{i=1}^{N} V_{t-1}(i) \cdot a_{ij} \cdot b_j(y_t)$$

- Compute $N \cdot T$ values for $V_t(j)$. Each computation step requires iteration over N predecessor states.

- Total runtime is $O(N^2 \cdot T)$, i.e.
  - linear in sentence length
  - quadratic in size of tag set

Question 3a: Supervised learning

- Given a set of POS tags and *annotated* training data $(x_1,y_1), \ldots, (x_T,y_T)$, compute parameters for HMM that maximize likelihood of training data.

DT	NN	VBD	NNS	IN	DT	NN
The	representative	put	chairs	on	the	table.

NNP	VBZ	VBN	TO	VB	NR
Secretariat	is	expected	to	race	tomorrow.

Maximum likelihood training

- Estimate bigram model for state sequence:

$$a_{ij} = \frac{C(X_t = q_i, X_{t+1} = q_j)}{C(X_t = q_i)} \qquad c_j = \frac{\#\text{ sentences with } X_1 = q_j}{\#\text{ sentences}}$$

- ML estimate for emission probabilities:

$$b_i(o) = \frac{C(X_t = q_i, Y_t = o)}{C(X_t = q_i)}$$

- Can apply smoothing as for ordinary n-gram models (e.g. increase all counts C by one).

Maximum likelihood estimation

Observed emissions y	3	1	3
Observed states x	H	H	C

C(H → H)	C(H → C)	C(H → 3)	C(H → 1)
1	1	1	1

$$a_{HH} = \frac{C(X_t = H, X_{t+1} = H)}{C(X_t = H)} = 0.5$$

$$b_H(3) = \frac{C(X_t = H, Y_t = 3)}{C(X_t = H)} = 0.5$$

$$a_{HC} = \frac{C(X_t = H, X_{t+1} = C)}{C(X_t = H)} = 0.5$$

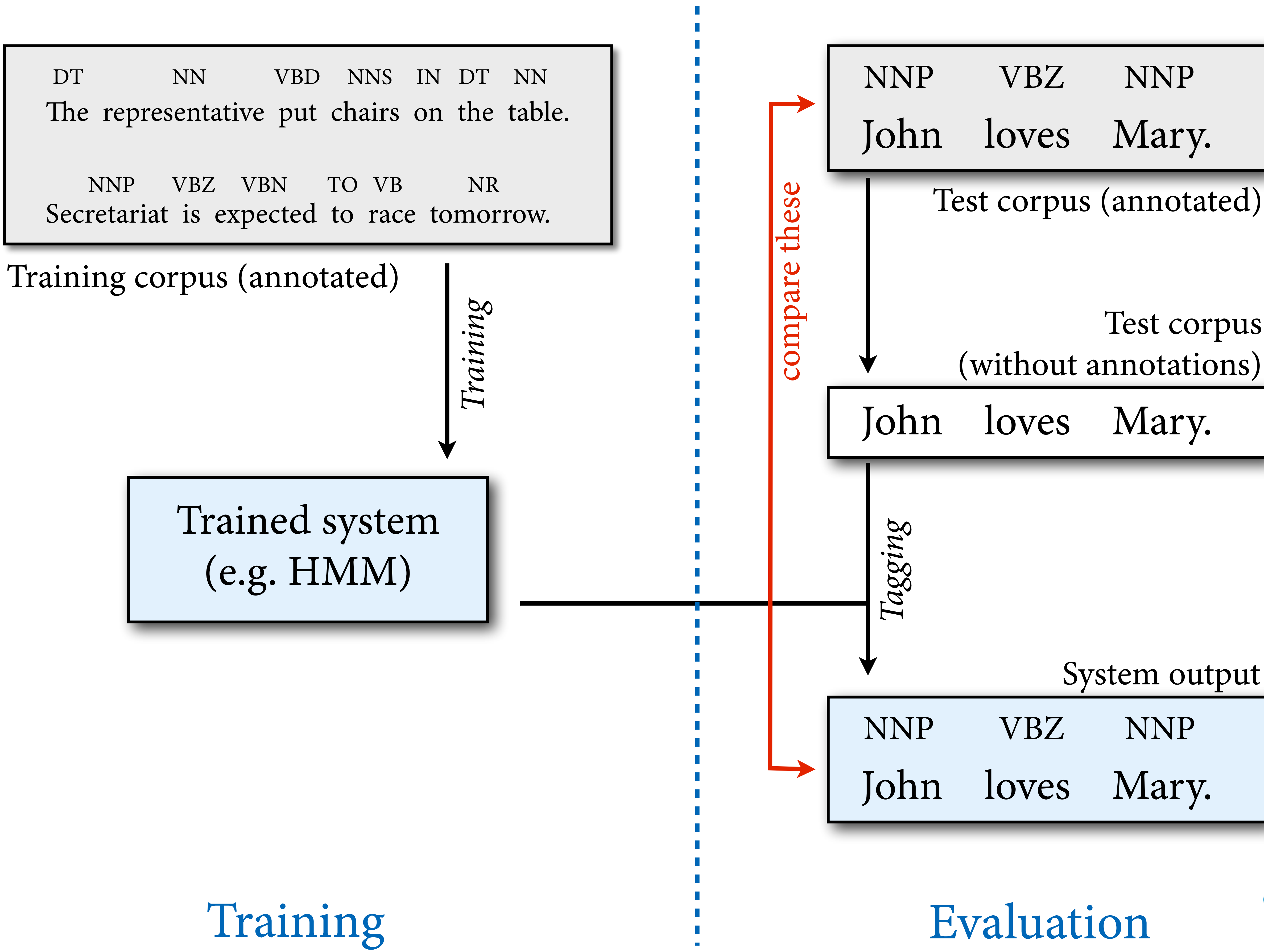
$$b_H(1) = \frac{C(X_t = H, Y_t = 1)}{C(X_t = H)} = 0.5$$

Evaluation

- How do you know how well your tagger works?

- Run it on *test data* and evaluate *accuracy*.
    - Test data: Really important to evaluate on unseen sentences to get a fair picture of how well tagger generalizes.
    - Accuracy: Measure percentage of correctly predicted POS tags.

