

## L01: the Expectation Operator

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31 January 2022

### 1 Probability: A refresh on some useful definitions

The random variables  $x$  and  $y$  are called independent if:

$$p(x, y) = p(x)p(y).$$

**Note:** In this course  $p(x)$  indicates probability density function (PDF). For a review on random variables and PDFs, check out the probability review notes.

Product rule

$$p(x, y) = p(x|y)p(y)$$

Total probability

$$p(x) = \int p(x, y)dy \quad (\text{marginalization})$$

$$p(x|z) = \int p(x, y|z)dy$$

Bayes Theorem

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\int p(x, y)dx}$$

Def: Expectation Operator

$$\mathbb{E}\{x\} = \int_{-\infty}^{+\infty} x \cdot p(x)dx = \mu_x$$

### 2 Properties

$\mathbb{E}\{\cdot\}$  is linear

- $\mathbb{E}\{A\} = A$
- $\mathbb{E}\{Ax\} = A\mathbb{E}\{x\}$
- $\mathbb{E}\{A + x\} = A + \mathbb{E}\{x\}$
- $\mathbb{E}\{x + y\} = \mathbb{E}\{x\} + \mathbb{E}\{y\}$ :

$$\begin{aligned}\mathbb{E}\{x + y\} &= \int \int (x + y)p(x, y)dx dy = \\ &= \int \int xp(x, y)dx dy + \int \int yp(x, y)dx dy = \\ &= \int x \left( \int p(x, y)dy \right) dx + \int y \left( \int p(x, y)dx \right) dy = \\ &= \int xp(x)dx + \int yp(y)dy = \mathbb{E}\{x\} + \mathbb{E}\{y\}\end{aligned}$$

## 3 Anti-properties

- $\mathbb{E}\{x, y\} \neq \mathbb{E}\{x\}\mathbb{E}\{y\}$  (in general)
- If  $x, y$  uncorrelated ( $\sigma_{xy} = 0$ )  $\Rightarrow \mathbb{E}\{x, y\} = \mathbb{E}\{x\}\mathbb{E}\{y\}$
- If  $x, y$  independent  $\Rightarrow x, y$  uncorrelated.

## 4 Expectation of multi-dimensional r.v.

$$\mathbb{E}\left\{\begin{bmatrix} x \\ y \end{bmatrix}\right\} = \begin{bmatrix} \mathbb{E}\{x\} \\ \mathbb{E}\{y\} \end{bmatrix}$$

## 5 Conditional expectation

$$\mathbb{E}\{x|y\} \triangleq \int_{-\infty}^{+\infty} x \cdot p(x|y) dx$$

## 6 Covariance. Scalar form

Autocovariance or covariance

$$\sigma_{xx}^2 = cov(x, x) = \mathbb{E}\{(x - \mathbb{E}\{x\})^2\} = \mathbb{E}\{x^2\} - (\mathbb{E}\{x\})^2$$

Cross-covariance

$$\sigma_{xy}^2 = cov(x, y) = \mathbb{E}\{(x - \mathbb{E}\{x\})(y - \mathbb{E}\{y\})\}$$

## 7 Covariance. Vectorial form

Covariance

$$\Sigma_x = cov(x, x) = cov(x) = \mathbb{E}\{(x - \mathbb{E}\{x\})(x - \mathbb{E}\{x\})^\top\}$$



Cross-covariance

$$\Sigma_{xy} = cov(x, y) = \mathbb{E}\{(x - \mathbb{E}\{x\})(y - \mathbb{E}\{y\})^\top\}$$

Example: Expand  $\Sigma_{xy}$ .

$$\begin{aligned} \Sigma_{xy} &= \mathbb{E}\{(x - \mathbb{E}\{x\})(y - \mathbb{E}\{y\})^\top\} \\ &= \mathbb{E}\{xy^\top + x(-\mathbb{E}\{y\})^\top - \mathbb{E}\{x\}y^\top + \mathbb{E}\{x\}\mathbb{E}\{y\}^\top\} = \\ &= \mathbb{E}\{xy^\top\} - \mathbb{E}\{x\mathbb{E}\{y\}^\top\} - \mathbb{E}\{\mathbb{E}\{x\}y^\top\} + \mathbb{E}\{\mathbb{E}\{x\}\mathbb{E}\{y\}^\top\} = \\ &= \mathbb{E}\{xy^\top\} - \mathbb{E}\{x\}\mathbb{E}\{y\}^\top - \mathbb{E}\{x\}\mathbb{E}\{y^\top\} + \mathbb{E}\{x\}\mathbb{E}\{y^\top\} = \\ &= \mathbb{E}\{xy^\top\} - \mathbb{E}\{x\}\mathbb{E}\{y^\top\} = \Sigma_{xy} \end{aligned}$$

**Col 1:**  $\Sigma_x = \Sigma_x^\top$  (is symmetric. But  $\Sigma_{xy}$  is not symmetric)

$$\Sigma_x = \mathbb{E}\{xx^\top\} = \mathbb{E}\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\right\} = \mathbb{E}\left\{\begin{bmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2x_2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3x_3 \end{bmatrix}\right\}$$

**Col 2:**  $\Sigma_x$  is Positive (Semi)definite or PSD:  $v^\top \Sigma_x v \geq 0 \forall v$

## 8 Sample mean and sample covariance

$x_i \sim p(x)$  iid

Sample mean

$$\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i$$

Sample covariance (unbiased estimate)

$$\bar{\Sigma}_x = \frac{1}{\underbrace{N-1}} \sum_{i=1}^N (x_i - \bar{x})(x_i - \bar{x})^\top$$