

L07: Particle Filter and Monte-Carlo localization

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1 Particle filter (PF)

$$\begin{cases} \overline{bel}(x_t) = \int p(x_t|x_{t-1}, u_t) bel(x_{t-1}) dx_{t-1} \\ bel(x_t) = \eta p(z|x_t) \overline{bel}(x_t) \end{cases} \quad (1)$$

Gaussian filters (Unimodal distributions):

- Kalman Filter Linear system
- Extended KF Non-Linear system
- Unscented KF Non-Linear system (more at extra notes on L06)

Non-parametric : Particle Filter (PF)

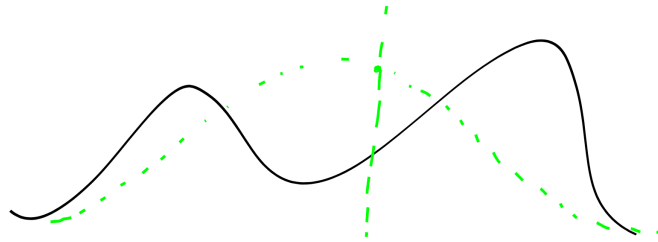


Figure 1: Non-parametric filters do not assume a unimodal distribution, such as KF which always approximates the solution to a Gaussian distribution (green line).

Particle set: $X_t = \{\langle x_t^{[1]}, \omega_t^{[1]} \rangle, \dots, \langle x_t^{[M]}, \omega_t^{[M]} \rangle\}$.

The particle set consists of M particles, each of them is a pair of a state $x^{[m]}$ and a weight $\omega^{[m]}$.



Figure 2: Example of a particle set. On top, samples from a 1D PDF, on the bottom, a 2D Gaussian PDF with few samples drawn and the $1\text{-}\sigma$ iso-contour plotted.

$$\begin{aligned} x^{[m]} &\sim p(x) && \text{Weighted samples. Particles become a good representation of PDFs} \\ \omega^{[m]} &= p_z(x^{[m]}) && (\text{if } M \text{ is large enough}) \end{aligned}$$

Q: What are the weights on sample mean and sample covariance?

1. Particle filter (X_{t-1}, u_t, z_t):
2. for $m = 1 : M$
3. $x_t^{[m]} \sim p(x_t | u_t, x_{t-1}^{[m]})$
4. $\omega_t^{[m]} = p(z_t | x_t^{[m]})$ propagation $\overline{bel}(x_t)$
5. $\overline{X}_t = \overline{X}_t \cup \langle x_t^{[m]}, \omega_t^{[m]} \rangle$ correction
6. $X_t = \text{resampe}^*(\overline{X}_t)$ (better correction) X are "down" from $bel(x_t)$ and not \overline{bel}

[Gordon] reading introduces resampling as a requirement for the PF to work property

2 Bayes filter for full states

$$bel(x_{0:t}) = p(x_{0:t} | u_{1:t}, z_{1:t})$$

particles $x_{0:t}^{[m]} = x_0^{[m]}, x_1^{[m]}, \dots, x_t^{[m]}$ Sequence of samples of states over time.

$$\begin{aligned} bel(x_{0:t}) &= \eta p(z_t | x_{0:t}, z_{1:t-1}, u_{1:t}) p(x_{0:t} | z_{1:t-1}, u_{1:t}) \\ (\text{Markov} + \text{Product rule}) &= \eta p(z_t | x_t) p(x_t | x_{0:t-1}, z_{1:t-1}, u_{1:t}) p(x_{0:t-1} | z_{1:t-1}, u_{1:t}) = \\ &= \eta p(z_t | x_t) p(x_t | x_{t-1}, u_t) \underbrace{p(x_{0:t-1} | z_{1:t-1}, u_{1:t-1})}_{bel(x_{0:t-1})} \end{aligned}$$

We have obtained a new recursive form of the Bayes filter, but now considering the state variable to be a sequence of all estimates at all instants of time, i.e., time $0 : t$.

$$\begin{aligned} \overline{bel}(x_{0:t}) &= p(x_t | x_{t-1}, u_t) bel(x_{0:t-1}) \\ bel(x_{0:t}) &= \eta p(z_t | x_t) \overline{bel}(x_{0:t}) \end{aligned}$$

2.1 Prediction step

From this full state Bayes (no marginalization) given a particle $x_{t-1}^{[m]} \sim bel(x_{0:t-1})$

$$\overline{bel}(x_{0:t}) \begin{cases} \overline{x}_t^{[m]} &\sim p(x_t | x_{t-1}^{[m]}, u_t) \cdot \omega_{t-1}^{[m]} && \text{Sample drawn from the previous sample.} \\ \overline{\omega}_{t-1}^{[m]} &= 1 \cdot \omega_{t-1}^{[m]} && \text{importance factor from } bel. \end{cases} \quad (2)$$