Evaluation on test data



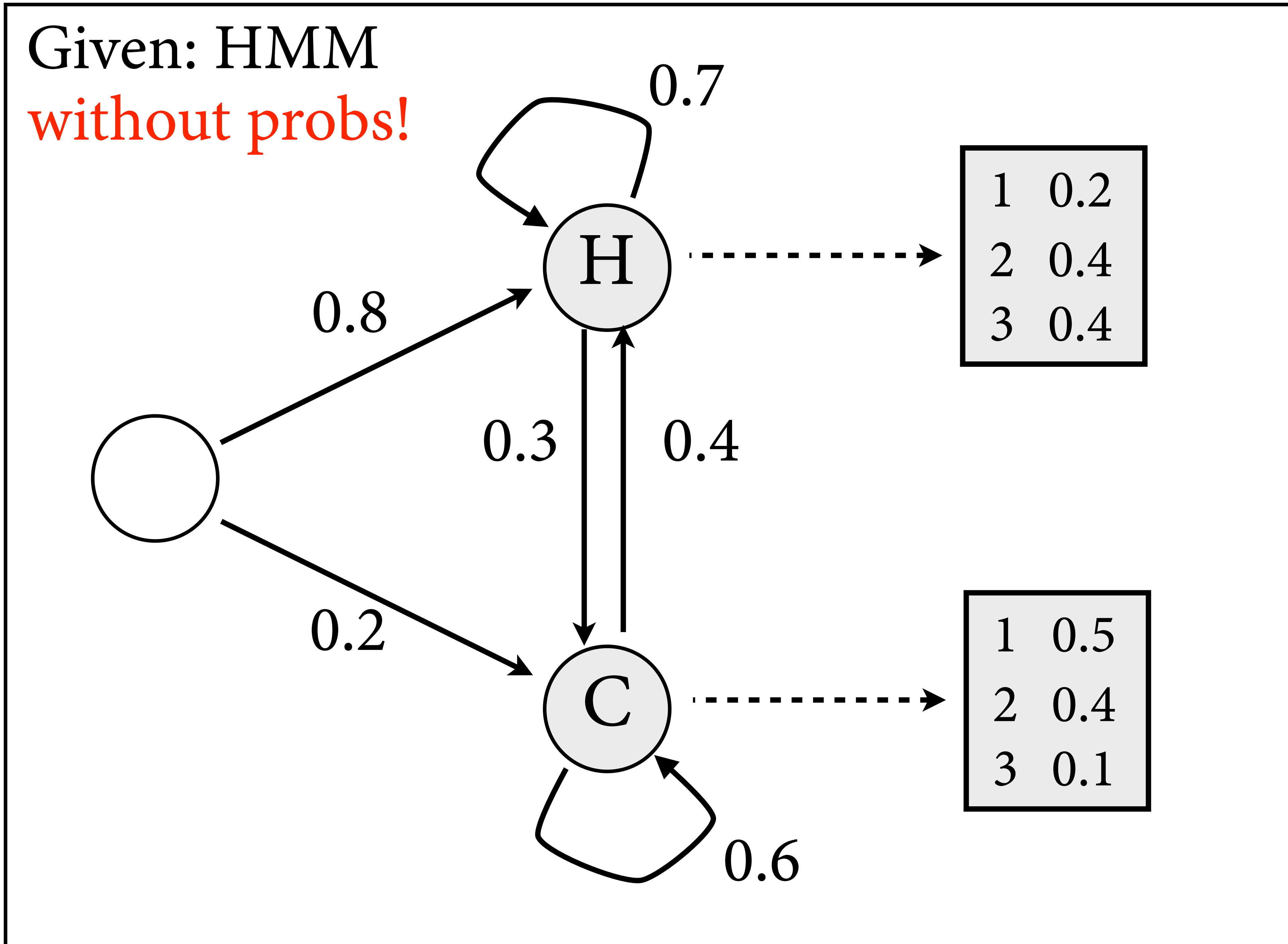
Question 3b: Unsupervised learning

- Given a set of POS tags and *unannotated* training data $y_1, \ldots, y_T$, compute parameters for HMM that maximize likelihood of training data.

- Relevant because annotated data is expensive to obtain, but raw text is really cheap.

The representative put chairs on the table.
Secretariat is expected to race today.

The setup



Observations: 2, 3, 3, 2, 3, 2, 3, 2, 2, 3, 1, 3, 3, …

The setup

- If we had counts of state transitions in corpus, we could simply use ML estimation.

$$a_{ij} = \frac{C(q_i \to q_j)}{C(q_i \to \bullet)}$$

- Idea: replace actual counts by *estimated* counts.

$$a_{ij} \approx \frac{\hat{C}(q_i \to q_j)}{\hat{C}(q_i \to \bullet)}$$

- How can we estimate counts?

Estimated counts

Observed emissions y	3	1	3	$C(H \rightarrow H)$
***Observed* states x**	H	H	H	**2**

Observed emissions y	3	1	3	$C(H \rightarrow H)$		$P(x \mid y)$
***Hidden* states x**	H	H	H	2	*	1
	H	H	C	+ 1	*	0
	H	C	H	+ 0	*	0
		…				
	C	C	C	+ 0	*	0
				=		**2**

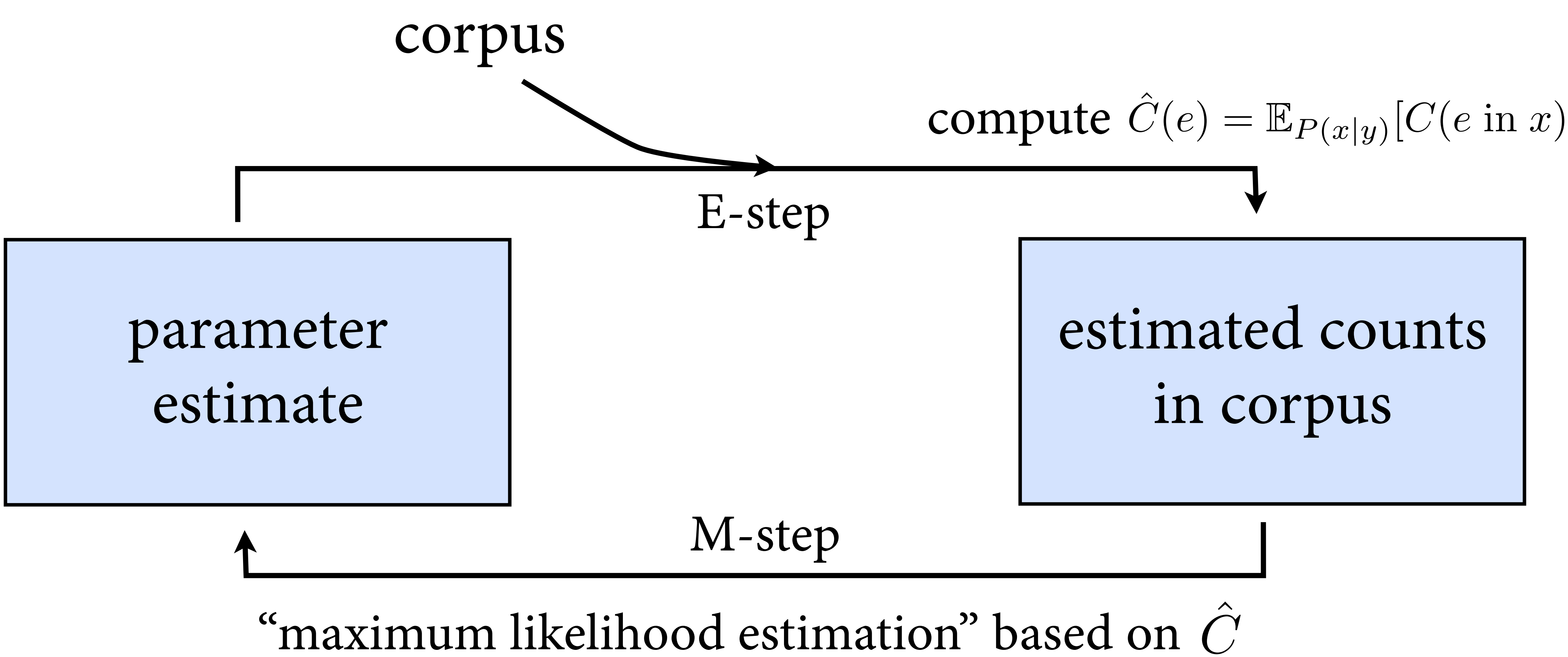
Estimated counts

Observed emissions y	3	1	3	$C(H \rightarrow H)$
***Observed* states x**	H	H	H	**2**

