

L03: Gaussians II

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1 Conditioning a joint Gaussian PDF

$$p(x_a, x_b) = \alpha \exp \underbrace{\left\{ -\frac{1}{2} \begin{bmatrix} x_a - \mu_a \\ x_b - \mu_b \end{bmatrix}^\top \Sigma^{-1} \begin{bmatrix} x_a - \mu_a \\ x_b - \mu_b \end{bmatrix} \right\}}_{\Delta}$$

Problem: $p(x_a|x_b)$, where $x_a \in \mathcal{R}^n, x_b \in \mathcal{R}^m$

$$\Sigma = \begin{bmatrix} \Sigma_a & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_b \end{bmatrix}, \quad \Sigma_{ab} = \Sigma_{ba}^\top \text{ (}\Sigma \text{ symmetric)}$$

$$\text{Information matrix } \Lambda = \Sigma^{-1}, \quad \Lambda = \begin{bmatrix} \Lambda_a & \Lambda_{ab} \\ \Lambda_{ba} & \Lambda_b \end{bmatrix}$$

Expand the exponent Δ :

$$\begin{aligned} \Delta = & -\frac{1}{2}(x_a - \mu_a)^\top \Lambda_a (x_a - \mu_a) - \frac{1}{2}(x_a - \mu_a)^\top \Lambda_{ab} (x_b - \mu_b) \\ & - \frac{1}{2}(x_b - \mu_b)^\top \Lambda_{ba} (x_a - \mu_a) - \frac{1}{2}(x_b - \mu_b)^\top \Lambda_b (x_b - \mu_b) \end{aligned} \quad (1)$$

Solution: Completing the square

Intuition \rightarrow we want an exponent to only depend on x_a since x_b is conditioned ("given")

$$\Delta = -\frac{1}{2}x_a^\top \Sigma_{a|b}^{-1} x_a + x_a^\top \Sigma_{a|b}^{-1} m - \frac{1}{2}m^\top \Sigma_{a|b}^{-1} m + \text{const} \quad (2)$$

Q: What happens to x_b and constant terms?

2nd order term: $x_a^\top \Sigma_{a|b}^{-1} x_a$,

$$\Sigma_{a|b}^{-1} = \Lambda_a \quad (3)$$

1st order term: $x_a^\top \underbrace{(\Lambda_a \mu_a - \Lambda_{ab}(x_b - \mu_b))}_{\Sigma_{a|b}^{-1} \mu_{a|b}} \Rightarrow$

$$\mu_{a|b} = \Sigma_{a|b} \underbrace{(\underbrace{\Lambda_a}_{\text{use (3)}} \mu_a - \Lambda_{ab}(x_b - \mu_b))}_{\text{use (3)}} = \mu_a - \Sigma_{a|b} \Lambda_{ab} (x_b - \mu_b) \quad (4)$$

We'll use the following matrix equality:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} M & -MBD^{-1} \\ -D^{-1}CM & D^{-1} + D^{-1}CMBD^{-1} \end{bmatrix} \quad (5)$$

Where $M = (A - BD^{-1}C)^{-1}$, (M^{-1} Schur complement)

Let $\begin{bmatrix} \Sigma_a & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_b \end{bmatrix}^{-1} = \begin{bmatrix} \Lambda_a & \Lambda_{ab} \\ \Lambda_{ba} & \Lambda_b \end{bmatrix}$, then

$$\Lambda_a = \overbrace{(\Sigma_a - \Sigma_{ab}\Sigma_b^{-1}\Sigma_{ba})}^{\Sigma_{a|b}}^{-1} \quad (6)$$

$$\Lambda_{ab} = -\Lambda_a \Sigma_{ab} \Sigma_b^{-1} \quad (7)$$

$$\begin{aligned} \Rightarrow \mu_{a|b} &= \mu_a - \Sigma_{a|b} \overbrace{\left(- \underbrace{\Lambda_a}_{\text{use (3)}} \Sigma_{ab} \Sigma_b^{-1} \right)}^{\Lambda_{ab}} (x_b - \mu_b) \\ &= \mu_a + \Sigma_{ab} \Sigma_b^{-1} (x_b - \mu_b) \end{aligned} \quad (8)$$

$$p(x_a|x_b) = \mathcal{N}(x_a; \mu_a + \Sigma_{ab}\Sigma_b^{-1}(x_b - \mu_b), \Sigma_a - \Sigma_{ab}\Sigma_b^{-1}\Sigma_{ba})$$

