CAIT
Mobile Robotics Lab
Perception in Robotics Course

# L06: EKF and Localization

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## 1 Extended Kalman filter

### Kalman filter: Linear system plus Gaussian prior

$$\left.\begin{aligned} \bar{\mu}_t &= A_t \mu_t + B_t u_t \\ \bar{\Sigma}_t &= A_t \Sigma_{t-1} A_t^T + R_t \end{aligned}\right\} \text{prediction (marginalize)}$$

$$\left.\begin{aligned} K_t &= \bar{\Sigma}_t C_t^T \left( C_t \bar{\Sigma}_t C_t^T + Q \right)^{-1} \\ \mu_t &= \bar{\mu}_t + K_t \left( z_t - C_t \bar{\mu}_t \right) \\ \Sigma_t &= \left( I - K_t C_t \right) \bar{\Sigma}_t \end{aligned}\right\} \text{correction (conditioning)}$$

### Motion model: first order Taylor expansion

$$x_t = g(x_{t-1}, u_t, \epsilon_t) \approx g(\mu_{t-1}, u_t, 0) + \frac{\partial g}{\partial x_{t-1}}\Big|_{\mu_{t-1}} (x_{t-1} - \mu_{t-1}) + \epsilon_t$$

In L05 discussed on how to model $g(\cdot)$ for different systems and how to obtain the probabilistic model.

### Sensor model

We observe features of landmarks (L05):

$$z_t = h(x_t, \eta_t) \approx h(\mu_t, 0) + \frac{\partial h}{\partial x_t}\Big|_{\mu_t} (x_t - \mu_t) + \eta_t$$

### Intuition on linearization

Linearizing assumes **errors**!

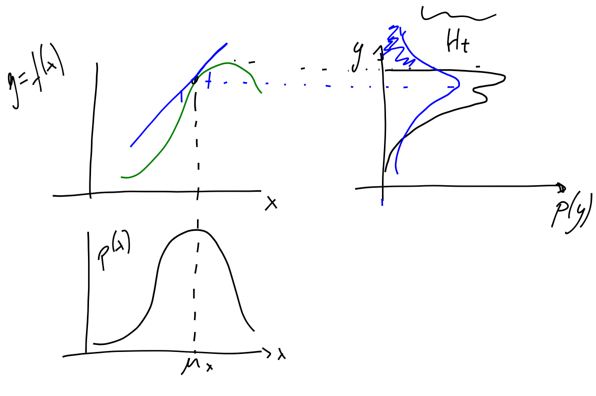


Figure 1: Linearization intuition. The random variable $x \sim p(x)$ is transformed by the function $y = f(x)$. The true distribution of $y \sim p(y)$ would not be Gaussian (black line in the top right), but after linearizing, the PDF is approximated as Gaussian.

### Equations of the Extended Kalman Filter

Inputs: $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$

1. $\bar{\mu}_t = g(\mu_{t-1}, u_t)$
2. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$
3. $K_t = \bar{\Sigma}_t H_t^T \left( H_t \bar{\Sigma}_t H_t^T + Q_t \right)^{-1}$
4. $\mu_t = \bar{\mu}_t + K_t \underbrace{(z_t - h(\bar{\mu}_t))}_{\Delta z}$ where $\Delta z$ is the innovation vector.
5. $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$
   return $\mu_t, \Sigma_t$ (actually $\mathcal{N}(\mu_t, \Sigma_t)$)

Properties:

- EKF is very efficient $O(k^{2.4} + n^2)$
- Not optimal, but in practice works well (depends on the non-linearities, some are more problematic)

Compact initial distribution reduces the error because we are "near" the linearization point $(O(||\Delta x||))$

## 2 Localization

Markov localization directly uses Bayes filter (see Figure 2):

$$\bar{bel}(x_t) = \int p(x_t | u_t, x_{t-1}, m)\ bel(x_{t-1})\ dx_{t-1}$$
$$bel(x_t) = \eta\ p(z_t | x_t, m)\ \bar{bel}(x_t)$$

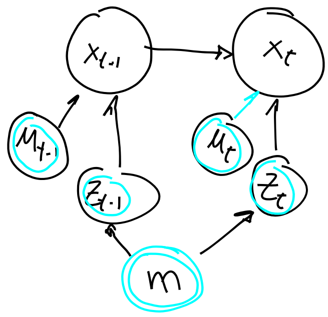

Figure 2: Bayes network corresponding to the localization problem. Observable variables are coloed in cyan and estimated variables $x$'s in black.