Observed emissions y	3	1	3	$C(H \rightarrow H)$		$P(x \mid y)$
***Hidden* states x**	H	H	H	2	*	0.408
	H	H	C	+ 1	*	0.034
	H	C	H	+ 0	*	0.272
		…				
	C	C	C	+ 0	*	0.136
				=		**0.864**

$$\hat{C}(q_i \rightarrow q_j) = \mathbb{E}_{P(x|y)}[C(q_i \rightarrow q_j \text{ in } x)] = \sum_x P(x \mid y) \cdot C(q_i \rightarrow q_j \text{ in } x)$$

Expectation Maximization



e: some event of interest, e.g. transition H → H

Counting by transition

Observed y	3	1	3	P(x | y)	$X_1 = H, X_2 = H$	$X_2 = H, X_3 = H$	$C(H \to H)$
Hidden x	H	H	H	0.408	1	1	2
	H	H	C	0.034	1	0	1
	C	H	H	0.014	0	1	1
		…					
	C	C	C	0.136	0	0	0

$P(X_1 = H, X_2 = H \mid y) = 0.442$

$P(X_2 = H, X_3 = H \mid y) = 0.422$

0.864

$E[C(H \to H)]$
$=$ **0.864**

$$\hat{C}(q_i \to q_j) = \mathbb{E}_{P(x|y)}[C(q_i \to q_j \text{ in } x)] = \sum_x P(x \mid y) \cdot C(q_i \to q_j \text{ in } x)$$
$$= \sum_{t=1}^{T-1} P(X_t = q_i, X_{t+1} = q_j \mid y)$$

Plan for computing E

$$\boxed{\mathbb{E}_{P(x|y)}[C(q_i \to q_j \text{ in } x)] = \sum_{t=1}^{T-1} P(X_t = q_i, X_{t+1} = q_j \mid y)}$$

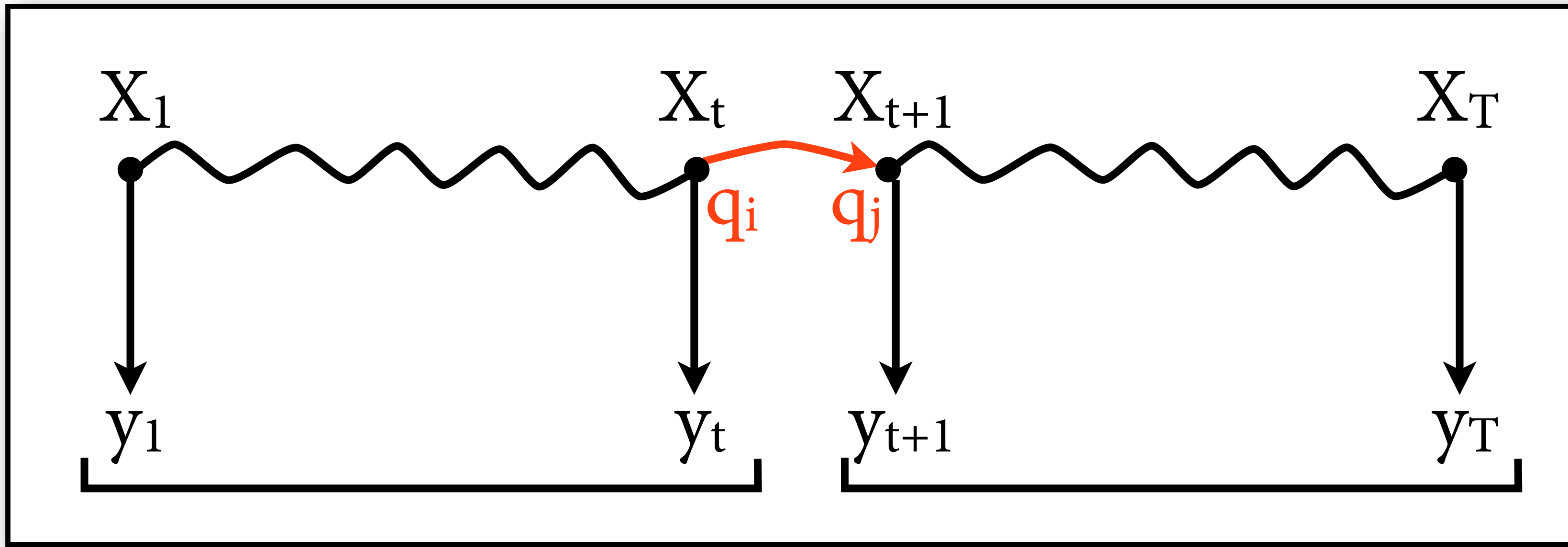
- How can we compute this efficiently?
  Challenge: Prob is conditioned on y!

- We compute $\xi_t(i,j) = P(X_t = q_i, X_{t+1} = q_j \mid y)$

$$= \frac{P(X_t = q_i, X_{t+1} = q_j, y)}{P(y)}$$

- Do it in two steps:
  - compute $\xi'_t(i,j) = P(X_t = q_i, X_{t+1} = q_j, y)$
  - compute P(y), using forward algorithm

$$\xi_t'(i,j) = P(X_t = q_i, X_{t+1} = q_j, y)$$



$$P(X_t = q_i, X_{t+1} = q_j, y)$$

$$= P(y_1, \ldots, y_t, X_t = q_i) \cdot P(y_{t+1}, X_{t+1} = q_j \mid y_1, \ldots, y_t, X_t = q_i)$$
$$\cdot P(y_{t+2}, \ldots, y_T \mid y_1, \ldots, y_{t+1}, X_t = q_i, X_{t+1} = q_j)$$

$$= P(y_1, \ldots, y_t, X_t = q_i) \cdot P(y_{t+1}, X_{t+1} = q_j \mid X_t = q_i) \cdot P(y_{t+2}, \ldots, y_T \mid X_{t+1} = q_j)$$

$$= \alpha_t(i) \cdot a_{ij} \cdot b_j(y_{t+1}) \cdot \beta_{t+1}(j)$$

forward prob:

$$\alpha_t(i) = P(y_1, \ldots, y_t, X_t = q_i)$$

backward prob:

$$\beta_t(i) = P(y_{t+1}, \ldots, y_t \mid X_t = q_i)$$

Backward probabilities

$$\beta_t(i) = P(y_{t+1}, \ldots, y_t \mid X_t = q_i)$$

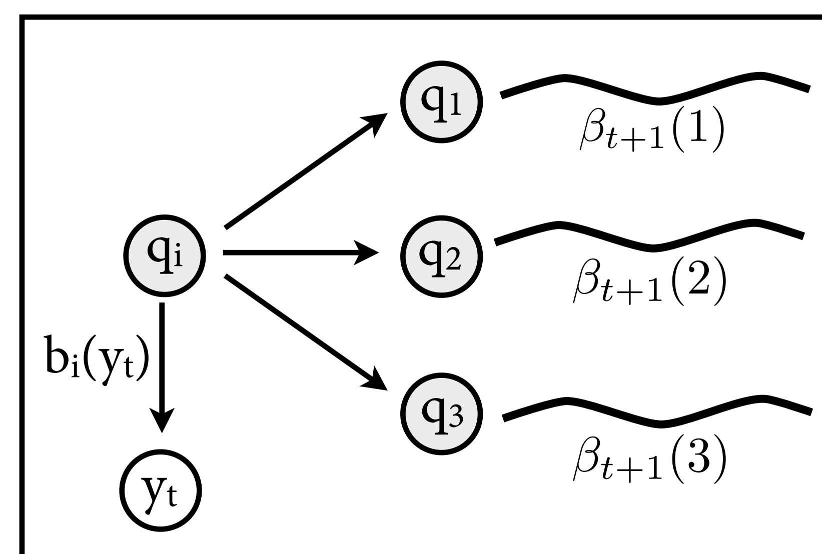
- Base case, t = T:

  $\beta_T(i) = 1$ for all i *

- Inductive case, compute for t = T-1, …, 1:

$$\beta_t(i) = \sum_{j=1}^{N} a_{ij} \cdot b_j(y_{t+1}) \cdot \beta_{t+1}(j)$$

- Exact mirror image of forward.



*) this is different in J&M because of q_F

Putting it all together

- Compute estimated transition counts for all i, j, t:

$$\xi_t(i,j) = \frac{\xi'_t(i,j)}{\hat{P}(y)} = \frac{\alpha_t(i) \cdot a_{ij} \cdot b_j(y_{t+1}) \cdot \beta_{t+1}(j)}{\sum_q \alpha_T(q)}$$

- Compute overall estimated transition counts:

$$\mathbb{E}_{P(x|y)}[C(q_i \to q_j \text{ in } x)] = \sum_{t=1}^{T-1} \xi_t(i,j)$$

- Revise estimate of transition probabilities:

$$a_{ij} := \frac{\mathbb{E}_{P(x|y)}[C(q_i \to q_j \text{ in } x)]}{\mathbb{E}_{P(x|y)}[C(q_i \to \bullet \text{ in } x)]}$$

The other parameters

- Revise initial and emission probabilities using estimated counts, in completely analogous way.

- Here's what it looks like for emission prob:

$$\gamma_t(j) = P(X_t = q_j \mid y) = \frac{\hat{P}(X_t = q_j, y)}{\hat{P}(y)} = \frac{\alpha_t(j) \cdot \beta_t(j)}{\hat{P}(y)}$$

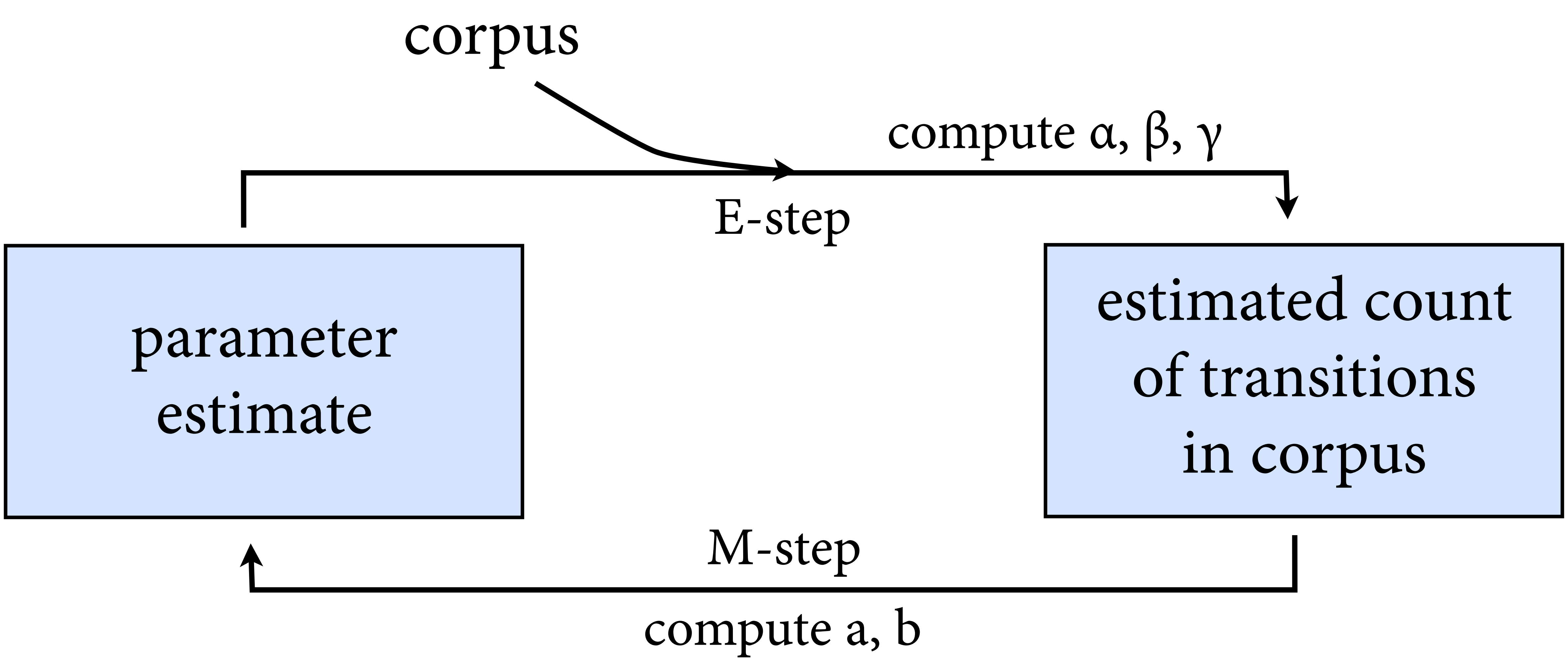
$$b_j(o) \approx \left(\sum_{\substack{t=1 \\ y_t = o}}^{T} \gamma_t(j) \right) \Big/ \sum_{t=1}^{T} \gamma_t(j)$$

