Confidence interval for user scores

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1 Introduction

There are many websites that allow users to rate certain items, like songs and products. Usually, each item gets a user score, an average of all the ratings. But when it comes to ranking a set of items from best to worst, how should it be done? We will investigate some of the ways that this is usually done, and present another way that involves exact confidence intervals of the user score.

2 Confidence interval for μ using the eCDF

Suppose we have a set of i.i.d. user ratings $x_1, x_2, \ldots, x_n$ that come from some unknown probability distribution $\mathcal{D}$ with unknown mean μ, unknown cumulative distribution function F, and **known** support $[a, b]$. Consider the empirical cumulative distribution function $\hat{F}_n$ given by

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(x_i \leq x), \tag{1}$$

where

$$\mathbf{1}(x_i \leq x) = \begin{cases} 1 & \text{if } x_i \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

The value $\hat{F}_n(x)$ is the proportion of samples that are less than x, and the value $n\hat{F}_n(x)$ is the number of samples that are less than x. Then

$$n\hat{F}_n(x) \sim \text{Binomial}(n, F(x)). \tag{2}$$

To see why (2) is true, use the fact that $\mathbf{1}(x_i \leq x)$ for $i = 1, 2, \ldots, n$ equals 1 if $x_i \leq x$ and 0 otherwise. Since each $x_i \sim \mathcal{D}$, the probability that $x_i \leq x$ is $F(x)$, so $\mathbf{1}(x_i \leq x)$ follows a Bernoulli distribution with probability of success $F(x)$. This means that $n\hat{F}_n(x) = \sum_{i=1}^{n} \mathbf{1}(x_i \leq x)$ is the sum of n i.i.d. Bernoulli random variables with success probability $F(x)$, which implies that $n\hat{F}_n(x) \sim \text{Binomial}(n, F(x))$.

We would like to find upper and lower bounds $U(x)$ and $L(x)$ such that

$$\mathbb{P}\left(\text{Binomial}(n, U(x)) \leq n\hat{F}_n(x)\right) = \frac{\alpha}{2}, \tag{3}$$

$$\mathbb{P}\left(\text{Binomial}(n, L(x)) \geq n\hat{F}_n(x)\right) = \frac{\alpha}{2} \tag{4}$$

for $x = 0, 1, \ldots, n$. These bounds form what is called a **Clopper-Pearson confidence interval** of $F(x)$, and they can be used to form what is called a **pointwise confidence band** of F.

The CDF of a binomial distribution with n trials and success probability p is

$$k \mapsto I_{1-p}(n-k, 1+k),$$

where I is the regularized incomplete beta function given by

$$I_x(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t$$

and Γ is the Gamma function given by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\mathrm{d}t.$$

We can rewrite (3) and (4) in terms of I to get

$$I_{1-U(x)}\left(n - n\hat{F}_n(x), 1 + n\hat{F}_n(x)\right) = \frac{\alpha}{2},$$

$$1 - I_{1-L(x)}\left(n - n\hat{F}_n(x) + 1, n\hat{F}_n(x)\right) = \frac{\alpha}{2}.$$

Using the fact that $I_{1-x}(q,r) = 1 - I_x(r,q)$ for all $q, r \in \mathbb{R}$, we get

$$1 - I_{U(x)}\left(1 + n\hat{F}_n(x), n - n\hat{F}_n(x)\right) = \frac{\alpha}{2},$$
$$I_{L(x)}\left(n\hat{F}_n(x), n - n\hat{F}_n(x) + 1\right) = \frac{\alpha}{2}.$$

Simplifying,

$$I_{U(x)}\left(1 + n\hat{F}_n(x), n\left(1 - \hat{F}_n(x)\right)\right) = 1 - \frac{\alpha}{2},$$
$$I_{L(x)}\left(n\hat{F}_n(x), n\left(1 - \hat{F}_n(x)\right) + 1\right) = \frac{\alpha}{2}.$$

