Linear Algebra for Machine Learning in Python

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High Performance Computing and Analytics Lab

Overview

Introcution

Essential operations

Linear curve fitting

System dynamics, and dimensions

Introcution

Motvating linear algebra

TODO

Matrices

m, n is a matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, a_{ij} \in \mathbb{R}. \tag{1}$$

Essential operations

Addition

TODO

Multiplication

TODO

Motivation of the determinant

TODO

Computing determinants in two or three dimensions

The two dimensional case:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21} \tag{2}$$

$$\tag{3}$$

Computing the determinant of a three dimensional matrix.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \tag{4}$$

Computing determinants in two or three dimensions

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Computing the determinant of a three dimensional matrix.

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Draw the sign pattern on the board:

$$\begin{vmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} \tag{5}$$

The determinant can be expandend along any column as long as the sign pattern is respected.

Determinants in n-dimensions

$$\begin{vmatrix} a_{11} & a_{21} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{m2} & \dots & a_{mn} \end{vmatrix} + a_{21} \begin{vmatrix} a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{m2} & \dots & a_{mn} \end{vmatrix}$$

$$-a_{m1} \begin{vmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \end{vmatrix}$$

The Transpose

TODO

The Inverse

TODO

Linear curve fitting

What is the best line connecting measurements?

The Pseudoinverse

$$\mathbf{A}^{\dagger} = (\mathbf{A}^{T}\mathbf{A})^{-1}\mathbf{A}^{T} \tag{6}$$

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$$\mathbf{A}^{\dagger} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T \tag{6}$$

Sometimes solving $\mathbf{Ax} + \mathbf{b} = 0$ is implossible. One the board, derive:

$$\min_x \frac{1}{2}|\mathbf{Ax} - \mathbf{b}|^2 \tag{7}$$

$$\tag{8}$$

At the optimum we expect,

$$\begin{aligned}
0 &= \nabla_x \frac{1}{2}|\mathbf{Ax} - \mathbf{b}|^2 && (9)\\
&= \nabla_x \frac{1}{2}(\mathbf{Ax} - \mathbf{b})^T(\mathbf{Ax} - \mathbf{b}) && (10)\\
&= (\mathbf{Ax} - \mathbf{b})\mathbf{A}^T && (11)\\
&= \mathbf{A}^T\mathbf{Ax} - \mathbf{A}^T\mathbf{b} && (12)\\
\mathbf{A}^T\mathbf{b} &= \mathbf{A}^T\mathbf{Ax} && (13)\\
(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} &= \mathbf{x} && (14)
\end{aligned}$$

System dynamics, and dimensions

Eigenvalues

TODO