estimated count of o emitted in state q_j

estimated count of state q_j

Forward-Backward Algorithm

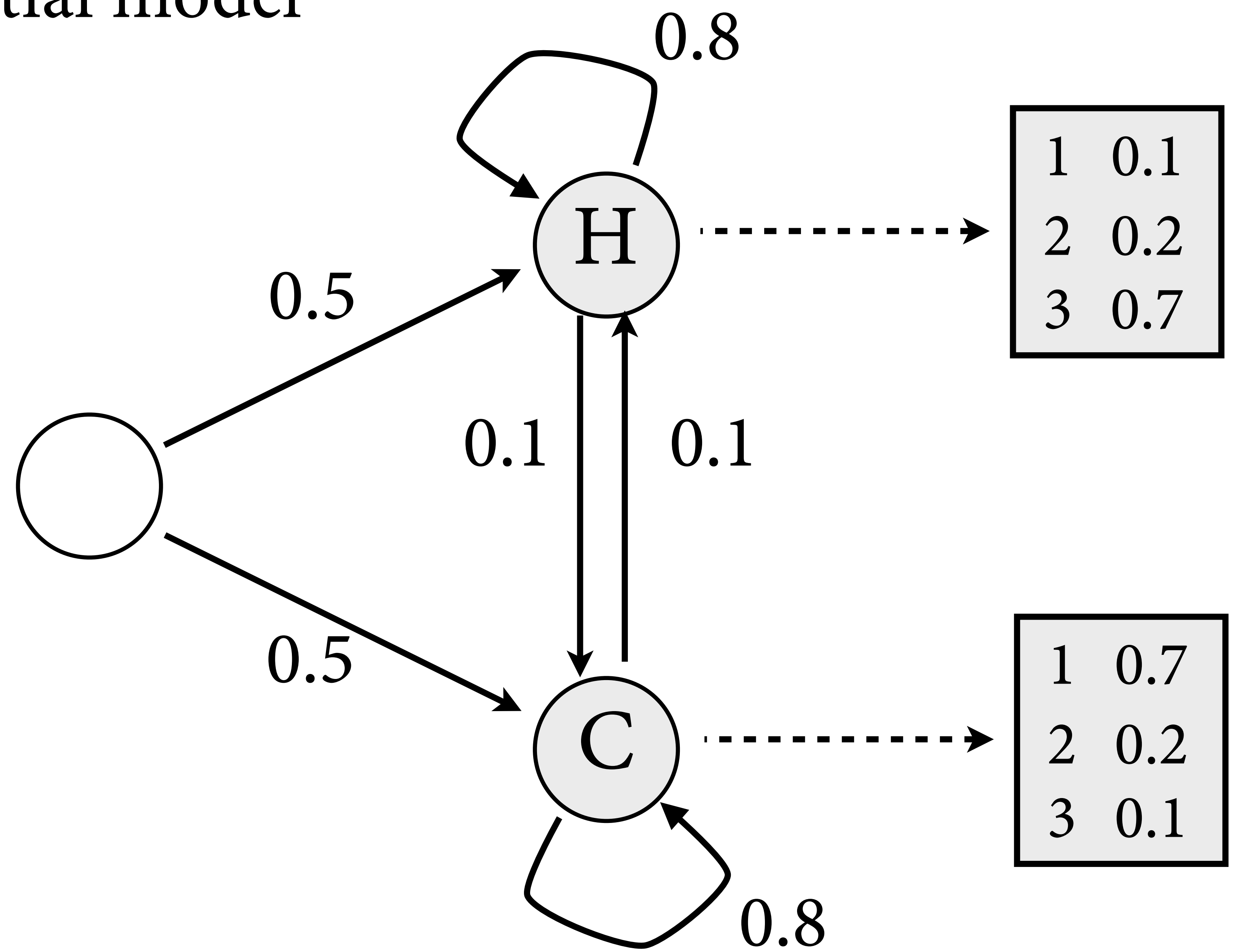
Initialization: start with some estimation of parameters.



Continue computation until parameters don’t change much.