In Fig. 3 is depicted an example of particles propagated (predicted)

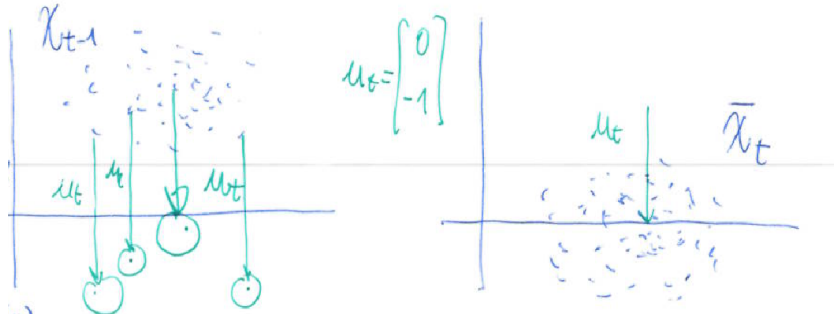


Figure 3: Example of the prediction step. For each particle, a propagation is sampled and this conforms the new particle set \bar{X}_t .

Importance sampling

We will briefly introduce the concept of *importance sampling* before we continue with the PF derivation, since it plays an essential role on it.

$$\begin{aligned} \mathbb{E}_{x \sim p(x)} \{I(x \in A)\} &= \int I(x \in A) p(x) dx = \int I(x \in A) \frac{p(x)}{q(x)} \cdot q(x) dx \\ &= \mathbb{E}_{x \sim q} \{I(x \in A) \cdot \omega(x)\}, \end{aligned} \quad (3)$$

where $\omega(x) = \frac{p(x)}{q(x)}$ is the Importance factor, $p(x)$ is the target distribution, which we usually can't use directly and $q(x)$ is the proposal distribution, more accessible and ready to use. The function $I()$ in this context is the indicator function.

Example: Probability of sample a 1d r.v X in the interval $[15, 17]$ if

$$p(x) = \mathcal{N}(0, 1) \quad A = \{x : 15 \leq x \leq 17\}$$

$$1) \quad p(x \in A) = \sum I(x^m \in A) p(x^m), \quad x^m \sim p(x^m)$$

$$2) \quad \text{Importance Sampling: } p(x \in A) = \sum I(x^m \in A) \underbrace{\frac{p(x^m)}{q(x^m)}}_{\omega^m} \cdot q(x^m), \quad x^m \sim q(x)$$

for instance $q(x) = \mathcal{N}(16, 1)$ (proposal distribution)

With this proposal distribution we don't need trillions of samples but only hundreds.

$$\omega^m = \frac{\mathcal{N}(x^m; 0, 1)}{\mathcal{N}(x^m; 16, 1)} \quad (4)$$

2.2 Correction step

$$\underbrace{bel(x_{0:t})}_{\text{target distribution}} = \eta p(z_t | x_t) \cdot \underbrace{\bar{bel}(x_{0:t})}_{\text{proposal distribution}}, \quad (5)$$

where \bar{X}_t is the particle set representing the belief PDF $\bar{bel}(x_{0:t})$, our proposal distribution.

In order to correctly characterize the posterior $bel(x_{0:t})$ we are going to weight the particles already drawn \bar{X}_t with proper weights ω (importance factors).

$$x_t^{[m]} = \bar{x}_t^{[m]} \quad (\text{previously drawn}) \text{ from proposal distribution.}$$

$$\omega_t^{[m]} = \frac{\text{target distribution}}{\text{proposal distribution}} = \frac{\eta p(z_t | x_t^{[m]}) \cdot \overline{bel}(x_t^{[m]})}{\overline{bel}(x_t^{[m]})} = \eta p(z_t | x_t^{[m]}) \quad (6)$$

Example: Correction step applied to the particle set \bar{X}_t

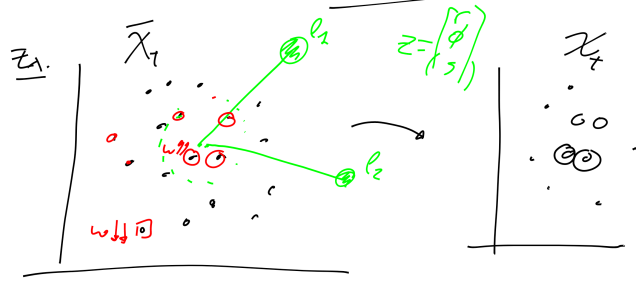


Figure 4: Example of correction step for a particle set.

Problem: creates an almost empty set of particles with weights non-zero and many particles with low weights \Rightarrow Degenerating over time.

3 PF Resampling

Resampling is the solution to the degeneracy occurring when propagating and correcting multiple times a particle set.