**Example**: 1D "3 doors" problem. On Figure 3 on the 1-st plot any of the three doors could have been detected. On the 4-th plot only propagation occurs. On the 5-th plot again a door is detected, but given the previous *bel* the robot is better localizing as it is shown on the 6-th plot.

### Localization problems (taxonomy)

- Local (position tracking) [$x_0$ given] vs Global [$x_0$ unkown, kidnapped problem]
- Static vs Dynamic [moving furniture, doors, snow...]
- Passive vs Active [exploration, belief planning]
- Single-robot vs Multi-robot

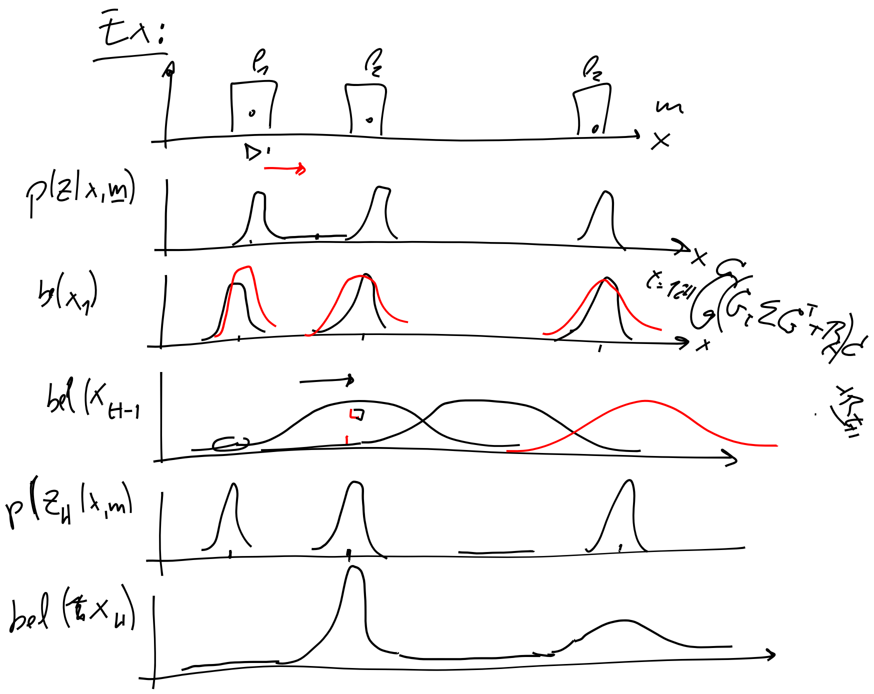

Figure 3: “3 doors” problem. In the top figure, any of the three doors could have been detected, but there is still ambiguiity in the solution, as denoted in $\text{bel}(x_1)$.

## 3 EKF localization

Gaussians are unimodal distribution and we have used 3 modes on the 3 doors problem. We need to solve the data association problem landmark-observation.

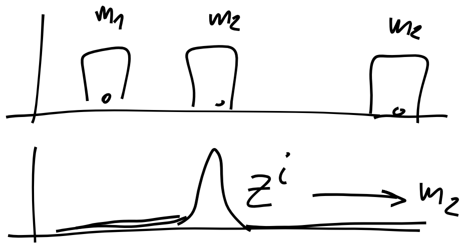

Figure 4: Unimodality of Gaussian in ”3 doors” problem

We will assume known correspondences: $c_i = j$ (from landmark $m_j$). For the case of 3 observations of each landmark:

$$p(z|x, m, c) = \prod_{j=1}^{3} p(z_j x, m, c_j)$$

### Algorithm: EKF localization with known correspondences

Inputs: $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, c_t, m$ (ProbRob 204)

1. $G_t = \frac{\partial g(x_{t-1}, u_t)}{\partial x_{t-1}}$, $V_t = \frac{\partial g(x_{t-1}, u_t)}{\partial u_t}$, $M_t^{arc} = \begin{bmatrix} \alpha_1 v_t^2 + \alpha_2 \omega_t^2 & 0 \\ 0 & \alpha_3 v_t^2 + \alpha_4 \omega_t^2 \end{bmatrix}$ ($M_t^{arc}$ is for arc circular model)

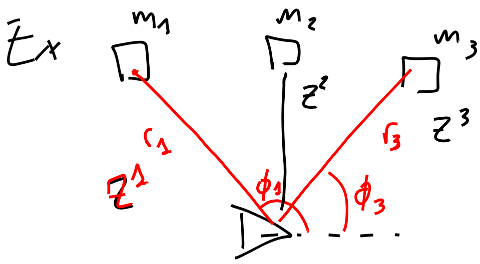


Figure 5: Data association. Now each door has a singature and we can distinguish them.