Example

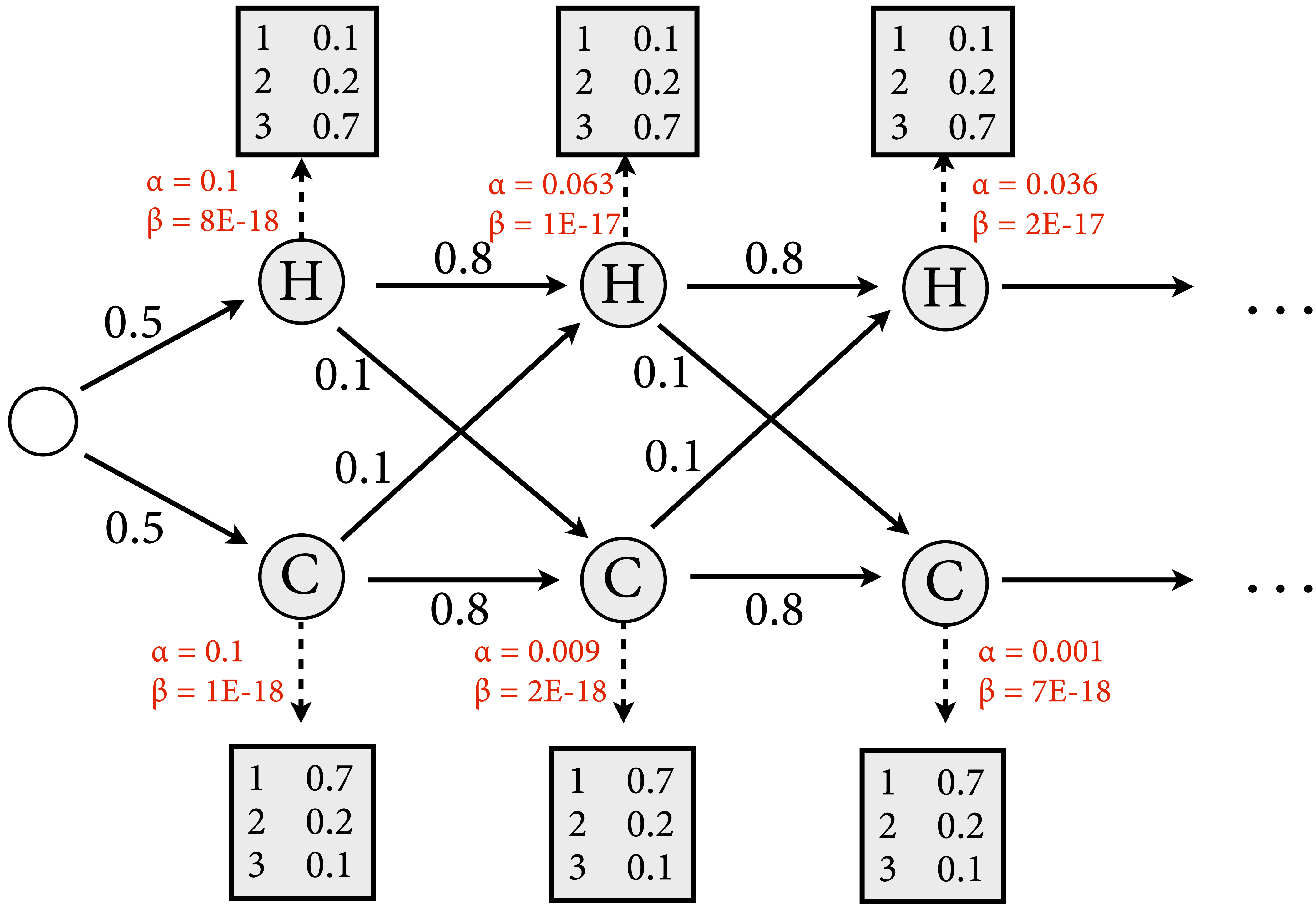
Step 1: initial model



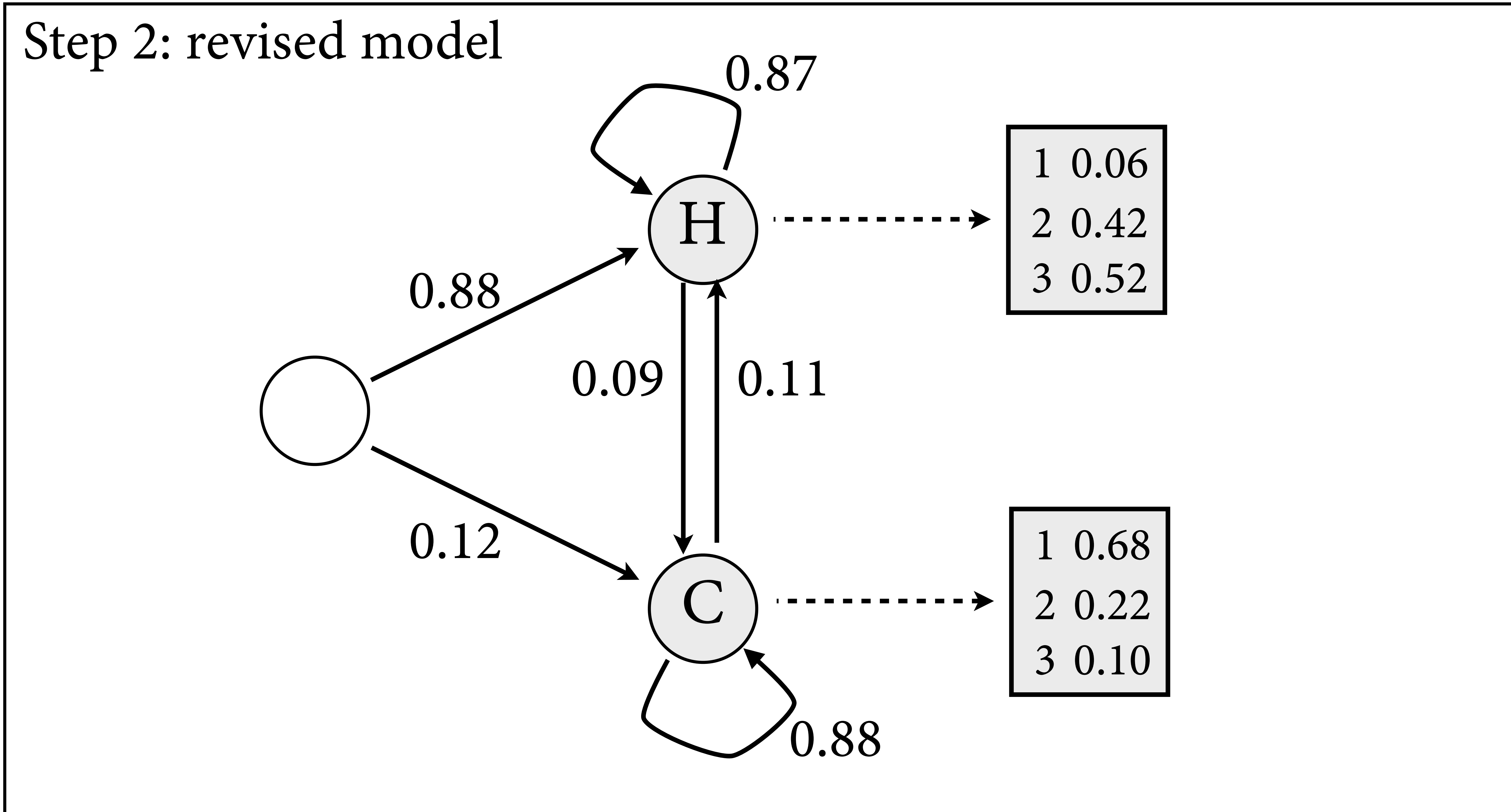
Probs do not sum to one because Eisner's HMMs have q_F.

2, 3, 3, 2, 3, 2, 3, 2, 2, 3, 1, 3, 3, 1, 1, 1, 2, 1, 1, 1, 3, 1, 2, 1, 1, 1, 2, 3, 3, 2, 3, 2, 2