We can then use the inverse regularized incomplete beta function I^{-1} to solve for $U(x)$ and $L(x)$:

$$U(x) = I^{-1}_{1-\frac{\alpha}{2}}\left(1 + n\hat{F}_n(x), n\left(1 - \hat{F}_n(x)\right)\right),$$
$$L(x) = I^{-1}_{\frac{\alpha}{2}}\left(n\hat{F}_n(x), n\left(1 - \hat{F}_n(x)\right) + 1\right).$$

It should be noted that $U(x)$ is undefined for $x \geq x_{(n)}$ and $L(x)$ is undefined for $x < x_{(1)}$. We can get around this by noting that

$$\lim_{r \to 0} I^{-1}_p(q,r) = 1,$$
$$\lim_{q \to 0} I^{-1}_p(q,r) = 0$$

for any $p \in [0,1]$ and $q, r > 1$. So we can redefine U as

$$U(x) := \begin{cases} I^{-1}_{1-\frac{\alpha}{2}}\left(1 + n\hat{F}_n(x), n\left(1 - \hat{F}_n(x)\right)\right) & \text{if } x < x_{(n)}, \\ 1 & \text{otherwise.} \end{cases} \tag{5}$$

and redefine L as

$$L(x) := \begin{cases} I^{-1}_{\frac{\alpha}{2}}\left(n\hat{F}_n(x), n\left(1 - \hat{F}_n(x)\right) + 1\right) & \text{if } x \geq x_{(1)}, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

We can now finally derive a confidence interval of μ. First, note that μ can be expressed in terms of the cumulative distribution function F using the identity

$$\mu = b - \int_a^b F(x)\mathrm{d}x.$$

Let μ_L and μ_U be the lower and upper confidence bounds of μ, respectively. If we consider all possible distribution functions on $[a,b]$ that are between L and U, then the CDF that minimizes the mean is U and the CDF that maximizes the mean is L. Therefore, the upper limit of a $100(1-\alpha)\%$ confidence interval of μ is