2 Marginalizing a joint Gaussian PDF

Problem: $p(x_a) = \int p(x_a, x_b) dx_b$

Solution: Same as before, we will expand the exponent and complete the square, now twice

$$\begin{aligned} \int e^{\Delta} dx_b &= e^{\Delta(x_a)} \underbrace{\int e^{\Delta(x_b)} dx_b}_{\eta} * \underbrace{e^{\Delta(const)}}_{\eta'} \\ \Delta &= -\frac{1}{2}(x_a - \mu_a)^T \Lambda_a (x_a - \mu_a) - \frac{1}{2}(x_a - \mu_a)^T \Lambda_{ab} (x_b - \mu_b) \\ &\quad - \frac{1}{2}(x_b - \mu_b)^T \Lambda_{ba} (x_a - \mu_a) - \frac{1}{2}(x_b - \mu_b)^T \Lambda_b (x_b - \mu_b) \\ &= -\frac{1}{2}x_b^T \Lambda_b x_b + x_b^T \underbrace{(\Lambda_b \mu_b - \Lambda_{ba}(x_a - \mu_a))}_{\Delta(x_b)} - \frac{1}{2}m^T \Lambda_b m \\ &\quad + \underbrace{\frac{1}{2}m^T \Lambda_b m - \frac{1}{2}x_a^T \Lambda_a x_a + x_a^T (\Lambda_a \mu_a + \Lambda_{ab} \mu_b)}_{\Delta(x_a)} + const \end{aligned} \quad (9)$$

$$\begin{aligned}
 \frac{1}{2}m^\top \Lambda_b m &= \frac{1}{2}(\mu_b - \Lambda_b^{-1} \Lambda_{ba}(x_a - \mu_a))^\top \Lambda_b (\mu_b - \Lambda_b^{-1} \Lambda_{ba}(x_a - \mu_a)) \\
 &= \frac{1}{2}x_a^\top \Lambda_{ab} \Lambda_b^{-1} \Lambda_b \Lambda_b^{-1} \Lambda_{ba} x_a - x_a^\top \Lambda_{ab} \Lambda_b^{-1} \Lambda_b (\mu_b + \Lambda_b^{-1} \Lambda_{ba} \mu_a) + const \\
 &= \frac{1}{2}x_a^\top \Lambda_{ab} \Lambda_b^{-1} \Lambda_{ba} x_a - x_a^\top \Lambda_{ab} (\mu_b + \Lambda_b^{-1} \Lambda_{ba} \mu_a) + const
 \end{aligned} \tag{10}$$

$$\begin{aligned}
 \Delta(x_a) &= -\frac{1}{2}x_a^\top \Lambda_a x_a + \frac{1}{2}x_a^\top \Lambda_{ab} \Lambda_b^{-1} \Lambda_{ba} x_a + x_a^\top (\Lambda_a \mu_a + \Lambda_{ab} \mu_b) \\
 &\quad - x_a^\top \Lambda_{ab} (\mu_b + \Lambda_b^{-1} \Lambda_{ba} \mu_a) + const \\
 &= -\frac{1}{2}x_a^\top \underbrace{(\Lambda_a - \Lambda_{ab} \Lambda_b^{-1} \Lambda_{ba})}_{\Sigma_a} x_a + x_a^\top (\Lambda_a - \Lambda_{ab} \Lambda_b^{-1} \Lambda_{ba}) \mu_a + const
 \end{aligned} \tag{11}$$

$$p(x_a) = \int p(x_a, x_b) dx_b = \mathcal{N}(x_a; \mu_a, \Sigma_a)$$

- Marginalizing a Gaussian is as simple as selecting the submatrix inside Σ and the corresponding μ !
- Gaussians are their self conjugate priors