E-Step

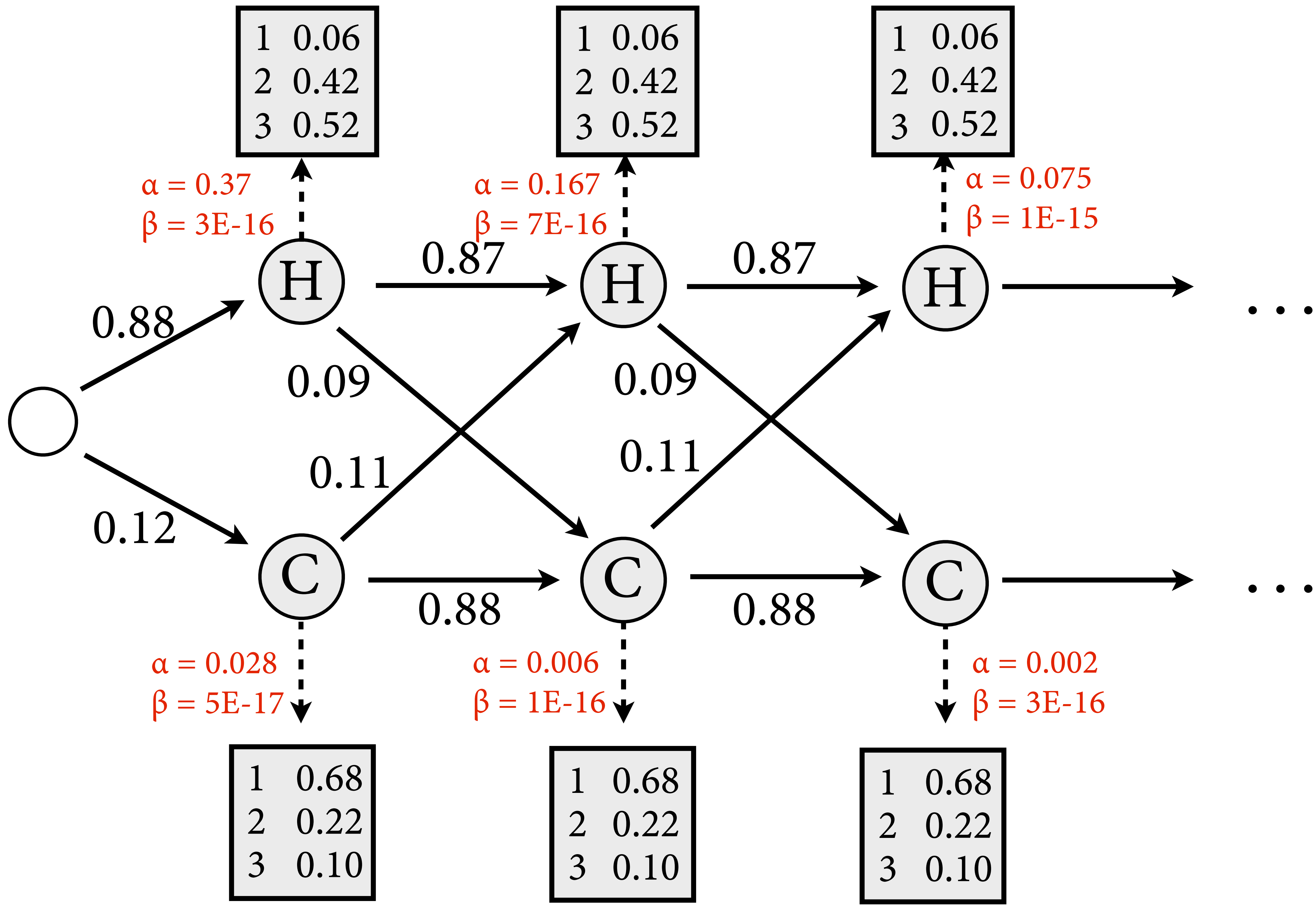

1 0.1
2 0.2
3 0.7
α = 0.1
β = 8E-18
α = 0.063
β = 1E-17
α = 0.036
β = 2E-17
H
0.8
0.8
0.5
0.1
0.1
0.1
0.1
0.5
C
0.8
0.8
α = 0.1
β = 1E-18
α = 0.009
β = 2E-18
α = 0.001
β = 7E-18
1 0.7
2 0.2
3 0.1
. . .

M-Step



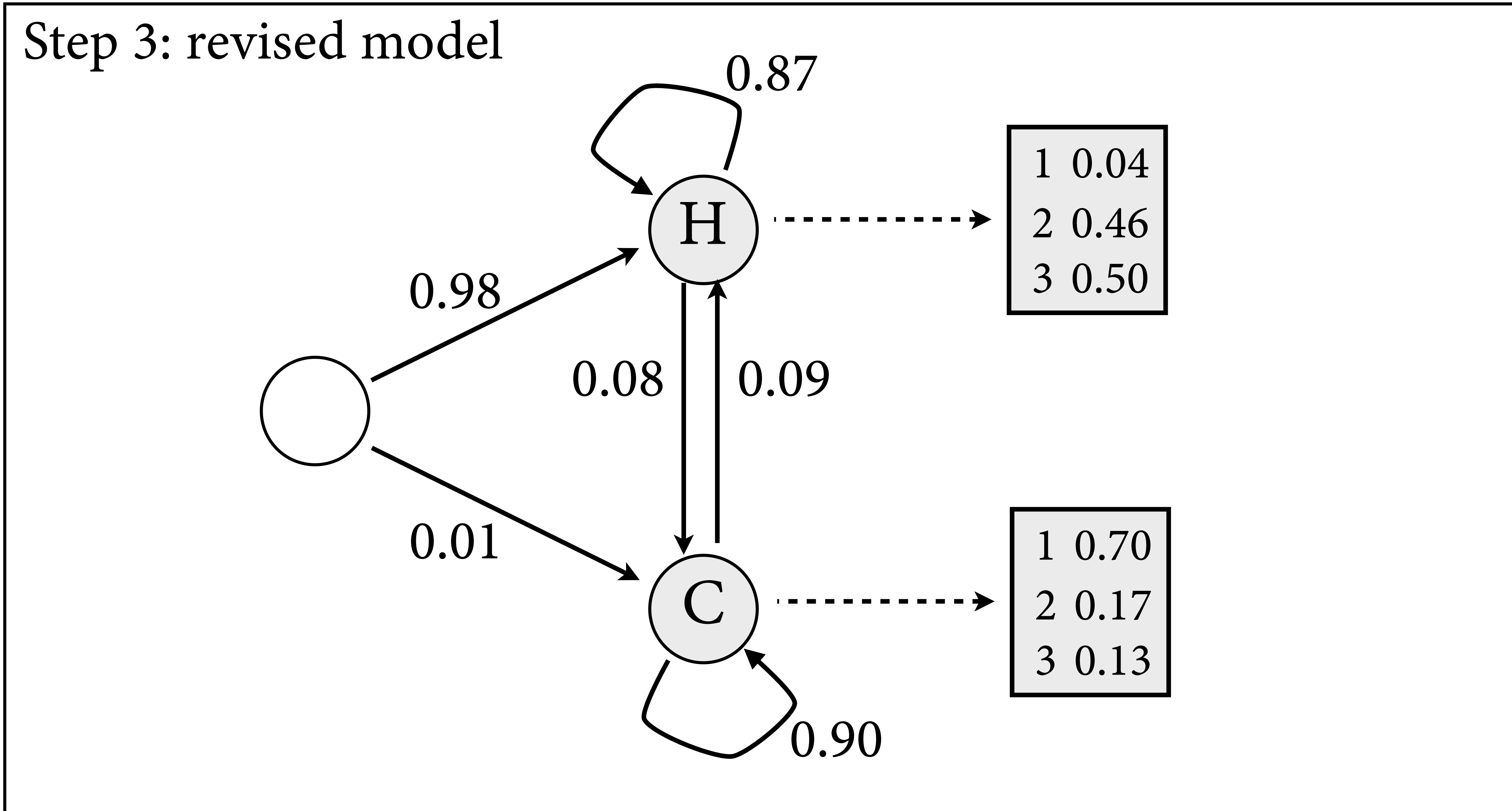
2, 3, 3, 2, 3, 2, 3, 2, 2, 3, 1, 3, 3, 1, 1, 1, 2, 1, 1, 1, 3, 1, 2, 1, 1, 1, 2, 3, 3, 2, 3, 2, 2