2. $\bar{\mu}_t = g(\mu_{t-1}, u_t)$
3. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + V_t M_t V_t^T = G_t \Sigma_{t-1} G_t^T + R_t$
4. $Q_t = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\phi^2 \end{bmatrix}$. Range and bearing observation noise. Known correspondences $\implies$ $\sigma_s^2 = 0$ (eliminate)
5. for $\{i : z_t^i = [r_t^i, \phi_t^i]^T\}$:
6. $\hat{z}_t^i = \begin{bmatrix} \sqrt{(m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2} \\ atan2(m_{j,y} - \bar{\mu}_{t,y}, m_{j,x} - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{bmatrix}$
7. $H_t^i = \frac{\partial h(x_t)}{\partial x_t}\Big|_{\bar{\mu}_t} = \begin{bmatrix} \frac{-(m_{j,x}-\bar{\mu}_{t,x})}{\sqrt{(m_{j,x}-\bar{\mu}_{t,x})^2+(m_{j,y}-\bar{\mu}_{t,y})^2}} & \frac{-(m_{j,y}-\bar{\mu}_{t,y})}{\sqrt{(m_{j,x}-\bar{\mu}_{t,x})^2+(m_{j,y}-\bar{\mu}_{t,y})^2}} & 0 \\ \frac{m_{j,y}-\bar{\mu}_{t,y}}{(m_{j,x}-\bar{\mu}_{t,x})^2+(m_{j,y}-\bar{\mu}_{t,y})^2} & \frac{-(m_{j,x}-\bar{\mu}_{t,x})}{(m_{j,x}-\bar{\mu}_{t,x})^2+(m_{j,y}-\bar{\mu}_{t,y})^2} & -1 \end{bmatrix}$
8. $S_t^i = H_t^i \bar{\Sigma}_t (H_t^i)^T + Q_t$
9. $K_t^i = \bar{\Sigma}_t (H_t^i)^T (S_t^i)^{-1}$
10. $\bar{\mu}_t = \bar{\mu}_t + K_t^i \left(z_t^i - \hat{z}_t^i\right)$ (innovation vector for $z_t^i$)
11. $\bar{\Sigma}_t = \left(I - K_t^i H_t^i\right) \bar{\Sigma}_t$
12. endfor
13. $\mu_t = \bar{\mu}_t \quad (\mu_t = \bar{\mu}_t + \sum_i K_t^i \left(z_t^i - \hat{z}_t^i\right))$
14. $\Sigma_t = \bar{\Sigma}_t$
15. return $\mu_t, \Sigma_t$

## Iterated EKF

Why are we updating the prediction belief $\bar{bel}(x_t)$ $I$ times?

If we assume that $\{z_t^i\}_{i=1}^I$ are independent:

$$\begin{aligned} bel(x_t) =& p(z|x,m,c) \cdot \bar{bel}(x_t) \\ =& \prod^I p(z^i|x,m,c) \cdot \bar{bel}(x_t) \\ =& p(z^1|x,m,c) \cdot \\ & p(z^2|x,m,c) \cdot \\ & \dots \\ & \underbrace{p(z^I|x,m,c) \cdot \bar{bel}(x_t)}_{\text{conditioning}} \end{aligned}$$

where we are iteratively conditioning a distribution, starting from $p(z^I|x,m,c) \cdot \bar{bel}(x_t)$ up to the first observation.

# 4 Summary

- Extended Kalman Filter and linearization errors.
- Localization $\rightarrow$ EKF localization as a uni-modal solution.
- Map of landmarks:

$$m = \left\{ \begin{bmatrix} m_{1,x} \\ m_{1,y} \end{bmatrix} \begin{bmatrix} m_{2,x} \\ m_{2,y} \end{bmatrix} \dots \begin{bmatrix} m_{i,x} \\ m_{j,y} \end{bmatrix} \dots \right\} \text{ map of known landmark} \tag{1}$$

  The localization problem becomes a state estimation problem $\Rightarrow$ EKF,UKF (in additional notes)

$$bel(x_t) = p(x_t|U, Z, m) \tag{2}$$

  Assume (for now) $c_t^i = j$ $(z^i \rightarrow m_j)$ known correspondences