Idea: “survival of the fittest”. Only the most likely particles ($\omega^m \uparrow$) ‘might’ survive

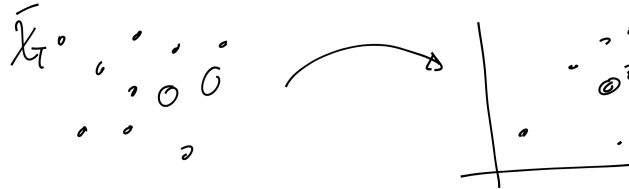


Figure 5: Resampling (the solution). Now, resampling guarantees that the highest values of ω^m are more likely to *survive* but it also give chances to the particles with small values of importance factors to represent the next particle set. From M samples on \bar{X}_t we get M samples on X_t (closest to the $bel(x_0)$)

3.1 Independent Resampling. First solution

We create a cumulative distribution function:

$$c_m = c_{m-1} + \omega^{[m]} \quad (\text{normalization should be considered}) \quad (7)$$

for $m = 1 : M$

$$a \sim U[0; 1] \quad (\text{uniform distribution})$$

$$j = \text{find}(c_m, a)$$

$$X_t = X_t \cup \langle x_t^{[j]}, \omega_t^{[j]} \rangle$$

Problem: over time, independent resampling induces a loss of diversity in the particle population X_t .

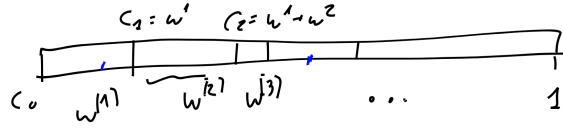


Figure 6: Independent Resampling.

3.2 Low-variance sampling

We create a similar distribution as in the independent sampling algorithm:

$$c_m = c_{m-1} + \omega^{[m]}.$$

The difference is that we no longer sample from this discrete distribution. Only an initial random configuration r is sampled, and then we add particles at intervals $1/M$ over the full set.

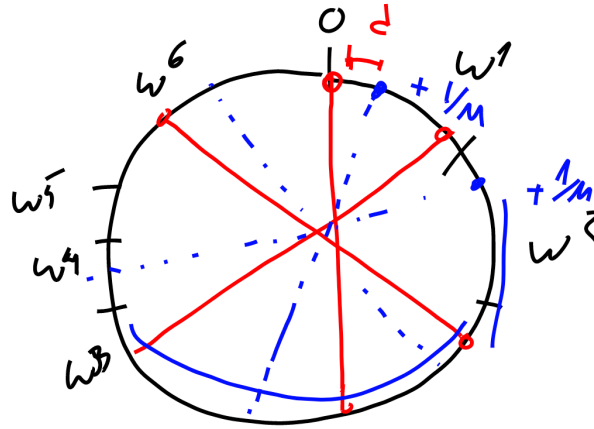


Figure 7: Low-variance sampling scheme. We select particles as equally spaced intervals.

Algorithm: low-variance sampling (\bar{X}_t): (ProbRob 110)

$X_t = \phi$, $c = \omega_t^{[1]}$, $i = 1$;

$r \sim U(0, 1/M)$;

for $m = 1:M$ **do**

$a = r + (m - 1) \cdot \frac{1}{M}$

while $a > c$ **do**

$i++$;

$c = c + \bar{\omega}^{[i]}$;

end

$X_t = X_t \cup \langle \bar{x}_t^{[i]}, \frac{1}{M} \rangle$;

end

return X_t

4 Monte-Carlo localization (MCL)

Reading: Dellaert'99.

We want to solve the Markov localization (L06) using PF.

$$bel(x_t) = p(x_t|U, Z, m) \rightarrow X_t, \quad (8)$$

where X_t is the particle set representing the posterior belief and m is the map of landmarks.

Algorithm: MCL (X_{t-1}, u_t, z_t, m) :

```
 $\bar{X}_t = X_t = \phi;$   
for  $m = 1:M$  do  
   $x_t^{[m]} = \text{sample\_motion\_model}(u_t, x_{t-1}^{[m]})$  (Section 2.1 and L05)  
   $\omega_t^{[m]} = \text{measurement\_model}(z_t, x_t^{[m]}, m)$  (Section 2.2 and L06)  
   $\bar{X}_t = \bar{X}_t \cup \langle x_t^{[i]}, \omega_t^{[m]} \rangle$   
end  
 $X_t = \text{low\_variance\_sampling}(\bar{X}_t)$ 
```

5 Summary

- Particle filter is a version of the Bayes filter for full sequences $x_{0:t}$.
- Prediction step $\bar{x}_t^{[m]} \sim p(x_t | x_{t-1}^{[m]}, u_t) \cdot \omega_{t-1}^{[m]}$.
- Correction step $\omega_t^{[m]} = \eta p(z_t | x_t^{[m]})$.
- Resampling regenerates the particle set.