E-Step



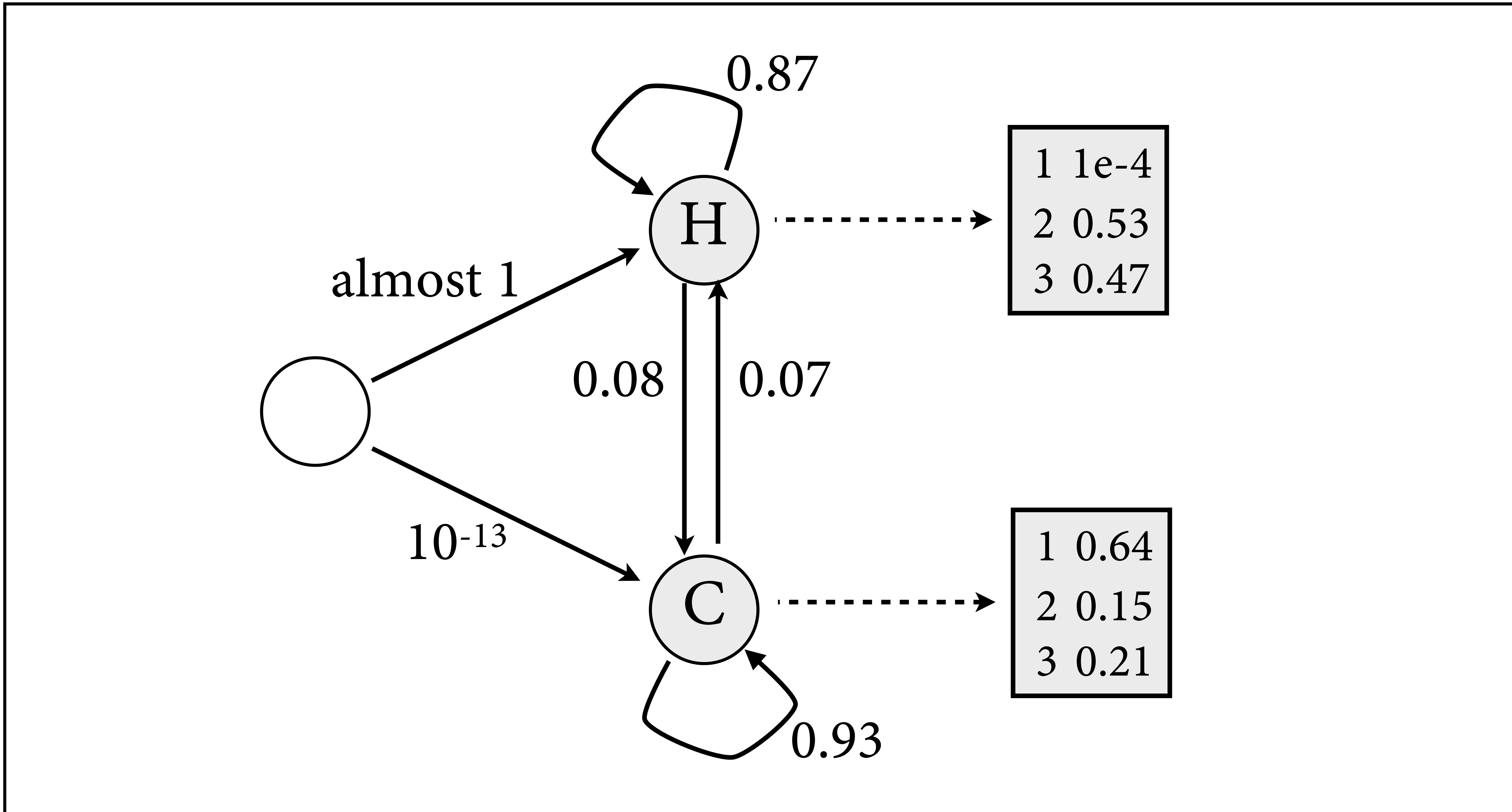
1 0.06
2 0.42
3 0.52
α = 0.37
β = 3E-16
α = 0.167
β = 7E-16
α = 0.075
β = 1E-15
H
0.87
0.87
0.88
0.09
0.09
0.11
0.11
0.12
C
0.88
0.88
α = 0.028
β = 5E-17
α = 0.006
β = 1E-16
α = 0.002
β = 3E-16
1 0.68
2 0.22
3 0.10

M-Step



2, 3, 3, 2, 3, 2, 3, 2, 2, 3, 1, 3, 3, 1, 1, 1, 2, 1, 1, 1, 3, 1, 2, 1, 1, 1, 2, 3, 3, 2, 3, 2, 2

Result after 10 iterations



2, 3, 3, 2, 3, 2, 3, 2, 2, 3, 1, 3, 3, 1, 1, 1, 2, 1, 1, 1, 3, 1, 2, 1, 1, 1, 2, 3, 3, 2, 3, 2, 2

Some remarks

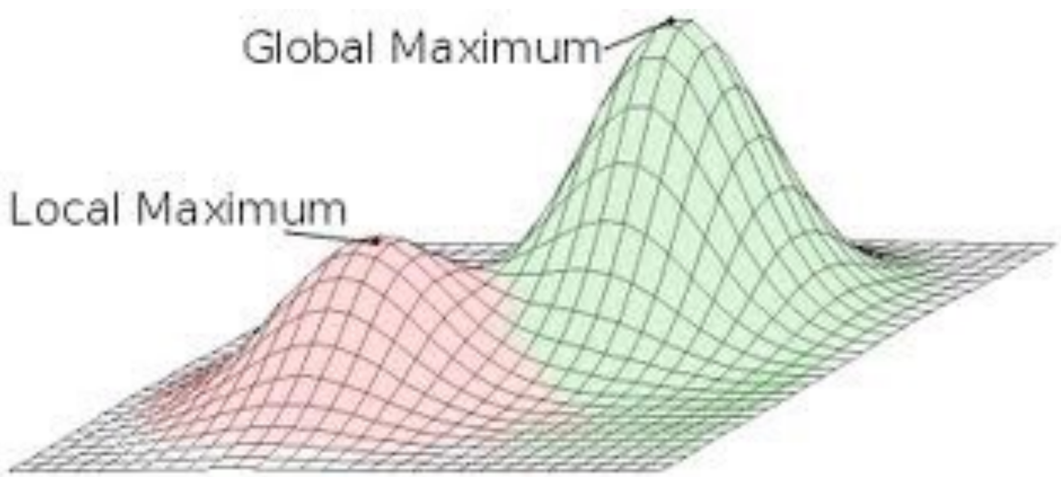
- Forward-backward algorithm also called *Baum-Welch Algorithm* after inventors.

- Special case of the *expectation maximization* algorithm:
  - E-Step: Compute expected values of relevant counts based on current parameter estimate.
  - M-Step: Adjust model based on estimated counts.

- Runtime of each iteration is $O(N^2 T)$.
  Most of the time goes into E-step.

Some remarks

- EM algorithm is guaranteed to improve likelihood of corpus in each iteration.
- However, can run into *local maxima:* would have to go through worse model to find globally best one.
- Extremely sensitive to initial parameter estimate. Only useful in practice if HMM structure very strongly constrained (e.g. speech recognition).



Conclusion

- Evaluate tagger on *accuracy* on *unseen* data.
- Training algorithms for HMM estimation:
  - *supervised* training from annotated data: maximum likelihood
  - *unsupervised* training from unannotated data: forward-backward