$$\mu_U = b - \int_a^b L(x)\mathrm{d}x,$$

and the lower limit is

$$\mu_L = b - \int_a^b U(x)\mathrm{d}x.$$

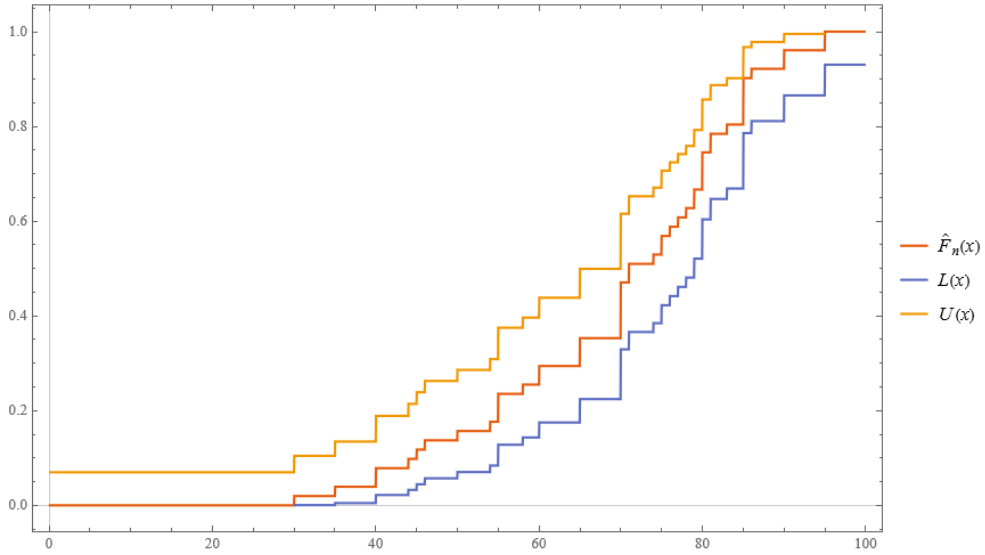
Using properties of U and L,

$$\begin{aligned} \mu_U &= b - \int_a^b L(x)\mathrm{d}x \\ &= b - \left(\int_a^{x_{(1)}} 0\mathrm{d}x + \int_{x_{(1)}}^b L(x)\mathrm{d}x\right) \\ &= b - \int_{x_{(1)}}^b L(x)\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} \mu_L &= b - \int_a^b U(x)\mathrm{d}x \\ &= b - \left(\int_a^{x_{(n)}} U(x)\mathrm{d}x + \int_{x_{(n)}}^b 1\mathrm{d}x\right) \\ &= x_{(n)} - \int_a^{x_{(n)}} U(x)\mathrm{d}x. \end{aligned}$$

For ease of computation, we can replace the integral with a sum by realizing that L and U are both step functions that jump at every unique value of x_i for $i = 1, 2, \ldots, n$:



The eCDF $\hat{F}_n(x)$ along with its pointwise 95% confidence bands for some set of data ($n = 51$) with support $\{0, 1, \ldots, 100\}$.

Let $\tilde{x}_{(1)}, \tilde{x}_{(2)}, \ldots, \tilde{x}_{(m)}$ be the *distinct* order statistics of $x_1, x_2, \ldots, x_n$, with $\tilde{x}_{(1)} = x_{(1)}, \tilde{x}_{(m)} = x_{(n)}$, and $m \leq n$. Then since L and U only jump at each $\tilde{x}_{(i)}$ for $i = 1, 2, \ldots, m$, we can get that

$$\mu_L = \int_{x_{(1)}}^{b} L(x)\mathrm{d}x = \left(b - x_{(n)}\right) L(x_{(n)}) + \sum_{i=1}^{m-1} \left(\tilde{x}_{(i+1)} - \tilde{x}_{(i)}\right) L(\tilde{x}_{(i)})$$

and

$$\mu_U = \int_{a}^{x_{(n)}} U(x)\mathrm{d}x = \left(x_{(1)} - a\right) U(a) + \sum_{i=1}^{m-1} \left(\tilde{x}_{(i+1)} - \tilde{x}_{(i)}\right) U(\tilde{x}_{(i)}).$$

With these formulas, we finally get that the $100(1-\alpha)\%$ confidence interval of μ is

$$[\mu_L, \mu_U]$$

where

$$\mu_L = x_{(n)} - \left(x_{(1)} - a\right) U(a) - \sum_{i=1}^{m-1} \left(\tilde{x}_{(i+1)} - \tilde{x}_{(i)}\right) U(\tilde{x}_{(i)}),$$
$$\mu_U = b - \left(b - x_{(n)}\right) L(x_{(n)}) - \sum_{i=1}^{m-1} \left(\tilde{x}_{(i+1)} - \tilde{x}_{(i)}\right) L(\tilde{x}_{(i)}).$$

3 Code

Here's a program in Rust that outputs the $100(1-\alpha)\%$ confidence interval of the mean for a dataset with support $[a, b]$.

Install git and run the command `git clone https://github.com/Fr0stium/mean_ci.git` to download the program from GitHub.

To run the program, enter the command `cargo run -- path alpha min_support max_support` inside the program's directory, where `path` is the path of a text file containing a set of numbers separated by commas, `alpha` is α, `min_support` is a, and `max_support` is b. Note that the path to the text file should not contain any spaces.

An example of a valid text file generated from a distribution with support $\{0, 0.5, \ldots, 5\}$ is

```
4.5,2.5,4,3.5,4.5,5,4.5,3.5,4.5,4,3,5,4.5,4,5,3,4,4,3,4,3,3,3.5,4,4,4,4,4.5,5,3.5,4,4.5,4,3.5,3,1,4.5
```

With this file, the program outputs the following:

```
Number of ratings: 37
Mean: 3.864864864864865
90% confidence interval: (3.3494896400115475, 4.177131394268508)
```